Math 54 Section Worksheet 12

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1 Virtually Together

Let's first start with abstract vector spaces... let's find the eigenvalues and corresponding eigenspaces of the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by $T(p(t)) = t\frac{\partial}{\partial t}p(t)$.

Now let's get to the complex stuff... let's find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}$.

2 Problems

1. Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.
   (a) Find the eigenvalues and eigenvectors of A. Is A diagonalizable?
   (b) Find the eigenvalues and eigenvectors of B. Is B diagonalizable?
   (c) (Tricky) Find a matrix P such that $A = PBP^{-1}$.
2. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Find the eigenvalues of A and for each eigenvalue, find a basis for the corresponding eigenspace. Is A diagonalizable?

3. Let $A = \begin{pmatrix} 1 & 1 & -5 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}$. Find the eigenvalues of A. For each eigenvalue, find a basis for the corresponding eigenspace. Is A diagonalizable?

4. All matrices below are $n \times n$. True or False?

   (a) If A is similar to B, and both are invertible, then A^{-1} is similar to B^{-1}.

   (b) If A and B have the same eigenvalues, then A is similar to B.

   (c) If A and B have the same characteristic polynomial, then A is similar to B.

   (d) If A and B are both diagonalizable, then A is similar to B.

   (e) If A and B are both diagonalizable and have the same eigenvalues, then A is similar to B.

   (f) If A and B are both diagonalizable and have the same characteristic polynomial, then A is similar to B.

   (g) If all the eigenvalues of A are 0, then A is the 0 matrix.

   (h) If A is a real matrix with eigenvalue $a + bi$, then $a - bi$ is also an eigenvalue of A.

   (i) If A is a real matrix with eigenvector $\mathbf{u} + i\mathbf{v}$, then $\mathbf{u} - i\mathbf{v}$ is also an eigenvector of A.

   (j) The matrix A always has n complex eigenvalues, counting multiplicity.

   (k) If A is diagonalizable, then A^2 is diagonalizable.

   (l) If A^2 is diagonalizable, then A is diagonalizable.

3 Extra (possibly difficult) Problems

5. Prove that any matrix of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ where $a, b \in \mathbb{R}$ with $b \neq 0$, is not diagonalizable.

6. Let V be the vector space $\text{Span}\{\cos t, \sin t\}$. Let $D : V \to V$ be the function $D(f) = \frac{\partial}{\partial t} f$.

   (a) Write down a matrix that represents D.

   (b) Can you find an alternate basis $\mathcal{E}$ so that when written in this basis, D is a diagonal matrix?

   Hint: You might find the following identities useful:

   $$e^{it} = \cos t + i \sin t \qquad e^{-it} = \cos t - i \sin t$$

7. Diagonalize $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.