Math 54 Section Worksheet 11 Solutions

GSI: Jeremy Meza
Office Hours: Mon 3:30-5:30pm, Zoom ID: 7621822286
Monday, March 16, 2020

1 Together

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(e_1) = (2, -2)^T$ and $T(e_2) = (-1, 3)^T$. In the standard basis, T is given by the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$$

Let's find the eigenvalues and eigenvectors of A. We want to find all scalars λ such that the linear system $Ax = \lambda x$ has a nontrivial solution. Equivalently, we are looking for all scalars λ such that...

(Circle ONE) (a) The matrix $(A - \lambda I)$ is invertible. (b) The matrix $(A - \lambda I)$ is not invertible.

(c) The matrix A is invertible. (d) The matrix A is not invertible.

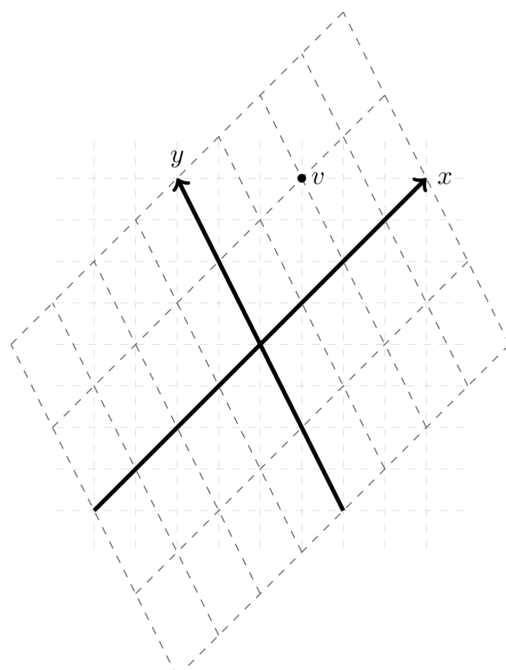
An equivalent condition to a matrix B being invertible is....

(Circle ONE) (a) B has no zero rows (b) $\operatorname{Col} B = \{0\}$ (c) $\det B \neq 0$ (d) 0 is a solution to $Bx = 0$.

After some work, we find that the eigenvalues of A are $\lambda = 1, 4$. Plugging these numbers back into our linear system $Ax = \lambda x$, we can solve for our eigenvectors...

After some more work, we find that the eigenvectors of A are $(1, 1)^T$ and $(-1, 2)^T$ for the eigenvalues 1, 4, respectively.

To the right is the Cartesian grid, drawn not with the usual x, y axes, but instead with our eigenvectors as the axes. How strange! A vector v is drawn on the grid in our basis of eigenvectors as the point $(2, 1)$. Let's plot the point $T(v)$.



We see that in the basis of eigenvectors, T is given by the matrix:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

In the standard basis, v is represented as the vector $(1, 4)^T$. Let's check that our answer for $T(v)$ is correct by calculating in the standard basis:

$$T(v) = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$$

Calculating $T(v)$ in the standard basis was so much more work than calculating $T(v)$ in our new basis of eigenvectors. Just imagine how much harder it would be in the standard basis if we had $n \times n$ matrices, as opposed to doing the calculation with an $n \times n$ diagonal matrix!

This is what eigenvalues and eigenvectors do for us. They give us a natural coordinate system to perform all our calculations in, and they make calculations as easy as possible by only having to multiply vectors by a diagonal matrix.

2 Problems

1. Define the matrix $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$. A has eigenvalues $\lambda = 2, 8$. Diagonalize A or explain why A is not diagonalizable.

   Just because A is 3×3 and there are only 2 distinct eigenvalues does NOT mean we can conclude A is not diagonalizable. We must check whether there is a basis of eigenvectors. As such, we find the eigenvectors:

   $\underline{\lambda = 2}$.
   $A - 2I = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$, which has null space = Span $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

   $\underline{\lambda = 8}$.
   $A - 8I = \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}$, which has null space = Span $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

   Since the geometric multiplicities add up to 3 (which is to say that there are 3 linearly independent eigenvectors of A), then we conclude that A is diagonalizable.

2. Suppose all matrices below are $n \times n$ matrices, unless otherwise specified. True or False?

   F (a) A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.
   T (b) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.
   F (c) A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.
   F (d) If A is diagonalizable, then A is invertible.
   F (e) If A is invertible, then A is diagonalizable.
   F (f) A is diagonalizable if A has n eigenvectors.
   F (g) If A is diagonalizable, then A has n distinct eigenvalues.
   T (h) If $AP = PD$, with D diagonal, then the nonzero columns of P must be eigenvectors of A.
   F (i) If A is 6×6 and has eigenvalues with algebraic multiplicities 3, 2, and 1, then A is diagonalizable.
   T (j) If A is 6×6 and has eigenvalues with geometric multiplicities 3, 2 and 1, then A is diagonalizable.
   T (k) If A has n eigenvalues all with algebraic multiplicity 1, then A is diagonalizable.
   T (l) If there exists an eigenvalue λ of A whose geometric multiplicity is less than its algebraic multiplicity, then A is not diagonalizable.

3. What is the characteristic polynomial of the identity transformation? What is the characteristic polynomial of the 0 transformation?

   In the standard basis, the associated matrix of the identity transformation is the identity matrix. This has characteristic polynomial $(1 - \lambda)^n$. Likewise, the 0 matrix has the characteristic polynomial λ^n. Both of these transformations are diagonalizable, with a single eigenvalue of algebraic multiplicity n and geometric multiplicity n.

3 Extra (possibly difficult) Problems

4. Let V be the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let $T : V \to V$ be defined by

$$(Tf)(x) = \int_0^x f(t)\, dt$$

   Show that T has no eigenvalues.

5. Let A be an $n \times n$ real matrix. Prove that if n is odd, then A must have at least one (real) eigenvalue. (Hint: look at the characteristic polynomial and use some property of continuous functions)

6. Recall the Fibonacci sequence F_n, defined recursively as $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0, F_1 = 1$. We can write this as

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \qquad \text{with} \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Reason to yourself that $\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors for A.

(b) Diagonalize A, i.e. write $A = PDP^{-1}$ for some diagonal matrix D.

(c) Calculate A^{n-1}.

(d) Derive Binet's formula for F_n:

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$