Math 54 Section Worksheet 10 Solutions

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1 Warm-up

- Turn to a virtual person next to you and define what an *eigenvalue* of a matrix A is. Have them define what an *eigenvector* is.

  An eigenvalue of a matrix A is a scalar λ such that there is a non-zero $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. We call $\mathbf{v}$ an eigenvector of A for the eigenvalue λ. Note that $\mathbf{v}$ must be non-zero (otherwise we could take $\mathbf{v} = 0$ and then every λ would satisfy $A\mathbf{0} = \lambda\mathbf{0}$!)

- What are the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$?

  The eigenvalues are $2, 3, -1$, with eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, respectively.

  Note that the eigenvectors are not unique, and we could have chosen any scalar multiples of the ones we did.

2 Virtually Together

Let's find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

An eigenvalue λ with eigenvector $\mathbf{v}$ satisfies the equation $A\mathbf{v} = \lambda\mathbf{v}$. Rewriting, it satisfies $(A - \lambda I)\mathbf{v} = 0$, which is to say that $\mathbf{v} \in \mathrm{Nul}(A - \lambda I)$.

We first find the eigenvalues. We want all values λ for which $A - \lambda I$ has a nonzero nullspace. An equivalent condition to a matrix having a non-zero null space is that the matrix is not invertible, which in turn is equivalent to $\det(A - \lambda I) = 0$. We call this the **characteristic polynomial**:

$$\text{The characteristic polynomial of } A = \det(A - \lambda I)$$

Let's find the characteristic polynomial for A above:

$$\det(A-\lambda I) = \det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda)-2 = 6-5\lambda+\lambda^2-2 = \lambda^2-5\lambda+4 = (\lambda-4)(\lambda-1)$$

We conclude that the eigenvalues of A are $\lambda = 4$ and $\lambda = 1$, because these are the values of λ that make the characteristic polynomial equal to 0. Now we find the eigenvectors, which we do separately:

$\underline{\lambda = 4}$: We search for $\mathbf{v}$ such that $A\mathbf{v} = 4\mathbf{v}$, so we solve the linear system $(A - 4I)\mathbf{v} = 0$.

$$(A - 4I)\mathbf{v} = \begin{pmatrix} 3-4 & 2 \\ 1 & 2-4 \end{pmatrix}\mathbf{v} = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\mathbf{v} = 0$$

We find that the **eigenspace for** $\lambda = 4$ is Span$\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$. The eigenspace of an eigenvalue λ is defined as the set of all eigenvectors for λ, along with the 0 vector.

$\underline{\lambda = 1}$: Similarly, if $\mathbf{v}$ is an eigenvector for $\lambda = 1$, then we have that

$$(A - 1I)\mathbf{v} = \begin{pmatrix} 3-1 & 2 \\ 1 & 2-1 \end{pmatrix}\mathbf{v} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\mathbf{v} = 0$$

and so the eigenspace for $\lambda = 1$ is Span$\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$.

3 Problems

1. Is $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ an eigenvector of the matrix $\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$? If so, what is the eigenvalue?

   To check that $\mathbf{v}$ is an eigenvector, we simply compute $A\mathbf{v}$:

   $$A\mathbf{v} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

   and hence $\mathbf{v}$ is an eigenvector of A, with eigenvalue $\lambda = -2$.

2. Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$. Describe the corresponding eigenspaces by giving vectors that span them.

   The characteristic polynomial is

   $$\det\begin{pmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{pmatrix}. = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)$$

   and so A has eigenvalues $\lambda = 3, 2$. The eigenspace for $\lambda = 3$ is Span$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ and the eigenspace for $\lambda = 2$ is Span$\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$, which we find by solving for the null spaces of $A - 3I$ and $A - 2I$, respectively.

3. Suppose all matrices A below are $n \times n$. True or False?

   T (a) If $Ax = \lambda x$ for some nonzero vector x, then λ is an eigenvalue of A.

F (b) If $Ax = \lambda x$ for some scalar λ, then x is an eigenvector of A.

T (c) A matrix A is not invertible if and only if 0 is an eigenvalue of A.

T (d) A number c is an eigenvalue of A if and only if the equation $(A-cI)x = 0$ has a nontrivial solution.

F (e) A matrix A can have infinitely many eigenvalues.

T (f) If A has an eigenvalue, then it has infinitely many eigenvectors.

T (g) 0 can be an eigenvalue of A.

F (h) 0 can be an eigenvector of A.

T (i) Finding an eigenvector of A may be difficult, but checking whether a given vector is in fact an eigenvector is easy.

F (j) To find the eigenvalues of A, reduce A to echelon form.

F (k) If v_1 and v_2 are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

F (l) The eigenvalues of a matrix are on its main diagonal.

T (m) An eigenspace of A is a null space of a certain matrix.

F (n) The determinant of A is the product of the diagonal entries in A.

F (o) An elementary row operation on A does not change the determinant.

T (p) $(\det A)(\det B) = \det AB$.

F (q) $\det(A + B) = \det A + \det B$.

F (r) If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.

T (s) The multiplicity of a root r of the characteristic equation of A is called the algebraic multiplicity of r as an eigenvalue of A.

F (t) A row replacement operation on A does not change the eigenvalues.