# Math 54 Section Worksheet 9

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## 1 Together

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $e_1 \mapsto (2, -2)^T$ and $e_2 \mapsto (-1, 3)^T$. In the standard basis, $T$ is given by the matrix

$$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Let's find the eigenvalues and eigenvectors of $A$. We want to find all scalars $\lambda$ such that the linear system $Ax = \lambda x$ has a nontrivial solution. Equivalently, we are looking for all scalars $\lambda$ such that...

(Circle ONE) (a) The matrix $(A - \lambda I)$ is invertible. (b) The matrix $(A - \lambda I)$ is not invertible.

(c) The matrix $A$ is invertible. (d) The matrix $A$ is not invertible.

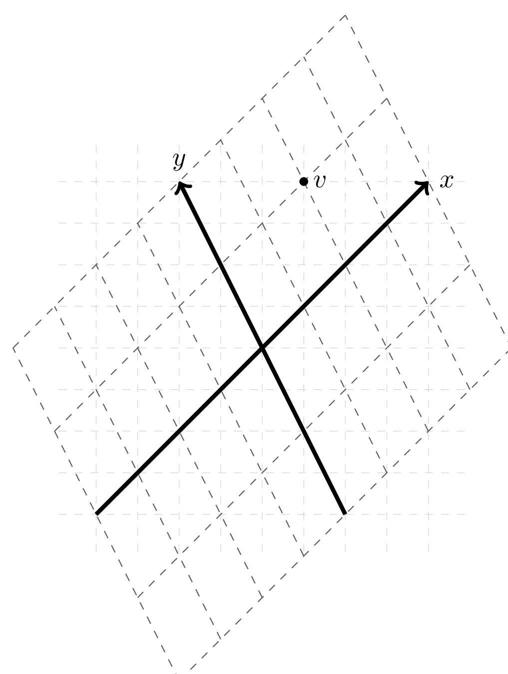
An equivalent condition to a matrix $B$ being invertible is....

(Circle ONE) (a) $B$ has no zero rows (b) $\operatorname{Col} B = \{0\}$ ($c$) $\det B \neq 0$ (d) 0 is a solution to $Bx = 0$.

After some work, we find that the eigenvalues of $A$ are $\lambda = 1, 4$. Plugging these numbers back into our linear system $Ax = \lambda x$, we can solve for our eigenvectors...

After some more work, we find that the eigenvectors of $A$ are $(1, 1)^T$ and $(-1, 2)^T$ for the eigenvalues 1, 4, respectively.

To the right is the Cartesian grid, drawn not with the usual $x, y$ axes, but instead with our eigenvectors as the axes. How strange! A vector $v$ is drawn on the grid in our basis of eigenvectors as the point $(2, 1)$. Let's plot the point $T(v)$.




We see that in the basis of eigenvectors, $T$ is given by the matrix:

$$D = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

In the standard basis, $v$ is represented as the vector $(1, 4)^T$. Let's check that our answer for $T(v)$ is correct by calculating in the standard basis:

$$T(v) = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

Calculating $T(v)$ in the standard basis was so much more work than calculating $T(v)$ in our new basis of eigenvectors. Just imagine how much harder it would be in the standard basis if we had $n \times n$ matrices, as opposed to doing the calculation with an $n \times n$ diagonal matrix!

This is what eigenvalues and eigenvectors do for us. They give us a natural coordinate system to perform all our calculations in, and they make calculations as easy as possible by only having to multiply vectors by a diagonal matrix.

## 2 Green Problems

1. (5.1 #21) $A$ is an $n \times n$ matrix. True or False.
   (a) If $Ax = \lambda x$ for some vector $x$, then $\lambda$ is an eigenvalue of $A$.
   (b) A matrix $A$ is not invertible if and only if 0 is an eigenvalue of $A$.
   (c) A number $c$ is an eigenvalue of $A$ if and only if the equation $(A - cI)x = 0$ has a nontrivial solution.
   (d) Finding an eigenvector of $A$ may be difficult, but checking whether a given vector is in fact an eigenvector is easy.
   (e) To find the eigenvalues of $A$, reduce $A$ to echelon form.

2. (5.1 #22) $A$ is an $n \times n$ matrix. True or False.
   (a) If $Ax = \lambda x$ for some scalar $\lambda$, then $x$ is an eigenvector of $A$.
   (b) If $v_1$ and $v_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
   (c) A steady-state vector for a stochastic matrix is actually an eigenvector.
   (d) The eigenvalues of a matrix are on its main diagonal.
   (e) An eigenspace of $A$ is a null space of a certain matrix.

3. (5.2 #21) $A, B$ are $n \times n$ matrices. True or False.
   (a) The determinant of $A$ is the product of the diagonal entries in $A$.
   (b) An elementary row operation on $A$ does not change the determinant.
   (c) $(\det A)(\det B) = \det AB$.
   (d) If $\lambda + 5$ is a factor of the characteristic polynomial of $A$, then 5 is an eigenvalue of $A$.

4. (5.2 #22) $A, B$ are $n \times n$ matrices. True or False.
   (a) If $A$ is $3 \times 3$, with columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, then $\det A$ equals the volume of the parallelepiped determined by $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$.
   (b) $\det A^T = (-1) \det A$.
   (c) The multiplicity of a root $r$ of the characteristic equation of $A$ is called the algebraic multiplicity of $r$ as an eigenvalue of $A$.
   (d) A row replacement operation on $A$ does not change the eigenvalues.

## 3 Extra (possibly difficult) Problems

5. What is the characteristic polynomial of the identity transformation? What is the characteristic polynomial of the 0 transformation?

6. Let $V$ be the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let $T : V \to V$ be defined by

$$(Tf)(x) = \int_0^x f(t)\, dt$$

   Show that $T$ has no eigenvalues.

7. Let $A$ be an $n \times n$ real matrix. Prove that if $n$ is odd, then $A$ must have at least one (real) eigenvector. (Hint: look at the characteristic polynomial and use some property of continuous functions)

8. A square matrix $A$ is anti-symmetric if $A^T = -A$ and tridiagonal if there are zeroes everywhere except on the main diagonal and the diagonals just above and below the main diagonal.

   Let $A$ be a real anti-symmetric tridiagonal matrix with elements $a_{i,i+1} = c_i$ for $1 \le i \le n$. Give a formula for $\det A$ when $n$ is even.