# Math 54 Section Worksheet 12

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October 11, 2018

## 1 Green Problems

1. (6.3 #21) True or False.

   (a) If $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and to $\mathbf{u}_2$ and if $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $z$ must be in $W^\perp$.

   (b) For each $\mathbf{y}$ and each subspace $W$, the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$ is orthogonal to $W$.

   (c) The orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto a subspace $W$ can sometimes depend on the orthogonal basis for $W$ used to compute $\hat{\mathbf{y}}$.

   (d) If $\mathbf{y}$ is in a subspace $W$, then the orthogonal projection of $\mathbf{y}$ onto $W$ is $\mathbf{y}$ itself.

   (e) If the columns of an $n \times p$ matrix $U$ are orthonormal, then $UU^T\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of $U$.

2. (6.3 #22) True or False.

   (a) If $W$ is a subspace of $\mathbb{R}^n$ and if $\mathbf{v}$ is in both $W$ and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

   (b) In the Orthogonal Decomposition Theorem, each term $\frac{\mathbf{y}\cdot\mathbf{u}_i}{\mathbf{u}_i\cdot\mathbf{u}_i}\mathbf{u}_i$ in the formula for $\hat{\mathbf{y}}$ is itself an orthogonal projection of $\mathbf{y}$ onto a subspace of $W$.

   (c) If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace of $W$ and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto $W$.

   (d) The best approximation to $\mathbf{y}$ by elements of a subspace $W$ is given by the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$.

   (e) If an $n \times p$ matrix $U$ has orthonormal columns, then $UU^T x = x$ for all $x$ in $\mathbb{R}^n$.

3. (6.4 #17) True or False.

   (a) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $W$, then multiplying $v_3$ by a scalar $c$ gives a new orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, c\mathbf{v}_3\}$.

   (b) The Gram-Schmidt process produces from a linearly independent set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ an orthogonal set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ with the property that for each $k$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span the same subspace as that spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_p$.

   (c) If $A = QR$, where $Q$ has orthonormal columns, then $R = Q^T A$.

4. (6.4 #18) True or False.

   (a) If $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthognal set in $W$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $W$.

   (b) If $\mathbf{x}$ is not in a subspace $W$, then $\mathbf{x} - \text{proj}_W \mathbf{x}$ is not zero.

   (c) In a $QR$ factorization, say $A = QR$ (when $A$ has linearly independent columns), the columns of $Q$ form an orthonormal basis for the column space of $A$.

# 2 Extra Problems

5. Use the Gram-Schmidt process to find an orthogonal basis for the column space of the matrix

$$A = \begin{pmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{pmatrix}$$

# 3 Challenge

6. Let $A$ be a real orthogonal $n \times n$ matrix. Prove that every eigenvalue of $A$ is either 1 or -1, and also that $\det A = \pm 1$.

7. Let $A$ be a $2 \times 2$ orthogonal matrix. Prove that the linear transformation $x \mapsto Ax$ is either a rotation about the origin or a reflection about a line through the origin. (Hint: $Ax$ is determined by $Ae_1$ and $Ae_2$).

8. Let $A$ be a $3 \times 3$ orthogonal matrix and suppose there is some vector $v$ such that $Av = v$. Make a guess as to what the linear transformation $x \mapsto Ax$ can be, geometrically, as in the problem above.