Math 54 Section Worksheet 11

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1 Warm-Up

1. Define the matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Find the eigenvalues of A and B and determine if either is diagonalizable by finding the dimension of the eigenspaces.
2. Let $\mathbf{a} = (1, -2, 3)^T$ and $\mathbf{b} = (4, 1, -1)^T$. Calculate $\mathbf{a} \cdot \mathbf{b}$. Calculate the *norm* $\|\mathbf{a}\|$.
3. Try to recall the following concepts:

orthogonal	orthonormal	orthogonal projection
orthogonal matrix	orthogonal complement	

2 Together

Last week we discussed how, given a diagonalizable linear transformation, eigenvectors give you a natural coordinate system to perform all your calculations in, and make it as easy as possible by only ever having to multiply by a diagonal matrix. The only problem was that if we wanted to apply the linear transformation to a vector in this basis, we needed to first know how to write the vector in the basis of eigenvectors.

This isn't always so easy, but it is easy when the basis has a very special quality that the standard basis has, but that not every basis of eigenvectors has. Can you guess what it is?

You guessed it! Orthogonality!

Let's say $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is an orthonormal set of vectors in $\mathbb{R}^n$.

(a) Is this set of vectors linearly independent?

(b) Does this set form a basis for $\mathbb{R}^n$?

(c) Show that for any vector $\mathbf{v}$ in $\mathbb{R}^n$, we can expand

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{b}_1)\mathbf{b}_1 + \cdots + (\mathbf{v} \cdot \mathbf{b}_n)\mathbf{b}_n$$

(d) What happens when $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is not orthonormal, but just orthogonal? Does this make much of a difference?

Our conclusion is that it's very easy to expand a vector as a linear combination of basis vectors *if* the basis vectors are orthonormal. (Stay tuned Thursday to see how to make a basis orthonormal using *Gram-Schmidt*).

This leads us to the natural question: what *is* the dot product? What does it *mean*? Well from part (c) above, we see that $\mathbf{v} \cdot \mathbf{b}_i$ is the coefficient of $\mathbf{b}_i$ in the expansion of $\mathbf{v}$, or something like the "amount" of $\mathbf{v}$ in the direction of $\mathbf{b}_i$. We call the term $(\mathbf{v} \cdot \mathbf{b}_i)\mathbf{b}_i$ the *orthogonal projection* of $\mathbf{v}$ onto $\mathbf{b}_i$. In general, if $\mathbf{u}$ is not unit length, then the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$ is

$$\text{proj}_{\mathbf{u}}\mathbf{v} =$$

For a concrete example, let's say we are in $\mathbb{R}^2$, $\mathbf{u}$ above is the unit vector $(1, 0)$ along the x-axis and $\mathbf{v}$ is just some vector in the first quadrant. Can you express $\mathbf{v} \cdot \mathbf{u}$ in terms of $\mathbf{v}, \mathbf{u}$ and the angle θ between $\mathbf{v}$ and $\mathbf{u}$?

$$\mathbf{v} \cdot \mathbf{u} = \text{the x-component of } \mathbf{v} =$$

3 Green Problems

4. (6.1 #19) True or False.
   (a) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.
   (b) For any scalar c, $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.
   (c) If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   (d) For a square matrix A, vectors in $\operatorname{Col} A$ are orthogonal to vectors in $\operatorname{Nul} A$.
   (e) If vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ span a subspace W and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \dots, p$, then $\mathbf{x}$ is in $W^\perp$.

5. (6.1 #20) True or False.
   (a) $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$.
   (b) For any scalar c, $\|c\mathbf{v}\| = \text{c}\ \|\mathbf{v}\|$.
   (c) If $\mathbf{x}$ is orthogonal to every vector in a subspace W, then $\mathbf{x}$ is in $W^\perp$.
   (d) If $\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   (e) For an $m \times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space of A.

6. (6.2 #23) True or False.
   (a) Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.
   (b) If $\mathbf{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix
   (c) If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.
   (d) A matrix with orthonormal columns is an orthogonal matrix.
   (e) If L is a line through 0 and if $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L.

7. (6.2 #24) True or False.
   (a) Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.
   (b) If a set $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set.
   (c) If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\mathbf{x}$ preserves lengths.
   (d) The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.
   (e) An orthogonal matrix is invertible.