Math 54 Section Worksheet 10

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1 Green Problems

1. (5.3 #21) A, B, P, D are $n \times n$ matrices. True or False.
   (a) A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.
   (b) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.
   (c) A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.
   (d) If A is diagonalizable, then A is invertible.
2. (5.3 #22) A, B, P, D are $n \times n$ matrices. True or False.
   (a) A is diagonalizable if A has n eigenvectors.
   (b) If A is diagonalizable, then A has n distinct eigenvalues.
   (c) If $AP = PD$, with D diagonal, then the nonzero columns of P must be eigenvectors of A.
   (d) If A is invertible, then A is diagonalizable.
3. (5.5 #23) Let A be an $n \times n$ real matrix with the property that $A^T = A$. Let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q = \overline{\mathbf{x}}^T A\mathbf{x}$. The equalities below show that q is a real number by verifying that $\bar{q} = q$. Give a reason for each step.
$$\bar{q} = \overline{\overline{\mathbf{x}}^T A\mathbf{x}} = \mathbf{x}^T \overline{A\mathbf{x}} = \mathbf{x}^T A\overline{\mathbf{x}} = (\mathbf{x}^T A\overline{\mathbf{x}})^T = \overline{\mathbf{x}}^T A^T \mathbf{x} = q$$

2 Extra Problems

4. Define the matrix
$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$
   A has eigenvalues $\lambda = 2, 8$. Diagonalize A or explain why A is not diagonalizable.
5. Let $\mathcal{D} = \{d_1, d_2\}$ and $\mathcal{B} = \{b_1, b_2\}$ be bases for vector spaces V, W, respectively. Let $T : V \to W$ be a linear transformation with the property that
$$T(d_1) = 2b_1 - 3b_2 \qquad T(d_2) = -4b_1 + 5b_2$$
   Find the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$.

6. Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $p(t)$ to the polynomial $p(t) + t^2 p(t)$.
   (a) Find the image of $p(t) = 2 - t + t^2$.
   (b) Show that T is a linear transformation.
   (c) Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

7. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

3 Challenge

8. Recall the Fibonacci sequence F_n, defined recursively as $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0, F_1 = 1$. We can write this as

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \qquad \text{with} \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

   Reason to yourself that $\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$.
   (a) Find the eigenvalues and corresponding eigenvectors for A.
   (b) Diagonalize A, i.e. write $A = PDP^{-1}$ for some diagonal matrix D.
   (c) Calculate A^{n-1}.
   (d) Derive Binet's formula for F_n:

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$