Hecke Algebras and Kazhdan-Lusztig Polynomials

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Abstract

I foolhardily attempt to give a 50-min introduction to Hecke Algebras and Kazhdan-Lusztig polynomials. In the first three sections I will (hopefully) sufficiently motivate Hecke algebras and introduce Kazhdan-Lusztig polynomials. The remainder will then be devoted to explaining, with examples, the connection between Kazhdan-Lusztig polynomials and representations of the associating Weyl group, with emphasis on $W = S_n$ in section 4 and type B in section 5. If time permits, I will mention unexpected connections to other areas of algebra, combinatorics, and geometry.

0 Disclaimer

As the abstract suggests, this talk will be an incomplete (albeit lengthy) primer on Hecke Algebras and Kazhdan-Lusztig polynomials. For the curious audience member, I suggest reading any of the well written expositions given in the references. In particular, [Cur79, Mar] are great for motivation of the Hecke algebra, [Hum92] goes deeper into the structure of the Hecke algebra and the R-polynomials, [BB06] expands on the particular representation theory of the Hecke Algebra for S_n, as first outlined by Kazhdan-Lusztig in their seminal paper [KL79], and applications can be found in [Shi06]. While many of the talking points here are adapted from these references, any and all errors are completely my own.

1 An Incomplete Motivation for Hecke Algebras

Representation theory, while admittedly vast and intangible, can be traced back to two guiding problems: first, the construction of the irreducible (or indecomposable) representations, and second how to decompose a representation into its irreducible constituents (this is of course assuming one is already interested in knowing how a group represents itself as acting on some space).

One such object that arose in this program was the *Iwahori-Hecke algebra*, whose irreducible representations will be in bijection with the irreducible components of an induced representation. Before we delve into this, I should warn that the following section will contain some constructions in representation theory that may not be familiar to the uninitiated representation theorist. Given our time constraint, I will be going fast and loose through these concepts, but I will be sure to highlight the main takeaway before we enter the next section. So don't fret, this part will all be over soon, with the world of Coxeter groups and combinatorics waiting ahead.

To adequately motivate the appearance of Hecke algebras, let's first consider the case where we have some finite group G and an irrep ψ of a subgroup $H \leq G$. We can view ψ as the $\mathbb{C}H$-module $\mathbb{C}He_\psi$ for some idempotent e_ψ (think projecting down from the regular representation to the irreducible constituent ψ). To construct a representation of G, we can look at the induced representation $\psi^G \simeq \mathbb{C}Ge_\psi$, which will in general not be irreducible. The *centralizer algebra* $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}Ge_\psi)$ will yield information about the decomposition of ψ^G (for example, Schur's lemma tells us that ψ^G is irreducible iff $\dim \mathrm{End}_{\mathbb{C}G}(\mathbb{C}Ge_\psi) = 1$. In general, see Mackey's Theorem). Some general nonsense from ring theory shows that $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}Ge_\psi)$ is isomorphic to the double coset $e_\psi\mathbb{C}Ge_\psi$.

Let's do an example. The idempotent $e_H = \frac{1}{|H|}\sum_{h\in H} h$ affords the trivial rep of H, and hence $\mathbb{C}Ge_H$ is the permutation representation on the cosets of H. Recall that $\mathbb{C}G$ may be identified with complex valued functions on G under convolution:

$$(fg)(x) = \sum_{y\in G} f(xy)g(y^{-1})$$

Under this identification, the centralizer algebra $e_H\mathbb{C}Ge_H$ is the subalgebra of functions constant on (H,H)-double cosets.

We define the *Hecke algebra* wrt to ψ to be $\mathcal{H}(G,H,\psi) = e_\psi\mathbb{C}Ge_\psi$.

Theorem 1.1. *The map $\chi \mapsto \chi|_{\mathcal{H}}$ defines a bijection from the irreducible components of ψ^G to the irreducible characters of $\mathcal{H}$.*

This is the big reason why one studies the representations of the Hecke Algebra: They tell you how to decompose induced representations.

If we now specialize G to be a Chevalley group (finite group of Lie type) over a finite field $\mathbb{F}_q$, and B its Borel, then Iwahori gave specific generators and relations for $\mathcal{H}(G,B,1_B)$ as a deformed group algebra over the Weyl group W of G.[1] It is this presentation that we will be employing.[2]

[1]For the curious reader, if we want to extend this theory to topological groups G with a closed subgroup K, the centralizer algebra which consists of complex-valued functions constant on (H,H)-double cosets becomes the space of K-biinvariant continuous functions of compact support $C_c(K\backslash G/K)$. This algebra, denoted $\mathcal{H}(G//K)$, is called the *Hecke Ring*.

[2]To produce the Hecke Algebra over an affine Weyl group, one can instead look at G a reductive algebraic group over a (non-Archimedean local) field and K an (Iwahori) subgroup.

2 A Modern Definition

Let's now move on to the world of Coxeter groups, as promised. Let (W, S) be a Coxeter system.

Definition 2.1. Let $A = \mathbb{Z}[q, q^{-1}]$. The Hecke Algebra $\mathcal{H} = \mathcal{H}_q(W)$ is the free A-module on the set W, with basis elements denoted T_w for $w \in W$, subject to the following relations:

$$T_s T_w = T_{sw} \text{ if } sw > w$$

$$(T_s - q)(T_s + 1) = 0$$

A few remarks are in order:

Remark 2.1. If we set $q = 1$, we get the group algebra $\mathbb{C}W$ with basis $\{T_w\}$. As such, we may call this a *q-deformed* group algebra.

Remark 2.2. The presence of the q^{-1} is needed so as to introduce inverses. Indeed, we have directly that

$$T_s^2 + (1-q)T_s - q = 0 \implies T_s^{-1} = q^{-1}T_s + (q^{-1} - 1)$$

and we can calculate T_w^{-1} since by the first relation $T_w = T_{s_1} \cdots T_{s_r}$ if $w = s_1 \cdots s_r$ is a reduced decomposition.

Remark 2.3. For those who didn't gloss over the first section, to see the connection between this presentation and the motivation above, we can let $T_w = \frac{1}{|B|}\sum_{x \in BwB} x$ be the characteristic functions on (B, B)-double cosets, which form a basis for the subalgebra $\mathcal{H}(G, B, 1_B)$ of functions constant on (B, B)-double cosets.

As an algebra $\mathcal{H}$ is given by the Coxeter-like generators and relations

$$\mathcal{H} = \langle T_i \mid T_i^2 = (q-1)T_i + q,\, T_iT_jT_i \cdots = T_jT_iT_j \cdots \rangle$$

Tits showed that $\mathcal{H} \otimes \mathbb{C}$ is isomorphic to $\mathbb{C}W$, but in order to write down an explicit bijection, we need to enlarge the ground ring to include $q^{1/2}$. As such, often the Hecke algebra has as its ground ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$, and the second relation may be written as

$$(T'_s - q^{1/2})(T'_s + q^{-1/2}) = 0 \tag{1}$$

Which is seen to be equivalent to the first definition by the substitution $T'_s = q^{-1/2}T_s$. (It's an annoying fact of the literature that this definition seems to be non-standardized. You will see either or both of these definitions, with the assumption that the reader is able to follow computations "up to a factor of $q^{1/2}$". In keeping with tradition, I too will present both scaled relations, i.e. my T_w may in fact carelessly refer to T'_w.)

Now, we want to understand the structure of H more closely. To start, let's note that (1) is somewhat arbitrary, in that we can map $q \mapsto q^{-1}$ and $T'_s \mapsto -T'_s$

and the relation still holds. Similarly, we can also map $T'_s \mapsto -(T'_s)^{-1}$ and again (1) holds:

$$\begin{aligned}(-(T'_s)^{-1} - q^{1/2})(-(T'_s)^{-1} + q^{-1/2}) &= (-T'_s - (q^{-1/2} - q^{1/2}) - q^{1/2})(-T'_s - (q^{-1/2} - q^{1/2}) + q^{-1/2}) \\ &= (-T'_s - q^{-1/2})(-T'_s + q^{1/2}) = 0\end{aligned}$$

Composing both these maps give us the following fact:

Proposition 2.1. *There exists a unique ring involution $\mathcal{H} \to \mathcal{H}$ sending $q \mapsto q^{-1}$ and $T_w \mapsto T_{w^{-1}}^{-1}$.*

This involution is called the *bar involution* and is denoted $\bar{\cdot}$. As the length of w increases, computing the inverse seems to become quite infeasible. It is the main property of this involution that the computation is not *as* bad as one might initially guess:

Proposition 2.2.

$$\overline{T_w} = T_{w^{-1}}^{-1} = q^{-\ell(w)} \sum_{x \leq w} R_{x,w}(q) T_x$$

where $R_{x,w}(q) \in \mathbb{Z}[q]$ is a polynomial in q of degree $\ell(w) - \ell(x)$ and $R_{w,w} = 1$.

The polynomials $R_{x,y}(q)$ above are referred to as *R-polynomials* and play an intermediary role in defining *Kazhdan-Lusztig polynomials*. They can be computed recursively using reduced decompositions of x, w and the lifting property of Coxeter groups.

3 The Kazhdan-Lusztig Basis

We now fast forward to the year 1979. In their exploration of the Springer correspondence, Kazhdan and Lusztig introduced a rather mysterious theorem/definition for a collection of polynomials.

Definition/Theorem 3.1. For any $w \in W$, there is a unique element $C_w \in \mathcal{H}$ such that

$$\begin{aligned}\overline{C_w} &= C_w \\ C_w &= q^{-\ell(w)/2} \sum_{x \leq w} P_{x,w}(q) T_x\end{aligned}$$

where $P_{x,w}(q) \in \mathbb{Z}[q]$ is a polynomial in q of degree $\leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$ for $x < w$ and $P_{w,w} = 1$.

The elements C_w are referred to as the *Kazhdan-Lusztig* basis elements and the polynomials $P_{x,w}$ are *Kazhdan-Lusztig* polynomials (hereby referred to as KL polynomials). In the words of Humphreys, "They [KL polynomials] are appreciably more subtle than the earlier $R_{x,w}$". For starters, their degrees are not even immediately evident.

Keep in mind that we are ultimately interested in the representations of $\mathcal{H}$. The natural place to look for representations is in the regular representation, and to understand why these polynomials were originally defined the way they are, it is essential to see the way in which they multiply. This structure can be obtained from their existence proof, and hence let's first delve into this.

The existence of the C_w is proved by induction. Take $s \in S$ with $sw < w$. If $P_{x,w}$ has maximal degree, that is, the degree $(\ell(w) - \ell(x) - 1)/2$ is achieved, we write $x \prec w$ and define $\mu(x,w)$ to be the coefficient of this highest power of q in $P_{x,w}$. Otherwise, we define $\mu(x,w) = 0$.

Now, to calculate the KL polynomials, we can take the second line of the definition of C_w, apply bar to both sides, and expand out $\overline{T_x}$ in terms of R-polynomials and T_y for $y \leq x$. Following this calculation, we get the following recursive definition for $P_{x,w}$:

$$P_{x,w}(q) := \begin{cases} qP_{sx,sw} + P_{x,sw} - \sum\limits_{sz<z} \mu(z,sw) q^{(\ell(w)-\ell(z))/2} P_{x,z} & : x < sx \\ P_{sx,sw} + qP_{x,sw} - \sum\limits_{sz<z} \mu(z,sw) q^{(\ell(w)-\ell(z))/2} P_{x,z} & : sx < x \end{cases} \tag{2}$$

This seems rather technical, so let's do an example and deduce some useful properties.

Example 3.1. Take $W = S_3$. First note that if $\ell(w) - \ell(x) \leq 2$, then $\deg P_{x,w} \leq (2-1)/2 = 1/2$, and so $P_{x,w}$ is a constant. Setting $q = 0$ in (2) and by induction, we see that $P_{x,w} = 1$. We only have left to compute $P_{e,s_1s_2s_1}$. Applying (2), we get

$$\begin{aligned} P_{e,s_1s_2s_1} &= qP_{s_1,s_2s_1} + P_{e,s_2s_1} - \sum_{\substack{z \leq s_2s_1 \\ s_1z<z}} \mu(z,s_2s_1) q^{(3-\ell(z))/2} P_{e,z} \\ &= q + 1 - \mu(s_1,s_2s_1) q P_{e,s_1} \\ &= 1 \end{aligned}$$

Hence, all Kazhdan-Lusztig polynomials are identically 1 for S_3.

Example 3.2. Take $W = I_n = \langle s,t \mid s^2 = t^2 = (st)^n = e \rangle$. The computation of the KL polynomials is greatly simplified in this case because of the behavior of the Bruhat order: $x < w \iff \ell(x) < \ell(w)$. We prove by induction on $\ell(w)$ that $P_{x,w} = 1$.

Consider any term in the sum of (2). Since $P_{z,sw} = 1$ by assumption, then $\mu(z,sw) \neq 0$ iff $(\ell(sw) - \ell(z) - 1)/2 = 0$ iff $\ell(sw) - \ell(z) = 1$. For $sw \neq e, s, t$, there are precisely two such z that satisfy this: one with a reduced expression starting with s and the other with a reduced expression starting with t. Only the former satisfies $sz < z$ and hence either we get $P_{x,w} = q + 1 - q = 1$ or $P_{x,w} = 1$, using the lifting property and depending on whether $sx < sw$ or not.

Following these examples, you might question whether there are ever non-constant KL polynomials. In fact, most are, but unfortunately we won't do any of these "non-trivial" examples in which $P_{x,w} \neq 1$, as the KL polynomials are difficult to compute in general. Brenti has a lot of work (not cited here) on the computation of these polynomials. Below I give a few useful properties of the KL polynomials, some of which were witnessed in the above examples.

Proposition 3.1. *Let $x \le w$ and $s \in S$. We have*

(i) $P_{x,w}(0) = 1$.

(ii) $P_{x,w} = 1$ *if* $\ell(w) - \ell(x) \le 2$.

(iii) If $sw < w$ *then* $P_{x,w} = P_{sx,w}$.

(iv) If W *has a longest element* w_0, *then* $P_{x,w_0} = 1$.

(v)

$$T_s C_w = \begin{cases} -C_w & : sw < w \\ qC_w + q^{1/2} C_{sw} + q^{1/2} \sum_{sz<z} \mu(z,w) C_z & : w < sw \end{cases}$$

Proposition 3.1(v) is of particular importance as we will soon see how it relates to representations of the Hecke algebra. For those purposes, we will extend our definition of $\mu(z,w)$ when $z > w$, in which case we set $\mu(z,w) = \mu(w,z)$.

Here is a curious non sequitur:

Theorem 3.1. *(Polo.) Any polynomial in q with positive integer coefficients and constant term 1 occurs as a KL polynomial $P_{y,w}$ for $y, w \in S_n$.*

4 Representations of the Hecke Algebra

Following Kazhdan and Lusztig, we will construct a representation of $\mathcal{H}$ in terms of certain *KL-graphs*. This representation will have as a basis some of the C_w's, with actions given by the $P_{y,w}$'s. They will be indexed by subsets called *cells* which are defined in terms of *preorders*.

A reflexive and transitive relation $\le$ is a called a preorder. To any preorder we can associate an equivalence relation given by $x \sim y$ if $x \le y \le x$. The preorder then induces a partial order on equivalence classes.

Preorders $\le_L$ and $\le_R$ on W are defined as follows: $\le_L$ (resp. $\le_R$) is the weakest relation such that for all w, the linear span of $\{C_v \mid v \le_L w\}$ (resp. $\{C_v \mid v \le_R w\}$) is a left (resp. right) ideal in $\mathcal{H}$. The transitive closure of $\le_L \cup \le_R$ is denoted $\le_{LR}$, and hence $\{C_v \mid v \le_{LR} w\}$ forms a 2-sided ideal in $\mathcal{H}$. The equivalence classes $\sim_L, \sim_R, \sim_{LR}$ are denoted *left, right* and *2-sided cells*, respectively.

Without giving a precise definition of $\le_L$ just yet, let's see if we can make this more explicit. Let's set $z \le_L w$ if there is $s \in D_L(z) \backslash D_L(w)$ and $\mu(z,w) \ne 0$. We note that if $w < sw$, then $sw \le_L w$ and every term in the sum of Proposition 3.1(v) is less than w in this left preorder. Thus, T_s takes C_w into the span of itself and various C_z for $z \le_L w$. After taking an appropriate transitive closure of our incomplete definition of $\leq_L$, we see that $\{C_v \mid v \leq_L w\}$ will then be a left ideal.

In keeping with this imprecise definition, let us fix a left cell $Z \subset W$ and define $\mathcal{I}_Z$ to be the span of all C_w $(w \in Z)$ together with C_z for $z \leq_L w$ $(w \in Z)$. Define $\mathcal{I}'_Z$ to be the span of all C_z for which $z \leq_L w$ for some $w \in Z$, but $z \notin Z$. Then, $\mathcal{I}_Z$ is a left ideal in $\mathcal{H}$ by the discussion above, and $\mathcal{I}'_Z$ is an ideal by transitivity. Hence the quotient $\mathcal{M} := \mathcal{I}_Z/\mathcal{I}'_Z$ affords a representation of $\mathcal{H}$, moreover with basis in natural correspondence with $C_w, w \in Z$.

If you didn't follow all this, that's okay. Everything discussed in the previous few paragraphs can be visualized and encoded in what Kazhdan and Lusztig call a *W-graph*, which is a more general notion of what we will call a *KL-graph*.

Definition 4.1. The *(left) colored Kazhdan-Lusztig graph* is the directed graph $\tilde{\Gamma}_{(W,S)} = (W, E)$ whose set E of labeled edges $x \xrightarrow[s]{\mu} y$ are of the following two types:

(i) $x \neq y,\ \mu = \mu(x,y) \neq 0,\ s \in D_L(x) \backslash D_L(y)$.

(ii) $x = y,\ s \in S,\ \mu = \begin{cases} q^{1/2} & s \notin D_L(x) \\ -q^{-1/2} & s \in D_L(x) \end{cases}$

Figure 1 shows the KL-graph for S_3. The first thing we might notice is that if $\ell(x) > \ell(y)$, then $x \to y$ is an edge in $\tilde{\Gamma}_{(W,S)}$ iff x is greater than y in the left weak Bruhat order, i.e. $x = sy$ for some $s \in S$. Let's now reformulate our notion of left cell.

Definition 4.2. Let $x, y \in W$. We say that $x \leq_L y$ if there exists a directed path in $\tilde{\Gamma}_{(W,S)}$ from x to y. Define $x \sim_L y$ if there is a directed path from $x \to y$ and from $y \to x$, i.e. x, y are in the same strongly connected component of $\tilde{\Gamma}_{(W,S)}$. In this case, we say that x and y are in the same *left cell*.

In Figure 1, the left cells are $\{e\}, \{w_0\}, \{s_1, s_2s_1\}, \{s_2, s_1s_2\}$. We should note that not all edges in $\tilde{\Gamma}_{(W,S)}$ may be edges in the Bruhat graph. For example, Figure 2 shows a left cell in S_5. As another example, let's compute the left cells and left KL-graph of the dihedral group.

Example 4.1. Set $W = I_n$. Firstly, the condition in $x \xrightarrow[s]{} y$ that $s \in D_L(x) \backslash D_L(y)$ means that there is no arrow from e to any other element, and so e is in its own left cell. Likewise, there is no arrow to w_0, and so $\{w_0\}$ is another left cell.

Now, for $x \neq e$ and $w \neq w_0$, from Example 3.2, we know that $P_{x,w} = 1$ for any $x \leq w$, and so $\mu(x,w) = 1$ iff $w = sx$ or $w = xs$ for some $s \in S$. Thus, $x \to w$ iff the reduced expressions of x and w start with different generators. There are then two remaining left cells, namely: $\{s, ts, sts, tsts, \ldots\}$ and $\{t, st, tst, stst, \ldots\}$. The KL graph is left as an exercise to the reader.

With our notion of KL-graph, the left regular actions of T_s on the basis C_w given by Proposition 3.1(v) can be combined and rewritten as

$$T_sC_w = q^{1/2} \sum_{z \xrightarrow[s]{\mu} w} \mu(z,w)C_z \tag{3}$$

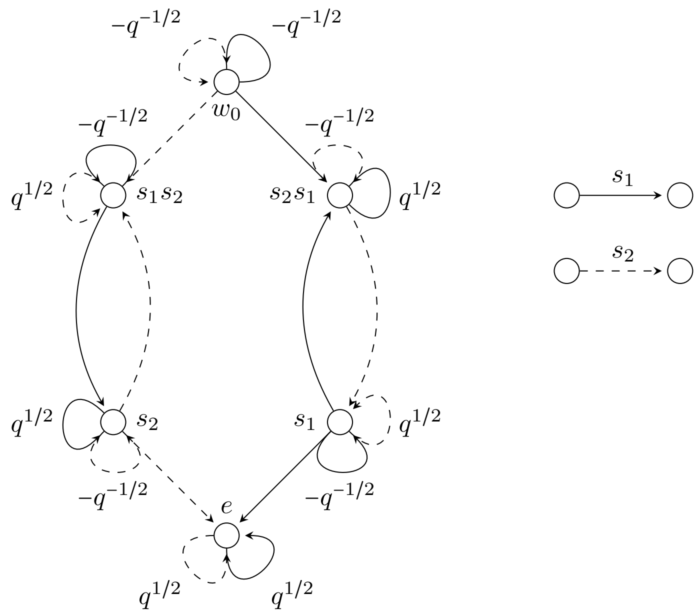

Figure 1: The (left) colored KL-graph $\tilde{\Gamma}_{(W,S)}$ of S_3. All labels $\mu(x,y) = 1$ are dropped for clarity.

In general, we can redefine $w \leq_L w'$ if C_w appears with non-zero coefficient in $T_s C_{w'}$ for some $s \in S$. Similarly, $w \leq_R w'$ if C_w appears with non-zero coefficient in $C_{w'} T_s$.

Recall that we have a partial order on the set of left cells induced by the left preorder: $\mathcal{C} \leq_L \mathcal{C}'$ if there exists (equivalently any) $x \in \mathcal{C}$ satisfying $x \leq_L y$ for some (equivalently any) $y \in \mathcal{C}'$. If W is finite, we can order the left cells $\mathcal{C}^1, \dots, \mathcal{C}^k$ so that if $\mathcal{C}^i \leq_L \mathcal{C}^j$, then $i < j$. As we have noted, (3) implies that T_s sends C_w for $w \in \mathcal{C}^j$ to a linear combination of elements C_z for $z \in \mathcal{C}^i$ with $i \leq j$. In other words, the regular representation of $\mathcal{H}$, when written as a matrix with respect to the C_w basis, is block upper triangular of the form:

$$A(s) = \begin{pmatrix} A_{\mathcal{C}^1}(s) & & & * \\ & A_{\mathcal{C}^2}(s) & & \\ & & \ddots & \\ & 0 & & \ddots \\ & & & & A_{\mathcal{C}^k}(s) \end{pmatrix}$$

For each left cell $\mathcal{C}$, we can define a representation $KL_{\mathcal{C}}$ which sends T_s to the matrix $A_{\mathcal{C}}(s)$. In other words, we quotient out by the ideal $\mathcal{I}$ of elements C_z for $z \notin \mathcal{C}$ as discussed earlier. Belaboring the point, if one wants to write down

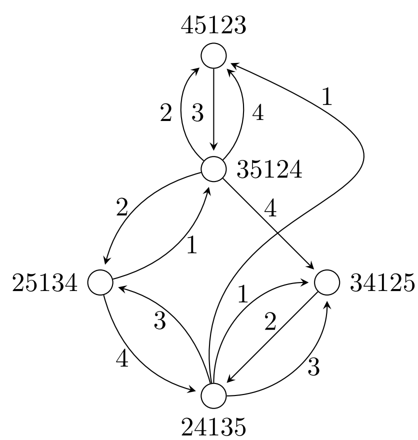

Figure 2: The KL-graph $\tilde{\Gamma}_{\mathcal{C}}$ for a left cell in S_5, without loops and labels $\mu(x,y)$. Labels i correspond to simple transpositions s_i. Note the edge from 24135 to 45123 that is not an edge in the Bruhat graph.

the matrix of T_s for the representation $KL_{\mathcal{C}}$, one can use (3) to expand T_sC_w for $w \in \mathcal{C}$ and zero out any terms C_z for $z \notin \mathcal{C}$.

To restate,

Theorem 4.1. *Let (W,S) be a finite Coxeter system. Then,*

$$Reg_{\mathcal{H}} \simeq \bigoplus_{\mathcal{C}} KL_{\mathcal{C}}$$

where $Reg_{\mathcal{H}}$ is the regular representation of $\mathcal{H}$, $KL_{\mathcal{C}}$ is the left cell representation defined above, and $\mathcal{C}$ runs over all left cells of W.

This theorem necessitates an example to fully understand what is happening. We will do just that for the case $q^{1/2} = 1$ and $W = S_n$.

4.1 Connection to Type A Combinatorics

In general, the KL representations are reducible. This can be quickly seen in the case of the dihedral group I_n, in which there are always 4 left cells for any n (see Example 4.1), but the number of irreps grows with n. However, perhaps one of the miracles of Type A is that the left cell representations are each isomorphic to a particular irreducible representation of S_n. The exact partition is in fact given by the RSK algorithm. (for a refresher on RSK and the representation theory of S_n in general, I suggest the excellent resources [Ful97] and [Sag13].) Let's see how this all unfolds.

We now set $q^{1/2} = 1$ and $W = S_n$ for the remainder of §4.1

Looking at Figure 1, we see that the left cells are $\{e\}, \{w_0\}, \{s_2, s_1s_2\}, \{s_1, s_2s_1\}$. This is eerily similar to the decompositions of the regular representation of S_n, in which case there is a 1-dimensional trivial rep, a 1-dim sign rep, and 2 copies of the 2-dim rep corresponding to the tableau $\ydiagram{1,2}$.

In fact, the left cell $\{e\}$ will always correspond to the trivial representation, since $T_sC_e = C_e = 1$ and $\{w_0\}$ will always correspond to the sign representation, since $sw < w$ for all s and hence $T_sC_{w_0} = -C_{w_0}$.

Let's write out the matrices for the left cell $\{s_1, s_2s_1\}$. Again looking at the KL-graph given in Figure 1 and using (3), we find

$$\begin{aligned} T_{s_1}C_{s_1} &= -C_{s_1} & T_{s_2}C_{s_1} &= C_{s_1} + C_{s_2s_1} \\ T_{s_1}C_{s_2s_1} &= C_{s_2s_1} + C_{s_1} + C_{w_0} & T_{s_2}C_{s_2s_1} &= -C_{s_2s_1} \end{aligned}$$

and hence after zeroing out any C_z for $z \notin \{s_1, s_2s_1\}$, the matrices in the basis $\{C_{s_1}, C_{s_2s_1}\}$ are given by

$$A(s_1) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A(s_2) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

For those not familiar with the representations of S_n, there is a natural basis for the irrep corresponding to a partition λ given by *Garnir* polynomials g_T indexed by standard Young tableaux T on λ. There are 2 standard Young tableaux of shape $(2,1)$, namely $\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$ and $\begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$. The Garnir polynomials are given respectively by $g_T = (x_1 - x_2)$ and $g_S = (x_1 - x_3)$. S_3 acts on these polynomials by permuting indices. The actions of s_1, s_2 are then given by

$$\begin{aligned} s_1 &: g_T \mapsto -g_T & s_2 &: g_T \mapsto (x_1 - x_3) = g_S \\ s_1 &: g_S \mapsto (x_2 - x_3) = g_S - g_T & s_2 &: g_S \mapsto (x_1 - x_2) = g_T \end{aligned}$$

In the basis $\{g_T, -g_T + g_S\}$, the matrices for s_1, s_2 are given by exactly the same matrices $A(s_1), A(s_2)$ above.

If you believe me that each left cell representation is in fact an irreducible representation, the natural question to ask next is what irreducible representation? The irreducible representations of S_n are indexed by partitions, whereas the KL representations are indexed by left cells, themselves composed of permutations of S_n. We therefore would like some way to go from permutations of S_n to partitions. Those acquainted with tableau combinatorics might recognize this as RSK! In fact, there is a deep connection at play.

Theorem 4.2. *Let $w, v \in S_n$ and suppose $w \leftrightarrow (P(w), Q(w))$ and $v \leftrightarrow (P(v), Q(v))$ under RSK. Let $\lambda = sh(P(w)) = sh(Q(w))$ and $\mu = sh(P(v)) = sh(Q(v))$. Then,*

1. *$w \simeq_L v$ iff $Q(w) = Q(v)$, i.e. $w \overset{dK}{\sim} v$, where $\overset{dK}{\sim}$ is a dual Knuth equivalence class. In other words, left cells are exactly dual Knuth equivalence classes.*

2. *$w \simeq_R v$ iff $P(w) = P(v)$, i.e. $w \overset{K}{\sim} v$, where $\overset{K}{\sim}$ is a Knuth equivalence class. In other words, right cells are exactly Knuth equivalence classes*

3. $w \simeq_{LR} v$ *iff* $\lambda = \mu$.

Moreover, if $w \in \mathcal{C}$ *for a left cell* $\mathcal{C}$*, then the representation* $KL_{\mathcal{C}}$ *is isomorphic to the irreducible representation* V_λ *of* S_n.

5 Modification for Type B

Time will certainly not permit for this section, so I will not go into as much detail as one might desire, but rather send the reader to [Shi06, §2.1]. To start off, recall that in general, 2-sided cells are not in bijection with irreducible representations (what they are actually in bijection with is a class of representations called *special* representations). To fix this, one modifies the parameters of the Hecke algebra, and in turn the notion of left cell, by introducing a function Φ from W to an infinite cyclic group.

Let Γ be the infinite cyclic group with generator $q^{1/2}$. Define $\Phi : W \to \Gamma$ by the condition that $\Phi(w) = \Phi(s_1)\cdots\Phi(s_r)$ if $w = s_1 \cdots s_r$ is a reduced expression, and $\Phi(s_i) = q^{m(s_i)/2}$ for some positive integers $m(s_i)$. Set $q_w^{1/2} = \Phi(w)$. For example, if $m(s_i) = 1$ for all i, then $q_w^{1/2} = q^{\ell(w)/2}$.

Now define the $\mathcal{H}_\Phi$ to be the generic Hecke algebra of W with respect to Φ: It is an algebra over $A = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ with basis T_w, $w \in W$ and relations

$$T_s T_w = T_{sw} \qquad \text{if } sw > w$$

$$(T_s - q_s^{1/2})(T_s + q_s^{-1/2}) = 0$$

Again, we note that if $m(s_i) = 1$, then $\mathcal{H}_\Phi$ is our original Hecke algebra. Following the same program, there exists unique elements C_w which form a basis of $\mathcal{H}_\Phi$ subject to the similar conditions as before, from which we get Kazhdan-Lusztig polynomials with respect to Φ. We transport our notion of left preorder and define $w \underset{L,\Phi}{\leq} w'$ if C_w appears with non-zero coefficient in $T_s C_{w'}$ for some $s \in S$.

Now, assume that (W', S') is a Weyl group of type A_n for $n \geq 3$. Let α be the unique automorphism of order 2 of (W', S'). The fixed point set W of α in W' is Coxeter group in its own right, with set of generators corresponding to the orbits of α in S': to an orbit $\mathcal{O}$ there corresponds the longest element in the subgroup generated by $\mathcal{O}$. One can show that (W, S) is a Weyl group of type $B_{n/2}$ if n even and of type $B_{(n+1)/2}$ if n odd.

Let Φ be the function on W which maps $w \mapsto q^{\ell(w)}$, where $\ell(w)$ is the length of w with respect to (W', S'). Then, the restriction of Φ to S has values $q^2, q^2, \ldots, q^2, q^3$ if n even and $q^2, q^2, \ldots, q^2, q$ if n odd. Lusztig proved the following result on left Φ-cells of W:

Theorem 5.1. *Let* $(W', S'), (W, S)$ *and* Φ *be defined as above. Then, each left* Φ*-cell of* W *affords an irreducible representation of* W*, and is the intersection of* W *with a left cell of* W'*. Moreover, all the irreducible representations of* W *arise in this way.*

6 Further applications

We now discuss some of the areas in which Kazhdan-Lusztig polynomials play a key role.

1. KOSTKA-FOULKES POLYNOMIALS (WEIGHT MULTIPLICITIES)
   Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let Q denote the root lattice Let $Q \subseteq P \subseteq \mathfrak{h}^*$ denote the root and weight lattices, respectively. Recall that for each dominant weight $\lambda \in P_+$ there is a unique finite dimensional irreducible representation V_λ with highest weight λ. Let χ_λ denote the character of V_λ. This representation decomposes into weight spaces for $\mathfrak{h}$ as

   $$V_\lambda = \bigoplus_{\mu \leq \lambda} \dim((V_\lambda)^\mu) V_\mu$$

   We define the *multiplicity* of μ in λ to be $K_{\lambda,\mu} = \dim(V_\lambda)^\mu$. In Type A, these are called *Kostka* numbers. They have the following alternative definitions:

   Proposition 6.1. *The following expressions are all equal to $K_{\lambda,\mu}$:*

   (a) $\sum_{w \in W} (-1)^w \mathcal{P}(w(\lambda+\rho)-(\mu+\rho))$ *where for $\nu \in P$, $P(\nu)$ is the number of ways to write ν as a sum of positive roots.*

   (b) The coefficient of m_μ in χ_λ, where m_μ is the appropriate monomial W-symmetric polynomial.

   (c) The coefficient of χ_λ in h_μ, where h_μ is the appropriate homogeneous W-symmetric polynomial.

   In Type A, we also have the following characterization:

   $K_{\lambda,\mu} =$ *The number of semistandard Young tableaux of shape λ and weight μ*

   The Kazhdan-Lusztig polynomials we consider are those for an affine Weyl group. Viewing W as a group of reflections about the linear hyperplanes orthogonal to the roots, we define the *(unextended) affine Weyl group W_a* to be the group of reflections about all the affine hyperplanes orthogonal to the roots. Note that a reflection about an affine hyperplane is equivalent to translating by a root, reflecting about the corresponding linear hyperplane, and translating back. As such, W_a can be written as the semidirect product $W_a = W \ltimes Q$. Enlarging even further to translations by the whole weight lattice, we define the *extended affine Weyl group* $\tilde{W} := W \ltimes P$.

   As a side note, a perhaps more conceptual way to visualize W_a is as a Coxeter group with generators $S \cup \{s_0\}$, where s_0 is an affine reflection about the plane perpendicular to the dominant short root. Viewing it this way, W_a will triangulate the ambient space into what are called *alcoves*. While $\tilde{W}$ is in general not a Coxeter group, it can still be visualized in the same manner.

With our presentations, the cosets $\tilde{W}/W$ are canonically identified with the weight lattice P, and hence the double cosets $W\backslash\tilde{W}/W$ identified with the W-orbits of P, i.e. with the dominant weights P_+.

For $\lambda \in P$, we denote $t(\lambda)$ for the element (e, λ) corresponding to translation by λ. Define w_λ be the unique maximal length representative of the double coset $Wt(\lambda)W$. Note that w_0 is indeed the longest element of W. Lusztig showed the following connection:

Theorem 6.1. *Let μ, λ be dominant integral weights with $\mu \le \lambda$. Then,*

$$K_{\lambda,\mu} = P_{w_\mu, w_\lambda}(1)$$

In fact, we can say much more about these KL polynomials than just their value at 1. There exists a q-analog of Kostka numbers, denoted $K_{\lambda,\mu}(q)$, and known as *Kostka-Foulkes polynomials.* They have the following equivalent characterizations:

Proposition 6.2. *The following expressions are all equal to $K_{\lambda,\mu}(q)$:*

(a) $\sum_{w \in W}(-1)^w \mathcal{P}_q(w(\lambda+\rho)-(\mu+\rho))$ where for $\nu \in P$, $\mathcal{P}_q(\nu) = \sum_{k\ge 0} r_k q^k \cdot$ with r_k = the number of ways to write ν as a sum of k positive roots.

(b) The coefficient of P_μ in χ_λ, where P_μ is a Hall-Littlewood polynomial.

(c) The coefficient of χ_λ in H_μ where H_μ is a transformed Hall-Littlewood polynomial.

In Type A, we also have the following characterization:

$$K_{\lambda,\mu}(q) = \sum_{T \in SSYT(\lambda,\mu)} q^{charge(T)}$$

With these in mind, Kato showed that the Kostka-Foulkes polynomials coincide with certain Kazhdan-Lusztig polynomials.

Theorem 6.2. *Let μ, λ be dominant integral weights with $\mu \le \lambda$. Then,*

$$K_{\lambda,\mu}(q) = q^{\langle \lambda-\mu, \rho^\vee \rangle} P_{w_\mu, w_\lambda}(q^{-1})$$

2. Intersection Cohomology of Schubert Varieties

Many properties known about Kazhdan-Lusztig polynomials, and proofs thereof, come from an underlying geometry at play. One particular long-standing conjecture was the notion of *positivity*:

Question: Are the coefficients of KL polynomials always non-negative?

In the case of Weyl groups, I will briefly explain how we get an answer in the affirmative by interpreting the coefficients of KL polynomials as dimensions of intersection cohomology groups.

Recall that for a reductive Lie group we have a Bruhat decomposition $G = \coprod_{w \in W} BwB$ and hence we have for the flag variety $G/B = \coprod_{w \in W} BwB/B$. The cell $X_w^\circ := BwB/B$ is called a *Schubert cell* in G/B and their closures

$$X_w := \overline{X_w^\circ} = \coprod_{y \leq w} X_y^\circ$$

are called *Schubert varieties.*

Theorem 6.3. *Let $H^i(X_w)$ denote the i^{th} cohomology sheaf of the intersection chain complex of X_w. Then,*

(a) The stalks $H^i_{p_y}(X_w)$ at all points $p_y \in X_y^\circ$ are isomorphic.

(b) $H^{2i+1}_{p_y}(X_w) = 0$ for all $y \leq w$ and all i.

(c) $P_{y,w}(q) = \sum_{i \geq 0} q^i \dim H^{2i}_{p_y}(X_w)$

Thus, the coefficient of q^i in $P_{y,w}(q)$ is the dimension of the local intersection cohomology group in degree $2i$ of X_w at a point in X_y°.

When the Coxeter group isn't a Weyl group, we don't have the geometry of the flag variety to utilize. However, positivity can still be achieved with the notion of perverse sheaves on flag varieties and using Beilinson-Bernstein-Deligne's celebrated decomposition theorem. Algebraic proofs have also recently been established by Elias and Williamson using the theory of Soergel bimodules, in which they give Hodge-theoretic proofs of the decomposition theorem.

Nevertheless, combinatorial proofs continue to elude. This can be illustrated in the following problem.

Conjecture 6.1. *If (W,S) and (W',S') are Coxeter systems and $y, w \in W$ and $y', w' \in W'$, then $P_{y,w} = P_{y',w'}$ whenever the intervals $[y,w]$ and $[y',w']$ are isomorphic posets.*

This innocuous conjecture is owed a remark. To a combinatorialist, this conjecture seems true, as the KL polynomials can be computed recursively using elements in the interval. However, to a geometer, this conjecture seems dubious. The following is quoted verbatim by Brenti [Bre03]:

> In effect, if the answer to the problem is yes, then this would mean that you could go to some geometer and say "Please compute the intersection homology of a Schubert variety", and at her reply "which Schubert variety?" you would say "Oh no..., sorry. I am not allowed to tell you that. I can only tell you, among all the Schubert cells contained in this Schubert variety, which pairs of cells touch each other, and in this case, which is the one of largest dimension". It is not unlikely that, at your reply, the geometer would probably never talk to you again about

mathematics. This is the reason, essentially, why most geometers think that the answer to this problem is no. Philosophically, it is thought that intersection homology is a deeper property than adjacency of Schubert cells. Yet, as some geometers have told me "There are many miracles that happen in Schubert varieties, and this could be one of them. It would certainly be one of the most amazing".

3. Intersection Homology of Unipotent Variety

   We have a similar setup as above, only now we specialize to $GL_n(\mathcal{F}_q)$ and take the variety $\mathcal{U}_\mu \subset GL_n(\mathcal{F}_q)$ of the conjugacy class of unipotent elements with Jordan blocks indexed by μ. If we let $IH^i_{p_\mu}(\mathcal{U}_\lambda)$ denote the local intersection homology of $\mathcal{U}_\lambda$ at any point $x \in \mathcal{U}_\mu$, then

   $$P_{w_\mu, w_\lambda}(q) = \sum_{i \geq 0} q^i \dim IH^{2i}_{p_\mu}(\overline{\mathcal{U}_\lambda})$$

   where w_μ, w_λ are the maximal length representatives of $Wt_\mu W$ and $Wt_\lambda W$, respectively, as first defined in the application to Kostka-Foulkes polynomials.

   We note that it is possible to extend this to general semisimple Lie groups, but one needs to adjust to account for *local systems*.

4. Composition Factors of Verma Modules

   Let's get right into it:

   Conjecture 6.2. *(Theorem 6.2) For $w \in W$, let M_w denote the Verma module $M_{-w(\rho)-\rho}$. Let L_w denote the irreducible quotient of M_w. The decomposition of M_w into simple modules L_w and vice versa is given as*

   $$ch(L_w) = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) ch(M_w) \qquad ch(M_w) = \sum_{y \leq w} P_{w_0 w, w_0 y}(1) ch(L_w)$$

   The Kazhdan-Lusztig conjectures were proven independently by Beilinson and Bernstein, and Brylinski and Kashiwara. Elaborating more on what was mentioned above with Schubert varieties, BB established a connection between highest weight representation theory and perverse sheaves using D-modules and Riemann-Hilbert correspondence. This led to developments in *geometric representation theory*.

5. Primitive Ideals in $\mathcal{U}(\mathfrak{g})$

6. Modular Representation Theory of Algebraic Groups

References

[BB06] Anders Bjorner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231. Springer Science & Business Media, 2006.

[Bre03] Francesco Brenti. Kazhdan-lusztig polynomials: history problems, and combinatorial invariance. *Séminaire Lotharingien de Combinatoire*, 49:B49b, 2003.

[Cur79] Charles W Curtis. Representations of finite groups of lie type. *Bulletin of the American Mathematical Society*, 1(5):721–757, 1979.

[Ful97] William Fulton. *Young tableaux: with applications to representation theory and geometry*, volume 35. Cambridge University Press, 1997.

[Hum92] James E Humphreys. *Reflection groups and Coxeter groups*, volume 29. Cambridge university press, 1992.

[KL79] David Kazhdan and George Lusztig. Representations of coxeter groups and hecke algebras. *Inventiones mathematicae*, 53(2):165–184, 1979.

[Mar] Eric Marberg. Coxeter systems and iwahori-hecke algebras. *https://www.math.ust.hk/ emarberg/Math6150F/index.html*.

[Sag13] Bruce E Sagan. *The symmetric group: representations, combinatorial algorithms, and symmetric functions*, volume 203. Springer Science & Business Media, 2013.

[Shi06] Jian-yi Shi. *The Kazhdan-Lusztig cells in certain affine Weyl groups*, volume 1179. Springer, 2006.