Complex Analysis II

- **Zeros and Poles** Identity principle. Removable singularity. Meromorphic functions. Pole. Laurent series. Essential singularity. Rational functions as meromorphic functions on $\mathbb{C} \cup \{\infty\}$.
- **Harmonic Functions** Harmonic conjugates. Mean value property for harmonic functions. Maximum principle for harmonic functions.
- **Integrals and Residues:** Contour integral. Homotopy of contour. Winding number. Residue theorem. Argument principle. Evaluation of definite integrals.
- **Common Approaches:**
  - Use (punctured) half circles for finitely many poles, rectangles for infinitely many poles, and keyholes for branches
  - Exponentials can be substituted for trigometric functions. Use bound $\int_\gamma |e^{iz}||dz| \leq$ constant where γ half circle of any radius.
  - Often keyhole contours can be replaced by half circles using change of coordinates $x \mapsto x^2$

(1) Find a harmonic function on $\{z \in \mathbb{C} : 0 < |z| < 1\}$ that is not the real part of a holomorphic function.

(2) Find four power series f_1, f_2, f_3, f_4 with radius of convergence 1 such that f_1, f_2 converge at 1 but f_3, f_4 do not, and the functions given by f_1, f_3 can be extended to functions holomorphic in a neighborhood of 1 but the functions given by f_2, f_4 cannot be.

(3) Let $f : B_0(1) - \{0\} \to \mathbb{C}$ be holomorphic where $B_0(1)$ is the unit ball. Assume $\int_{B_0(1)} |f|^2 dxdy < \infty$. Show that f extends to a holomorphic function on $B_0(1)$.

(4) True/False: A function $f(z)$ analytic on $|z - a| < r$ and continuous on $|z - a| \leq r$ extends for some $\delta > 0$ to a function analytic on $|z - a| < r + \delta$?

Past Exam Problems:

(5) Evaluate the integral

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{(z+2)^2}{z^2(2z-1)}dz$$

where the orientation is counterclockwise.

(6) Let D be the set consisting of the open unit disk and the point 1. Show that the power series

$$\sum_{n>0}\frac{z^{3^n}}{n}-\frac{z^{2\cdot 3^n}}{n}$$

converges at all points of D. By examining points with argument of the form $\pi/3^k$ show that the function it converges to is not continuous.

(7) For $0<a<b$ evaluate the integral

$$\frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{|ae^{i\theta}-b|^4}dz.$$

(8) Let $u:\mathbb{R}^2\to\mathbb{R}$ be harmonic. Show that if $u(z)>0$ for all $z\in\mathbb{R}^2$, then u is constant.

(9) Evaluate the integral

$$\int_{-\infty}^{\infty}\frac{e^{itx}}{(x+i)^2}dx$$

where $t\in\mathbb{R}$

(10) Evaluate

$$\int_0^{\infty}\frac{x^C}{x(1+x)}dx$$

where $0<C<1$.