Complex Analysis I

- **Derivatives and power series in $\mathbb{C}$:** Holomorphic functions. Conformal map. Rational functions, square root, exponential, logarithm. Cayley transform. Schwarz lemma. Automorphisms of the upper half plane and unit disk.
- **Properties of holomorphic functions:** Mean value property. Open mapping theorem. Maximum principle. Cauchy theorem. Morera theorem. Rouché theorem. Liouville theorem.
- **Common Approaches:**
  - Recall the inequality in Rouché's theorem is strict
  - $\frac{1+z}{1-z}$ is a holomorphic bijection from the unit ball to $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$
  - Any three points in $\mathbb{C}$ can be taken to any other three points in $\mathbb{C}$ by a fractional linear transformation.

1 Show that the polynomial $p(z) = z^{47} - z^{23} + 2z^{11} - z^5 + 4z^2 + 1$ has at least one root in the disk $|z| < 1$.

2 Suppose that f is analytic on the open upper half plane and satisfies $f(i) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbb{C}$. How large can $|f(2i)|$ be under these conditions?

3 Let the function f be analytic on $\mathbb{C}$. Suppose that $\frac{f(z)}{z} \to 0$ as $|z| \to \infty$. Show that f is constant.

4 Let $c_0, \ldots, c_{n-1}$ be complex numbers. Show that the zeros of the polynomial

$$z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$$

lie in the open disk with center 0 and radius

$$(1 + |c_0|^2 + \cdots + |c_{n-1}|^2)^{1/2}$$

Past Exam Problems:

(5) Let $a, b \in \mathbb{C}$ have nonpositive real parts. Show $|e^a - e^b| \leq |a - b|$.

(6) Let f and g be analytic functions on $\mathbb{C}$ with the property that $h(z) = f(g(z))$ is a nonconstant polynomial. Show that f and g are polynomials.

(7) How many zeros (counting multiplicities) does the polynomial

$$2z^5 - 6z^3 + z + 1$$

have in the annular region $1 \leq |z| \leq 2$?

(8) Suppose $f : \mathbb{C} \to \mathbb{C}$ is continuous. Assume f is holomorphic on $\mathbb{C} - \{z : \operatorname{Im} z = 0\}$. Show that f extends to a holomorphic function on $\mathbb{C}$.

(9) Let f be a nonconstant entire function whose values on $\mathbb{R}$ are real and nonnegative. Prove that all zeros of f on $\mathbb{R}$ have even order.

(10) Show the coefficients of the power series expansion of $\dfrac{z}{e^z - 1}$ are rational.