Metric Spaces

- **Properties of sets:** Metric. Open sets and open balls. Open cover. Refinement of cover and subcover. Limit points. Closure and closed sets. Bounded sets. Countable and uncountable. Compactness. Compactness in function spaces. Connectedness.
- **Topology of $\mathbb{R}^n$:** Least upper bound property of $\mathbb{R}$. ℓ^p metrics on $\mathbb{R}^n$ for $p \geq 1$. Heine-Borel theorem. Characterization of continuous functions. Bolzano theorem. Max/min of continuous function on compact set. Uniform continuity of continuous function on compact set. Arzela-Ascoli theorem. Stone-Weierstrauss theorem. Baire category theorem.
- **Common Approaches:**
  - Use diagonal argument to refine sequence.
  - Use Cantor set as example of nowhere closed set with every point an accumulation point
  - Use Cantor function as example of uniformly continuous but not absolutely continuous

(1) Let $X \subset \mathbb{R}^n$ be compact. Let $f : X \to \mathbb{R}$ be continuous. Show that for any $\varepsilon > 0$ there exists $M > 0$ such that $|f(y) - f(z)| \leq M|y - z| + \varepsilon$ for all $y, z \in X$.

(2) Is $\mathbb{Q}$ the intersection of countably many open subsets of $\mathbb{R}$?

(3) Show that every infinite closed subset of $\mathbb{R}^2$ is the closure of a countable set.

(4) Let $C^0([0,1])$ denote continuous functions on $[0,1]$. Set

$$d(f,g) := \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx.$$

Show that d is a metric on $C^0([0,1])$. Show that d is not a complete metric.

Past Exam Problems:

5 Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Set $g : [0,1] \ni x \mapsto \max\{f(x,y) : y \in [0,1]\} \in \mathbb{R}$. Show that g is continuous.

6 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with the following properties

1. For each $x_0 \in \mathbb{R}$ the function $\mathbb{R} \ni y \mapsto f(x_0, y) \in \mathbb{R}$ is continuous
2. For each $y_0 \in \mathbb{R}$ the function $\mathbb{R} \ni x \mapsto f(x, y_0) \in \mathbb{R}$ is continuous
3. For $K \subset \mathbb{R}^2$ compact have $f(K) \subset \mathbb{R}$ compact

Show that f is continuous.

7 Let $X \subset \mathbb{R}^n$ be a closed set. For $r > 0$ take $Y := \{p \in \mathbb{R}^n : |p - x| = r \text{ for some } x \in X\}$. Show Y is closed.

8 Give an example of a set $S \subset \mathbb{R}$ with uncountably many connected componenets. Can you choose S open? Can you choose S closed?

9 Let U be a nonempty, proper, open subset of $\mathbb{R}^n$. Construct a function $f : \mathbb{R}^n \to \mathbb{R}$ that is discontinuous at each point of U and continuous at each point of $\mathbb{R}^n \setminus U$.

10 Prove that there exists no continuous bijection from $(0,1)$ to $[0,1]$.

11 Let X be a totally ordered set (i.e. equipped with non-reflexive, transitive binary relation such that any two elements related). Let $L(X)$ denote collection of subsets, $S \subset X$, with property that for all $y \in S$ we have $x \in S$ for any $x < y$.

Find a countably infinite totally ordered set X for which $L(X)$ has the smallest possible cardinality. Same question with largest possible cardinality.