Differential Equations


\[
\begin{bmatrix}
x'(t) \\
y'(t)
\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
\]


- **Common Approaches:**
  - Use integrating factor to simplify differential equation/inequality.
  - Study dominant terms in differential equation to determine limiting behavior of solution.
  - Note \( \hat{f}'(n) = in\hat{f}(n) \). Use Fourier series to turn derivatives into multiplication.

1. Let \( a, b \in \mathbb{R} \) be positive. Let \( f : [0, a] \to \mathbb{R} \) satisfy \( f(x) \geq b \int_0^x f(t)dt \) for all \( x \in [0, a] \). Show that \( f \) is nonnegative on \([0, a]\).

2. Show that the initial value problem

\[ y'(x) = 2 + 3 \sin(y(x)), \quad y(0) = 4 \]

has a solution defined for \(-\infty < x < \infty\).

3. Show that there does not exist a unique differentiable function \( y : \mathbb{R} \to \mathbb{R} \) satisfying \( \frac{dy}{dx} = y^{1/2} \) and \( y(0) = 0 \). Find all solutions.

4. The Bessel function \( J_1(x) = a_0 + a_1x + a_2x^2 + \ldots \) satisfies the differential equation

\[ x^2 \frac{d^2 J_1}{dx^2}(x) + x \frac{dJ_1}{dx}(x) + (x^2 - 1)J_1(x) = 0 \]

with \( \frac{dJ_1}{dx}(0) = 1 \). Find the coefficients \( \{a_n\}_{n \geq 0} \)

5. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Assume that

\[ f(x) = f(x + 1) = f(x + \sqrt{2}) \]

for all \( x \in \mathbb{R} \). Show that \( f \) is constant.
Past Exam Problems:

6 Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Assume that $f(0) = 0$, $f'(0) > 0$ and $f(x) \leq f''(x)$ for all $x \geq 0$. Show that $f(x) > 0$ for all $x > 0$.

7 Let $n > 0$ be an integer. Let $\alpha$ and $\varepsilon$ be real numbers with $\varepsilon > 0$. Show that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

1. $|f^{(k)}(x)| \leq \varepsilon$ for $k = 0, 1, \ldots, n - 1$ and all $x \in \mathbb{R}$
2. $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, n - 1$
3. $f^{(n)}(0) = \alpha$

8 Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable. Assume that for some $n > 0$

$$f(0) = f(1) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = 0.$$ 

Show that $f^{(n+1)}(x) = 0$ for some $x \in (0, 1)$.

9 Find all $c \in \mathbb{R}$ such that the differential equation with boundary conditions

$$f'' - cf' + 16f = 0, \quad f(0) = f(1) = 1$$

has no solution.

10 Let $f : [0, \infty) \to \mathbb{R}$ be a continuous non-increasing function. Assume that $f(x) \geq 0$ with $\lim_{x \to \infty} f(x) = 0$. Show that

$$\lim_{R \to \infty} \int_0^R f(x) \sin(x) dx$$

exists.

11 Let $y(t)$ be a $\mathbb{R}$-valued solution defined for $0 < t < \infty$ of the differential equation

$$\frac{dy}{dt} = e^{-y} - e^{-3y} + e^{-5y}.$$ 

Show that $\lim_{t \to \infty} y(t) = \infty$.

12 Let $g : [0, 1] \to \mathbb{R}$ be continuous. Show that there exists $f : [0, 1] \to \mathbb{R}$ continuous such that

$$f(x) - \int_0^x f(x-t)e^{-t^2} dt = g(x)$$

for all $x \in [0, 1]$.