Differential Equations

- **Differential equations and sequences/series of functions:** Linear with constant coefficients. First order with continuous coefficients. Sturm-Liouville equation for second order with continuous coefficients. Separation of variables. Series solution. Fourier series.
- **Existence and behavior of solutions:** Contraction mapping theorem. Lipschitz function. Existence and uniqueness of ordinary differential equations. Behavior of system
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$
Riemann-Lebesgue lemma. Bessel inequality and Parseval theorem. Dirichlet-Jordan theorem on pointwise convergence.
- **Common Approaches:**
  - Use integrating factor to simplify differential equation/inequality.
  - Study dominant terms in differential equation to determine limiting behavior of solution.
  - Note $\widehat{f'}(n) = in\hat{f}(n)$. Use Fourier series to turn derivatives into multiplication

1 Let $a, b \in \mathbb{R}$ be positive. Let $f : [0, a] \to \mathbb{R}$ satisfy $f(x) \geq b \int_0^x f(t)dt$ for all $x \in [0, a]$. Show that f is nonnegative on $[0, a]$.

2 Show that the initial value problem

$$y'(x) = 2 + 3\sin(y(x)), \quad y(0) = 4$$

has a solution defined for $-\infty < x < \infty$.

3 Show that there does not exist a unique differentiable function $y : \mathbb{R} \to \mathbb{R}$ satisfying $\frac{dy}{dx} = y^{1/2}$ and $y(0) = 0$. Find all solutions.

4 The Bessel function $J_1(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ satisfies the differential equation

$$x^2 \frac{d^2 J_1}{dx^2}(x) + x \frac{dJ_1}{dx}(x) + (x^2 - 1)J_1(x) = 0$$

with $\frac{dJ_1}{dx}(0) = 1$. Find the coefficients $\{a_n\}_{n \geq 0}$

5 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Assume that

$$f(x) = f(x + 1) = f(x + \sqrt{2})$$

for all $x \in \mathbb{R}$. Show that f is constant.

Past Exam Problems:

6. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Assume that $f(0) = 0$, $f'(0) > 0$ and $f(x) \leq f''(x)$ for all $x \geq 0$. Show that $f(x) > 0$ for all $x > 0$.

7. Let $n > 0$ be an integer. Let α and ε be real numbers with $\varepsilon > 0$. Show that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

   1. $|f^{(k)}(x)| \leq \varepsilon$ for $k = 0, 1 \ldots, n-1$ and all $x \in \mathbb{R}$
   2. $f^{(k)}(0) = 0$ for $k = 0, 1 \ldots, n-1$
   3. $f^{(n)}(0) = \alpha$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable. Assume that for some $n > 0$

$$f(0) = f(1) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = 0.$$

Show that $f^{(n+1)}(x) = 0$ for some $x \in (0, 1)$.

9. Find all $c \in \mathbb{R}$ such that the differential equation with boundary conditions

$$f'' - cf' + 16f = 0, \quad f(0) = f(1) = 1$$

has no solution.

10. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous non-increasing function. Assume that $f(x) \geq 0$ with $\lim_{x \to \infty} f(x) = 0$. Show that

$$\lim_{R \to \infty} \int_0^R f(x) \sin(x) dx$$

exists.

11. Let $y(t)$ be a $\mathbb{R}$-valued solution defined for $0 < t < \infty$ of the differential equation

$$\frac{dy}{dt} = e^{-y} - e^{-3y} + e^{-5y}.$$

Show that $\lim_{t \to \infty} y(t) = \infty$.

12. Let $g : [0, 1] \to \mathbb{R}$ be continuous. Show that there exists $f : [0, 1] \to \mathbb{R}$ continuous such that

$$f(x) - \int_0^x f(x - t) e^{-t^2} dt = g(x)$$

for all $x \in [0, 1]$