Real Analysis II

- **Multivariable calculus and series:** Derivative. Gradient, curl and divergence. Minimum/Maximum. Lagrange multiplier. Jacobian determinant. Line integral. Surface integral. Absolute and conditional convergence. Rearrangement. Summation by parts. Power series. Multiplication of series. Sequence/series of functions.

- **Relating integrals over interior and boundary, and tests for convergence:** Inverse and implicit function theorem. Rank theorem. Green, Stokes, Gauss theorem. Contraction mapping theorem. Limit comparison. Integral comparison. Ratio and root tests. Alternating series. Dirichlet test.

- **Common Approaches:**
  - Root test is better than ratio test.
  - Use implicit function theorem to extend solution of differential equation.
  - Use interpolation to turn a sequence into a function. Use power series to turn a series into a function.

1 Discuss the solvability of the differential equation

$$(e^x \sin y)(y')^3 + (e^x \cos y)y' + e^y \tan x = 0$$

with the initial condition $y(0) = 0$. Does a solution exist in some interval about 0? If so, is it unique?

2 Let $M_{2\times 2}$ be the space of 2×2 matrices over $\mathbb{R}$ identified with $\mathbb{R}^4$. Define a function $F : M_{2\times 2} \ni X \to X + X^2 \in M_{2\times 2}$. Prove that the range of F contains a neighborhood of the origin.

3 Let a and x_0 be positive numbers. Recursively define a sequence $\{x_n\}_{n>0}$ by $x_n := \frac{1}{2}\big(x_{n-1} + \frac{a}{x_{n-1}}\big)$. Prove that this sequence converges. Find the limit.

4 Let $\{b_n\}_{n>0}$ be a sequence of real numbers with $b_k \geq b_{k+1}$ for all k with $\lim_{k\to\infty} b_k = 0$. Prove that the power series $\sum_{k=1}^{\infty} b_k z^k$ converges for $|z| \leq 1$ with $z \neq 1$.

Past Exam Problems:

5. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, 0) = 0$ and

$$f(x,y) = \left(1 - \cos\frac{x^2}{y}\right)\sqrt{x^2+y^2}.$$

1. Show that f is continuous at $(0, 0)$
2. Calculate all the directional derivatives of f at $(0, 0)$.
3. Show that f is not differentiable at $(0, 0)$.

6. Let x_n be the sequence of real numbers such that $\lim_{n\to\infty} 2x_{n+1} - x_n = x$. Show that $\lim_{n\to\infty} x_n = x$.

7. Suppose that a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ converges uniformly on $\mathbb{R}$ to a function $f : \mathbb{R} \to \mathbb{R}$, and that $c_n = \lim_{x\to\infty} f_n(x)$ exists for each positive integer n. Prove that $\lim_{n\to\infty} c_n$ and $\lim_{x\to\infty} f(x)$ both exist and are equal.

8. Prove that an $\mathbb{R}$-valued C^3 function f on $\mathbb{R}^2$ with

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

everywhere positive cannot have a local maximum.

9. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous. Show that

$$\frac{1}{3}f(\xi) = \int_0^1 x^2 f(x)dx$$

for some $\xi \in [0, 1]$.

10. Let P_2 denote the set of real polynomials of degree ≤ 2. Define the map

$$J : P_2 \ni f \to \int_0^1 f(x)^2 dx \in \mathbb{R}.$$

Let $Q = \{f \in P_2 : f(1) = 1\}$. Show that J attains a minimum value on Q and determine where the minimum occurs.