# Real Analysis I

- **Single variable calculus and sequences:** (Cauchy) Sequences. Limits. Infimum/Supremum and liminf/limsup. Derivatives. (In)definite integrals. Increasing/Decreasing functions. (Uniform) continuity. Polynomials. Interpolation polynomials. Inequalities.
- **Relations between derivatives and integrals, and approximation of functions by polynomials:** Fundamental theorem of calculus. Mean value theorem. L'Hospital's rule. Lagrange polynomial. Taylor expansion with remainder. Arithmetic mean geometric mean inequality. Cauchy-Schwarz inequality. Stirling's formula.
- **Common Approaches:**
  - Fundamental theorem of calculus better than mean value theorem.
  - Continuous functions can be approximated by differentiable functions.
  - Vary constants in integrand and differentiate under the integral.

1 Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(0) = 0$. Assume for $0 < x < 1$ $f$ is differentiable and $0 \leq f'(x) \leq 2f(x)$. Prove that $f \equiv 0$.

2 Show that $e$ is irrational.

3 Let $f$ be a continuous $\mathbb{R}$-valued function on $[0,1]$. Check that $\alpha \int_0^1 x^{\alpha-1} f(x)dx \xrightarrow[\alpha \searrow 0]{} f(0)$.

4 Compute the definite integral $\int_0^1 \frac{x^2-1}{\log x} dx$.

5 Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and $x$ with $0 < x < 1$ satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number $y$ with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

**Past Exam Problems:**

6 Is there a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 1$ such that $f'(x) \geq f(x)^2$ for all $x \in \mathbb{R}$?

7 Suppose that $f$ is a twice differentiable $\mathbb{R}$-valued function on $\mathbb{R}$. Assume that $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all $x \in \mathbb{R}$. Find a constant $b$ such that $|f'(x)| < b$ for all $x \in \mathbb{R}$.

8 Show that the improper integral

$$I = \int_0^\pi \log(\sin x) dx$$

converges. Compute $I$. Hint: Use the substitution $x = 2u$.

9 Let $0 \leq a \leq 1$. Can there exist a nonnegative continuous $\mathbb{R}$-valued function $f$ on $[0, 1]$ which satisfies the three conditions

1. $\int_0^1 f(x)dx = 1$
2. $\int_0^1 xf(x)dx = a$
3. $\int_0^1 x^2 f(x)dx = a^2$

10 Prove or supply a counterexample. If $f$ and $g$ are continuously differentiable functions on the interval $0 < x < 1$ which satisfy the conditions

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$$

and

$$\lim_{x\to 0} \frac{f(x)}{g(x)}$$

exists (as a finite limit), then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}.$$

11 Let $\{a_n\}_{n>0} \subset \mathbb{R}$ be a sequence such that $a_{n+m} \leq a_n + a_m$ for all $n, m$. Assume that $\{\frac{a_n}{n}\}_{n>0}$ bounded from below. Show existence of $\lim\limits_{n\to\infty} \frac{a_n}{n}$.