Throughout this page, all rings are commutative with $1 \neq 0$. You may assume that for any ring R, the set of nilpotent elements in R equals the intersection of all prime ideals of R.

1 Ring theory

1. Let k be a field.

   i) Does $k[[x]][y] = k[y][[x]]$?

   ii) Does $k((x))((y)) = k((x, y))$?

2. Let R be a Noetherian local ring.

   i) If $r \in R$ is a nonunit, show that $\bigcap_{n \geq 1} (r^n) = 0$.

   ii) If $r \in R$ is a nonzerodivisor such that $R/(r)$ has no nilpotents, show that R has no nilpotents.

3. Let R be a ring, and $R[x]$ the polynomial ring over R in 1 variable.

   i) If $f \in R[x]$ is a zerodivisor, show that there exists $0 \neq a \in R$ with $af = 0$.

   ii) Show that every maximal ideal in $R[x]$ contains a nonzerodivisor.

   iii) Show that $(\operatorname{nil} R)[x] = \operatorname{nil} R[x] = \operatorname{Rad} R[x]$, and deduce that $R[x]$ has infinitely many maximal ideals (here nil denotes the nilradical = set of nilpotents).

4. Let R be a ring. If every prime ideal of R is principal, show that every ideal is principal (in this case, R is a PID iff R has no nilpotents or idempotents).

5. For a ring R, let $\operatorname{Min}(R)$ be the set of minimal primes of R, i.e. prime ideals that are minimal with respect to inclusion. Notice that by Zorn's lemma, $\operatorname{Min}(R) \neq \emptyset$.

   i) Show that $\bigcup_{p \in \operatorname{Min}(R)} p \subseteq \{\text{zerodivisors in } R\}$.

   ii) Show that equality holds in (i) if R has no nilpotents.

2 Past exams

6. (6.12.26) Is there a ring that has exactly 5 units?

7. i) (2012) Let R be a ring. Show that R is a field iff there is a monic polynomial $f \in R[x]$ such that (f) is a maximal ideal in $R[x]$.

   ii) Show by example that i) need not hold if f is not required to be monic.

8. (6.11.17, 6.9.3) Find all ring automorphisms of (i) $\mathbb{Z}[x]$, and (ii) $\mathbb{R}$.

9. (6.10.7) Let R be a ring, $a \in R$, $n, m \in \mathbb{N}$, and $d := \gcd(n, m)$. Show that $(a^n - 1, a^m - 1) = (a^d - 1)$.

10. (6.9.13) Show that there is no ring with additive group isomorphic to $\mathbb{Q}/\mathbb{Z}$.