1 Linear algebra

1. Let k be a field, $\operatorname{char} k \neq 2$. For $n \geq 2$, show that there is a basis of $M_n(k)$ (the ring of $n \times n$ matrices over k) consisting only of non-diagonalizable matrices.

2. Let k be a field, and $D : M_n(k) \to k$ a multiplicative function, i.e. $D(AB) = D(A)D(B)$ for all $A, B \in M_n(k)$.

   i) Show that the following are equivalent:

   a) $D(0) \neq D(I)$

   b) $D(0) = 0$, $D(I) = 1$

   c) D vanishes on a proper nonempty subset of $M_n(k)$.

   ii) If the conditions in (i) are satisfied, show that the vanishing set of D is precisely the set of singular matrices, i.e. matrices with determinant 0. Deduce that the set of maps D satisfying (i) is in bijection with group homomorphisms from $GL_n(k)$ to $k^\times$.

3. i) Let φ be an endomorphism of a finite-dimensional k-vector space. Suppose $g, h \in k[x]$ satisfy $(g, h) = 1$, $g(\varphi)h(\varphi) = 0$. Show that $\ker g(\varphi) \cap \operatorname{im} g(\varphi) = 0$.

   ii) Show that one can always satisfy the hypotheses of (i) by taking $g = x^n$ for some $h \in k[x]$, $n \geq \dim \ker \varphi$.

4. Let V be a vector space, $f_i \in \operatorname{End}(V)$ pairwise commuting endomorphisms, and $E_i := \ker f_i$. Show that $\sum_i E_i = \bigoplus_i E_i$ iff $E_i \cap E_j = 0$ for all $i \neq j$.

5. Let V a finite-dimensional $\mathbb{Q}$-vector space. Show that there exists $\varphi \in \operatorname{End}(V)$ with no non-trivial proper invariant subspaces.

2 Past exam problems

6. (6.10.2) Let k be a field. Show that the only two-sided ideals of $M_n(k)$ are 0 and $M_n(k)$.

7. (7.7.7) Let $A \in GL_2(\mathbb{Z})$. If $A^n = I$ for some n, show that $A^{12} = I$.

8. (7.7.12) Let $A \in M_n(\mathbb{C})$. Is A similar to A^t?

9. (7.4.31) Let V be the vector space of polynomials in one variable over $\mathbb{R}$ of degree ≤ 10, and let $D : V \to V$ be differentiation.

   i) Show that $\operatorname{tr} D = 0$.

   ii) Find all eigenvectors of D and e^D.

10. (7.5.21) If $A \in M_n(\mathbb{C})$, and $f \in \mathbb{C}[x]$, show that the eigenvalues of $f(A)$ are precisely $f(\lambda)$, where λ is an eigenvalue of A.