1 Intro/review

1. Let G be a group of order n.

   i) Show that G is isomorphic to a subgroup of A_{n+2}.

   ii) Show that if G has no subgroup of index 2, G is isomorphic to a subgroup of A_n.

2. Let k be a field, and $A \in M_n(k)$. If B is a nilpotent matrix commuting with A, show that $\det(A+B) = \det A$.

3. Let R be a local ring, i.e. a ring with a unique maximal ideal. Show that the only idempotents in R are 0 and 1.

4. Let $K/\mathbb{Q}$ be a field extension, and $n \in \mathbb{N}$ odd. Show that K contains a primitive n^{th} root of unity iff K contains a primitive $(2n)^{\text{th}}$ root of unity.

5. Let K be the splitting field of $x^4 - 10x^2 + 1$ over $\mathbb{Q}$. Determine $\text{Gal}(K/\mathbb{Q})$, and find α such that $K = \mathbb{Q}(\alpha)$.

2 Past exam problems

6. i) (6.4.5) Show that no group of order 112 is simple.

   ii) Let $p < q < r$ be primes. Show that no group of order pqr is simple.

7. i) (6.11.21) Show that $x^{n-1} + x^{n-2} + \ldots + x + 1$ is irreducible over $\mathbb{Q}$ iff n is prime.

   ii) (6.11.25) Show that $x^4 + x^3 + x^2 + 6x + 1$ is irreducible over $\mathbb{Q}$.

8. (7.6.25) Let $A \in M_n(\mathbb{C})$, with characteristic and minimal polynomials χ and μ, respectively. If $\chi(\lambda) = \mu(\lambda)(\lambda - i)$, $\mu(\lambda)^2 = \chi(\lambda)(\lambda^2 + 1)$, find the Jordan canonical form of A.

9. (6.12.6) Let k be a field. Show that k is finite iff the additive group of k is finitely generated.

10. (7.6.41) Let p be a prime. Show that every element of $GL_2(\mathbb{F}_p)$ has order dividing either $p^2 - 1$ or $p(p-1)$.