1 Group theory

1. Let G be a finite group and p a prime dividing $|G|$. If H is a p-subgroup of G, show that $p \bigm| \big([G:H] - [N_G(H):H]\big)$.

2. Let G be a finite group, and let p be the smallest prime dividing $|G|$.

   i) Show that any subgroup of index p is normal.

   ii) Let P be a Sylow p-subgroup of G, and $H \leq Z(P)$ a cyclic subgroup. Show that $H \trianglelefteq G$ iff $H \leq Z(G)$.

3. Find the order of the group with presentation $\langle x, y \mid x^4 = y^3 = 1, xy = y^2x^2 \rangle$.

4. Let G be a group with $G/Z(G)$ abelian, and let $m \in \mathbb{N}$ be odd. Prove that $G^m := \{x^m \mid x \in G\}$ is a normal subgroup of G.

5. Let $n \in \mathbb{N}$, $n \neq 2$. Show that no group of order $2^n \cdot 3 \cdot 5$ is simple.

2 Past exam problems

6. (6.4.16) Let G be a group of order $2m$, with m odd. Show that G has a unique subgroup of order m.

7. (6.1.3) Does there exist a group G with a normal subgroup H such that G/H is not isomorphic to any subgroup of G? What if G is finite? Abelian?

8. i) (6.8.20) If G is abelian, show that $|\operatorname{Aut}(G)|$ is odd iff $|\operatorname{Aut}(G)| = 1$.

   ii) (6.2.8) For any group G, show that $|\operatorname{Aut}(G)| = 1$ iff $|G| \leq 2$.

   iii) (6.1.9) If G is finite, show that $\operatorname{Aut}(G)$ acts transitively on $G \setminus \{e\}$ iff $G \cong (\mathbb{Z}/p\mathbb{Z})^n$ for some p prime, $n \in \mathbb{N}$.

9. (6.8.21) For $n \in \mathbb{N}$, show that there is a unique group of order n iff $\gcd(n, \phi(n)) = 1$ (here ϕ is the Euler phi function).

10. (6.7.7) Let F_n be the free group on n generators. Show that $F_n \cong F_m$ iff $n = m$. (Remark: however, for any n, m, there exist injections $F_n \hookrightarrow F_m$ and $F_m \hookrightarrow F_n$.)