1 Group theory

1. Let $G$ be a finite group and $p$ a prime dividing $|G|$. If $H$ is a $p$-subgroup of $G$, show that $p \mid ([G : H] - [N_G(H) : H]).$

2. Let $G$ be a finite group, and let $p$ be the smallest prime dividing $|G|$.
   i) Show that any subgroup of index $p$ is normal.
   ii) Let $P$ be a Sylow $p$-subgroup of $G$, and $H \leq Z(P)$ a cyclic subgroup. Show that $H \trianglelefteq G$ iff $H \leq Z(G)$.

3. Find the order of the group with presentation $\langle x, y \mid x^4 = y^3 = 1, xy = y^2x^2 \rangle$.

4. Let $G$ be a group with $G/Z(G)$ abelian, and let $m \in \mathbb{N}$ be odd. Prove that $G^m := \{x^m \mid x \in G\}$ is a normal subgroup of $G$.

5. Let $n \in \mathbb{N}$, $n \neq 2$. Show that no group of order $2^n \cdot 3 \cdot 5$ is simple.

2 Past exam problems

6. (6.4.16) Let $G$ be a group of order $2m$, with $m$ odd. Show that $G$ has a unique subgroup of order $m$.

7. (6.1.3) Does there exist a group $G$ with a normal subgroup $H$ such that $G/H$ is not isomorphic to any subgroup of $G$? What if $G$ is finite? Abelian?

8. i) (6.8.20) If $G$ is abelian, show that $|\text{Aut}(G)|$ is odd iff $|\text{Aut}(G)| = 1$.
    ii) (6.2.8) For any group $G$, show that $|\text{Aut}(G)| = 1$ iff $|G| \leq 2$.
    iii) (6.1.9) If $G$ is finite, show that $\text{Aut}(G)$ acts transitively on $G \setminus \{e\}$ iff $G \cong (\mathbb{Z}/p\mathbb{Z})^n$ for some $p$ prime, $n \in \mathbb{N}$.

9. (6.8.21) For $n \in \mathbb{N}$, show that there is a unique group of order $n$ iff $\gcd(n, \phi(n)) = 1$ (here $\phi$ is the Euler phi function).

10. (6.7.7) Let $F_n$ be the free group on $n$ generators. Show that $F_n \cong F_m$ iff $n = m$. (Remark: however, for any $n, m$, there exist injections $F_n \hookrightarrow F_m$ and $F_m \hookrightarrow F_n$.)