The Obstruction Theory Of Kähler-Einstein Metrics
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N.B This brief note is intended to sketch some recent work in Kähler theory. I am not a specialist in this field and I skipped details here and there for the sake of clarity and length. Please read the original papers cited herein if you would like a complete picture.

Introduction The obstruction theory of geometric structures on manifolds and varieties is a fundamental part of modern geometry, in every incarnation. In general, the goal of such theories is to characterize the condition of existence or non-existence of a structure in terms of easily computed or ostensibly unrelated data about the underlying space. Here are three elementary examples:

Example 1. Degree and Genus: The genus of a closed surface Σ is the count of its holes, and the degree of a line bundle L counts the degree of a divisor D corresponding to L. Together this data measure the obstruction to the existence of a global holomorphic section of L on a Riemann surfaces Σ via the Riemann-Roch formula.

Example 2. Stieffel-Whitney Classes: These are distinguished cohomology classes \( w_i(ξ) \in H^i(M, \mathbb{Z}/2) \) detecting properties of a real vector bundle ξ on a manifold M. \( w_1(TM) \) measures the obstruction to an orientation on M and \( w_2(TM) \) measures the obstruction to a spin structure.

Example 3. The Euler Characteristic: \( χ(Σ) \) is the alternating sum of the dimensions of the homology groups of Σ. When Σ is a compact oriented surface, Gauss-Bonnet tells us that \( χ(Σ) \) is a topological measure of the existence of a positive, zero or negatively curved metric via the formula \( \int Rdv_g = 2πχ(Σ) \).

Note that the first example here can be viewed as a linear geometric PDE problem and that the obstruction is (famously) rooted in algebraic geometry and topology. The second problem falls firmly into the realm of geometric and algebraic topology. A more difficult “non-linear” problem, which is in many ways the proper successor to the last example, is the following:

Question. What is the obstruction to a compact Kähler manifold M admitting a Kähler-Einstein metric?

Note that the Ricci tensor \( Ric_g \) corresponding to a Kähler metric g yields a representative of the first Chern class via \( c_1(M) = \frac{i}{2π} Ric_g(J, J) \in H^2(M, \mathbb{Z}) \) (with J the complex structure on M). Thus, the positive definiteness of g and the Einstein equations \( Ric_g = λg \) imply that any Kähler M admitting a Kähler-Einstein metric must have first Chern class with a well-defined sign, i.e \( c_1(M) \) is either positive, negative or 0 evaluated against any complex curve in M, independent of the curve.

For \( c_1(M) < 0 \), it was proven independently by Yau and Aubin that any such M admits a unique negatively curved Kähler-Einstein metric, while for \( c_1(M) = 0 \) the analogous statement was proven by Yau in his celebrated series of papers starting with [Yau78]. The more subtle \( c_1(M) > 0 \), or Fano case has been a longstanding problem, which was recently solved by Chen, Donaldson and Sun (CDS) in the papers [CDS15i]-[CDS15iii]. It is this series of papers and results that we will focus on here.

K-stability of a Fano The primary reason for the increased difficulty of the positive case is that the simple, obvious criterion for the existence of an Einstein metric, i.e \( c_1(M) \) having a definite sign, was quickly found to be insufficient. In the 80’s it was conjectured that the proper condition for a \( c_1 > 0 \) variety was
for it to possess no global $c_1(M)$ vector-fields, but even then Gang Tian demonstrated in [Tian97] that sub-variety of $G_1\mathbb{C}^T$ with positive Chern class admitted no Einstein-Kähler metric, even though they met this stronger criterion.

In the same paper, Tian formulated an early version of a necessary algebraic stability condition called $K$-stability. We will introduce the definition used in [CDS15ii] now, and motivate its original introduction in [Tian97] afterwards. In what follows, let $X$ be a Fano manifold of complex dimension $n$.

**Definition 1.** A test configuration $(\mathcal{X}, i)$ of $X$ is a flat family (fibration) $\pi: \mathcal{X} \to \mathbb{C}$ and a fibration compatible embedding $i: \mathcal{X} \to \mathbb{C}P^N \times \mathbb{C}$ for some $N$, satisfying the following conditions.

1. $i(\mathcal{X})$ is invariant under a $\mathbb{C}^*$ action on $\mathbb{C}P^N \times \mathbb{C}$ that covers the standard action on $\mathbb{C}$.
2. The fiber of $\mathcal{X}$ at 1 is $X$ and the (central) fiber at 0, $X_0$, is a normal variety with log terminal singularities. This condition controls how singular $X_0$ can be. In particular, if $f: Y \to X$ is a resolution of $Y$, then the pullback of the canonical bundle $f^*K_{X_0}$ will agree with $K_X$ up to a sum of divisors $\sum \delta_i E_i$ with $\delta_i > -1$. I do not have enough expertise to give a good interpretation of why this particular condition is important.
3. The embedding $X \hookrightarrow \mathcal{X} \to \mathbb{C}P^N$ is given by complete linear system of $K_X^m$ for some $m > 0$. In other words, the embedding $X \to \mathbb{C}P^N$ is the natural one constructed from global sections of some tensor power of the anti-canonical bundle, which is ample because $X$ is Fano. We also need the embedding $X_0 \to \mathbb{C}P^N$ to arise this way.

Because $X_0$ is invariant under the $\mathbb{C}^*$ action, the dimension $d(k)$ vector space of sections $H^0(X_0, L^k)$ (with $L = K_{X_0}^{-m}$) inherit a $\mathbb{C}^*$ action with some weight $w(k)$. By “general theory” (the phrasing of [CDS15iii]) we know that for large $k$, $d(k)$ and $w(k)$ are given by polynomials of degree $n$ and $n + 1$ respectively. Thus we can expand:

$$\frac{w(k)}{kd(k)} = F_0 + F_1k^{-1} + O(k^{-2})$$

**Definition 2.** We define the Futaki invariant as $Fut(\mathcal{X}, i) = -F_1$.

**Definition 3.** A Fano $X$ is called $K$-stable when $Fut(\mathcal{X}, i) > 0$ for all test configurations $(\mathcal{X}, i)$ such that $X_0 \not\cong X$ and $Fut(\mathcal{X}, i) \geq 0$ if $\mathcal{X}$ is not the trivial family.

In [Tian97], Tian proved that any compact Kähler-Einstein $M$ must satisfy a form of $K$-stability. He used an equivalent integral formulation of $Fut(\mathcal{X}, i)$ in terms of a total integral over $W$ of an expression in $\text{Ric}(\omega)$ and $\omega$, the Ricci 2-form and the Kähler 2-form. A version of this formula is provided on p. 264-265 of [CDS15iii]. Utilizing this formula, Tian looked at $X_t$ as $t$ varied from 1 to 0 (so from $X$ to the central fiber $X_0$). He illustrated that if $X_0 \not\cong X_1$ then the $C^0$ norm of the Kähler potential $\phi_t$ blows up in the $t \to 0$ limit, where $\phi_t$ is introduced as the variation of the KE metric $\omega_{KE}$ on $X_1$ as one moves along the fibers $X_t$ with metrics $\omega_t = \omega_{KE} + \partial\partial\bar{\partial}\phi_t$. Tian was then able to use $\phi_t$ to estimate the integral expression for $Fut(\mathcal{X})$ from below and demonstrate non-negativity, due to $\phi_t$’s roll in said expression.

Conceptually, the relationship between the algebro-geometric notion of $K$-stability and the analytic notion of Kähler-Einstein seems to come from the fact that the existence of such a nice metric on $X$ allows one to control quantities with formulae in terms of the metric on an algebraic family containing $X$. This paradigm extends even to metric derived quantities which are ultimately metric independent (i.e algebraic or topological invariants), in particular the Futaki invariant.
Later work (see [Ber12]) established that the form of $K$-stability given above and used in [CDS15i] - [CDS15iii] was necessary.

**Theorem 4.** Every Fano Kähler-Einstein manifold is $K$-stable.

The converse statement was the focus of CDS. The proof relies on a program, formulated by Donaldson, where a continuity argument is implemented on the existence of singular Kähler-Einstein metrics of cone angle $2\pi \beta$ along a divisor $D$. We will now outline the steps in this proof. Fix a K-stable Fano $X$.

1. Fix a $\lambda > 0$ and a smooth divisor $D$ in the linear system $|−\lambda K_X|$. Such a choice is possible for $\lambda$ large due to Bertini’s theorem. Consider Kähler-Einstein metrics on $X$ with cone angle $2\pi \beta$ along $D$, which satisfy: \[ \text{Ric}(\omega_\beta) = (1 − (1 − \beta)\lambda)\omega_\beta + 2\pi(1 − \beta)[D]. \] Such metrics $\omega_\beta$ are defined to satisfy the Kähler-Einstein equations on $X − D$ and to have Kähler potential in $C^{2,\alpha,\beta}$ for some $\alpha \in (0, \beta^{-1} − 1)$.

2. Let $I$ be the set of $\beta \in (0, 1]$ such that such an $\omega_\beta$ like those described above exists.

3. $I$ is non-empty. If we choose $\beta = 1/N$ with $N$ large enough, we can show that the existence of such a Kähler-Einstein metric on $X$ is equivalent to the existence of such a metric on an orbifold $\hat{X}$ constructed out of $X$ with an orbifold singularity about $D$. This orbifold K-E metric is negative curvature, and as in the smooth case the existence theory was solved by Yau and Aubin. Thus $I$ is non-empty.

4. Similarly, the open-ness of $I$ follows from linear elliptic estimates similar to those in the smooth case. In particular, an inverse function theorem on Banach spaces can be used in the context of Kähler-metrics with cone singularities as long as a small additional assumption is met.

5. The closed-ness of $I$ is, as in the Calabi-Yau case, the difficult part and this is where the $K$-stability comes in. Let $g_i$ be a sequence of K-E metrics on $X$ with cone-singularity $\beta_i$ converging to $\beta_\infty$. By older theory of cone-singular Kähler-Einstein metrics, we can assume $\beta_\infty > 1 − \lambda^{-1}$ since otherwise one can show that the curvature of the resulting metric must be negative, and the existence theory of such metrics is well-established.

In the $\beta_\infty > 1 − \lambda^{-1}$ case, the papers [CDS15i] - [CDS15ii] illustrate that a sequence of singular metrics $g_i$ with cone angle $\beta_i$ along $D$ smoothly Gromov-Hausdorff converge to a limiting manifold $W$. CDS show that $W$ is in fact a $\mathbb{Q}$-Fano variety with a Weil divisor $\Delta$ that satisfies the assumptions on the central fiber $X_0$ of a test configuration. They also illustrate that $W$ has a weak cone K-E metric with singularity of angle $2\pi \beta_\infty$ along $\Delta$. Using the theory of Luna slices, CDS thus construct a test configuration $\mathcal{X}$ with central fiber $X_0 = W$ such that $\Delta$ is the flat limit of $D$. They then illustrate that the associated Futaki invariant $\mathcal{F}$ vanishes, thus implying by the stability assumption that the central fiber $(W, \Delta)$ is isomorphic to $(X, D)$. The cone singular metric on $W$ can then be pulled back to a cone metric of angle $2\pi \beta_\infty$. This illustrates closeness.

Step 5 here is the most technically difficult and required a large amount of original analysis by CDS. In particular, [CDS15i] - [CDS15ii] focused primarily on establishing the algebro-geometric properties if the G-H limit of $(X, D, g_i)$.

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1Technically, a small generalization of the Futaki invariant to families with divisors is used in the end. Of course, the $K$-stability assumption carries over to this extended case.
**Possible Questions**  As a student and a non-specialist, it is difficult for me to muse productively about the future applications and implications of the techniques developed by CDS. However, there is a point here that I believe deserves some attention. I was originally drawn to this body of work because of the beautiful interplay between the PDE analysis, Riemannian geometry and algebraic geometry. This is a general feature of complex geometry and Kähler theory, but here the authors utilize a less conventional technique that embody this interaction: G-H limits in the algebraic context.

Studying the algebro-geometric properties of smooth Gromov-Hausdorff limits of Kähler manifolds under various analytic assumptions could provide an interesting general set of problems. Here the assumption is extremely strong: CDS study sequences of K-E metrics, where powerful bootstrapping estimates are available. One could conceivably attempt to generalize their results, examining the algebro-geometric properties of geometric limits of Kähler metrics with lower-bounded Ricci curvature over some fixed variety $X$. Perhaps such limits can often be described and characterized in terms of properties of algebraic families containing $X$.

**References**


[CDS15ii] Chen, Xiuxiong; Donaldson, Simon; Sun, Song “Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than $2\pi$. “ J. Amer. Math. Soc. 28 (2015), no. 1, 199234.

[CDS15iii] Chen, Xiuxiong; Donaldson, Simon; Sun, Song “Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches $2\pi$ and completion of the main proof.” J. Amer. Math. Soc. 28 (2015), no. 1, 235278.
