Section 4/14:

Tuesday, April 14, 2020 11:01 AM

Topics: L'Hopital's Rule (12.7), Improper Integrals (8.4) and (if time) Newton's method (12.6). 1.3373$(1 + x^2)$(1+x^2)

Announcements:

- Youtube playlist with recordings of sections: Math 16B Sp20 Sections.
- Midterm grading is almost done! The students who come to section seem to have done pretty well, but I don't want to speak to soon.

12.7: L'Hopital's Rule. Say you want to evaluate a limit of the form

$$(*) \qquad \lim_{x \to a} \frac{f(x)}{g(x)}$$

But unfortunately, the top and bottom are not defined at a. For instance, you could be in the following situation.

$$(1) \qquad f(a) = 0 \qquad g(a) = 0$$

So, the fraction $f(a)/g(a)$ is not well-defined because you're dividing by zero! L'Hopital's rule tells you something you can try in this situation.

L'Hopital's Rule (Simple Version): Suppose that $f(x)$ and $g(x)$ satisfy (1), and also suppose that $f'(a)$ and $g'(a)$ are finite and non-zero. Then

$$\lim \frac{f(x)}{} = \lim \frac{f'(x)}{} = \frac{f'(a)}{}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

Example 1. Find the following limit using L'Hopital's Rule.

$$\lim_{x \to 0} \frac{e^x - 1}{2}$$

Solution 1. Here $f(x) = e^x - 1$ and $g(x) = 2x - 2$. The derivative are given by $f'(x) = e^x$ and $g'(x) = 2$. Thus by L'Hopital's rule

$$\lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

There is a more general version of L'Hopital's rule that you should also know how to use. Here is how that one works.

L'Hopital's Rule (General Version): Suppose that $f(x)$ and $g(x)$ satisfy one of the following conditions.

(1) $\lim_{x \to a} f(x) = 0$ $\quad$ $\lim_{x \to a} g(x) = 0$

(2) $\lim_{x \to a} f(x) = \pm\infty$ $\quad$ $\lim_{x \to a} g(x) = \pm\infty$

Then the limit of (*) can be calculated as follows.

a) If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = C$ for some constant C, then $\lim_{x \to a} \frac{f(x)}{g(x)} = C$ also.

b) If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \pm\infty$ then $\lim_{x \to a} \frac{f(x)}{g(x)}$ also diverges to $\pm\infty$.

Example 2. Evaluate the following limits

$$\lim_{x\to\infty} \frac{e^x - 1}{2x - 2} \qquad \lim_{x\to\infty} \frac{\ln(x)^2}{x}$$

Solution 2. For the first limit, we use L'Hopital's rule and observe that

$$\lim_{x\to\infty} \frac{e^x - 1}{2x - 2} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

In particular, the limit doesn't exist and diverges to ∞.

$$\lim_{x\to\infty} \frac{\ln(x)^2}{x} = \lim_{x\to\infty} \frac{2\ln(x)/x}{1} = \lim_{x\to\infty} \frac{2\ln(x)}{x}$$

Now we have to apply L'Hopital **for a second time**.

$$\lim_{x\to\infty} \frac{2\ln(x)}{x} = \lim_{x\to\infty} \frac{2/x}{1} = 0$$

Exercises 1: Evaluate the following limits.

$$(a)\ \lim_{x\to 0} \frac{xe^{-x}}{2e^{-x} - 2} \qquad (b)\ \lim_{x\to 0} \frac{\ln(e^x + 1)}{2e^{-x} - 2} \qquad (c)\ \lim_{x\to 0} \left(\frac{e^x}{x^2} - \frac{1}{x^2} - \frac{1}{x}\right)$$

8.4: Indefinite Integrals: Indefinite integrals are integrals where the bounds of integration are infinite

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Indefinite Integrals: Suppose that $f(x)$ is a function with anti-derivative $F(x)$. Then we define the integrals

$$\int_a^\infty f(x)dx := \lim_{b\to\infty}\int_a^b f(x)dx = \lim_{b\to\infty} F(b) - F(a)$$

If the limit on the left exists, we say that the integral **converges.** Otherwise, we say that the integral **diverges.** We define the indefinite integrals

$$\int_{-\infty}^b f(x)dx \qquad\qquad \int_{-\infty}^\infty f(x)dx$$

in the analogous way.

Example 3. Find the following indefinite integrals.

$$\int_2^\infty \frac{1}{x^2}dx \qquad\qquad \int_{-\infty}^0 e^x dx$$

Solution 3. Using the definition of indefinite integrals, we can calculate that

$$\int_2^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x^2}dx = \lim_{b\to\infty}\frac{-1}{b} - \frac{-1}{2} = \frac{1}{2}$$

$$\int_{-\infty}^0 e^x dx = \lim_{b\to-\infty}\int_b^0 e^x dx = \lim_{b\to\infty} 1 - e^b = 1$$

Exercise 2. Determine if the following indefinite integrals converge or diverge. If it converges, find the integral.

$$\int_2^\infty \frac{3}{x^3}\,dx \qquad\qquad \int_{-\infty}^\infty \frac{2x}{1+x^2}\,dx$$

Newton's Method: Suppose you want to find a solution to the equation

$$f(x) = 0 \text{ for } a \le x \le b$$

But it's hard to solve the equation explicitly. You can find an approximate solution using Newton's method.

Newton's Method. Let $f(x)$ be a function and suppose that c_n is a convergent sequence that satisfies

$$c_n = c_{n-1} - \frac{f(c_n)}{f\prime(c_{n-1})}$$

Here we assume that $f'(c_n)$ is non-zero for each n and that c_n has a limit. Then the limit $c = \lim_{n\to\infty} c_n$ satisfies $f(c) = 0$. In particular, the sequence c_n approximates a zero of f.

So to approximate

Example 4. Approximate a solution to the equation

$$3x^3 - x^2 + 5x - 12 = 0 \text{ for } 1 \le x \le 2$$

with 4 steps of Newton's method.

Solution 4. The derivative is given by

$$f'(x) = 9x^2 - 2x + 5$$

Start by picking $c_1 = 1$. Then calculate 4 more elements of the sequence c_n.

$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \left(-\frac{5}{12}\right) = 1.4167$ (-5/12) = -0.4167

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 1.3373$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 1.33333$$