Worksheet 6: 10.1-10.4

This worksheet is about solving ordinary differential equations (or ODEs). An *ODE* for an unknown function $y(x)$ of a variable x is an equation that y satisfies in terms of y and its derivatives y', y'' etc. An example of a differential equation is

$$y'' = x + e^x$$

A *solution* $y(x)$ of an ODE is a function y that satisfies the equation. A solution to the above example is

$$y(x) = \frac{1}{6}x^3 + e^x$$

Generally, ODEs have many solutions. The first general method for solving ODEs is separation of variables.

Separation Of Variables Suppose you are given a differential equation of the form

$$\frac{dy}{dx} = \frac{p(x)}{q(y)}$$

Then you can multiply both sides by $q(y)$ to get

$$q(y)\frac{dy}{dx} = p(x) \qquad \text{or} \qquad q(y)dy = p(x)dx$$

If we integrate p and q to get

$$Q(y) = \int q(y)dy \qquad \text{and} \qquad P(x) = \int p(x)dx$$

then the equation becomes $Q(y) = P(x)$. Usually we can rearrange this to get an equation $y = R(x)$, and this is the solution that we're looking for.

Example Suppose we're solving $\frac{dy}{dx} = \frac{y^2}{x^3}$. Then we can use $p(x) = x^{-3}$ and $q(y) = y^{-2}$ as out p and q. The integrals are $P(x) = \int p(x)dx = \frac{-1}{2x^2} + C$ and similarly $Q(y) = \int q(y)dy = \frac{-1}{y} + C$. So using separation of variables tells us that

$$\frac{-1}{y} = \frac{-1}{2x^2} + C$$

Now we can do algebra to get y in terms of x.

$$\frac{-1}{y} = \frac{-1 + 2Cx^2}{2x^2} \implies \boxed{y(x) = \frac{2x^2}{1 - 2Cx^2}}$$

Exercise 1 (§10.1 # 13, 15) Solve the following ODE using separation of variables.

(a) $$\frac{dy}{dx} = \frac{y^2 + 6}{2y}$$ (b) $$\frac{dy}{dx} = \frac{11e^y}{e^x}$$

Don't move on to the next page until you've tried this problem.

Solution 1 (a) To apply separation of variables to the first ODE, we use $q(y) = \frac{2y}{y^2+6}$ and $p(x) = 1$. We can rearrange the ODE to get

$$\frac{2y}{y^2+6} \cdot \frac{dy}{dx} = 1$$

Writing this in terms of differentials, we have

$$\frac{2y}{y^2+6} dy = dx$$

Now we integrate this, to get

$$\ln(y^2+6) = \int \frac{2y}{y^2+6} dy = \int 1 dx = x + c$$

Rearranging this and letting $C = e^c$, we get

$$y^2 + 6 = Ce^x \implies \boxed{y = \sqrt{Ce^x - 6}}$$

(b) The next one is similar. We rearrange and write things in

$$\frac{dy}{dx} e^{-y} = 11e^{-x} \implies e^{-y} dy = 11e^{-x} dx$$

Integrating, we get

$$-e^{-y} = \int e^{-y} dy = \int 11e^{-x} dx = -11e^{-x} + C$$

Doing some algebra, we get

$$-y = \ln(11e^{-x} + C) \implies \boxed{y = -\ln(11e^{-x} + C)}$$

Integration Factors The next method that we'll talk about is integration factors. This is a method for solving first order linear ODE, which are ODE of the form

$$y' + p(x)y = f(x) \tag{0.1}$$

To solve this ODE, we can use the *integration factor*, which is just the integral of $p(x)$.

$$I(x) = \int p(x)dx \qquad \text{so that} \qquad I'(x) = p(x)$$

The thing to notice here is that if y is a solution to 0.1, then

$$(e^{I(x)}y)' = e^{I(x)}y' + I'(x)e^{I(x)}y = e^{I(x)}(y' + p(x)y) = e^{I(x)}f(x)$$

Or in short $(e^{I(x)}y)' = e^{I(x)}f(x)$. Integrating this equation gets us

$$e^{I(x)}y = \int e^{I(x)}f(x)dx \implies y(x) = e^{-I(x)}\int e^{I(x)}f(x)$$

So the conclusion is the following. **the general solution to 0.1 is**

$$y(x) = e^{-I(x)}\int e^{I(x)}f(x) \qquad \text{where} \qquad I(x) \text{ satisfies } I'(x) = p(x)$$

The indefinite integral introduces a constant C into the general solution: if we are given an *initial condition* that fixes the value of $y(a)$ for some fixed a

$$y(a) = b$$

then we pick a value of C to get this equation to be satisfied.

Example Suppose we are given the ODE

$$y' + \frac{1}{x}y = 5 \qquad y(1) = 0$$

This equation is of the form 0.1, so we can use integration factors. Here $p(x) = \frac{1}{x}$. So we can take $I(x) = \ln(x)$ so that $I'(x) = \frac{1}{x}$ (in general we would integrate to find $I(x)$). Then we can use the general solution to get

$$y(x) = e^{-\ln(x)}\int e^{\ln(x)} \cdot 5 = x^{-1} \cdot \int 5x dx = x^{-1} \cdot (\frac{5}{2}x^2 + C) = \frac{5}{2}x - Cx^{-1}$$

To get $y(1) = 0$, we see that $\frac{5}{2} - C = 0$, so we should choose $C = \frac{5}{2}$. So with the initial conditions, the solution is

$$\boxed{y(x) = \frac{5}{2}x - \frac{5}{2}x^{-1} = \frac{5}{2}(x - x^{-1})}$$

Exercise 2 Use integration factors to solve the following first order linear ODEs.

$$\text{(a)}\quad x\frac{dy}{dx} - y - x = 0 \quad y(1) = 0 \qquad \text{and} \qquad \text{(b)}\quad 2xy + x^3 = x\frac{dy}{dx}$$

You need to rearrange these to directly apply integration factors.
Don't move on to the next page until you've tried this problem.

Solution 2 Starting with (a), we rearrange the ODE into the nice standard form.

$$\frac{dy}{dx} - \frac{1}{x}y = 1$$

This is pretty similar to the example. We now have $p(x) = -\frac{1}{x}$ and $I(x) = -\ln(x)$. Then the general solution is

$$y(x) = e^{\ln(x)} \int e^{-\ln(x)} \cdot 1dx = x \int \frac{1}{x} dx = x(\ln(x) + C)$$

With the initial conditions $y(1) = 0$, we find that $y(1) = C$, so we should take $C = 1$. The final solution is thus

$$\boxed{y(x) = x \cdot (\ln(x) + 1)}$$

For (b), we proceed similarly. Rearranging, we find that

$$\frac{dy}{dx} - 2y = x^2$$

Our $p(x)$ is now -2, so $I(x) = -2x$. Thus our general solution is

$$y(x) = e^{2x} \cdot \int e^{-2x} \cdot x^2 dx$$

We can use integration by parts (tabular or otherwise) to integrate the right-hand side and get

$$\boxed{y(x) = e^{2x} \cdot (\frac{-1}{4} e^{-2x} \cdot (2x^2 + 2x + 1) + C) = \frac{-1}{2}x^2 - \frac{1}{2}x - \frac{1}{4} + Ce^{2x}}$$

We weren't given any initial conditions $y(a) = b$, so no need to fix the value of C.