1. Six Easy Steps to Solving The Heat Equation

In this document I list out what I think is the most efficient way to solve the heat equation. A heat equation problem has three components.

A Differential Equation: For \(0 < x < L, 0 < t < \infty\)
\[
\frac{\partial u}{\partial t} = \beta^2 \frac{\partial^2 u}{\partial x^2}
\]

Boundary values: For \(0 < t < \infty\)
\[
u(0,t) = u(L,t) = 0
or
\frac{\partial u}{\partial x}(0,t) = u(L,t) = 0
\]

Initial values: For \(0 < x < L\)
\[
u(x,0) = f(x)
\]

Step 1: Write down the solution by separating the variables. Write
\[
u(x,t) = \sum_n X_n(x)T_n(t)
\]
You can rescale \(X, T\) to assume that \(T_n(0) = 1\).

Step 2: Plug the initial values into the equation for \(\nu\) to get
\[
f(x) = \nu(x,0) = \sum_n X_n(x)
\]
Note that this will be a Fourier series for \(f(x)\).

Step 3: Look at the boundary values to determine if your Fourier series should be sines or cosines. If you’re given that \(\nu(0,t) = 0\) then each \(X_n(0) = 0\), so each \(X_n\) should be a sine. If you’re given that \(\frac{\partial \nu}{\partial x}(0,t) = 0\) then the derivative of \(X_n(0)\) is 0, so each \(X_n\) should be a cosine.

Step 4: Compute \(X_n(x)\). Based on your deduction for step 3, either \(X_n = b_n \sin(n\pi x/L)\) or \(X_n = a_n \cos(n\pi x/L)\) (except \(X_0 = a_0/2\)). The Fourier coefficients are computed by
\[
b_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) dx
\]
\[
a_n = \frac{2}{L} \int_0^L f(x) \cos(n\pi x/L) dx
\]

Once you compute these coefficients you have your functions \(X_n(x)\).

Step 5: Compute \(T_n(t)\). Each pair \(X_n(x)T_n(t)\) should be a solution to the differential equation
\[
\frac{\partial \nu}{\partial t} = \beta^2 \frac{\partial^2 \nu}{\partial x^2}
\]
Plug in \(u = X_n(x)T_n(t)\), isolate the variables, and plug in the function \(X_n(x)\) you obtained in Step 4. This yields a differential equation for \(T_n(t)\). You can combine this with the initial value \(T_n(0) = 1\) to solve for \(T_n(t)\).

Step 6: Once you have \(X_n(x)\) and \(T_n(t)\) you can write down the final solution
\[
u(x,t) = \sum_n X_n(x)T_n(t)
\]