FOURIER SERIES: SOLVING THE HEAT EQUATION

BERKELEY MATH 54, BRERETON

1. Six Easy Steps to Solving The Heat Equation

In this document I list out what I think is the most efficient way to solve the heat equation. A heat equation problem has three components.

A Differential Equation: For $0 < x < L$, $0 < t < \infty$

$$\frac{\partial u}{\partial t} = \beta^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary values: For $0 < t < \infty$

$$u(0,t) = u(L,t) = 0$$

or

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

Initial values: For $0 < x < L$

$$u(x,0) = f(x)$$

Step 1: Write down the solution by separating the variables. Write

$$u(x,t) = \sum_n X_n(x)T_n(t)$$

You can rescale X, T to assume that $T_n(0) = 1$.

Step 2: Plug the initial values into the equation for u to get

$$f(x) = u(x,0) = \sum_n X_n(x)$$

Note that this wil be a fourier series for $f(x)$.

Step 3: Look at the boundary values to determine if your fourier series should be sines or cosines. If you're given that $u(0,t) = 0$ then each $X_n(0) = 0$, so each X_n should be a sine. If you're given that $\frac{\partial u}{\partial x}(0,t) = 0$ then the derivative of $X_n(0)$ is 0, so each X_n should be a cosine.

Step 4: Compute $X_n(x)$. Based on your deduction for step 3, either $X_n = b_n \sin(n\pi x/L)$ or $X_n = a_n \cos(n\pi x/L)$ (except $X_0 = a_0/2$). The fourier coefficients are computed by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(n\pi x/L)dx$$

Once you compute these coefficients you have your functions $X_n(x)$.

Step 5: Compute $T_n(t)$. Each pair $X_n(x)T_n(t)$ should be a solution to the differential equation $\frac{\partial u}{\partial t} = \beta^2 \frac{\partial^2 u}{\partial x^2}$. Plug in $u = X_n(x)T_n(t)$, isolate the variables, and plug in the function $X_n(x)$ you obtained in Step 4. This yields a differential equation for $T_n(t)$. You can combine this with the intial value $T_n(0) = 1$ to solve for $T_n(t)$.

Step 6: Once you hve $X_n(x)$ and $T_n(t)$ you can write down the final solution

$$u(x,t) = \sum_n X_n(x)T_n(t)$$