ON THE HOMOGENEITY CONJECTURE

JOSEPH A. WOLF

Dedicated to Prof. Toshi Kobayashi on the occassion of his 60th birthday

Abstract. Consider a connected homogeneous Riemannian manifold (M, ds^2) and a Riemannian covering $(M, ds^2) \to \Gamma\backslash(M, ds^2)$. If $\Gamma\backslash(M, ds^2)$ is homogeneous then every $\gamma \in \Gamma$ is an isometry of constant displacement. The Homogeneity Conjecture suggests the converse: if every $\gamma \in \Gamma$ is an isometry of constant displacement on (M, ds^2) then $\Gamma\backslash(M, ds^2)$ is homogeneous. We survey the cases in which the Homogeneity Conjecture has been verified, including some new results, and suggest some related open problems.

1. Introduction.

Let (M', ds'^2) be a connected locally homogeneous Riemannian manifold. We are going to study a simple geometric condition for (M', ds'^2) to be (globally) homogeneous. The obvious conditions for this are (i) that (M', ds'^2) is complete and (ii) that the universal Riemannian covering manifold (M, ds^2) of (M', ds'^2) is complete, connected and locally homogeneous. Then (M, ds^2) is homogeneous, specifically

Lemma 1.1. *If ξ_U is a Killing vector field on an open subset $(U, ds^2|_U)$ of (M, ds^2), then ξ_U extends uniquely to a Killing vector field ξ on (M, ds^2), and ξ generates a one parameter group $\{\exp(t\xi) | t \in \mathbb{R}\}$ of isometries of (M, ds^2). In particular local homogeneity of (M', ds'^2) results in global homogeneity of (M, ds^2).*

The special case $U = M$ of Lemma 1.1 says

Lemma 1.2. *Suppose that (M', ds'^2) is homogeneous. Consider the largest connected group of isometries, $G' = \mathbf{I}^0(M', ds'^2)$. Then there is a Lie group covering $G \to G'$ such that the action of G' on M' lifts to an effective transitive isometric action of G on (M, ds^2).*

Now we can formulate a simple, but basic, observation.

Proposition 1.3. ([33, Theorems 1 and 2].) *Suppose that (M', ds'^2) is homogeneous. Express $M' = \Gamma\backslash M$ where Γ is a discontinuous group of fixed point free isometries of (M, ds^2). Let $G \subset \mathbf{I}(M, ds^2)$ as in Lemma 1.2. Then G centralizes Γ, and every $\gamma \in \Gamma$ is an isometry of constant displacement on (M, ds^2).*

Proof. As constructed, every element of G sends Γ–orbits to Γ-orbits, so G normalizes Γ. But G is connected and Γ is discrete, so G centralizes Γ.

Let $x, y \in M$ and $\gamma \in \Gamma$. Choose $g \in G$ with $g(x) = y$. Let ρ denote distance in (M, ds^2). Then the displacement $\rho(x, \gamma(x)) = \rho(gx, g\gamma(x)) = \rho(gx, \gamma g(x)) = \rho(y, \gamma(y))$. □

The "Homogeneity Conjecture" includes the converse:

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Homogeneity Conjecture. *Let* (M, ds^2) *be a connected simply connected homogeneous Riemannian manifold and* $\pi : (M, ds^2) \to (M', ds'^2)$ *a Riemannian covering. Express* $M' = \Gamma\backslash M$ *where* Γ *is a discontinuous group of isometries of* (M, ds^2)*. Then* (M', ds'^2) *is homogeneous if and only if every* $\gamma \in \Gamma$ *is an isometry of constant displacement on* (M, ds^2)*.*

This paper surveys the published cases for which the Homogeneity Conjecture has been verified, and also includes a number of new results. In the cases where I recall published results I tried to indicate the steps in the proof, with precise references to the original papers for the reader who wants to see the details. In the cases of new results I indicated complete proofs. I also tried to be precise about cases where verification of the Homogeneity Conjecture still is an open problem.

In order to have uniform notation to the extent possible, some notation has been changed from that in the references. This is especially noticeable when dealing with isotropy split fibrations. However the structure of the isometry group and the definition of the group G can change from section to section as we look at various classes of Riemannian manifolds.

In **Part I** we sketch the verification of the Homogeneity Conjecture for Riemannian symmetric spaces, and then we indicate how this verification extends to Finsler symmetric spaces. Part I consists of

Section 2: Spaces of Constant Curvature. These were the first instances of results that led to the Homogeneity Conjecture. The euclidean case (curvature $K = 0$) is elementary. The hyperbolic case ($K < 0$) illustrates the fact that for noncompact manifolds one can work more generally with isometries of bounded displacement; it depends on the fact there that any two distinct geodesics diverge. The elliptic case ($K > 0$) is the hard case because the round sphere has so many symmetries, and it requires some nontrivial finite group theory.

Section 3: Riemannian Symmetric Spaces with Simple Isometry Group. As one expects from the cases of constant curvature, the noncompact case is straightforward and the compact case is delicate. It is relatively straightforward to study isometries γ of constant displacement when γ is contained in the identity component $G = \mathbf{I}^0(M, ds^2)$ of the isometry group, but is much less straightforward when γ is in a component that involves outer automorphisms of G.

Section 4: Riemannian Symmetric Spaces that are Group Manifolds. These are the ones where $\mathbf{I}^0(M, ds^2)$ is not simple and the symmetric space (M, ds^2) is a group manifold G with bi-invariant Riemannian metric ds^2. Then $M = (G \times G)/\{diag\, G\}$; G acts by left and right translations and the isotropy subgroup of $\mathbf{I}^0(M, ds^2)$ acts on G by inner automorphisms. The delicate points are the cases where G has outer automorphisms.

Section 5: A Classification Free Approach. In this later development, isometries of constant displacement are characterized as preserving a minimizing geodesic from a point to its image. Combining this with Proposition 3.7 one proves that if an isometry $g \in \mathbf{I}(M, ds^2)$ is of constant displacement then its centralizer in $\mathbf{I}(M, ds^2)$ is transitive on M. That is enough to verify the Homogeneity Conjecture for Γ cyclic, and it sidesteps the case by case argument parts of the proofs of Theorems 3.11 and 4.6 that involve close looks at outer automorphisms.

Section 6: Extension to Finsler Symmetric Spaces. It makes perfect sense to consider isometries of constant displacement on a metric space (M, ρ), and if $\Gamma\backslash(M, \rho)$ is a homogeneous metric space then every $\gamma \in \Gamma$ is an isometry of constant displacement. Thus one can consider the Homogeneity Conjecture for metric spaces, in particular for Finsler manifolds. In this section we indicate the proof of the Homogeneity Conjecture for Finsler symmetric spaces.

In **Part II** we sketch the verification (or progress toward the verification) of the Homogeneity Conjecture for various geometrically defined classes of compact homogeneous Riemannian manifolds. Part II consists of

Section 7: Isotropy Splitting Fibrations. This section develops a tool for tracing constant displacement isometries along a certain class of fibrations, modeled on the canonical projections $SO(k+\ell)/SO(k) \to SO(k+\ell)/[SO(k) \times SO(\ell)]$ of a Stieffel manifold over a Grassmann manifold.

Section 8: Manifolds of Positive Euler Characteristic. This section consists of new results. We look at Riemannian manifolds (M, ds^2) where $M = G/H$, G is a compact connected Lie group, ds^2 is a G–invariant Riemannian metric on M, and the Euler characteristic $\chi(M) \neq 0$. This last condition is equivalent to $\operatorname{rank} H = \operatorname{rank} G$, and then $\chi(M)$ is the quotient $|W_G/W_H|$ of the orders of the Weyl groups. The proofs quickly reduce to the case where G is simple. The main result here is Theorem 8.1. That verifies the Homogeneity Conjecture when, for every $\gamma \in \Gamma$, $\mathrm{Ad}(\gamma)$ is an inner automorphism on G. It is an open problem to deal with outer automorphisms.

Section 9: Compact Group Manifolds. This section also contains new results. We look at Riemannian manifolds (M, ds^2) on which a connected Lie group G acts simply transitively by isometries. That reduces to Riemannian manifolds (G, ds^2) where ds^2 is a Riemannian metric invariant under the left translations $\ell(g), g \in G$. The identity component of the isometry group has form $\ell(G) \times r(H)$ where $r(H)$ consists of the right translations $r(h)$ that preserve ds^2, i.e. such that $\mathrm{Ad}(h)$ preserves the inner product on $\mathfrak{g}$ defined by ds^2. The main result here is Theorem 9.1, which says that a finite group Γ of constant displacement isometries is contained either in $r(H)$ or $\ell(G)$. In the $r(H)$ case Γ centralizes $\ell(G)$, so $\Gamma\backslash(G, ds^2)$ is homogeneous and the Homogeneity Conjecture is verified. In the $\ell(G)$ case one needs more information on the elements of Γ.

Section 10: Positive Curvature Manifolds. In this section we verify the Homogeneity Conjecture for Riemannian homogeneous spaces (M, ds^2), $M = G/H$, such that M admits Riemannian metric dt^2 of strictly positive sectional curvature. We rely on the classification of those spaces M and the structure of their isometry groups, carry that over to the more general spaces (M, ds^2), and use various tools to complete the verification there of the Homogeneity Conjecture.

In **Part III** we sketch the verification of the Homogeneity Conjecture for several classes of noncompact homogeneous Riemannian manifolds. In these noncompact cases "bounded" can replace "constant displacement" and the result becomes independent of the choice of Riemannian metric. The results are all based on [36] and [28]. Part III consists of

Section 11: Negative Curvature. This section recalls the results of [36], which were applied to Riemannian symmetric spaces in Subsection 3A.

Section 12: Semisimple Groups. This section deals with the cases where a real semisimple group G, without any compact factors, is transitive on (M, ds^2). Combining ideas from [36] and [28] it is shown that (M, ds^2) has no nontrivial bounded isometries, and in particular the Homogeneity Conjecture is verified for (M, ds^2). This applies, in particular, to the flag domains that appear in automorphic function theory.

Section 13: Bounded Automorphisms. This section recalls the results of Jacques Tits [28] on bounded automorphisms. They give results on bounded isometries. For reasons of clarity I have quoted them in the original.

Section 14: Exponential Solvable Groups. This section combines results on exponential solvable groups ([41], [43] and semisimple groups with no compact factors (from Section 12. It is shown that if such a group is transitive then there are no nontrivial bounded isometries.

In **Part IV** we summarize the results and pose some open problems. Part IV consists of

Section 15: Open Problems. This section mentions five open problems related to the Homogeneity Conjecture. They are

- to complete the results on manifolds (M, ds^2) with $\chi(M) > 0$,
- to complete the results on group manifolds with left invariant Riemannian metric,
- to verify the Homogeneity Conjecture for weakly symmetric Riemannian manifolds, or even geodesic orbit Riemannian manifolds,
- to study the Homogeneity Conjecture for Finsler manifolds, and
- to study an appropriate variation on the Homogeneity Conjecture for pseudo–Riemannian manifolds.

Much of the material in Sections 8, 9 and 14 is new, except of course where it is cited from one of the references.

Part I. Riemannian Symmetric Spaces.

In three Sections 2 through 4 we will sketch the proof of the Homogeneity Conjecture for the cases where (M, ds^2) is a Riemannian symmetric space. In Section 2 we carry this out for the cases of constant sectional curvature; they illustrate the issues that must be addressed in general, and in particular for symmetric spaces. In fact the case of constant positive curvature is the most difficult case, and it requires a bit of finite group theory.

After I published these results on Riemannian symmetric spaces, H. Freudenthal and V. Ozols gave shorter proofs for some special cases. Freudenthal [12] gave a proof for the case where Γ is contained in the identity component $\mathbf{I}^0(M, ds^2)$, and Ozols [22] gave a classification–free proof for the case where Γ is cyclic. The result of Ozols is sketched in Section 5.

Section 6 sketches the proof of the Homogeneity Conjecture for Finsler symmetric spaces. This is a bit technical; the strategy is to develop tools of Finsler geometry that allow one to reduce considerations to the Riemannian case.

The main results in Part I are Theorem 2.1 for constant curvature spaces, Theorem 3.1 (for Riemannian symmetric spaces) and Theorem 6.1 (for Finsler symmetric spaces).

2. Spaces of Constant Curvature.

In this section we indicate the proof of the Homogeneity Conjecture for the cases where (M, ds^2) has constant sectional curvature K. This material comes from [33] and [34]. The statement is

Theorem 2.1. *Let (M, ds^2) be a connected simply connected homogeneous Riemannian manifold of constant sectional curvature K. Let $\pi : (M, ds^2) \to (M', ds'^2) = (\Gamma\backslash M, ds'^2)$ be a Riemannian covering. Then (M', ds'^2) is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement on (M, ds^2).*

Theorem 2.1 will follow directly from Proposition 1.3, Lemmas 2.2 and 2.3, and Proposition 2.11.

2A. Constant Nonpositive Curvature.

The hyperbolic space case $K < 0$ is easy because any two distinct geodesics diverge. If γ is an isometry of bounded displacement and g is a geodesic of (M, ds^2), this says that $\gamma(g) = g$. Since any point $x \in M$ can be described as the intersection of two geodesics this says $\gamma(x) = x$. Thus

Lemma 2.2. *Any isometry of bounded displacement on real hyperbolic space is the identity transformation. If (M', ds'^2) is a homogeneous Riemannian manifold of constant negative curvature, then it is isometric to the real hyperbolic space $\mathbb{H}^n(\mathbb{R})$.*

The euclidean space case $K = 0$ is elementary because any two straight lines diverge unless they are parallel. If γ is an isometry of bounded displacement and σ is a geodesic of (M, ds^2), this says that $\gamma(\sigma)$ is parallel to σ. A rigid motion of euclidean space that sends every straight line to a parallel line must be a pure translation. Thus

Lemma 2.3. *Any isometry of bounded displacement on euclidean space is a pure translation. If (M', ds'^2) is a homogeneous Riemannian manifold of constant sectional curvature zero, then it is isometric to a product $\Gamma\backslash\mathbb{E}^n \cong \mathbb{T}^k \times \mathbb{E}^{n-k}$ of a locally euclidean torus with an euclidean space.*

2B. Constant Positive Curvature: Binary Dihedral and Binary Polyhedral Groups.

The round sphere case $K > 0$ is the hard case. It is worked out in [34] and uses some nontrivial finite group theory [26]. It was more or less conjectured by G. Vincent, at least for the case where Γ is cyclic or binary dihedral, in the last sentence of [29, §10,5]. Here is a sketch of the proof from [34].

We first describe the homogeneous quotients of $\mathbb{S}^n$ so that we know the structure of the groups Γ that occur here. We'll need some finite group preliminaries.

Recall that the *dihedral group* $\mathbb{D}_m$ has order $2m > 4$ and is the symmetry group in $SO(3)$ of a regular m–gon, the *tetrahedral group* $\mathbb{T}$ has order 12 and is the symmetry group in $SO(3)$ of a regular tetrahedron, the *octahedral group* $\mathbb{O}$ has order 24 and is the symmetry group in $SO(3)$ of a regular octahedron, and the *icosahedral group* $\mathbb{I}$ has order 60 and is the symmetry group in $SO(3)$ of a regular icosahedron. The last three are the *polyhedral groups*. Every finite subgroup of $SO(3)$ is a cyclic, dihedral, tetrahedral, octahedral or icosahedral group. If two finite subgroups of $SO(3)$ are isomorphic they are conjugate in $SO(3)$.

Let $p : Sp(1) \to SO(3)$ denote the universal covering group. It is 2–sheeted, and $Sp(1)$ is the group of unit quaternions. The *binary dihedral*, *binary tetrahedral*, *binary octahedral* and *binary icosahedral* groups are the

$$\mathbb{D}_k^* = p^{-1}(\mathbb{D}_k),\ \ \mathbb{T}^* = p^{-1}(\mathbb{T}),\ \ \mathbb{O}^* = p^{-1}(\mathbb{O}) \text{ and } \mathbb{I}^* = p^{-1}(\mathbb{I}).$$

The last three are the *binary polyhedral* groups. Every finite subgroup of $Sp(1)$ is a cyclic, binary dihedral, binary tetrahedral, binary octahedral or binary icosahedral group. If two finite subgroups of $Sp(1)$ are isomorphic they are conjugate in $Sp(1)$.

Suppose that $(M', ds'^2) = \Gamma\backslash\mathbb{S}^n$ is a homogeneous Riemannian manifold of constant sectional curvature $K > 0$. As in Lemma 1.2 the identity component $G' = \mathbf{I}^0(M', ds'^2)$ of the centralizer of Γ in the isometry group $\mathbf{I}(\mathbb{S}^n) = O(n+1)$ is transitive on $\mathbb{S}^n$. In particular it is irreducible on the ambient $\mathbb{R}^{n+1}$. Thus Schur's Lemma says that the centralizer $\mathbb{A}$ of G' in the algebra of linear transformations of $\mathbb{R}^{n+1}$ is a real division algebra. Now there $\mathbb{R}^{n+1}$ is a left $\mathbb{A}$ vector space, $\Gamma \subset \mathbb{A} \cap O(n+1)$, and there are three cases.

(2.4)

1. $\mathbb{A} = \mathbb{R}$, so $\Gamma \subset (\mathbb{R} \cap O(n+1)) = O(1) = \{z \in \mathbb{R} \mid |z = 1\} = \{\pm I\}$.
Then (M', ds'^2) is the round sphere or the real projective space.

2. $\mathbb{A} = \mathbb{C}$, so $\Gamma \subset (\mathbb{C} \cap O(n+1)) = U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$.
Then Γ is a cyclic group $\mathbb{Z}_k, k > 2$, with generator $\mathrm{diag}\{J_k, \dots, J_k\}$
where $J_k = \begin{pmatrix} e^{2\pi\sqrt{-1}/k} & 0 \\ 0 & e^{-2\pi\sqrt{-1}/k} \end{pmatrix}$, and (M', ds'^2) is a "lens space".

3. $\mathbb{A} = \mathbb{Q}$, so $\Gamma \subset (\mathbb{Q} \cap O(n+1)) = Sp(1) = \{z \in \mathbb{Q} \mid |z| = 1\}$.
Then $\Gamma \subset Sp(1)$ is a binary dihedral or binary polyhedral group.

2C. Constant Positive Curvature.

Now we assume that Γ is a finite non–cyclic group that has a faithful unitary representation $\varphi : \Gamma \to U(\ell)$ such that every $\varphi(\gamma)$ is an isometry of constant displacement on the unit sphere $\mathbb{S}^{2\ell-1}$ in $\mathbb{C}^\ell$.

Lemma 2.5. ([34, Lemma 1]) *With Γ as just specified,*

(1) *Every abelian subgroup of Γ is cyclic.*

(2) *Given primes p and q, every subgroup of Γ of order pq is cyclic.*

(3) *Γ has a unique element of order 2. It generates the center of Γ.*

(4) *If α and its transpose α^t are conjugate elements of Γ, then either $\alpha = \alpha^t$ or $\alpha^{-1} = \alpha^t$.*

Sketch of Proof. Statements (1), (2) and the uniqueness of elements of order 2 in Γ follow from the fact that Γ has a free action on a sphere. As Γ has even order [29, §10.5], (3) follows when we show that any central element $\gamma \in \Gamma$ has order 2.

Looking at characters, one sees that the irreducible components of φ are equal and inherit the property that every element in the image is an isometry of constant displacement on the unit sphere in the representation space. Thus we may assume that φ is irreducible. If $\gamma \neq 1$ is central in Γ, Schur's Lemma shows that $\varphi(\gamma)$ is scalar, so its eigenvalues satisfy $\lambda = \overline{\lambda}$, in other words $\lambda = \pm 1$. As $\gamma \neq 1$ now $\varphi(\gamma) = -1$. That proves (3).

If α and α^t are conjugate they have the same eigenvalues, $\{\lambda, \overline{\lambda}\} = \{\lambda', \overline{\lambda'}\}$. If $\lambda = \lambda'$ then $\alpha = \alpha^t$, and if $\lambda = \overline{\lambda'}$ then $\alpha^{-1} = \alpha^t$. That proves (4). ♢

Lemma 2.6. ([34, Lemma 2]) *Let Γ_1 be a normal subgroup of Γ, assume Γ_1 cyclic or binary dihedral $\mathbb{D}_k^*$ ($k \neq 2$), and suppose Γ generated by Γ_1 and some element $\tau \in \Gamma$. Then Γ is cyclic or binary dihedral*

Proof. This is a typical verification based on the structure of binary dihedral and binary polyhedral groups in terms of generators and relations. First, one supposes that $\Gamma_1 = \langle\alpha\rangle$, cyclic. Then $\tau\alpha\tau^{-1}$ is α or α^{-1} by Lemma 2.5. If $\tau\alpha\tau^{-1} = \alpha$ then Γ is abelian, hence cyclic by Lemma 2.5. If $\tau\alpha\tau^{-1} = \alpha^{-1} \neq \alpha$ then τ has order 4 and Γ is binary dihedral.

Now suppose $\Gamma_1 = \mathbb{D}^*_m$ with $m > 2$: $\alpha^m = \beta^4 = 1$ and $\beta\alpha\beta^{-1} = \alpha^{-1}$. As $m \neq 2$, $\langle\alpha\rangle$ is a characteristic subgroup of Γ_1, hence normal in Γ, so $\tau\alpha\tau^{-1}$ is α or α^{-1}. β^2 is central in Γ because it has order 2. Thus $\Gamma' := \langle\alpha, \beta^2, \tau\rangle$ or $\Gamma' := \langle\alpha, \beta^2, \tau\beta\rangle$ is abelian, hence cyclic, and $\Gamma = \langle\Gamma', \beta\rangle$. $\tau\beta\tau^{-1}$ has order 4 so it has form $\beta\alpha^u$ or $\beta^3\alpha^u$. Thus $\beta^{-1}\tau\beta$ has form $\alpha^u\tau$ or $\alpha^u\tau\beta^2$, and $\beta^{-1}(\tau\beta)\beta$ has form $\alpha^u(\tau\beta)$ or $\alpha^u(\tau\beta)\beta^2$. Thus Γ_1 is normal in Γ, and the first paragraph of the proof shows that Γ is binary dihedral. □

Now we need a result of G. Vincent [29, Théorème X] which implies that if Γ has all Sylow subgroups cyclic then it is either cyclic or binary dihedral $\mathbb{D}^*_m$ (m odd). We will also need a procedure of H. Zassenhaus [45, proof of Satz 7], which depends on his result [45, Satz 6]: If Γ is solvable and of order not divisible by 2^{s+1}, and if Γ has an element of order 2^{s-1}, $s > 1$, then Γ has a normal subgroup Γ_1 with cyclic Sylow 2–subgroup, such that Γ/Γ_1 is the cyclic group $\mathbb{Z}_2$, the alternating group $\mathbb{A}_4$, or the symmetric group $\mathbb{S}_3$. While [45] has errors, they are corrected in [46] and [47], and they have no consequences for our results here.

Lemma 2.7. *If* Γ *is solvable then it is cyclic, binary dihedral, binary tetrahedral or binary octahedral.*

Proof. The odd Sylow subgroups of Γ are cyclic and the 2–Sylow subgroups are either cyclic or generalized quaternionic (binary dihedral $\mathbb{D}^*_m$ where $m > 1$ is a power of 2), because every abelian subgroup of Γ is cyclic. If the 2–Sylow subgroups of Γ are cyclic, we are done by the above-mentioned result Vincent. Otherwise, Γ has order $2^s n$ with n odd and $s > 2$, and an element of order 2^{s-1}. From the above-mentioned result of Zassenhaus we have a normal subgroup $\Gamma_1 \subset \Gamma$ with all Sylow subgroups cyclic and Γ/Γ_1 equal to $\mathbb{Z}_2, \mathbb{A}_4$ or $\mathbb{S}_4$, and Γ_1 is either cyclic of $\mathbb{D}^*_m, m$ odd by the result of Vincent.

If $\Gamma/\Gamma_1 = \mathbb{Z}_2$ then Γ is cyclic or binary dihedral by Lemma 2.6.

If $\Gamma/\Gamma_1 = \mathbb{A}_4$ with Γ_1 cyclic one argues that Γ is binary dihedral or binary tetrahedral. If $\Gamma/\Gamma_1 = \mathbb{A}_4$ with Γ_1 binary dihedral one argues that Γ is binary dihedral. These arguments are a little bit complicated.

If $\Gamma/\Gamma_1 = \mathbb{S}_4$ with projection $\psi : \Gamma \to \mathbb{S}_4$ then $\Gamma' := \psi^{-1}(\mathbb{A}_4)$ is binary dihedral or binary tetrahedral. If $\Gamma' = \mathbb{D}^*_q$ then $\Gamma = \mathbb{D}^*_{2q}$ by Lemma 2.6. If $\Gamma' = \mathbb{T}^*$ then $\Gamma_1 = \mathbb{Z}_2$ and $\Gamma = \mathbb{O}^*$. □

Now we need only show that if Γ is not solvable then $\Gamma = \mathbb{I}^*$. For that we use $\mathbb{I}^* \cong SL(2,5)$, the group of 2×2 unimodular matrices over the field $\mathbb{F}_5$, and M. Suzuki's theorem [26, Theorem E] that a non–solvable group with every abelian subgroup cyclic has a normal subgroup isomorphic to some $SL(2,p)$ with $p > 3$ prime.

Lemma 2.8. *If* $\Gamma \cong SL(2,p)$ *then* $p = 3$ *or* $p = 5$.

Proof. Let ω generate the multiplicative group of non-zero elements of the field of p elements, and

$$\nu = \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{smallmatrix}\right) \text{ and } \alpha = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \text{ in } SL(2,p)$$

Then $\nu\alpha\nu^{-1} = \alpha^{\omega^2}$ so $\omega^2 = \pm 1 \mod p$ by Lemma 2.5, so $\omega^4 = 1$, so $p-1$ divides 4. Thus p is 2, 3, or 5, and $p \neq 2$ because $SL(2,2)$ has several éléments of order 2. □

Lemma 2.9. *If* Γ *has a normal subgroup* $\Gamma_1 \cong SL(2,5)$ *then* $\Gamma = \Gamma_1$.

Idea of Proof. One argues that $\Gamma/\Gamma_1 \subset \mathrm{Aut}(\Gamma_1)/\mathrm{Int}(\Gamma_1)$. That group has order 2, and its nontrivial element $\mathrm{Ad}(\sigma)$ is represented by conjugation by $\left(\begin{smallmatrix} 0 & -1 \\ 2 & 0 \end{smallmatrix}\right)$.

Suppose $\Gamma \neq \Gamma_1$. We can assume σ^2 central in Γ, so $\sigma^2 = -1 \in SL(2,5)$. Denote $\alpha = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\beta = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in $SL(2,5)$. Then $\mathrm{Ad}(\sigma)\alpha = \beta^3$ and $\gamma\alpha\gamma^{-1} = \beta^{-1}$ so β is conjugate in *Gamma* to $\beta^{-3} = \beta^2$. That would say $\beta = I$ or $\beta^3 = I$, which is a contradiction. ♢

Lemma 2.10. *If* Γ *is not solvable then* $\Gamma \cong \mathbb{I}^*$.

Proof. Lemmas 2.8 and 2.9, and the result of Suzuki [26, Theorem E], show $\Gamma \cong SL(2,5)$. $\square$

Combining Lemmas 2.7 and 2.10 we have the first part of

Proposition 2.11. *Consider a Riemannian manifold $(M', ds'^2) = \Gamma\backslash\mathbb{S}^n$ of constant positive curvature. Suppose that every element $\gamma \in \Gamma$ is an isometry of constant displacement on $\mathbb{S}^n$. Then the group Γ is cyclic, binary dihedral, binary tetrahedral, binary octahedral or binary icosahedral, and (M', ds'^2) is homogeneous.*

Idea of Proof. Since every $\gamma \in \Gamma$ is of constant displacement on $\mathbb{S}^n$, we know the structure of Γ from Lemmas 2.7 and 2.10, and we run through the cases to see in each case that Γ is given as in (2.4). This is immediate for Γ cyclic of order 1 or 2. For Γ cyclic of order $m > 2$ a generator is given by J_m in 2.4, so Γ is given by Case 2 of 2.4.

If $\Gamma = \mathbb{D}^*_m$ with $m > 2$ even it has a normal subgroup $\Gamma_1 = \mathbb{Z}_{2m}$ with generator γ as in Case 2 of 2.4 with $k = 2m$, and another generator β with $\beta\gamma\beta^{-1} = \gamma^{-1}$. Then $\beta^2 = \gamma^m = -1$ so β acts as $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in block form on the two eigenspaces of γ. Thus Γ is given by Case 3 of 2.4.

If $\Gamma = \mathbb{D}^*_m$ with $m > 2$ odd, the argument is similar.

If Γ is of form $\mathbb{T}^*$ or then $\mathbb{O}^*$ one builds it up from a cyclic subgroups as for the even binary dihedral cases. As was implicit there, the building blocks have exactly two joint eigenspaces and an element of the normalizer exchanges them.

If $\Gamma = \mathbb{I}^*$ there are only two irreducible representations that give isometries of constant displacement on the corresponding spheres. They have degree 2 and are $O(4)$–conjugate, so we may assume that $\Gamma \to U(\frac{n+1}{2})$ is a sum of copies of just one of those irreducibles. Thus Γ is given by Case 3 of 2.4. $\diamondsuit$

Theorem 2.1 follows by combining Lemmas 2.2 and 2.3 with Proposition 2.11. $\square$

3. Riemannian Symmetric Spaces with Simple Isometry Group.

In Sections 3 and 4 we describe our original proof the Homogeneity Conjecture for the cases where (M, ds^2) is a Riemannian symmetric space. That uses É. Cartan's classification of symmetric spaces. Some of the results along the way are valid more generally for Riemannian homogeneous spaces. We indicate that situation in the notation as well as the statements, in some cases writing (L, dt^2) and (L', dt'^2) instead of (M, ds^2) and (M', ds'^2). Then in Section 5 we describe Ozols' argument which minimizes the use of classification in the proof of Theorem 3.11.

Theorem 3.1. *Let (M, ds^2) be a connected simply connected Riemannian symmetric space. Let $\pi : (M, ds^2) \to (M', ds'^2) = (\Gamma\backslash M, ds'^2)$ be a Riemannian covering. Then (M', ds'^2) is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement on (M, ds^2).*

There are three surprises here. One is that the divergent geodesic argument of Lemma 2.2 works (with some modification) for symmetric spaces of noncompact type. The second is that the round sphere, as in Proposition 2.11, is the hard case for symmetric spaces of compact type. Perhaps that is because the sphere has so many isometries. The third is that the result holds for Finsler symmetric spaces [10].

If (M, ds^2) is a Riemannian product, $(M, ds^2) = (M_1, ds_1^2) \times (M_2, ds_2^2)$ with no Euclidean factor, and if γ is an isometry of constant displacement on (M, ds^2), then $\gamma = \gamma_1 \times \gamma_2$ where each γ_i is an isometry of constant displacement on (M_i, ds_i^2). Thus, in the de Rham decomposition

$$(M, ds^2) = (M_0, ds_0^2) \times (M_1, ds_1^2) \times \cdots \times (M_\ell, ds_\ell^2),$$

(M_0, ds_0^2) euclidean and the other (M_i, ds_i^2) irreducible, we have $\gamma = \gamma_0 \times \gamma_1 \times \cdots \times \gamma_\ell$ where each γ_i is an isometry of constant displacement on (M_i, ds_i^2). Consequently we need only prove Theorem 3.1 for each of the γ_i. By Lemma 2.3 we already know it for γ_0, so in the proof of Theorem 3.1 we may (and do) assume that (M, ds^2) is an irreducible Riemannian symmetric space.

3A. Symmetric Spaces of Nonpositive Curvature.

Proposition 3.2. ([36, Theorem 1]) *If (L, dt^2) is a complete connected simply connected Riemannian manifold of sectional curvature $\geqq 0$, with no euclidean factor in its de Rham decomposition, then every bounded isometry of (L, dt^2) is trivial. In particular if a Riemannian quotient $(L', dt'^2) := \Gamma\backslash(L, dt^2)$ is homogeneous then $\Gamma = \{1\}$ and $(L', dt'^2) = (L, dt^2)$.*

The symmetric space case is a special case of Proposition 3.2:

Proposition 3.3. *Any isometry of bounded displacement on an irreducible Riemannian symmetric space (M, ds^2) of noncompact type is the identity transformation. In particular, if $(M', ds'^2) = \Gamma\backslash(M, ds^2)$ is homogeneous then $(M', ds'^2) = (M, ds^2)$.*

We now assume that (M, ds^2) is an irreducible Riemannian symmetric space of compact type. We will use two results of Élie Cartan: the classification and the description of the full group of isometries. The description in question as formulated in [37, Theorem 8.8.1] is

Proposition 3.4. (É. Cartan) *Let $(M = G/K, ds^2)$ be an irreducible simply connected Riemannian symmetric space where $G = \mathbf{I}^0(M, ds^2)$. Let s be the symmetry. Define $K'' = K \cup s \cdot K$ and $G'' = G \cup s \cdot G$. Let $\mathrm{Aut}(K)^G$ (resp. $\mathrm{Int}(K)^G$) denote the group of all (resp. all inner) automorphisms of K that extend to automorphisms of G. Let K' be the isotropy subgroup of $\mathbf{I}(M, ds^2)$. Then $K' = \bigcup(k_i \cdot K'')$ and $\mathbf{I}(M, ds^2) = \bigcup(k_i \cdot G'')$, disjoint unions where $\mathrm{Aut}(K)^G = \bigcup \left(\mathrm{Ad}(k_i)|_K \cdot \mathrm{Int}(K)^G\right)$ disjoint.*

3B. Compact Simple Isometry Group Using Classification.

We run through the steps in the argument for Theorem 3.1 with $\mathbf{I}(M, ds^2)$ compact and simple.

Lemma 3.5. ([35, Theorem 5.2.2]) *Let (L, dt^2) be a connected homogeneous Riemannian manifold such that the identity components of the isotropy subgroups of $(L, d\tau^2)$ are irreducible on the tangent spaces. Suppose that $\beta \in \mathbf{I}(L, dt^2)$ centralizes $\mathbf{I}^0(L, dt^2)$, $g \in \mathbf{I}(L, dt^2)$ has a fixed point on L, and $\gamma = g\beta$ is an isometry of constant displacement on (L, dt^2). Then $\gamma = \beta$, i.e. $g = 1$.*

Lemma 3.6. ([35, Corollaries 5.2.3 and 5.2.4]) *Suppose that (L, dt^2) is a compact homogeneous Riemannian manifold of Euler characteristic $\chi(L) > 0$. Suppose further that the identity components of the isotropy subgroups of $(L, d\tau^2)$ are irreducible on the tangent spaces. Finally suppose that $\mathbf{I}(L, dt^2) = \bigcup \left(\beta_j \cdot \mathbf{I}^0(L, dt^2)\right)$ where the β_j centralize $\mathbf{I}^0(L, dt^2)$. (This is automatic if $\mathbf{I}^0(L, dt^2)$ has no outer automorphism, in other words if $\mathbf{I}^0(L, dt^2)$ is not of type $A_n (n > 1)$, $D_n (n > 3)$ nor E_6.) Let Γ be a group of isometries of constant displacement on (L, dt^2). Then Γ centralizes $\mathbf{I}^0(L, dt^2)$, so $(L', dt'^2) := \Gamma\backslash(L, dt^2)$ is homogeneous.*

Now we need a result of Jean de Siebenthal [25, pp. 57–58].

Proposition 3.7. *Let L^0 be the identity component of a compact Lie group L, $x \in L$, and T a maximal torus of the centralizer of $Z_L(x)$. Write $\langle T, x\rangle$ for the group generated by T and x. Then every element of the component xL^0 of x is $\mathrm{Ad}(L^0)$–conjugate to an element of xT. If $x' \in Lx$ and T' is a maximal torus of $Z_L(x')$ then $\langle T', x'\rangle$ is $\mathrm{Ad}(L^0)$–conjugate to $\langle T, x\rangle$.*

to show that in certain components of certain $\mathbf{I}(M, ds^2)$ every element has a fixed point, in particular those components do not contain any isometries of constant displacement.

Lemma 3.8. ([35, Lemma 5.3.1]) *Let σ be a symmetry of a compact connected Riemannian symmetric space (M, ds^2). Then every element of $\sigma \cdot \mathbf{I}^0(M, ds^2)$ has a fixed point on M.*

Proof. Write σ for the symmetry at $x_0 \in M$. Let T be a maximal torus of the isotropy subgroup K' of $\mathbf{I}(M, ds^2)$ at x_0. K' and the centralizer of σ in $\mathbf{I}(M, ds^2)$ have the same identity component. Thus de Siebenthal's theorem shows that every element of $\sigma \cdot \mathbf{I}^0(M, ds^2)$ is conjugate to an element of $(\sigma \cdot T) \subset K'$. The Lemma follows. □

The argument of Lemma 3.8 proves

Lemma 3.9. ([35, Lemma 5.3.2]) *Let (L, dt^2) be a compact connected Riemannian homogeneous manifold, K' the isotropy subgroup of $\mathbf{I}(L, dt^2)$ at $x_0 \in L$, $k \in K'$, and $g \in k \cdot \mathbf{I}^0(L, dt^2)$. Suppose that K' contains a maximal torus of the centralizer of k in $\mathbf{I}(L, dt^2)$. Then g has a fixed point on L.*

Lemma 3.10. ([35, Lemma 5.3.3]) *Let (M, ds^2) be a compact connected irreducible Riemannian symmetric manifold such that a connected isotropy subgroup of $\mathbf{I}(M, ds^2)$ admits no outer automorphism. Let γ be an isometry of (M, ds^2) which has no fixed point. Then $\gamma \in \mathbf{I}^0(M, ds^2)$.*

Proof. $\mathbf{I}(M, ds^2) = \mathbf{I}^0(M, ds^2) \cup \sigma \cdot \mathbf{I}^0(M, ds^2)$ by Proposition 3.4. Now apply Lemma 3.8. □

Now we come to the main result of this section.

Theorem 3.11. ([35, Theorem 5.5.1]) *Let (M, ds^2) be a compact connected simply connected irreducible Riemannian symmetric manifold with $\mathbf{I}^0(M, ds^2)$ simple. Let Γ be a group of isometries of constant displacement on (M, ds^2).*

If Γ is finite, then $(M', ds'^2) = \Gamma\backslash(M, ds^2)$ is a Riemannian homogeneous manifold. If (M, ds^2) is neither an odd dimensional sphere, nor a space $SU(2m)/Sp(m)$ with $m > 1$, nor a complex projective space of odd complex dimension > 1, nor a space $SO(4n+2)/U(2n+1)$ with $n > 0$, then Γ is finite and centralizes $\mathbf{I}^0(M, ds^2)$, and (M', ds'^2) is a Riemannian symmetric manifold; otherwise (M, ds^2) has finite groups of constant displacement isometries and the corresponding quotients are Riemannian homogeneous but not Riemannian symmetric.

If (M, ds^2) is neither an odd dimensional sphere nor a space $SU(2m)/Sp(m)$ with $m > 1$, then Γ is finite; otherwise (M, ds^2) has one–parameter groups of constant displacement isometries.

Indication of Proof. Let $G = \mathbf{I}^0(M, ds^2)$ and let K be the isotropy subgroup at $x \in M$. By Lemma 3.6 we need only check the cases where G is of type $A_n (n > 1)$, $D_n (n > 3)$ or E_6. As the statements are known for spheres (see Section 2), É. Cartan's classification of symmetric spaces shows that we need only check the cases

(AI) $SU(n)/SO(n), n > 2$; (AII) $SU(2n)/Sp(n), n > 1$; (AIII) $SU(p+q)/\{S(U(p) \times U(q))\}, pq > 1$;

(DI) $SO(p+q)/SO(p) \times SO(q)$ with $p \geqq 2, q \geqq 2, p+q > 4, p+q$ even; (DIII) $S0(2n)/U(n), n > 1$;

(EI) $E_6/\mathrm{Ad}(C_4)$; (EII) $E_6/\{A_5 \times A_1\}$; (EIII) $E_6/\{D_5 \times T_1\}$; (EIV) E_6/F_4.

The cases where G is a classical group involve quite a lot of matrix calculation, and the E_6 cases depend on classical subgroups of E_6. ♢

4. **Riemannian Symmetric Spaces that are Group Manifolds.**

In this section we indicate the proof of the Homogeneity Conjecture for compact Riemannian group manifolds with bi–invariant metric. Those symmetric spaces were introduced by É. Cartan in [7] and led to his development of symmetric space theory.

Fix a compact connected simply connected simple Lie group G with bi–invariant Riemannian metric ds^2. Let $T(G)$ denote the set of all isometries of the form

$$(g_1, g_2) : x \mapsto g_1^{-1} x g_2 \text{ where } x,\ g_1,\ g_2 \in G.$$

Then $T(G)$ is the identity component of the isometry group $\mathbf{I}(G, ds^2)$.

Lemma 4.1. ([35, Lemmas 4.2.1, 4.2.2]) *If $(g_1, g_2) \in T(G)$ is an isometry of constant displacement on (G, ds^2) then every conjugate of g_1 commutes with every conjugate of g_2, so either g_1 or g_2 is central in G.*

The symmetry $\sigma : x \mapsto x^{-1}$ satisfies $\sigma \cdot (g_1, g_2) \cdot \sigma^{-1} = (g_2, g_1)$. Let $I'(G, ds^2)$ denote the group $T(G) \cup \sigma \cdot T(G)$ of isometries of (G, ds^2).

Proposition 4.2. ([35, Lemma 4.2.3, Theorem 4.2.4]) *Every element of* $\sigma \cdot T(G)$ *has a fixed point on* G. *Every subgroup* $\Gamma \subset I'(G, ds^2)$ *of isometries of constant displacement is conjugate in* $I'(G, ds^2)$ *to a group of left translations of* G.

Lemma 4.3. ([35, Lemmas 4.3.1, 4.3.2]) *Let* α *be an automorphism of* G. *Let* $g \in G$ *and suppose that* $x \mapsto g\alpha(x)$ *is an isometry of constant displacement on* (G, ds^2). *Then* $\alpha(ug\alpha(u^{-1})) = ug\alpha(u^{-1})$ *for ever* $u \in G$. *Let* B *denote the identity component of the centralizer of* g *in* G. *Then* $\alpha(B) = B$, *and if* $\alpha|_B$ *is an inner automorphism of* B *then* α *is inner on* G.

Proposition 4.4. ([35, Theorem 4.3.3]) *Let* $\alpha \in \mathrm{Aut}(G)$, $(g_1, g_2) \in T(G)$, *and* $\gamma = (g_1, g_2) \cdot \alpha$. *Suppose that both* γ *and* γ^2 *are isometries of constant displacement on* (G, ds^2). *Then* α *is an inner automorphism of* G.

Lemma 4.5. ([35, Lemma 4.3.4]) *Let* $\alpha \in \mathrm{Aut}(G)$, σ *the symmetry* $x \mapsto x^{-1}$ *and* $g \in G$. *Suppose that* $\gamma = (g, 1) \cdot \alpha \cdot \sigma$ *is an isometry of constant displacement on* (G, ds^2). *Then* $(ug\alpha^2(u^{-1}))^{-1} = \alpha(ug\alpha^2(u^{-1}))$ *for every* $u \in G$.

Now we come to the main results of this section:

Theorem 4.6. ([35, Theorem 4.5.1]) *Let* G *be a compact connected Lie group and* ds^2 *a bi–invariant Riemannian metric on* G. *Let* Γ *be a group of isometries of constant displacement on* (G, ds^2). *Then* Γ *is conjugate in* $\mathbf{I}(G, ds^2)$ *to a group of left translations on* G. *Thus* $\Gamma\backslash(G, ds^2)$ *is homogeneous.*

In particular, the Homogeneity Conjecture holds for Riemannian symmetric spaces that are group manifolds.

Combining Theorems 3.11 and 4.6 we have verified the Homogeneity Conjecture (3.1) for Riemannian symmetric spaces.

Remark 4.7. In Theorem 4.6 the metric ds^2 on G is bi-invariant. Later we will see the extent to which the result holds more generally for left–invariant metrics on Lie groups.

5. **A Classification Free Approach.**

In this section we look at the characterizations of constant displacement isometries in terms of invariant geodesics. We follow the notation of [22], slightly adjusted for consistency with the other sections of this paper. Let (M, ds^2) be a complete Riemannian manifold and $\sigma : \mathbb{R} \to M$ a geodesic parameterized proportional to arc length. Then an isometry $\gamma \in \mathbf{I}(M, ds^2)$ *preserves* σ if there is a constant c such that $\gamma(\sigma(t)) = \sigma(t + c)$ for all $t \in \mathbb{R}$. We say that

γ satisfies P_x if γ preserves at least one minimizing geodesic from x to $\gamma(x)$,

$Pres(\gamma) = \{x \in M \mid \gamma \text{ satisfies } P_x\}$, and

$Crit(\gamma) = \{x \in M \mid x \text{ is a critical point of the square of the displacement function of } \gamma\}$.

In [21] it was shown that if γ has small displacement, i.e. if $\gamma(x)$ is not in the cut locus of x for any $x \in M$, then $Crit(\gamma) = Pres(\gamma)$, and from that isometries of small constant displacement have transitive centralizers. The results of [22] extend that. The main results of this section, taken from [22], are the following.

Proposition 5.1. [22, Theorem 1.6] *Let* (M, ds^2) *be a complete connected* C^∞ *Riemannian manifold. Given an isometry* $\gamma \in \mathbf{I}(M, ds^2)$ *the following are equivalent.*

1. γ *is an isometry of constant displacement.*
2. *If* $x \in M$ *then* γ *preserves some minimizing geodesic from* x *to* $\gamma(x)$.
3. *If* $x \in M$ *then* γ *preserves every minimizing geodesic from* x *to* $\gamma(x)$, *i.e.* $Pres(\gamma) = M$.

Proposition 5.2. [22, Theorem 2.6] *Let* (M, ds^2) *be a connected simply connected Riemannian symmetric space of compact type. If* $\gamma \in \mathbf{I}(M, ds^2)$ *then* $Pres(\gamma) = \bigcup Z_G^0(\gamma) \cdot x_i$ *where* $\{x_i\}$ *is a set of representatives of the components of* $Pres(\gamma)$.

Corollary 5.3. [22, Corollary 2.7] *Let (M, ds^2) be a connected simply connected Riemannian symmetric space of compact type. If $\gamma \in \mathbf{I}(M, ds^2)$ then γ is an isometry of constant displacement if and only if $Z_G^0(\gamma)$ is transitive on M.*

Corollary 5.4. [22, Corollary 2.9] *Let (M, ds^2) be a connected simply connected Riemannian symmetric space of compact type. Let Γ be a finite cyclic group of isometries of constant displacement. Then $\Gamma\backslash(M, ds^2)$ is homogeneous.*

6. **Extension to Finsler Symmetric Spaces.**

In this section we indicate the steps in the proof the Finsler symmetric space analog of Theorem 3.1. The result is

Theorem 6.1. (Deng–Wolf [10, Theorem 1.1]) *Let Γ be a properly discontinuous group of isometries of a connected simply connected globally symmetric Finsler space (M, F). Then $\Gamma\backslash(M, F)$ is a homogeneous Finsler space if and only if Γ consists of isometries of constant displacement. Further, if $\Gamma\backslash(M, F)$ is homogeneous, and if in the decomposition of (M, F) as the Berwald product of Minkowski space and irreducible symmetric Finsler spaces, none of whose factors is*

a compact Lie group with a bi-invariant Finsler metric,
an odd-dimensional sphere with the constant curvature Riemannian metric,
a complex projective of odd complex dimension > 1 with the standard Riemannian metric,
$\mathrm{SU}(2n)/\mathrm{Sp}(n)$*, $n \geq 2$ with a possibly non–Riemannian* $\mathrm{SU}(2n)$*-invariant Finsler metric, nor*
$\mathrm{SU}(4n+2)/\mathrm{U}(2n+1)$*, $n \geqq 1$, with a possibly non–Riemannian* $\mathrm{SU}(4n+2)$*-invariant Finsler metric,*

then $\Gamma\backslash(M, F)$ is a Finsler symmetric space.

This uses some essential background on Finsler spaces, all of which can be found in the background expository parts of S. Deng's treatise [9]. Then the notion of Minkowski Lie Algebra is introduced as a convenience in studying affine symmetric Berwald spaces, leading to a Finsler symmetric space variation on the Riemannian manifold de Rham decomposition. That applies to affine symmetric Berwald spaces. As in the Riemannian case, isometries of constant displacement decompose accordingly as products. That uses the ideas behind Ozols' description [22] of constant displacement isometries in terms of invariant minimizing geodesics, and Crittenden's description [8] of cut locus *vs* conjugate locus in symmetric spaces.

At that point the connected simply connected affine symmetric Berwald space (M, F) is a Berwald product $(M, F) = (M_0, F_0) \times (M_1, F_1) \times \cdots \times (M_r, F_r)$ with (M_0, F_0) flat and the other (M_i, F_i) irreducible, and the group Γ decomposes accordingly as $\Gamma = \Gamma_0 \times \Gamma_1 \times \cdots \times \Gamma_r$. The Homogeneity Conjecture is easy for (M_0, F_0) and more or less similar to the Riemannian case for (M_i, F_i) of noncompact type. For (M_i, F_i) of compact type one needs the result of Szabó [27], that if (M, F) is a Berwald space, then there exists a Riemannian metric Q on M whose Levi-Cività connection coincides with the linear connection of (M, F). With that one can reduce the Homogeneity Conjecture for (M_i, F_i) of compact type to the Riemannian result, and the main part of Theorem 6.1 follows. As in the Riemannian case, the part on symmetric space quotients is easily extracted.

Part II. Geometric Classes of Compact Riemannian Manifolds.

In this Part we verify the Homogeneity Conjecture for several classes of compact Riemannian homogeneous spaces. Those are manifolds of Euler charactristic $\chi(G/H) \neq 0$ in Section 8, compact group manifolds in Section 9 and manifolds of positive curvature in Section 10. But first we develop some tools in Section 7 that we will need for those geometrically defined classes of manifolds.

7. **Isotropy Splitting Fibrations.**

In this section we develop and apply a tool for reducing cases of the Homogeneity Conjecture to Riemannian symmetric spaces and other cases that we will have verified. The tool is based on the idea of the classical fibration $SO(k+\ell)/SO(k) \to (SO(k)\times SO(\ell))$ of a Stieffel manifold over a Grassmann manifold. It applies to real, complex and quaternionic Stieffel manifolds and will be a key to considering Riemannian homogeneous spaces of the classes mentioned above.

7A. Definition and Goal.

Our basic setup here is

$$\begin{aligned}&G \text{ is a compact connected simply connected Lie group,}\\ &K = HN \text{ where } H \text{ and } N \text{ are closed connected subgroups of } G \text{ such that}\\ &\quad\text{(i) } K=(H\times N)/(H\cap N),\ \text{(ii) Lie algebras } \mathfrak{h}\perp\mathfrak{n} \text{ and (iii) } \dim H \neq 0 \neq \dim N\,,\\ &\text{the centralizers } Z_G(H)=Z_H N^\flat \text{ and } Z_G(N)=H^\flat Z_N \text{ with } H=(H^\flat)^0 \text{ and } N=(N^\flat)^0\\ &M=G/H \text{ and } M''=G/K \text{ are normal Riemannian homogeneous spaces of } G.\end{aligned} \tag{7.1}$$

We may assume that the the metrics ds^2 on M and ds''^2 on M'' are the normal metrics given by the negative of the Killing form of $\mathfrak{g}$. Also, since G is simply connected, and H and N are connected, M and M'' are simply connected. We refer to the fibration $\pi : M \to M''$ as an *isotropy splitting fibration.* Since N normalizes H, $G\times N$ acts on M by left and right translations, $\ell(g)r(n)(xH) = gxHn^{-1} = gxn^{-1}H$. The main result on isotropy splitting fibrations is

Theorem 7.2. *Suppose that $M = G/N$ is compact, connected and simply connected, that* $\operatorname{rank} K = \operatorname{rank} G$ *and that $\{G, H, N\}$ satisfies* (7.1). *If Γ is a group of isometries of constant displacement on (M, ds^2) then $\Gamma \subset (\ell(Z_G)\times r(N))$ where Z_G denotes the center of G. Conversely, if $\Gamma \subset (\ell(Z_G)\times r(N))$ then every $\gamma\in\Gamma$ is an isometry of constant displacement on (M, ds^2).*

Examples 7.3. *Before indicating the proof of* Theorem 7.2 *we mention some interesting examples.*

1. G/K is an irreducible hermitian symmetric space and $K = Z_K^0 K'$.
   - $G/K' \to G/K$ is a circle bundle over G/K ($H = K'$ and $N = Z_K^0$)
   - $G/Z_K^0 \to G/K$ is a principal K'–bundle over G/K ($H = Z_K^0$ and $N = K'$)
   - $SU(s+t)/SU(s) \to SU(s+t)/S(U(s)U(t))$ and $SU(s+t)/U(s) \to SU(s+t)/S(U(s)U(t))$
2. G/K is a quaternion–kaehler symmetric space and $K = Sp(1)K'$.
   - $G/K = SO(s+t)/SO(s)SO(t)$ with $s = 3, 4$, $s \leqq t$, st even
   - $G/K = Sp(s+t)/Sp(s)Sp(t)$ with $1 \leqq s \leqq t$
   - $G/K = G_2/A_1A_1\,, F_4/A_1C_3\,, E_6/A_1A_5\,, E_7/A_1D_6$ or E_8/A_1E_7
3. G/K is a nearly–kaehler 3–symmetric space and $K = SU(3)K'$ so G/K is G_2/A_2, F_4/A_2A_2, $E_6/A_2A_2A_2$, E_7/A_2A_5, or E_8/A_2E_6. In the F_4 case the $A_2 = SU(3)$ can be given by a long root or a short root. The corresponding fibrations are $G/K' \to G/K$ principal $SU(3)$ bundle and $G/SU(3) \to G/K$ principal K' bundle.
4. G/K is the 5–symmetric space E_8/A_4A_4, yielding to two principal $SU(5)$ bundles $E_8/A_4 \to E_8/A_4A_4$.
5. G/K is an odd dimensional real Grassmann manifold and $K = SO(2s+1)\times SO(2t+1)$. Then $G/K_1 \to G/K$ is $SO(2s+2t+2)/SO(2s+1) \to SO(2s+2t+2)/(SO(2s+1)\times SO(2t+1))$.

7B. The Full Isometry Group.

Here we run through the arguments of [42]. The corresponding part of Section 9 will correspond to a limiting case $\mathfrak{n} = 0$. As we saw dealing with compact symmetric spaces, verification of the Homogeneity Conjecture requires that we find the full group of isometries of (M, ds^2). We do that now. As more or less noted earlier,

Lemma 7.4. *The right action of $N^\flat$ on M, given by $r(n)(gH) = gHn^{-1} = gn^{-1}H$, is a well defined action by isometries. The fiber of $\pi : M \to M''$ through gH is $F := r(N)(gH)$.*

Now we have larger (than G) transitive groups of isometries of M given by

$$G^\flat = \ell(G)\cdot r(N^\flat) \text{ and } (G^\flat)^0 = \ell(G)\cdot r(L) \text{ acting by } (\ell(g), r(n)) : xH \to g(xH)n^{-1} = gxn^{-1}H\,. \tag{7.5}$$

Every $g^\flat := (\ell(g), r(n))$ sends fiber to fiber and induces an isometry $\ell(g) \in (M'', ds''^2)$. Specializing a theorem of Reggiani [23, Corollary 1.3] for the first assertion we have

Proposition 7.6. *If the Riemannian manifold (M, ds^2), $M = G/H$, is irreducible for its de Rham decomposition, then $(G^\flat)^0$ is the identity component $\mathbf{I}^0(M, ds^2)$ of its isometry group. Consequently*

(1) *The algebra of all Killing vector fields on (M, ds^2) is $d((\ell, r)(\mathfrak{g}^\flat)) := d\ell(\mathfrak{g}) \oplus dr(\mathfrak{n})$.*

(2) *Define $\mathfrak{t} = \{\xi \in \mathfrak{g} \mid \xi$ defines a constant length Killing vector field on $(M'', ds''^2)\}$. Then the set of all constant length Killing vector fields on (M, ds^2) is $d\ell(\mathfrak{t}) \oplus dr(\mathfrak{n})$.*

(3) *If $\operatorname{rank} K = \operatorname{rank} G$ then $\mathfrak{t} = 0$ and $dr(\mathfrak{n})$ is the set of all constant length Killing vector fields on (M, ds^2).*

Corollary 7.7. *If $H \not\cong N$ then every isometry of (M, ds^2) normalizes $r(N)$ and sends fiber to fiber of $\pi : M \to M''$.*

The normalizer of H in G also normalizes the centralizer of N so it normalizes K as well. Denote

$$
\begin{aligned}
&\operatorname{Aut}(G,H) = \{\alpha \in \operatorname{Aut}(G) \mid \alpha(H) = H\}, \ \operatorname{Int}(G,H) = \{\alpha \in \operatorname{Aut}(G,H) \mid \alpha|_H \text{ is inner}\},\\
&\qquad\text{and } \operatorname{Out}(G,H) = \operatorname{Aut}(G,H)/\operatorname{Int}(G,H);\\
&\operatorname{Aut}(G,H,ds^2) = \{\alpha \in \operatorname{Aut}(G,H) \mid \alpha^* ds^2 = ds^2\}, \ \operatorname{Int}(G,H,ds^2) = \operatorname{Aut}(G,H,ds^2) \cap \operatorname{Int}(G,H),\\
&\qquad\text{and } \operatorname{Out}(G,H,ds^2) = \operatorname{Aut}(G,H,ds^2)/\operatorname{Int}(G,H,ds^2),
\end{aligned} \tag{7.8}
$$

and similarly for $\operatorname{Aut}(G,K), \operatorname{Int}(G,K), \operatorname{Out}(G,K)$ and $xxx(G,K,ds''^2)$. $\operatorname{Out}(G,H) \subset \operatorname{Out}(G,K)$ because H is a local direct factor of K. In many cases $\operatorname{Out}(G,H) = \operatorname{Out}(G,K)$ because $\mathfrak{n}$ is the $\mathfrak{g}$–centralizer of $\mathfrak{h}$ and $\mathfrak{h} \not\cong \mathfrak{n}$. But there are exceptions, such as orthocomplementation (which exchanges the two factors of K) in the cases of Stieffel manifold fibrations

$$
\begin{aligned}
&SO(2k)/SO(k) \to SO(2k)/[SO(k) \times SO(k)],\\
&SU(2k)/U(k) \to SU(2k)/S(U(k) \times U(k)) \text{ and}\\
&Sp(2k)/Sp(k) \to Sp(2k)/[Sp(k) \times Sp(k)].
\end{aligned}
$$

There are other exceptions, including $E_6/[A_2A_2A_2]$, but neither F_4/A_2A_2 nor E_8/A_4A_4 is an exception.

Lemma 7.9. *Suppose that $\operatorname{rank} K = \operatorname{rank} G$. Let $\alpha \in \operatorname{Aut}(G,H)$ (and thus also $\alpha \in \operatorname{Aut}(G,N)$, so $\alpha \in \operatorname{Aut}(G,K)$). Then the following conditions are equivalent:* (i) *$\alpha|_K$ is an inner automorphism of K,* (ii) *as an isometry, $\alpha \in \mathbf{I}^0(M, ds^2)$, and* (iii) *as an isometry, $\alpha \in \mathbf{I}^0(M'', ds''^2)$.*

Denote base points $x_0 = 1H \in G/H = M$ and $x_0'' = 1K \in G/K = M''$. We reformulate Lemma 7.9 as

Lemma 7.10. *Let $\operatorname{rank} K = \operatorname{rank} G$. Recall that ds^2 is the normal metric on $M = G/H$ defined by the negative of the Killing form of $\mathfrak{g}$. The isotropy subgroup $\mathbf{I}(M, ds^2)_{x_0}$ has identity component*

$$\mathbf{I}^0(M, ds^2)_{x_0} = H \cdot \{(\ell(n), r(n)) \in G^\flat \mid n \in N\}$$

and

$$\mathbf{I}(M, ds^2)_{x_0} = \bigcup_{\alpha \in \operatorname{Out}(G,H)} H\alpha \cdot \{(\ell(n), r(n)) \in G^\flat \mid n \in N^\flat\} \cong \bigcup_{\alpha \in \operatorname{Out}(G,H)} H\alpha \cdot N^\flat. \tag{7.11}$$

Given $\alpha \in \operatorname{Out}(G,H)$ the component $H\alpha \cdot N^\flat = H\alpha' \cdot N^\flat$ if and only if $\alpha = \alpha'$ modulo inner automorphisms.

We now define two subgroups $G^\dagger \subset \mathbf{I}(M, ds^2)$ and $G''^\dagger \subset \mathbf{I}(M'', ds''^2)$ of the isometry groups by

$$G^\dagger = \bigcup_{\alpha \in \operatorname{Out}(G,H)} G\alpha \cdot N^\flat \subset \mathbf{I}(M, ds^2) \text{ and } G''^\dagger = \bigcup_{\beta \in \operatorname{Out}(G,K)} G\beta \subset \mathbf{I}(M'', ds^2)\,. \tag{7.12}$$

Here $(g\alpha, n)$ acts on M by $xH \mapsto g\alpha(x)n^{-1}H$ and $g\beta$ acts on M'' by $xK \mapsto g\beta(x)K$.

Theorem 7.13. *Let $\pi : M \to M''$, be an isotropy–split fibration with $M = G/H$ and $M'' = G/K$ as in* (7.1)*, and $\operatorname{rank} K = \operatorname{rank} G$. Recall that ds^2 is the normal Riemannian metric on M defined by the negative of the Killing form of $\mathfrak{g}$. Then the identity component $\mathbf{I}^0(M, ds^2) = G^\flat$ and the full isometry group $\mathbf{I}(M, ds^2) = G^\dagger$.*

7C. Isometries of Constant Displacement.

We first accumulate a few simple observations:

(i) If $\text{rank}\, K = \text{rank}\, G$ and $g^\flat = (g, r(n)) \in G^\flat$ there is a $g^\flat$–invariant fiber $xF = \pi^{-1}(xK)$ of $M \to M''$,

(ii) the metrics ds^2 on $M = G/H$ and ds''^2 on $M'' = G/K$ are naturally reductive relative to G,

(iii) the isotropy–split manifold (M, ds^2) is a geodesic orbit space,

(iv) the fiber F of $\pi : M \to M''$ is totally geodesic in (M, ds^2),

(v) $(F, ds^2|_F)$ is a geodesic orbit space.

The splitting fibration construction, starting with Proposition 7.14 just below, is used for a flat rectangle argument: ξ_1 and ξ_2 are commuting Killing vector fields, typically $\xi_2 \in dr(\mathfrak{n})$ and $\xi_1 \in \mathfrak{g}$, such that $\xi_1 \perp \mathfrak{n}$ and both $g \times r(l) = \exp(\xi_1 + \xi_2)$ and $r(l) = \exp(\xi_2)$ have the same constant displacement. Then the $\exp(t_1\xi_1 + t_2\xi_2)(1H)$, for $0 \leqq t_{\leqq} 1$, form a flat rectangle. There $r(l)$ is displacement along one side while $g \times r(l)$ is displacement along the diagonal. Since these displacements are the same one argues that $\xi_1 = 0$. The result is

Proposition 7.14. *Suppose that* $\text{rank}\, K = \text{rank}\, G$. *Let* Γ *be a subgroup of* $G^\flat$ *such that every* $\gamma \in \Gamma$ *is an isometry of constant displacement on* (M, ds^2). *Then* $\Gamma \subset (Z_G \times r(N^\flat))$ *where* Z_G *is the center of* G.

Proposition 7.14 applies, in particular, to Examples 7.3(1) through Examples 7.3(4). Now we look for the components of the full isometry group.

Lemma 7.15. [42, Lemma 5.5] *Suppose that* $\text{rank}\, K = \text{rank}\, G$. *Let* $\alpha \in \text{Out}(G, H)$ *and* $\gamma \in G^\flat\alpha$ *such that both* γ *and* γ^2 *are isometries of constant displacement on* (M, ds^2). *Then* $\alpha|_H$ *is an inner automorphism of* H *and* $\gamma \in (Z_G \times r(N^\flat))$.

Theorem 7.16. *Let* $\pi : (M, ds^2) \to (M'', ds''^2)$, *be an isotropy–split fibration with* $M = G/H$ *and* $M'' = G/K$ *as in* (7.1), *and* $\text{rank}\, K = \text{rank}\, G$. *Recall that* ds^2 *is the normal Riemannian metric on* M *defined by the negative of the Killing form of* $\mathfrak{g}$. *Let* $(M, ds^2) \to \Gamma\backslash(M, ds^2)$ *be a Riemannian covering whose deck transformation group* Γ *consists of isometries of constant displacement. Then* $\Gamma \subset (Z_G \times r(N^\flat))$ *and* $\Gamma\backslash(M, ds^2)$ *is homogeneous. In other words the Homogeneity Conjecture is verified for* (M, ds^2).

7D. Odd Real Stieffel Manifolds.

The idea of isotropy split fibrations came from the fibrations of Stieffel manifolds over Grassmann manifolds. The cases $\chi(M'') \neq 0$ are covered by Theorem 7.16. That leaves just one case, Example 7.3(5), the case of the odd dimensional real Grassmann manifold where $\chi(M'') = 0$:

(7.17) $$M = SO(2s + 2t + 2)/SO(2s + 1) \to SO(2s + 2t + 2)/(SO(2s + 1) \times SO(2t + 1)) = M''.$$

We know $\mathbf{I}^0(M, ds^2) = G \times r(N)$ by Proposition 7.6. The following variation on Proposition 7.14 uses an argument of Cámpoli [5].

Proposition 7.18. *Let* $\pi : M \to M''$ *as in* (7.1) *with* $\chi(M) = 0$. *If* Γ *is a group of isometries of constant displacement on* (M, ds^2), *and if* $\Gamma \subset \mathbf{I}^0(M, ds^2)$, *then* $\Gamma \subset (Z_G \times r(N))$. *Conversely if* $\Gamma \subset (Z_G \times r(N))$ *then every* $\gamma \in \Gamma$ *is an isometry of constant displacement on* (M, ds^2).

A look at the relevant group structure shows that $\mathbf{I}(M, ds^2) = \mathbf{I}^0(M, ds^2) \cup \sigma \cdot \mathbf{I}^0(M, ds^2)$ where σ is the symmetry on the symmetric space (M'', ds''^2). A short matrix calculation shows that every element of $\sigma \cdot \mathbf{I}^0(M, ds^2)$ has a fixed point on M. In view of Theorem 7.18 we can adapt the $\chi(M) > 0$ argument to the odd Stieffel manifold split fibration, as follows.

Proposition 7.19. *Let* $\pi : (M, ds^2) \to (M'', ds''^2)$ *as in* (7.17). *If* Γ *is a group of isometries of constant displacement on the Stieffel manifold* (M, ds^2) *then* $\Gamma \subset (\{\pm I\} \times r(SO(1 + 2t)))$, *and* $(M, ds^2) = \Gamma\backslash(M, ds^2)$ *is homogeneous. Thus the Homogeneity Conjecture holds for the Stieffel manifold fibration* (7.17) *over an odd dimensional real Grassmann manifold.*

Since the odd dimensional real Grassmann manifolds are the only irreducible Riemannian symmetric spaces $M = G/K$, G simple, that fit the decomposition $K = HN$ of (7.1), Proposition 7.19 can be made to appear to be more general:

> Let $\pi : (M, ds^2) \to (M'', ds''^2)$ as in (7.1) where $M'' = G/K$, G simple, is a Riemannian symmetric space. Then the Homogeneity Conjecture holds for (M, ds^2).

This will become more interesting when we look at the group manifold case.

8. Manifolds of Positive Euler Characteristic.

In this section we describe the current state of research on the Homogeneity Conjecture for manifolds (M, ds^2) of nonzero Euler characteristic. If $M = G/K$ and the metric ds^2 is normal relative to G then the Homogeneity Conjecture was verified in Theorem 7.16. But if ds^2 is not normal there are some open problems. The current result for ds^2 that need not be normal is as follows.

Theorem 8.1. *Suppose that $M = G/K$ is compact, connected and simply connected, with $\chi(M) > 0$ i.e.* $\operatorname{rank} K = \operatorname{rank} G$. *Suppose further that G is simple, that ds^2 is a G–invariant Riemannian metric on M* (not necessarily normal)*, and that $G = \mathbf{I}^0(M, ds^2)$.*

Let Γ be a group of isometries of constant displacement on (M, ds^2), such that each $\mathrm{Ad}(\gamma)$*, $\gamma \in \Gamma$, is an inner automorphism of G.* {This is automatic unless G is of type A_n ($n \geqq 2$), D_n ($n \geqq 4$) or E_6.}. *Then Γ centralizes G, each component of $\mathbf{I}(M, ds^2)$ contains at most one element of Γ, and the Riemannian quotient manifold $\Gamma\backslash(M, ds^2)$ is homogeneous. Conversely, of course, if $\Gamma\backslash(M, ds^2)$ is homogeneous then every $\gamma \in \Gamma$ is an isometry of constant displacement on (M, ds^2).*

Theorem 8.1 will be an immediate consequence of Corollary 8.10 and Proposition 8.11. At this time it is not known whether we really need the restriction that each $\mathrm{Ad}(\gamma)$, $\gamma \in \Gamma$, be an inner automorphism of G. Following Lemma 7.15 that was automatic for ds^2 normal.

First we describe the isometry group $\mathbf{I}(M, ds^2)$ where $M = G/K$, $\operatorname{rank} G = \operatorname{rank} K$, as described just above. The description of $\mathbf{I}(M, ds^2)$ is of the form $G^\dagger = \bigcup Gk$, with isotropy subgroup $K^\dagger = \bigcup Kk$, with k specified by (7.8) and (8.5). below. Our description is inspired by Cartan's description ([6], [7]) of the full isometry group of a symmetric space — or see [35] or [37].

8A. Basic Setup.

The first step in describing the isometry group $\mathbf{I}(G/H, ds^2)$ is to have an idea of the structure of the maximal possible group $\mathbf{I}^0(G/H, dt^2)$, especially when dt^2 is the normal Riemannian metric. Then one works out $G = \mathbf{I}^0(G/H, ds^2)$ from the specific structure of $\mathbf{I}^0(G/H, dt^2)$. This is especially useful when $\chi(M) > 0$, when $(G/H, ds^2)$ is a group manifold with simply transitive group G, and when $(G/H, ds^2)$ has strictly positive curvature. So it is convenient to introduce the definition:

(8.2) if $G = \mathbf{I}^0(G/H, ds^2)$ then G is *isometry-maximal* for $\mathbf{I}(G/H, ds^2)$.

In practice it is not so difficult to check (8.2). For example let M be the group manifold $SU(2) = Sp(1) = S^3$ and let $\{\omega_1, \omega_2, \omega_3\}$ be the left invariant Maurer–Cartan forms. The left invariant metrics ds^2, leading to isomorphism classes of isometry groups, are represented by

(1) $ds^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$ for $\mathbf{I}(S^3, ds^2) = O(4)$, constant curvature,
(2) $ds^2 = \omega_1^2 + \omega_2^2 + a\omega_3^2$ with $0 < a < 1$ for $\mathbf{I}(S^3, ds^2) = \{SU(2) \times U(1)\}/\{\pm(1, 1)\}$, and
(3) $ds^2 = \omega_1^2 + b\omega_2^2 + a\omega_3^2$ with $0 < a < b < 1$ for $\mathbf{I}(S^3, ds^2)\{SU(2) \times [\pm(1, 1)]\}/\{\pm(1, 1)\}$.

A less trivial example: $Sp(n)$ is transitive on the complex projective space $P^{2n-1}(\mathbb{C}) = SU(2n)/U(2n-1)$, so if ds^2 is the Fubini–Study metric on $P^{2n-1}(\mathbb{C})$ then $Sp(n)$ is not isometry–maximal. But if ds^2 is not the Fubini–Study metric then $Sp(n)$ is isometry-maximal for $(Sp(n)/Sp(n-1)U(1), ds^2)$. Two other classical cases are from $SO(2r+2)/U(r+1) = SO(2r+1)/U(r)$ and $SO(7)/(SO(5) \cdot SO(2)) = G_2/U(2)$. See Onischik's paper [19], especially Table 7, for a more comprehensive list. Or see [13, Part II, Section 4].

The trivial $Sp(1)$ example just above, shows that if ds^2 is not the normal metric on G/K then there can be inner automorphisms of G that act on G/K but do not preserve ds^2.

8B. The Isometry Group

We start with the base point $x_0 = 1K \in G/K = M$. Then as in (7.8) we have $\mathrm{Aut}(G,K)$, $\mathrm{Int}(G,K)$ and $\mathrm{Out}(G,K)$, and $\mathrm{Aut}(G,K,ds^2)$, $\mathrm{Int}(G,K,ds^2)$ and $\mathrm{Out}(G,K,ds^2)$. Consider the subgroups

(8.3) $$G' = \bigcup \{Gk \mid k \in \mathbf{I}(M,ds^2)_{x_0} \text{ and } \mathrm{Ad}(k)|_K \text{ is inner}\} \text{ and } K' = G' \cap \mathbf{I}(M,ds^2)_{x_0}\ , \text{ and}$$

(8.4) $$G'' = \bigcup \{Gk \mid k \in \mathbf{I}(M,ds^2)_{x_0} \text{ and } \mathrm{Ad}(k) \text{ is inner on } G\} \text{ and } K'' = G'' \cap \mathbf{I}(M,ds^2)_{x_0}\ ,$$

(8.5) $$G^\dagger = \left(\bigcup\nolimits_{\mathrm{Ad}(k)\in \mathrm{Out}(G,K,ds^2)} G'k\right) \subset \mathbf{I}(M,ds^2) \text{ and } K^\dagger = G^\dagger \cap \mathbf{I}(M,ds^2)_{x_0}\ .$$

Here gk acts on M by $xK \mapsto gkxK = g\,kxk^{-1}K$.

Here are some obvious key remarks that will apply to G, G', G'' and $G^\dagger$.

Lemma 8.6. *Suppose $\chi(M) > 0$ and $k \in \mathbf{I}(M,ds^2)_{x_0}$. If $\mathrm{Ad}(k)$ is inner on $\mathbf{I}^0(M,ds^2)_{x_0}$, then it is inner on $\mathbf{I}^0(M,ds^2)$. In other words, if $\mathrm{Ad}(k)$ is outer on $\mathbf{I}^0(M,ds^2)$ it is outer on $\mathbf{I}^0(M,ds^2)_{x_0}$. If $\mathrm{Ad}(k)$ is inner on $\mathbf{I}^0(M,ds^2)$ then there exists $\gamma \in \mathbf{I}^0(M,ds^2)k$ that centralizes $\mathbf{I}^0(M,ds^2)$.*

{Remark. The case $S^{2n} = G/K = SO(2n+1)/SO(2n)$, where $\mathbf{I}(M,ds^2) = O(2n+1)$, and k the diagonal matrix $\mathrm{diag}\{-1,-1,+1,\dots,+1\}$, shows that one can have $\mathrm{Ad}(k)$ inner on G but outer on K.}

Proof. If $\mathrm{Ad}(k)$ is inner on $\mathbf{I}^0(M,ds^2)_{x_0}$ then its centralizer there contains a torus T' that is a maximal torus in $\mathbf{I}^0(M,ds^2)$, so $\mathrm{Ad}(k)$ is inner on $\mathbf{I}^0(M,ds^2)$. If $\mathrm{Ad}(k)$ is inner on $\mathbf{I}^0(M,ds^2)$ we have $\gamma = gk \in \mathbf{I}^0(M,ds^2)k$ that centralizes $\mathbf{I}^0(M,ds^2)$. □

Finally, we have a modified form of [42, Theorem 3.12], again following Section 4 there, as follows.

Theorem 8.7. *Suppose that $\chi(M) > 0$ and $G = \mathbf{I}^0(M,ds^2)$. Then $G^\dagger$ is the full isometry group $\mathbf{I}(M,ds^2)$, and the respective isotropy subgroups of G and $G^\dagger$ are K and $K^\dagger$.*

8C. Inner Automorphisms and Constant Displacement Isometries.

Lemma 8.8. *Let K be a compact connected Lie group of linear transformations on a real vector space V. Suppose that the representation of K on V does not have any trivial summands. Let $0 \neq w \in V$. Then the orbit $K(w)$ is not contained in a half–space.*

Proof. Suppose that $K(w) \subset U$ where U is a half space in V. So $K(w) \subset U = \{v \in V \mid \langle u, v\rangle < 0\}$ for some nonzero $u \in V$. The center of gravity $\bar{w} := \int_K k(w)dk \in U$ and $K(\bar{w}) = \bar{w}$. Now $\bar{w}\mathbb{R}$ defines a trivial summand for the representation of K on V. □

Now, with only the obvious changes, we have a very slight extension to [35, Theorem 5.2.2]:

Proposition 8.9. *Let (M,ds^2) be a connected Riemannian homogeneous manifold and suppose that:*

(a) *The representations of the connected linear isotropy subgroups $\mathbf{I}(M,ds^2)_x$ on the tangent spaces $T_x(M)$ satisfy the conditions on K and V in* Lemma 8.8.

(b) *Suppose that $\beta \in \mathbf{I}(M,ds^2)$ centralizes $\mathbf{I}^0(M,ds^2)$, $g \in \mathbf{I}(M,ds^2)$ has a fixed point on M, and $\gamma = g\beta$ is an isometry of constant displacement on (M,ds^2).*

Then $\gamma = \beta$, i.e. $g = 1$.

In order to apply Proposition 8.9 we need to know that the action of $\mathbf{I}^0(M,ds^2)_x$ on $T_x(M)$ has no trivial summands. For $\chi(M) > 0$ that is implicit in the root space calculations of [4]. Thus

Corollary 8.10. *Let (M, ds^2) be a connected Riemannian homogeneous manifold of Euler characteristic $\chi(M) > 0$. Then $G'' = G \times Z$ where Z is the centralizer of G in $G^\dagger$. Specifically, $G'' = \bigcup G\beta_j$ with $Z = \{\beta_j\}$. If Γ is a group of isometries of constant displacement on (M, ds^2), and if $\Gamma \subset G''$, then $\Gamma \subset Z$ and $\Gamma\backslash(M, ds^2)$ is homogeneous.*

Among other things, Corollary 8.10 says that if two isometries γ_1 and γ_2 of constant displacement belong to the same component of the isometry group, so $\gamma_1\gamma_2^{-1} \in G$, then $\gamma_1 = \gamma_2$ because the center of G is reduced to the identity.

We complete the proof of Theorem 8.1 as follows. For clarity of exposition we repeat the key statements.

Proposition 8.11. *Suppose that $G^\dagger = G''$, for example that G is not of type A_n $(n \geqq 2)$, D_n $(n \geqq 4)$ nor E_6. Then every component Gk of $G^\dagger$ contains exactly one isometry of constant displacement of (M, ds^2). In particular, the Homogeneity Conjecture holds for (M, ds^2).*

8D. Outer Automorphisms and Constant Displacement Isometries

It is an open problem to drop the outer automorphism condition $\Gamma \subset G''$ in Theorem 8.1 and Proposition 8.11. It only applies to the cases where G of type A_n $(n \geqq 2)$, D_n $(n \geqq 4)$ or E_6. In the Riemannian symmetric space case [35] those cases were addressed by listing the possibilities for G/K and doing explicit calculations for each of them. But there are too many nonsymmetric cases, so we need a new idea. In brief, there are no definitive results yet. I won't try to describe the current fragmentary results because they are part of an active (at least on my part) research effort, and anything I write will quickly be obsolete.

9. Compact Group Manifolds.

In this section we consider the Homogeneity Conjecture for Riemannian manifolds (M, ds^2) on which there is a compact simply transitive group G of isometries. We already saw this for symmetric spaces, where $\mathbf{I}^0(M, ds^2)$ is of the form $(G \times G)/\{\mathrm{diag}(G)\}$. There M was identified with the group manifold G, and $(g_1, g_2) \in (G \times G)$ acted by $\ell(g_1)r(g_2) : x \mapsto g_1 x g_2^{-1}$. The isotropy subgroup $\mathrm{diag}(G)$ of $\mathbf{I}(G, ds^2)$ all inner automorphisms of G, possibly also some outer automorphisms, and conjugation $(g_1, g_2) \mapsto (g_2, g_1)$ by the symmetry at the identity. We take that as a model for isometry groups of group manifolds, and we apply the result to finite groups of isometries of constant displacement. Everything breaks up as a product under the decomposition of G as a product of simple Lie groups, so we may assume that G is simple. The result is not complete; at this time it is

Theorem 9.1. *Let (M, du^2) be a connected simply connected Riemannian manifold on which a compact simple Lie group G acts simply transitively by isometries. Then there is a G–equivariant isometry $(M, du^2) \cong (G, ds^2)$ where the Riemannian metric ds^2 is invariant under left translations $\ell(g) : x \mapsto gx$. Let Γ be a group of isometries of constant displacement on (G, ds^2), and let $r(G, ds^2)$ be the group of all right translations $r(g') : x \mapsto xg'^{-1}$ that are isometries. Then either $\Gamma \subset \ell(G)\ell(A)$ or $\Gamma \subset r(G, ds^2)r(A)$, where A is a finite set of isometries a such that the* $\mathrm{Ad}(a)$ *define outer automorphisms of G. If $\Gamma \subset r(G, ds^2)r(A)$ then the centralizer of Γ in $\mathbf{I}(G, ds^2)$ contains $\ell(G)$, so the Homogeneity Conjecture holds for Γ.*

See more details on A in Proposition 9.13 at the end of Section 9. One would like a more precise analysis of the case $\Gamma \subset \ell(G)\ell(A)$ to test the Homogeneity Conjecture there.

The result $(M, du^2) \cong (G, ds^2)$ is due to Ozeki [20, Theorems 2 and 3], extending work of Ochiai and Takahashi [18], so we need only look at the assertion that either $\Gamma \subset \ell(G)\ell(A)$ or $\Gamma \subset r(G, ds^2)r(A)$. That is Proposition 9.13 below. The theory does have some open problems concerning homogeneity when $\Gamma \subset \ell(G)$.

9A. The Isometry Group.

We start with a connected Lie group G. $\ell(G)$ will denote the group of left translations $\ell(g) : x \mapsto gx$, $r(G)$ the right translations $r(g) : x \mapsto xg^{-1}$, and ds^2 an $\ell(G)$–invariant Riemannian metric on G. Obviously $\ell(G)$ is contained in the identity component $\mathbf{I}^0(G, ds^2)$.

Proposition 9.2. ([18, Theorems 1 and 2]) *If G is compact then $\mathbf{I}^0(G, ds^2) \subset \ell(G)r(G)$, and $\ell(G)$ is a normal subgroup of $\mathbf{I}^0(G, ds^2)$.*

Write $\langle \, , \rangle$ for the inner product on $\mathfrak{g}$ defined by ds^2 and $r(G, ds^2) = \{r(g) \in r(G) \mid \mathrm{Ad}(g) \text{ preserves } \langle \, , \rangle\}$. Obviously $r(G) \cap \mathbf{I}(G, ds^2) \subset r(G, ds^2)$. Conversely let $r(v) \in r(G, ds^2)$. As $\mathrm{Ad}(v)$ preserves $\langle \, , \rangle$ at $1 \in G$ it preserves ds^2 at every $g \in G$, so $v \in \mathbf{I}(G, ds^2)$. Thus

Corollary 9.3. *If G is compact then $\mathbf{I}^0(G, ds^2) = \ell(G)r(G, ds^2)$.*

An automorphism of a compact connected Lie group either preserves each simple factor or permutes the simple factors. Since $\ell(G)$ is normal in $\mathbf{I}^0(G, ds^2)$, now

Corollary 9.4. *Suppose that G is compact and simple, and that $a \in \mathbf{I}(G, ds^2)$. If $\mathrm{Ad}(a)(\ell(G)) \neq \ell(G)$ then $\mathrm{Ad}(a)(\ell(G)) = r(G)$. In that case (G, ds^2) is a compact simple Lie group with bi–invariant Riemannian metric, so it is an irreducible Riemannian symmetric space.*

Let dt^2 denote a bi-invariant Riemannian metric on G. Let $L(G, ds^2)$ and $L(G, dt^2)$ denote the respective isotropy subgroups of $\mathbf{I}(G, ds^2)$ and $\mathbf{I}(G, dt^2)$ at $x_0 := 1 \in G$. They act on the tangent space $\mathfrak{g}$ at x_0 by $dh(\xi) = \mathrm{Ad}(h)\xi$. From Élie Cartan's papers [6] and [7], $L(G, dt^2) \cap \mathbf{I}^0(G, dt^2) = \{\ell(h)r(h) \mid h \in G\}$, and $L(G, dt^2)$ is generated by (i) $L(G, dt^2) \cap \mathbf{I}^0(G, dt^2)$, (ii) the symmetry s_{x_0} and (iii) the automorphisms $\alpha \in \mathrm{Aut}(G)$ that preserve and are outer on the identity component $L^0(G, dt^2)$ of $L(G, dt^2)$. If (G, ds^2) is not a symmetric space, i.e. if $ds^2 \neq dt^2$ (up to multiplication by a real constant), now $L(G, ds^2)$ is generated by (i) $L^0(G, ds^2)$ and (ii) $\{\alpha \in \mathrm{Aut}(G) \mid \alpha \text{ preserves both } L(G, ds^2) \text{ and } ds^2\}$. In particular,

$$\mathbf{I}(G, ds^2) = \ell(G)L(G, ds^2) \subset \ell(G)L(G, dt^2) = \mathbf{I}(G, dt^2). \tag{9.5}$$

Adapting (7.8) to our situation, we denote

$$\begin{aligned} &\mathrm{Int}(G, ds^2) = \{\alpha \in \mathrm{Aut}(G) \mid \alpha(L(G, ds^2)) = L(G, ds^2),\ \alpha|_{L(G,ds^2)} \text{ is inner, and } \alpha \text{ preserves } ds^2\} \\ &\mathrm{Out}(G, ds^2) = \{\alpha \in \mathrm{Aut}(G) \mid \alpha \text{ preserves both } L(G, ds^2) \text{ and } ds^2\}/\mathrm{Int}(G, ds^2) \end{aligned} \tag{9.6}$$

Express

$$\mathrm{Out}(G, ds^2) = \mathrm{Int}(G, ds^2)\alpha_1 \cup \cdots \cup \mathrm{Int}(G, ds^2)\alpha_k \tag{9.7}$$

where the $\alpha_i \in \mathrm{Aut}(G)$ form a set of representatives modulo inner automorphisms, and further express

$$\alpha_i = \mathrm{Ad}(a_i) = \ell(a_i)r(a_i) \text{ for } 1 \leqq i \leqq k. \tag{9.8}$$

Then we have the full isometry group as follows.

Theorem 9.9. *Let G be a compact connected simple Lie group and ds^2 a left-invariant Riemannian metric on G. Suppose that (G, ds^2) is not a symmetric space. Then $\mathbf{I}(G, ds^2) = \bigcup_{1 \leqq i \leqq k} \{\ell(G)\ell(a_i) \times r(G, ds^2)r(a_i)\}$ where $a_1 = 1$ and $\{a_1, \ldots, a_k\}$ is given by (9.6), (9.7) and (9.8).*

9B. Constant Displacement Isometries.

From [35], if Γ_{dt^2} is a group of constant displacement isometries on (G, dt^2) then either $\Gamma_{dt^2} \subset \ell(G)r(Z_G)$ or $\Gamma_{dt^2} \subset \ell(Z_G)r(G)$. Now let Γ_{ds^2} be a group of constant displacement isometries on (G, ds^2). Then certainly Γ_{ds^2} is a group of fixed point free isometries on (G, dt^2) but the issue is whether every $\gamma \in \Gamma_{ds^2}$ is of constant displacement on (G, ds^2).

We will write $\bar{r}(G, ds^2)$ for $\{w \in G \mid r(w) \in r(G, ds^2)\}$ and $\bar{r}(w)$ when $r(w) \in r(G, ds^2)\}$.

Lemma 9.10. *Let $\gamma = \ell(ua)r(va)$ be an isometry of constant displacement on (G, ds^2) where $u \in G$, $v \in \bar{r}(G, ds^2)$, and $a \in \{a_1, \ldots, a_k\}$. Then va commutes with every G–conjugate of ua, and ua commutes with every G–conjugate of va.*

Proof. Calculate the displacement $\rho(1,\gamma(1)) = \rho(uv^{-1}, 1)$ so γ has constant displacement $c_\gamma = \rho(1, uv^{-1})$r, and $c_\gamma = \rho(g,\gamma(g)) = \rho(g, uaga^{-1}v^{-1}) = \rho(gv, uaga^{-1}) = \rho(gva, uag) = \rho(va, \mathrm{Ad}(g^{-1})(ua))$. But $\mathrm{Ad}(g^{-1})(ua) = u'_g a \in Ga$ and $\gamma'_g = \ell(u'_g a)r(va)$ is $\ell(G)$–conjugate to γ, so it has the same constant displacement c_γ. Now the distance from va to any $\ell(G)$–conjugate of ua is the same constant c_γ.

Take a minimizing geodesic $\sigma(t) = va \cdot \exp(t\xi)$ from va to $guag^{-1}$. Then $\sigma'(0)$ is orthogonal to the orbit $\mathrm{Ad}(G)(ua)$ because that orbit lies in the sphere (boundary of the solid sphere) of radius c_γ with center va. As σ is orthogonal to some $\mathrm{Ad}(G)$–orbit it is orthogonal to every $\mathrm{Ad}(G)$–orbit that it meets. Thus $\sigma'(1)$ is tangent to the centralizer of $guag^{-1}$, which is totally geodesic so it contains $\sigma(0) = va$. Now va commutes with every G–conjugate of ua. But $va \cdot guag^{-1} = guag^{-1} \cdot va$ for $g \in G$ so $g^{-1}(va \cdot guag^{-1})g = g^{-1}(guag^{-1} \cdot va)g$ and $(g^{-1}vag)u = ua(g^{-1}vag)$. Thus ua commutes with every G–conjugate of va. □

Lemma 9.11. *Suppose that B is a simple Lie group and that B^0a is one of its topological components. Let $ua, va \in B^0a$ such that va commutes with every B^0–conjugate of ua. Then either ua or va belongs to the centralizer $Z_B(B^0)$.*

Proof. Suppose $va \notin Z_B(B^0)$. Then the centralizer $Z_B(va)$ is a proper subgroup of lower dimension in B that contains $\mathrm{Ad}(B^0)(ua)$. But $\mathrm{Ad}(B^0)(ua)$ generates a closed normal subgroup E of B contained in $Z_B(va)$. If $ua \notin Z_B(B^0)$ then $\dim E < \dim B$, contradicting simplicity of B (as a Lie group). □

Combining Lemmas 9.10 and 9.11 we have

Lemma 9.12. *Let $u \in G$, $v \in \bar{r}(G, ds^2)$ and $a \in \{a_1, \dots, a_k\}$ such that $\gamma := \ell(ua)r(va)$ is an isometry of constant displacement on (G, ds^2). Write $\alpha = \mathrm{Ad}(a) = \ell(a)r(a)$, so $\gamma = \ell(u)r(v)\alpha$. Then there are two cases:*

(1) *$ua \in Z_{\mathbf{I}(G,ds^2)}(G)$. Then $\ell(G)$ centralizes γ.*

(2) *$va \in Z_{\mathbf{I}(G,ds^2)}(G)$. Then $d[\ell(va)r(va)]$ is the identity on the tangent space $\mathfrak{g}$ so its action on G is trivial and $\gamma = \ell(uv^{-1})$.*

Now we see that these cases cannot be mixed.

Proposition 9.13. *Let Γ be a finite group of isometries of constant displacement on (G, ds^2). There are two cases:*

(1.) *$\Gamma \subset \bigcup_{1 \leqq i \leqq k} \ell(Ga_i)$ where $\{\alpha_1, \dots, \alpha_k\}$ and $\{a_1, \dots, a_k\}$ are given by (9.6), (9.7) and (9.8), and*

(2.) *$\Gamma \subset \bigcup_{1 \leqq i \leqq k} r((G, ds^2)a_i)$.*

In Case 2 *the centralizer of Γ in $\mathbf{I}(G, ds^2)$ contains $\ell(G)$, so the Homogeneity Conjecture holds for Γ.*

Proof. If $\Gamma \not\subset \bigcup_{1 \leqq i \leqq k} r((G, ds^2)a_i)$ then Γ contains an element $\gamma_1 = \ell(u'_1 a_i)r(v'_1 a_i)$ with $u'_1 a_i \notin Z_{\mathbf{I}(G,ds^2)}(G)$. Then $v'_1 a_i \in Z_{\mathbf{I}(G,ds^2)}(G)$ and $\gamma_1 = \ell(u_1)$ where $u_1 = (u'_1 a_1)(v'_1 a_i)^{-1} = u'_1 v'^{-1}_1 \notin Z_G$. Then, if we have $\gamma_2 = \ell(u'_2 a_j)r(v'_2 a_j) \in \Gamma$ with $\gamma'_2 a_j \notin Z_G$, $\gamma_2 = r(v'_2 u'^{-1}_2) = r(v_2)$. Thus $\gamma_3 := \gamma_1\gamma_2 = \ell(u_1)r(v_2)$ where $v_2 \in Z_G$ because $u_1 \notin Z_G$. Thus $\gamma_3 = \ell(u_1 v_2^{-1}) \in \ell(G)$ and $\gamma_1 = \ell(u_1) \in \ell(G)$ so also $\gamma_2 = \gamma_1^{-1}\gamma_3 \in \ell(G)$. But $\gamma_2 = r(v_2) \notin \ell(G)$. This contradicts our hypothesis $\Gamma \not\subset \bigcup_{1 \leqq i \leqq k} r((G, ds^2)a_i)$. We conclude that $\Gamma \subset \bigcup_{1 \leqq i \leqq k} \ell(Ga_i)$. □

10. **Positive Curvature Manifolds.**

In this section we verify the Homogeneity Conjecture for Riemannian manifolds (M, ds^2) such that there is some Riemannian metric dt^2 of strictly positive sectional curvature on M. Specifically, we combine Propositions 10.3, 10.9, 10.12 and 10.16 with the comments at the beginning of Subsection 10C,

Theorem 10.1. *Let $M = G/H$ be a connected, simply connected homogeneous space, and ds^2 a G–invariant Riemannian metric on M, where ds^2 is not required to be the normal Riemannian metric. Suppose that*

M admits another invariant Riemannian metric dt^2 of strictly positive curvature. Then the Homogeneity Conjecture is valid for (M, ds^2).

10A. The Classification.

The connected simply connected homogeneous Riemannian manifolds of positive sectional curvature were classified by Marcel Berger [3], Nolan Wallach [30], Simon Aloff and Nolan Wallach [1], and Lionel Bérard-Bergery [2]. Their isometry groups were worked out by Krishnan Shankar [24]. The spaces and the isometry groups are listed in the first two columns of Table 10.2 below. When there is a fibration that will be relevant to our verification of the Homogeneity Conjecture, it will also be listed in the first column.

Table 10.2
Isometry Groups of CSC Homogeneous Manifolds of Positive Curvature and Fibrations over Symmetric Spaces

	$M = G/H$	$\mathbf{I}(M, ds^2)$
1	$S^n = SO(n+1)/SO(n)$	$O(n+1)$
2	$P^m(\mathbb{C}) = SU(m+1)/U(m)$	$PSU(m+1) \rtimes \mathbb{Z}_2$
3	$P^k(\mathbb{H}) = Sp(k+1)/(Sp(k) \times Sp(1))$	$Sp(k+1)/\mathbb{Z}_2)$
4	$P^2(\mathbb{O}) = F_4/Spin(9)$	F_4
5	$S^6 = G_2/SU(3)$	$O(7)$
6	$P^{2m+1}(\mathbb{C}) = Sp(m+1)/(Sp(m) \times U(1))$ $P^{2m+1}(\mathbb{C}) \to P^m(\mathbb{H})$	$(Sp(m+1)/\mathbb{Z}_2) \times \mathbb{Z}_2$
7	$F^6 = SU(3)/T^2$ $F^6 \to P^2(\mathbb{C})$	$(PSU(3) \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$
8	$F^{12} = Sp(3)/(Sp(1) \times Sp(1) \times Sp(1))$ $F^{12} \to P^2(\mathbb{H})$	$(Sp(3)/\mathbb{Z}_2) \times \mathbb{Z}_2$
9	$F^{24} = F_4/Spin(8)$ $F^{24} \to P^2(\mathbb{O})$	F_4
10	$M^7 = SO(5)/SO(3)$	$SO(5)$
11	$M^{13} = SU(5)/(Sp(2) \times_{\mathbb{Z}_2} U(1)$ $M^{13} \to P^4(\mathbb{C})$	$PSU(5) \rtimes \mathbb{Z}_2$
12	$N_{1,1} = (SU(3) \times SO(3))/U^*(2)$	$(PSU(3) \rtimes \mathbb{Z}_2) \times SO(3)$
13	$N_{k,\ell} = SU(3)/U(1)_{k,\ell}$ $(k,\ell) \neq (1,1), 3 \mid (k^2+\ell^2+k\ell)$ $N_{k,\ell} \to P^2(\mathbb{C})$	$(PSU(3) \rtimes \mathbb{Z}_2) \times (U(1) \rtimes \mathbb{Z}_2)$
14	$N_{k,\ell} = SU(3)/U(1)_{k,\ell}$ $(k,\ell) \neq (1,1), 3 \nmid (k^2+\ell^2+k\ell)$ $N_{k,\ell} \to P^2(\mathbb{C})$	$U(3) \rtimes \mathbb{Z}_2$
15	$S^{2m+1} = SU(m+1)/SU(m)$ $S^{2m+1} \to P^m(\mathbb{C})$	$U(m+1) \rtimes \mathbb{Z}_2$
16	$S^{4m+3} = Sp(m+1)/Sp(m)$ $S^{4m+3} \to P^m(\mathbb{H})$	$Sp(m+1) \rtimes_{\mathbb{Z}_2} Sp(1)$
17	$S^3 = SU(2)$ $S^3 \to P^1(\mathbb{C}) = S^2$	$O(4)$
18	$S^7 = Spin(7)/G_2$	$O(8)$
19	$S^{15} = Spin(9)/Spin(7)$ $S^{15} \to S^8$	$Spin(9)$

10B. The Normal Metric Case.

The first four spaces $M = G/H$ of Table 10.2 are Riemannian symmetric spaces with $G = \mathbf{I}(M)^0$. The fifth space is $S^6 = G_2/SU(3)$, where the isotropy group $SU(3)$ is irreducible on the tangent space, so the only invariant metric is the one of constant positive curvature; thus it is isometric to a Riemannian symmetric space. In view of [35], the Homogeneity Conjecture is valid for the entries (1) through (5) of Table 10.2.

Entries (6), (7), (8) and (9) of Table 10.2 have $\operatorname{rank} G = \operatorname{rank} H$. Each is isotropy–split with fibration over a projective (thus Riemannian symmetric) space, as defined in [42, (1.1)]. The Homogeneity Conjecture follows, for these (M, ds^2) where ds^2 is the normal Riemannian metric.

The argument for entries (6), (7), (8) and (9) applies with only obvious changes to a number of other table entries, using [42, Theorem 6.1] instead of [42, Corollary 5.7]. And the Homogeneity Conjecture is immediate for (18), where $H = G_2$ acts irreducibly on the tangent space of $S^7 = Spin(7)/G_2$, so ds^2 is the standard constant positive curvature metric. That verifies the Homogeneity Conjecture for the entries (11), (13), (14), (15), (16), (17), (18) and (19) of Table 10.2, where ds^2 is the normal Riemannian metric on M. We have to take a closer look at table entries (10) and (12)

In case (10), where $M^7 = G/H = SO(5)/SO(3)$, H acts irreducibly on the tangent space, so ds^2 is normal and naturally reductive. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, $\mathfrak{h} \perp \mathfrak{m}$ as usual, and γ an isometry of constant displacement. Since $\mathbf{I}(M, ds^2) = SO(5)$ is connected we have $\eta \in \mathfrak{m}$ such that $\sigma(t) = \exp(t\xi)x_0\,, 0 \leqq t \leqq 1$, is a minimizing geodesic in (M, ds^2) from $x_0 = 1H$ to $\gamma(x_0)$. One argues that the corresponding Killing vector field X on (M, ds^2) has constant length. But there is no such nonzero vector field [44]. Thus there is no isometry $\neq 1$ of constant displacement for entry (10) of Table 10.2, so the Homogeneity Conjecture is immediate in that case.

In case (12), where $N_{1,1} = G/H = (SU(3) \times SO(3))/U^*(2)$, let γ be an isometry of constant displacement $d > 0$ and suppose that γ^2 also is an isometry of constant displacement. The argument for table entry (18) shows that $\gamma \notin \mathbf{I}^0(N_{1,1}, ds^2)$, so $\gamma = (g_1, g_2)\nu$ where $g_1 \in SU(3)$, $g_2 \in SO(3)$, $\nu^2 = 1$, $\mathrm{Ad}(\nu)$ is complex conjugation on $SU(3)$, and $\mathrm{Ad}(\nu)$ is the identity on $SO(3)$. It also shows $\gamma^2 \in \mathbf{I}^0(N_{1,1}, ds^2)$ so $\gamma^2 = 1$.

The centralizer of ν is $K := ((SO(3) \times SO(3)) \cup (SO(3) \times SO(3))\nu$. Using de Siebenthal [25] we reduce our considerations to the cases where g_1 is either the identity matrix I_3 or the matrix $I_3' := \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{smallmatrix}\right)$, and also g_2 is either I_3 or I_3'.

Recall that $H = U^*(2)$ is the image of $U(2) \hookrightarrow (SU(3) \times SO(3))$, given by $h \mapsto (\alpha(h), \beta(h))$ where $\alpha(h) = \left(\begin{smallmatrix} h & 0 \\ 0 & 1/\det(h) \end{smallmatrix}\right)$ and β is the projection $U(2) \to U(2)/(center) \cong SO(3)$. Further, $\mathbf{I}(L_{1,1}) = G \cup G\nu$ and its isotropy subgroup is $H \cup H\nu$. Observe that

if $(g_1, g_2) = (I_3\,, I_3)$ then $\gamma = (\alpha(I_2), \beta(I_2))\nu \in H\nu$, and

if $(g_1, g_2) = (I_3'\,, I_3)$ then $\gamma = (\alpha(-I_2), \beta(-I_2))\nu \in H\nu$.

Replace I_3' by $I_3'' = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$ and set $I_2'' = \left(\begin{smallmatrix} -1 & 0 \\ 0 & +1 \end{smallmatrix}\right)$, so

if $(g_1, g_2) = (I_3''\,, I_3'')$ then $\gamma = (\alpha(I_2''), \beta(I_2''))\nu \in H\nu$.

When $\gamma \in H\nu$ it cannot be of nonzero constant displacement. We have reduced our considerations to the case $\gamma = (I_3\,, I_3')\nu$, or equivalently to one of its conjugates. Compute

$$\left(\left(\begin{smallmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right), I_3\right) \cdot (I_3\,, I_3')\nu \cdot \left(\left(\begin{smallmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right), I_3\right)^{-1} = (I_3'\,, I_3')\nu \in H\nu,$$

so again γ cannot be of nonzero constant displacement. Thus there is no isometry $\neq 1$ of constant displacement for entry (12) of Table 10.2, so the Homogeneity Conjecture is immediate in that case.

Summarizing this section, we have proved

Proposition 10.3. *Let $M = G/H$ where G is a compact connected Lie group and M is simply connected. Let ds^2 be the normal Riemannian metric on M. Suppose that M has a Riemannian metric dt^2 for which every sectional curvature is > 0. Then the Homogeneity Conjecture is valid for (M, ds^2).*

10C. Dropping the Normality Condition.

In this subsection we see how to drop the normality condition on ds^2 in Proposition 10.3. In several cases this is automatic because the adjoint action of H on the tangent space $\mathfrak{g}/\mathfrak{h}$ is irreducible; there every invariant Riemannian metric on $M = G/H$ is normal. Those are the spaces given by the entries (1) (2), (3), (4), (5), (10), (11), (12) and (18) of Table 10.2. To consider most of the others one needs a variation on (7.1).

(10.4)

G is a compact connected simply connected Lie group,

$H \subset K$ are closed connected subgroups of G, $M' = G/K$ and $F = H\backslash K$, and

ds^2 is a G–invariant Riemannian metric on $M = G/H$, $ds'^2 = ds^2|_{M'}$ and $ds^2_F = ds^2|_F$, such that

(i) $\pi : M \to M'$ by $\pi(gH) = gK$, right action of K ,

(ii) (M', ds'^2) and (F, ds^2_F) are Riemannian symmetric spaces, and

(iii) the tangent spaces $\mathfrak{m}'$ for M', $\mathfrak{m}''$ for F and $(\mathfrak{m}' + \mathfrak{m}'')$ for M satisfy $\mathfrak{m}' \perp \mathfrak{m}''$.

That leads to a modification of [42, Lemma 5.2]:

Lemma 10.5. *Assume* (10.4). *Then the fiber F of $M \to M'$ is totally geodesic in M In particular it is a geodesic orbit space, and any geodesic of M tangent to F at some point is of the form $t \mapsto \exp(t\xi)x$ with $x \in F$ and $\xi \in \mathfrak{m}''$.*

That facilitates a variation on the arguments of [42, Proposition 5.4] and [42, Lemma 5.5] to show that Γ centralizes G, where Γ is a group of isometries of constant displacement on (M, ds^2), as follows

Lemma 10.6. *Suppose that $M = G/H$ is an entry of* Table 10.2 *for which $G = \mathbf{I}^0(M, ds^2)$, that (M, ds^2) satisfies* (10.4), *and that $\pi : (M, ds^2) \to (M', ds'^2)$ is a Riemannian submersion. Then the Homogeneity Conjecture holds for (M, ds^2).*

With some adjustments, especially for (6) and (19), Lemma 10.6 applies to (6) (7), (8), (9), (13), (14) and (19). The remaining three cases are addressed by direct computation. The simplest, (17), is the group manifold $SU(2) = Sp(1) = S^3$, described just after (8.2). There, (1) is the limit of (2) as $a \uparrow 1$ and of (3) as $a, b \uparrow 1$. Writing $(g, h) := \ell(g)r(h)$ for the transformation $x \mapsto gxh^{-1}$,

Lemma 10.7. *Let $\Gamma \subset \mathbf{I}(S^3, ds^2)$ be a finite group of constant displacement isometries of (S^3, ds^2). If $\gamma = \pm(g, h) \in \Gamma$ and $g \neq \pm 1$ then $h = \pm 1$.*

Corollary 10.8. *Let $\Gamma \subset \mathbf{I}(S^3, ds^2)$ be a finite group of constant displacement isometries of (S^3, ds^2). Then either $\Gamma \subset [SU(2) \times \{\pm 1\}]/[\pm(1, 1)]$ or $\Gamma \subset [\{\pm 1\} \times H]/[\pm(1, 1)]$.*

Proposition 10.9. *Let ds^2 be a left $SU(2)$–invariant Riemannian metric on S^3. Let Γ be a finite group of isometries of constant displacement on (S^3, ds^2). Then the centralizer of Γ in $\mathbf{I}(S^2, ds^2)$ is transitive on S^3, so the quotient Riemannian manifold $\Gamma\backslash(S^3, ds^2)$ is homogeneous. In other words, the Homogeneity Conjecture is valid for (S^3, ds^2).*

The second remaining case, (15), is the sphere $G/H = SU(m + 1)/SU(m) = S^{2m+1}, m \geqq 2$, total space of a circle bundle $S^{2m+1} = G/H \to G/K = P^m(\mathbb{C})$. The fiber over $z_0 = 1K$ is the center Z_K of $U(m)$. G/H has tangent space $\mathfrak{v} \oplus \mathfrak{z}_K$ where $\mathfrak{v}$ is the tangent space $\mathbb{C}^m$ of G/K and $\mathfrak{z}_K$ is the center of $\mathfrak{k}$; $\mathfrak{v}$ and $\mathfrak{z}_K$ are the (two) irreducible summands of the isotropy representation of H. Let ds^2 be an $SU(m + 1)$–invariant Riemannian metric on $M = S^{2m+1}$. Then $\mathbf{I}(M, ds^2)$ is either the orthogonal group $O(2m + 2)$ or $[U(m + 1) \cup \nu U(m + 1)]$ where $\mathrm{Ad}(\nu)$ is complex conjugation on $U(m + 1)$. In the first case (M, ds^2) is the constant curvature $(2m+1)$–sphere, where we know that the Homogeneity Conjecture is valid. In the second case ds^2 is given by

$$ds^2|_{\mathfrak{v}} = b'\kappa|_{\mathfrak{v}}\ ,\ \ ds^2|_{\mathfrak{z}_K} = b''\kappa|_{\mathfrak{z}_K}\ ,\ \ \text{and}\ \ ds^2(\mathfrak{t}', \mathfrak{z}_K) = 0 \qquad (10.10)$$

for some $b', b'' > 0$. The displacement satisfies $c^2 = b'\kappa(\eta', \eta') + b''\kappa(\eta'', \eta'')$. The normal metric is given by $b' = b''$. After some more computation one arrives at

Lemma 10.11. *Let $\Gamma \subset \mathbf{I}(M, ds^2)$ be a subgroup such that every $\gamma \in \Gamma$ is an isometry of constant displacement. If $\gamma = \nu g \in \Gamma \cap \nu U(m+1)$ then $m+1$ is even, $\gamma^2 = -I \in U(m+1)$, and Γ is $SU(m+1)$–conjugate to the binary dihedral group whose centralizer in $U(m+1)$ is $Sp(\frac{m+1}{2})$.*

Proposition 10.12. *Let ds^2 be an $SU(m+1)$–invariant Riemannian metric on $S^{2m+1}, m \geqq 2$. Let Γ be a finite group of isometries of constant displacement on S^{2m+1}. Then the centralizer of Γ in $\mathbf{I}(S^{2m+1}, ds^2)$ is transitive on S^{2m+1}, so the Riemannian quotient manifold $\Gamma\backslash(SU(m+1), ds^2)$ is homogeneous. In other words, the Homogeneity Conjecture is valid for (S^{2m+1}, ds^2).*

The third and most delicate remaining case, (16), is the sphere $G/H = S^{4m+3}$, total space of an S^3 bundle $G/H = Sp(m+1)/Sp(m) \to Sp(m+1)/\{Sp(m) \times Sp(1)\} = G/K$. G/H has tangent space $\mathfrak{v} + \mathfrak{w}$ where $\mathfrak{v}$ is the tangent space $\mathbb{H}^m$ of G/K and $\mathfrak{w}$ is the tangent space $\mathrm{Im}\,\mathbb{H}$ of the fiber of $S^{4m+3} \to P^m(\mathbb{H})$. The isotropy representation of H is the natural representation of $Sp(m)$ on $\mathbb{H}^m = \mathfrak{v}$, and on $\mathfrak{w}$ it is three copies of the trivial representation.

Write κ for the negative of the Killing form of $\mathfrak{g}$. Let $\kappa' = \kappa|_{\mathfrak{v}}$ and $\kappa'' = \kappa|_{\mathfrak{w}}$ where $\kappa(\mu, \nu) = -\mathrm{Re}\,\mathrm{trace}\,(\mu\,\overline{\nu})$ with trace taken in $Sp(m+1)$. Let $\{e_1, e_2, e_3\}$ be a κ''–orthonormal basis of $\mathfrak{w}$ and split $\kappa'' = \kappa_1 + \kappa_2 + \kappa_3$ accordingly. Then

$$ds^2|_{\mathfrak{v}} = b_0\kappa',\ ds^2|_{\mathfrak{w}} = b_1\kappa_1 + b_2\kappa_2 + b_3\kappa_3\,,\ ds^2(\mathfrak{v}, \mathfrak{w}) = 0 \text{ and } ds^2(e_i, e_j) = 0 \text{ for } i \neq j \tag{10.13}$$

for some positive numbers b_0, b_1, b_2 and b_3.

Lemma 10.14. *Let ds^2 be an $Sp(m+1)$–invariant Riemannian metric on $M = S^{4m+3}$, $m \geqq 1$. Then either ds^2 is invariant under $SU(2m+2)$, or $\mathbf{I}(M, ds^2) = Sp(m+1) \cdot L = (Sp(m+1) \times L)/\{\pm(I_{m+1}, I_3)\}$ where L is one of the following.*

(1) *$L = Sp(1)$ acting on $Sp(m+1)/Sp(m)$ on the right. L acts on the tangent space as multiplication by quaternion unit scalars on $\mathfrak{v}$ and the adjoint representation of $Sp(1)$ on $\mathfrak{w}$. This is the case $b_1 = b_2 = b_3$.*

(2) *$L = O(2) \times \mathbb{Z}_2$ acting on $Sp(m+1)/Sp(m)$ on the right. L acts the tangent space as multiplication by an $O(2) \times \mathbb{Z}_2$ (essentially circle) group of quaternion unit scalars on $\mathfrak{v}$, $O(2)$–rotation on the (e_1, e_2)–plane in $\mathfrak{w}$, and the $\mathbb{Z}_2$–action $e_3 \mapsto \pm e_3$ on $\mathfrak{w}$. This is the case where two, but not all three, of the b_i are equal, for example where $b_1 = b_2 \neq b_3$.*

(3) *$L = \mathbb{Z}_2^3$ acting on $Sp(m+1)/Sp(m)$ on the right. L acts on the tangent space by ± 1 on $\mathfrak{v}$ and the $e_i \mapsto \pm e_i$ on $\mathfrak{w}$. This is the case where b_1, b_2 and b_3 are all different.*

Making use of (10.4), the particular fibration here, and properties of $P^m(\mathbb{H})$, one arrives at

Lemma 10.15. *If $\gamma \in Sp(m+1)$ has constant displacement $c > 0$ on S^{4m+3}, $m \geqq 2$, then γ belongs to the centralizer of $Sp(m+1)$ in $\mathbf{I}(S^{4m+3}, ds^2)$.*

Proposition 10.16. *Let $\Gamma \subset \mathbf{I}(S^{4m+3}, ds^2)$ be a subgroup such that every $\gamma \in \Gamma$ is an isometry of constant displacement. Suppose $m \geqq 2$ and that ds^2 is not $SU(2m+2)$–invariant. Then Γ centralizes $Sp(m+1)$ in $\mathbf{I}(M, ds^2)$.*

As noted at the beginning of Section 10, Propositions 10.3, 10.9, 10.12 and 10.16, together with the comments at the beginning of Subsection 10C, combine to give the main result of this section, Theorem 10.1.

Part III. Noncompact Homogeneous Riemannian Manifolds.

In the next four sections we will sketch the proof of the Homogeneity Conjecture for several classes of noncompact Riemannian homogeneous spaces. In these noncompact cases "bounded" can replace "constant displacement" and the result becomes independent of the choice of Riemannian metric.

11. Negative Curvature.

If γ is a bounded isometry of a connected, simply connected, Riemannian manifold (M, ds^2) of sectional curvature $\leqq 0$, then [36] γ is an ordinary translation along the euclidean factor in the de Rham decomposition of (M, ds^2). Thus, as we noted in Proposition 3.2, ([36, Theorem 1]) If (L, dt^2) is a complete connected simply connected Riemannian manifold of sectional curvature $\leqq 0$, with no euclidean factor in its de Rham decomposition, then every bounded isometry of (L, dt^2) is trivial. In particular if a Riemannian quotient $(L', dt'^2) := \Gamma\backslash(L, dt^2)$ is homogeneous then $\Gamma = \{1\}$ and $(L', dt'^2) = (L, dt^2)$.

The Homogeneity Conjecture follows immediately for Riemannian manifolds of sectional curvature $\leqq 0$. There is an extension, due to Druetta, to manifolds without focal points [11], and the Homogeneity Conjecture is immediate for those spaces as well.

12. Semisimple Groups.

Suppose that G' is a connected real semisimple Lie group without compact local factors. Consider a Riemannian manifold (M, ds^2) on which G' acts transitively and effectively by isometries, in other words $G' \subset \mathbf{I}(M, ds^2)$ is transitive on M. An isometry γ of (M, ds^2) is *bounded* if the displacement function $c_\gamma(x) := \rho(x, \gamma(x))$ is bounded.

Proposition 12.1. [17, Theorem 2.1]. *The centralizer $B := Z_{\mathbf{I}(M,ds^2)}(G')$ of G' in $\mathbf{I}(M, ds^2)$ is the set of all bounded isometries of (M, ds^2). In particular every bounded isometry of (M, ds^2) centralizes G' and thus is of constant displacement on (M, ds^2). Thus the Homogeneity Conjecture holds for (M, ds^2).*

The structure of B in Proposition 12.1 is given as follows [17, Section 2]. Let G denote the closure of G' in $\mathbf{I}(M, ds^2)$. Then G' is the derived group of G, and G is reductive. Express $M = G/H$ where H is the isotropy subgroup at some point $x_0 \in M$. H is compact because G is transitive on M and is closed in $\mathbf{I}(M, ds^2)$. Write $N_G(H)$ for the normalizer of H in G and consider the right translations

$$r(u) : gH \mapsto gu^{-1}H \text{ for } u \in N_G(H) \text{ and } U = \{u \in N_G(H) \mid r(u) \in \mathbf{I}(M, ds^2)\}.$$

Of course $r(U) = \{r(u) \mid u \in U\}$.

Proposition 12.2. *If $\gamma \in \mathbf{I}(M, ds^2)$ then the following conditions are equivalent.*

(1.) *γ is an isometry of constant displacement on (M, ds^2).*
(2.) *γ is an isometry of bounded displacement on (M, ds^2).*
(3.) *$\gamma \in r(U)$.*
(4.) *The centralizer $Z_{\mathbf{I}(M,ds^2)}(\gamma)$ is transitive on M.*

In particular, if Γ is a discrete subgroup of $\mathbf{I}(M, ds^2)$ consisting of isometries of constant displacement then $\Gamma\backslash(M, ds^2)$ is homogeneous; so the Homogeneity Conjecture holds for (M, ds^2).

Proposition 12.1 follows from Proposition 12.2. The proof of Proposition 12.2 makes use of [14, Theorem 4.4] and a variation on some results of Tits [28] which we will describe in Section 13.

Let $G_\mathbb{C}$ denote a complex reductive Lie group, G a real form of $G_\mathbb{C}$ and Q a parabolic subgroup of $G_\mathbb{C}$. Then $Z = G_\mathbb{C}/Q$ is a *complex flag manifold*. The number of G–orbits on Z is finite, so there are open orbits. The open orbits are *flag domains* and their structure is G/L where $L_\mathbb{C}$ is the reductive part of Q. See [38] for details, [39] or [42] for applications to the representation theory of real semisimple Lie groups, and [32], [40] and [31] for applications to automorphic cohomology theory. From either Proposition 12.1 or 12.2,

Corollary 12.3. *Let $D = G/L$ be a flag domain with L compact, and let ds^2 be any G–invariant Riemannian metric on D. Then the Homogeneity Conjecture holds for (D, ds^2).*

13. Bounded Automorphisms.

An automorphism α of a locally compact group G is *bounded* if there is a compact subset $C \subset G$ such that $\alpha(g)g^{-1} \in C$ for every $g \in G$. If $g \in G$ the inner automorphism $\mathrm{Ad}(g) : t \mapsto gtg^{-1}$ is bounded if and only if the conjugacy class $\mathrm{Ad}(G)g$ is relatively compact. We write $B(G)$ for the set of all such elements of G. It is a subgroup. The result of Jacques Tits mentioned in Section 12 is

Proposition 13.1. [28, Théorème (1)] *Soit G un groupe de Lie sans sous-groupe invariant compact non discret, et soit N son plus grand sous-groupe invariant nilpotent connexe. Alors, $B(G)$ est contenu dans le centralisateur $Z_G(M)$ du groupe M engendré par N et par tous les sous-groupes simples non compacts de G. Les composantes connexes de l'élément neutre dans $B(G)$ et $Z_G(M)$, soient $B^0(G)$ et $Z^0_G(M)$, sont des sous-espaces vectoriels du centre de N (en particulier, $Z^0_G(M)$ est le centre connexe de $Z_G(M)$); de plus, $B^0(G) = B(G) \cap N$. Enfin, si G est connexe et si $C(G)$ designe son centre, $B(G) = B^0(G) \cdot C(G)$.*

We extract the part that is relevant for us here:

Corollary 13.2. [28, Corollaire (2)] *Soit G un groupe de Lie connexe. Si le radical R de G est nilpotent et simplement connexe, et si G/R n'a pas de sous-groupe invariant compact non discret, G n'a pas d' automorphisme non trivial à déplacement borné.*

14. **Exponential Solvable Groups.**

Complementing Corollary 13.2,

Proposition 14.1. ([41, Theorem 2.5] and [43]). *Let (M, d) be a metric space on which an exponential solvable Lie group S acts effectively and transitively by isometries. Let $G = \mathbf{I}(M, d)$. Then G is a Lie group, any isotropy subgroup K is compact, and $G = SK$. If $g \in G$ is a bounded isometry then g is a central element in S.*

We combine Corollary 13.2 and Proposition 14.1:

Theorem 14.2. *Let α be a bounded automorphism of a connected Lie group G. Suppose that the solvable radical of G is exponential solvable. Then $G/\mathrm{Ker}\,(\alpha)$ is compact. If the semisimple group G/R has no compact simple factor then α is the identity.*

Proof. Consider a Levi-Whitehead decomposition $G = RS$ where R is the solvable radical and S is a semisimple complement. Then $\alpha(R) = R$ and $\alpha|_R$ is bounded, so $\alpha|_R = 1$ by Proposition 14.1. Now α passes down to $\bar{\alpha} \in \mathrm{Aut}(G/R)$, where it is a bounded automorphism.

Split $S = S_n S_c$ where S_n is the product of the noncompact simple normal subgroups and S_c is the product of the compact ones. Then $G/R = \bar{S}_n \bar{S}_c$ where $\bar{S}_n$ is the image of S_n under $G \to G/R$ and $\bar{S}_c$ is the image of S_c. The two factors of G/R are $\bar{\alpha}$–invariant, and $\bar{\alpha}|_{\bar{S}_n} = 1$ by Proposition 14.1.

Now consider a basis $\{u_i, v_j, w_k\}$ of $\mathfrak{g}$ where $\{u_i\}$ is a basis of $\mathfrak{s}_n$, $\{v_j\}$ is a basis of $\mathfrak{s}_c$, and $\{w_k\}$ is a basis of $\mathfrak{r}$. The corresponding block form of the matrix of $d\alpha$ on $\mathfrak{g}$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{2,2} & 0 \\ A_{3,1} & A_{3,2} & 0 \end{pmatrix}$ and of $d\alpha|_{\mathfrak{j}}$ on $\mathfrak{j} := \mathfrak{s}_n + \mathfrak{r}$ is $\begin{pmatrix} 0 & 0 \\ A_{3,1} & 0 \end{pmatrix}$. That is nilpotent. Thus a bounded automorphism of $S_n R$ is unipotent, and consequently is the identity. Now $A_{3,1} = 0$, so $(\mathfrak{s}_n + \mathfrak{r}) \subset \mathrm{Ker}\,(d\alpha)$. In other words $\mathfrak{g}/\mathrm{Ker}\,(d\alpha)$ is a quotient of $\mathfrak{s}_c$ and $G/\mathrm{Ker}\,(\alpha)$ is a quotient of the compact group S_c. The theorem follows. □

If $g \in \mathbf{I}(M, ds^2)$ and (M, ds^2) is homogeneous, then g is a bounded isometry of (M, ds^2) if and only if $\mathrm{Ad}(g)$ is a bounded automorphism of $\mathbf{I}^0(M, ds^2)$. Thus an immediate consequence of Theorem 14.2 is

Theorem 14.3. *Let (M, ds^2) be a Riemannian manifold on which a connected Lie group G acts effectively and transitively by isometries. Suppose that the solvable radical R of G is exponential solvable and that the semisimple quotient G/R has no compact simple factor. Let Γ the a discrete group of isometries of bounded displacement on (M, ds^2). Then G centralizes Γ in $\mathbf{I}(M, ds^2)$, so $\Gamma\backslash(M, ds^2)$ is homogeneous, and the Homogeneity Conjecture holds for (M, ds^2).*

Part IV. Some Problems.

15. **Open Problems.**

We mention a few open problems in connection with the Homogeneity Conjecture. Of course we would welcome solid general proof and we hardly need mention that. In this section we describe five problems that are related to the cases where we do have a proof. The first two are simply to fill gaps in our knowledge, but the last three would certainly require some new ideas.

15A. Outer Automorphisms and Non-Normal Metrics For $\chi(M) > 0$.

Consider the case $\chi(M) > 0$ of Section 8. There $M = G/K$ with $\operatorname{rank} G = \operatorname{rank} K$, and the case of a normal (from the negative of the Killing form of $\mathfrak{g}$) metric dt^2 was settled in Theorem 7.16. As long as no outer automorphisms of G occur in $\mathbf{I}(M, ds^2)$ the Homogeneity Conjecture is proved in the affirmative; see Theorem 8.1. We don't yet know how to deal with outer automorphisms that might occur in isometries of constant displacement, but here is one possibility.

There is an $\mathrm{Ad}(K)$–invariant reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, and since $\mathfrak{k}$ contains a Cartan subalgebra of $\mathfrak{g}$ we can express $\mathfrak{m}$ as a sum of root planes $\mathfrak{g}^\varphi := (\mathfrak{g}_{\mathbb{C}}^\varphi + \mathfrak{g}_{\mathbb{C}}^{-\varphi}) \cap \mathfrak{g}$ for an appropriate set of positive roots φ. The Borel-de Siebenthal theory [4], applied recursively, should make this explicit and should give a useful description of ds^2 on $\mathfrak{m}$, perhaps using some of the decomposition techniques from [15] and [16].

15B. Outer Automorphisms for Group Manifolds.

We are in the case of a compact group manifold G with a left invariant Riemannian metric ds^2. If dt^2 is the normal metric defined by a negative multiple of the Killing form of $\mathfrak{g}$, then (G, dt^2) is a Riemannian symmetric space, and we know the groups Γ of constant displacement isometries from [35, Theorem 4.5.1]; see Theorem 4.6 in Section 4. More generally for (G, ds^2), in the notation of Proposition 9.13 and Theorem 9.1, either $\Gamma \subset \bigcup \ell(G)\ell(a_i)$ or $\Gamma \subset \bigcup r((G, ds^2)a_i)$. In this second case Γ centralizes the transitive group $\ell(G)$, but in the first case we need a better understanding of $\ell(\gamma)$ for $\gamma \in \mathbf{I}(G, ds^2)$ of constant displacement. A general G–invariant metric ds^2 can be diagonalized relative to the normal metric dt^2 by diagonalizing the corresponding inner product $\langle\, ,\, \rangle_s$ on $\mathfrak{g}$, say $\mathfrak{g} = \sum_{1 \leqq i \leqq r} a_i \mathfrak{m}_i$ where the $\mathfrak{m}_i$ are mutually orthogonal relative to the inner product $\langle\, ,\, \rangle_t$ from dt^2 and $0 < a_1 < \cdots < a_r$. Let x_0 denote the base point 1 and consider a constant displacement isometry $\gamma \in \mathbf{I}(M, ds^2)$. Let $\xi \in \mathfrak{g}$ such that $t \mapsto \exp(t\xi)x_0$ is a shortest (for $\xi \in \mathfrak{g}$) curve from x_0 to $\gamma(x_0)$. Decompose $\xi = \sum \xi_i \in \mathfrak{g}$ with $\xi_i \in \mathfrak{m}_i$. Each $\exp(t\xi_i)x_0$, $0 \leqq t \leqq 1$, is an (M, ds^2)–geodesic, and it should be possible to fit them together to describe minimizing geodesics in (M, ds^2) from $x_0 = 1K$ to $\gamma(x_0)$.

15C. Weakly Symmetric Spaces and Geodesic Orbit Spaces.

Without going into the definitions, weakly symmetric spaces form a very interesting class of homogeneous Riemannian manifolds, and geodesic orbit spaces form a larger interesting class of homogeneous Riemannian manifolds. It would be important to verify the Homogeneity Conjecture for them.

15D. Extension to Finsler Manifolds.

The notion of "constant displacement" makes perfect sense for isometries of metric spaces. Furthermore, the proof of the obvious part of the Homogeneity Conjecture

> Consider a connected homogeneous Riemannian manifold (M, ds^2) and a Riemannian covering $(M, ds^2) \to \Gamma\backslash(M, ds^2)$. If $\Gamma\backslash(M, ds^2)$ is homogeneous then every $\gamma \in \Gamma$ is an isometry of constant displacement.

holds for metric spaces:

> Let $x, y \in M$ and $\gamma \in \Gamma$. Choose $g \in G$ with $g(x) = y$. Let ρ denote distance in (M, ds^2). Then the displacement $\rho(x, \gamma(x)) = \rho(gx, g\gamma(x)) = \rho(gx, \gamma g(x)) = \rho(y, \gamma(y))$.

Thus one can conjecture the converse, as for the Riemannian case. But that certainly is asking too much, and one should start by asking whether the Homogeneity Conjecture is valid for Finsler spaces, or even Finsler spaces of Berwald type or of Randers type. Much of the Riemannian geometry machinery behind

Ozols' result [22, Theorem 1.6] (see Proposition 5.1 above) is available for Finsler spaces and is conveniently set out in [9, Chapter 1].

15E. Extension to Pseudo-Riemannian Manifolds.

Here the notion of distance – and thus of displacement – does not make sense, but geodesics and the notion of preserving a geodesic are the same as in the Riemannian case. The notion of "minimizing geodesic" is not available, so in view of [22, Theorem 1.6] (Proposition 5.1) one would look at isometries $\gamma \in \mathbf{I}(M, ds^2)$ that preserve every geodesic $\overline{x, \gamma(x)}$, $x \in M$. The proof the (2) $\Leftrightarrow$ (3) in [22, Theorem 1.6] would need some adjustment, and it might be necessary to restrict attention to geodesic orbit spaces.

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Department of Mathematics, University of California, Berkeley, CA 94720–3840, U.S.A.

Email address: `jawolf@math.berkeley.edu`