# FAMILIES OF GEODESIC ORBIT SPACES AND RELATED PSEUDO–RIEMANNIAN MANIFOLDS

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Abstract. Two homogeneous pseudo–riemannian manifolds $(G/H, ds^2)$ and $(G'/H', ds'^2)$ belong to the same *real form family* if their complexifications $(G_{\mathbb{C}}/H_{\mathbb{C}}, ds^2_{\mathbb{C}})$ and $(G'_{\mathbb{C}}/H'_{\mathbb{C}}, ds'^2_{\mathbb{C}})$ are isometric. The point is that in many cases a particular space $(G/H, ds^2)$ has interesting properties, and those properties hold for the spaces in its real form family. Here we prove that if $(G/H, ds^2)$ is a geodesic orbit space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then the same holds all the members of its real form family. In particular our understanding of compact geodesic orbit riemannian manifolds gives information on geodesic orbit pseudo–riemannian manifolds. We also prove similar results for naturally reductive spaces, for commutative spaces, and in most cases for weakly symmetric spaces. We end with a discussion of inclusions of these real form families, a discussion of D'Atri spaces, and a number of open problems.

## 1. Introduction

Let $(G/H, ds^2)$ be a homogeneous pseudo–riemannian manifold. For convenience of exposition we assume that $M$ and $G$ are connected. We have the complexification $(G_{\mathbb{C}}/H_{\mathbb{C}}, ds^2_{\mathbb{C}})$ where $ds^2$ is extended by complex bilinearity on every tangent space. $(G_{\mathbb{C}}/H_{\mathbb{C}}, ds^2_{\mathbb{C}})$ is pseudo–riemannian of signature $(n, n)$ where $n = \dim M$. The **real form family** of $(G/H, ds^2)$ consists of all pseudo–riemannian manifolds $(G'/H', ds'^2)$ with the same (up to isometry) complexification $(G_{\mathbb{C}}/H_{\mathbb{C}}, ds^2_{\mathbb{C}})$. Following [13] we write $\{\{(G/H, ds^2)\}\}$ for the real form family of $(G/H, ds^2)$.

There is some ambiguity in the literature. In this paper $(G_{\mathbb{C}}/H_{\mathbb{C}}, ds^2_{\mathbb{C}})$ does not belong to $\{\{(G/H, ds^2)\}\}$; we refer to $(G_{\mathbb{C}}/H_{\mathbb{C}}, ds^2_{\mathbb{C}})$ as the **crown** of $\{\{(G/H, ds^2)\}\}$ and write $(G_{\mathbb{C}}/H_{\mathbb{C}}, ds^2_{\mathbb{C}}) = \mathbf{cr}\{\{(G/H, ds^2)\}\}$.

From now on, suppose that we have a reductive decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathrm{Ad}(H)\mathfrak{m} = \mathfrak{m}$. Then $\mathfrak{m}$ represents the tangent space at $o = 1H$ and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathfrak{m}$ defined by $ds^2$.

If $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$ then $\{\{(G/H, ds^2)\}\} = \{\{(G'/H', ds'^2)\}\}$. In particular $(G/H, ds^2)$ and $(G'/H', ds'^2)$ have isometric complexifications, $(G_{\mathbb{C}}/H_{\mathbb{C}}, ds^2_{\mathbb{C}}) \cong (G'_{\mathbb{C}}/H'_{\mathbb{C}}, ds'^2_{\mathbb{C}})$, i.e. $\mathbf{cr}\{\{(G/H, ds^2)\}\} \cong \mathbf{cr}\{\{(G'/H', ds'^2)\}\}$. Further, we have a reductive decomposition $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$ stable under an involutive isometry $\theta'$ of $G'$ and an isomorphism $f : G \cong G'$ such that $f(H) = H'$ and $f^*(ds'^2) = ds^2$. One can exchange the rôles of $(G/H, ds^2)$ and $(G'/H', ds'^2)$ here. Thus, it is an equivalence relation for $(G/H, ds^2)$ and $(G'/H', ds'^2)$ to belongs to the same real form family.

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*Date*: file last edited 11 February 2023.

2010 *Mathematics Subject Classification.* 22E45, 43A80, 32M15, 53B30, 53B35.

*Key words and phrases.* geodesic orbit space, real form family, weakly symmetric space, naturally reductive space, commutative space, D'Atri space, pseudo–riemannian manifold, homogeneous manifold.

Research partially supported by a Simons Foundation grant.

A nonzero element $\eta \in \mathfrak{g}$ is a **geodesic vector** (at $o$) if $t \mapsto \exp(t\eta)o$ is a geodesic. A geodesic $t \mapsto \gamma(t)$ is called **homogeneous** if it comes from a geodesic vector as above. $(G/H, ds^2)$ is a **geodesic orbit space** or **GO space** if every geodesic on $(G/H, ds^2)$ is homogeneous.

**Proposition 1.1. Geodesic Lemma** *Let $(G/H, ds^2)$ be a homogeneous pseudo–riemannian manifold with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Then $(G/H, ds^2)$ is a geodesic orbit space if and only if*

(1.2) *given $\xi \in \mathfrak{m}$ there exist $\alpha \in \mathfrak{h}$ and $c \in \mathbb{R}$ such that $\langle [\xi + \alpha, \zeta]_{\mathfrak{m}}, \xi\rangle = c\,\langle \xi, \zeta\rangle$ for all $\zeta \in \mathfrak{m}$.*

*Further, if $\langle \xi, \xi\rangle \neq 0$ in* (1.2) *then $c = 0$, so if $c \neq 0$ in* (1.2) *then the geodesic is null.*

For the notion of homogeneous geodesic and the formula (1.2) characterizing geodesic vectors in the riemannian case see [9]. The pseudo–riemannian case appeared in [4] and [10], but without a proof. The correct mathematical formulation with the proof was given in [3].

We are going to study the structure of real form families of geodesic orbit spaces (in §3) using the form of the Geodesic Lemma. Then we look at corresponding structural matters for naturally reductive spaces (in §4), for commutative spaces (in §5), for weakly symmetric spaces (in §6), and for D'Atri spaces (in $ 7), ending with a list of open problems.

## 2. The Moduli Spaces

The moduli spaces $\Omega$ and $\Omega_{\mathbb{C}}$ will allow us to carry the *GO* property between various spaces in a real form family. Define real and complex polynomials

$$\begin{aligned}&\varphi : \mathfrak{m} \oplus \mathfrak{h} \oplus \mathfrak{m} \oplus \mathbb{R} \to \mathbb{R} \text{ by } \varphi(\xi, \alpha, \zeta, c) = \langle [\xi + \alpha, \zeta]_{\mathfrak{m}}, \xi\rangle - c\langle \xi, \zeta\rangle \text{ and}\\ &\varphi_{\mathbb{C}} : \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathbb{C} \to \mathbb{C} \text{ by } \varphi_{\mathbb{C}}(\xi, \alpha, \zeta, c) = \langle [\xi + \alpha, \zeta]_{\mathfrak{m}_{\mathbb{C}}}, \xi\rangle - c\langle \xi, \zeta\rangle\end{aligned} \tag{2.1}$$

where $\langle \cdot, \cdot\rangle$ extends from $\mathfrak{m} \oplus \mathfrak{m}$ to $\mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}$ by complex bilinearity. Note that $\varphi_{\mathbb{C}}(\xi, \alpha, \zeta, c)$ is linear in $\zeta$, in $\alpha$, and in $c$; and it is quadratic in $\xi$. Define subvarieties

$$\begin{aligned}&\Omega = \{(\xi, \alpha, c) \in (\mathfrak{m} \oplus \mathfrak{h} \oplus \mathbb{R}) \mid \varphi(\xi, \alpha, \zeta, c) = 0 \text{ for every } \zeta \in \mathfrak{m}\} \text{ and}\\ &\Omega_{\mathbb{C}} = \{(\xi, \alpha, c) \in (\mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}) \mid \varphi_{\mathbb{C}}(\xi, \alpha, \zeta, c) = 0 \text{ for every } \zeta \in \mathfrak{m}_{\mathbb{C}}\}.\end{aligned} \tag{2.2}$$

As one might guess from the notation we have

**Proposition 2.3.** *The real affine variety $\Omega = \Omega_{\mathbb{C}} \cap (\mathfrak{m} \oplus \mathfrak{h} \oplus \mathfrak{m} \oplus \mathbb{R})$, and it is a real form of the complex affine variety $\Omega_{\mathbb{C}}$. In other words $\Omega_{\mathbb{C}}$ is the complexification of $\Omega$. In particular, if $f$ is a holomorphic function on $\Omega_{\mathbb{C}}$ and $f|_{\Omega} \equiv 0$, then $f \equiv 0$.*

*Proof.* As $\varphi$ and $\varphi_{\mathbb{C}}$ are linear in $\zeta$ we can replace the "every $\zeta$" conditions in (2.2) by "$\{\zeta_1, \dots, \zeta_\ell\}$" conditions, where $\{\zeta_1, \dots, \zeta_\ell\}$ is a basis of $\mathfrak{m}$. In other words $\Omega$ is defined by the $\ell$ real polynomial functions $\varphi_j : (\xi, \alpha, c) \mapsto \varphi(\xi, \alpha, \zeta_j, c)$ on $\mathfrak{m} \oplus \mathfrak{h} \oplus \mathbb{R}$, and $\Omega_{\mathbb{C}}$ is defined by the $\ell$ complex polynomial functions $\varphi_{j;\mathbb{C}} : (\xi, \alpha, c) \mapsto \varphi_{\mathbb{C}}(\xi, \alpha, \zeta_j, c)$ on $\mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}$. As $\varphi_j = \varphi_{j;\mathbb{C}}|_{(\mathfrak{m} \oplus \mathfrak{h} \oplus \mathbb{R})}$ the Proposition is immediate. □

## 3. Geodesic Orbit Spaces

We start by pinning down the moduli spaces $\Omega$ and $\Omega_{\mathbb{C}}$ for the geodesic orbit case. From the definitions

**Lemma 3.1.** *In the notation of* (2.2), *$\xi + \alpha$ is a geodesic vector for $(G/H, ds^2)$ if and only if $\xi + \alpha \in \Omega$, and $\xi + \alpha$ is a geodesic vector for the crown $\mathbf{cr}\{\{(G/H, ds^2)\}\}$ if and only if $\xi + \alpha \in \Omega_{\mathbb{C}}$.*

Thus we have a minor reformulation of the Geodesic Lemma (Proposition 1.1), as follows.

**Proposition 3.2.** *In the notation of* (2.2), $(G/H, ds^2)$ *is a geodesic orbit space if and only if, for every* $\xi \in \mathfrak{m}$ *there is an* $\alpha \in \mathfrak{h}$ *with* $(\xi, \alpha) \in \Omega$*, and the crown* $\mathbf{cr}\{\{(G/H, ds^2)\}\}$ *is a geodesic orbit space if and only if, for every* $\xi \in \mathfrak{m}_{\mathbb{C}}$ *there is an* $\alpha \in \mathfrak{h}_{\mathbb{C}}$ *with* $(\xi, \alpha) \in \Omega_{\mathbb{C}}$*.*

Now we combine Propositions 2.3 and 3.2:

**Theorem 3.3.** *In the notation of* (2.2), $(G/H, ds^2)$ *is a geodesic orbit space if and only if the crown* $\mathbf{cr}\{\{(G/H, ds^2)\}\}$ *is a geodesic orbit space. In particular, if* $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$*, then* $(G/H, ds^2)$ *is a geodesic orbit space if and only if* $(G'/H', ds'^2)$ *is a geodesic orbit space.*

*Proof.* Proposition 3.2 says that $(G/H, ds^2)$ is a geodesic orbit space if and only if the projection $\pi : \Omega \to \mathfrak{m}$, by $\pi(\xi, \alpha) = \xi$, is surjective; and also $(G_{\mathbb{C}}/H_{\mathbb{C}}, ds^2_{\mathbb{C}})$ is a geodesic orbit space if and only if the projection $\pi_{\mathbb{C}} : \Omega_{\mathbb{C}} \to \mathfrak{m}_{\mathbb{C}}$, by $\pi_{\mathbb{C}}(\xi, \alpha) = \xi$, is surjective. But Proposition 2.3 ensures that $\pi$ is surjective if and only if $\pi_{\mathbb{C}}$ is surjective. That proves the first assertion. Since $\{\{(G'/H', ds'^2)\}\} = \{\{(G/H, ds^2)\}\}$ the corresponding $\pi' : \Omega' \to \mathfrak{m}'$ is surjective if and only if $\pi_{\mathbb{C}}$ is surjective. The second assertion follows. □

## 4. Naturally Reductive Spaces

A homogeneous space $(G/H, ds^2)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathrm{Ad}(H)\mathfrak{m} = \mathfrak{m}$, is called **naturally reductive** if

$$\text{if } \xi \in \mathfrak{m} \text{ then } t \mapsto \exp(t\xi)H \text{ is a complete geodesic in } (G/H, ds^2). \tag{4.1}$$

The Lie algebra formulation of (4.1) is

$$\langle [\xi, \eta]_{\mathfrak{m}}, \zeta \rangle + \langle \eta, [\xi, \zeta]_{\mathfrak{m}} \rangle = 0 \text{ for all } \xi, \eta, \zeta \in \mathfrak{m}. \tag{4.2}$$

The case $\zeta = \xi$ is $\langle [\xi, \eta]_{\mathfrak{m}}, \zeta \rangle + \langle \eta, [\xi, \zeta]_{\mathfrak{m}} \rangle = \langle [\xi, \eta]_{\mathfrak{m}}, \xi \rangle + \langle \eta, [\xi, \xi]_{\mathfrak{m}} \rangle = \langle [\xi, \eta]_{\mathfrak{m}}, \xi \rangle$, so naturally reductive spaces are geodesic orbit spaces. Or one can see this by noting that (4.1) is the case $\alpha = 0$ of (1.2).

As in (2.1) one has corresponding polynomials

$$\begin{aligned} &\psi : \mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m} \to \mathbb{R} \text{ by } \psi(\xi, \eta, \zeta) = \langle [\xi, \eta]_{\mathfrak{m}}, \zeta \rangle + \langle \eta, [\xi, \zeta]_{\mathfrak{m}} \rangle \text{ and} \\ &\psi_{\mathbb{C}} : \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \to \mathbb{C} \text{ by } \psi_{\mathbb{C}}(\xi, \eta, \zeta) = \langle [\xi, \eta]_{\mathfrak{m}_{\mathbb{C}}}, \zeta \rangle + \langle \eta, [\xi, \zeta]_{\mathfrak{m}_{\mathbb{C}}} \rangle. \end{aligned} \tag{4.3}$$

As in (2.2) those polynomials define corresponding moduli spaces

$$\begin{aligned} &\Psi = \{(\xi, \eta, \zeta)) \in (\mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m}) \mid \psi(\xi, \eta, \zeta) = 0\} \text{ and} \\ &\Psi_{\mathbb{C}} = \{(\xi, \eta, \zeta) \in (\mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}) \mid \psi_{\mathbb{C}}(\xi, \eta, \zeta) = 0\}. \end{aligned} \tag{4.4}$$

Then we have the analog of Proposition 2.3:

**Proposition 4.5.** *The real affine variety* $\Psi = \Psi_{\mathbb{C}} \cap (\mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m})$*, and it is a real form of the complex affine variety* $\Psi_{\mathbb{C}}$*. In other words* $\Psi_{\mathbb{C}}$ *is the complexification of* $\Psi$*. In particular, if* $f$ *is a holomorphic function on* $\Psi_{\mathbb{C}}$ *and* $f|_{\Psi} \equiv 0$*, then* $f \equiv 0$*.*

We reformulate the definition (4.2) of naturally reductive space:

**Proposition 4.6.** *In the notation of* (4.4), $(G/H, ds^2)$ *is a naturally reductive space if and only if* $\psi(\xi, \eta, \zeta) \in \Psi$ *whenever* $(\xi, \eta, \zeta)) \in (\mathfrak{m} \oplus \mathfrak{m} \oplus \mathfrak{m})$*. The crown* $\mathbf{cr}\{\{(G/H, ds^2)\}\}$ *is a naturally reductive space if and only if* $\psi(\xi, \eta, \zeta) \in \Psi_{\mathbb{C}}$ *whenever* $(\xi, \eta, \zeta)) \in (\mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}})$*.*

Now we combine Propositions 4.5 and 4.6:

**Theorem 4.7.** *In the notation of* (4.4), $(G/H, ds^2)$ *is a naturally reductive space if and only if the crown* $\mathbf{cr}\{\{(G/H, ds^2)\}\}$ *is a naturally reductive space. In particular, if* $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$*, then* $(G/H, ds^2)$ *is a naturally reductive space if and only if* $(G'/H', ds'^2)$ *is a naturally reductive space.*

## 5. Commutative Spaces

Consider a pseudo-riemannian manifold $(G/H, ds^2)$ where $G$ is the identity component of the group of all isometries. The $G$–invariant differential operators on $G/H$ form an associative algebra $\mathcal{D}(G,H)$. We say that $(G/H, ds^2)$ is **commutative** if the algebra $\mathcal{D}(G,H)$ is commutative. This is the usual definition when $H$ is compact and $(G/H, ds^2)$ is riemannian, but it makes perfectly good sense (and is appropriate for us) in any signature.

We will discuss D'Atri spaces in Section 7, but the point here is that commutative spaces are D'Atri spaces [6]. In dimensions $\leqq 5$ one can say a bit more. There, a homogeneous riemannian manifold is commutative if and only if it is naturally reductive.

Without loss of generality we suppose that $G/H$ is connected and simply connected, and that $G = I^0(G/H, ds^2)$. Then also $H$ is connected. As before we start with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Identify $\mathfrak{m}$ with the tangent space $T_{x_0}(G/H)$ at the base point $x_0 = 1H \in G/H$. That gives an obvious $\mathrm{Ad}(H)$–equivariant bijection between $\mathcal{D}(G,H)$ and the $\mathrm{Ad}(H)$–invariants [1] $S(\mathfrak{m})^H$ in the symmetric algebra $S(\mathfrak{m})$. See Helgason, [5, Ch. II, Theorem 4.6], for the details.

Given any real basis $\{\zeta_1, \dots, \zeta_\ell\}$ of $\mathfrak{h}$, $S(\mathfrak{m})^H$ is the intersection $\bigcap_{1 \leqq j \leqq \ell} S(\mathfrak{m})^{\zeta_j}$ of null spaces of the $\mathrm{ad}\,(\zeta_j)|_{S(\mathfrak{m})}$. As $\{\zeta_j\}$ is a complex basis of $\mathfrak{m}_{\mathbb{C}}$ we have

**Lemma 5.1.** *The algebra* $\mathcal{D}(G_{\mathbb{C}}, H_{\mathbb{C}})$ *is the complexification* $\mathcal{D}(G,H)_{\mathbb{C}}$ *of the algebra.* $\mathcal{D}(G,H)$.

Now $\mathcal{D}(G_{\mathbb{C}}, H_{\mathbb{C}})$ is commutative if and only if $\mathcal{D}(G,H)$ is commutative. We apply this to real form families.

**Theorem 5.2.** *The pseudo–riemannian manifold* $(G/H, ds^2)$ *is commutative if and only if its complexification* $(G_{\mathbb{C}}/H_{\mathbb{C}}, ds^2_{\mathbb{C}})$ *is commutative. If* $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$*, then* $(G'/H', ds'^2)$ *is commutative if and only if* $(G/H, ds^2)$ *is commutative.*

## 6. Weakly Symmetric Spaces

Recall that a pseudo–riemannian manifold $(M, ds^2)$ is **weakly symmetric** if, given $x \in M$ and a tangent vector $\xi \in T_x(M)$, there is an isometry $s_{x,\xi} \in I(M, ds^2)$ such that $s_{x,\xi}(x) = x$ and $ds_{x,\xi}(\xi) = -\xi$. The familiar special case: $(M, ds^2)$ is symmetric if, given $x \in M$ there is an isometry $s_x \in I(M, ds^2)$ such that $s_x(x) = x$ and $ds_x(\xi) = -\xi$ for every $\xi \in T_x(M)$.

Riemannian weakly symmetric spaces were introduced by Selberg [11] in the context of harmonic analysis and algebraic geometry. One of his results was that riemannian weakly symmetric spaces are commutative. In view of Theorem 5.2,

**Corollary 6.1.** *Let* $(G/H, ds^2)$ *be a riemannian weakly symmetric space. Then* $\mathbf{cr}\{\{(G/H, ds^2)\}\}$ *is commutative, and every* $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$ *is commutative.*

[1] by an abuse of notation we write $\mathrm{Ad}(H)$ instead of $S(\mathrm{Ad}(H))$ for the symmetric powers that form the action of $H$ on $S(\mathfrak{m})$.

Weakly symmetric pseudo–riemannian manifolds are geodesic orbit spaces [1, Theorem 4.2]. Thus, if $(G/H, ds^2)$ is weakly symmetric then, by Theorem 3.3, every $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$ is a geodesic orbit space.

There are $\aleph_0$ examples in the tables of [2] and [13]. Tables 3.6, 4.12, 5.1, 5.2 and 5.3 in [2] list various classes of real form families $\{\{(G/H, ds^2)\}\}$ with $(G/H, ds^2)$ weakly symmetric, $G$ semisimple and $H$ reductive. The Tables in [13] list various classes of real form families $\{\{(G/H, ds^2)\}\}$ for which $(G/H, ds^2)$ is a weakly symmetric nilmanifold with $G = N \rtimes H$.

The question here is just when weak symmetry of $(G/H, ds^2)$ implies weak symmetry for the members of its real form family $\{\{(G/H, ds^2)\}\}$. A partial answer is implicit in a result of Akhiezer and Vinberg [12, Theorem 12.6.10]; see [12, Corollary 12.6.12]:

**Proposition 6.2.** *Let $(G/H, ds^2)$ be a weakly symmetric pseudo–riemannian manifold with $G$ connected and reductive, and $H$ reductive in $G$. Then every $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$ is weakly symmetric.*

See [12, Section 15.4] for a discussion of commutativity for weakly symmetric riemannian nilmanifolds.

## 7. D'Atri Spaces

We say that a pseudo–riemannian manifold $(M, ds^2)$ is a **D'Atri space** if its local geodesic symmetries $\sigma_x : \exp(t\xi) \mapsto \exp(-t\xi)$, $\xi \in T_x(M)$ and $t$ reasonably small, are volume preserving. This is the standard definition when $(M, ds^2)$ is riemannian, but it makes perfectly good sense (and is appropriate for us) in any signature.

A geodesic orbit riemannian manifold is a D'Atri space [7, Theorem 1]. That argument of Kowalski and Vanhecke goes through *mutatis mutandis* for pseudo–riemannian manifolds, using the definition introduced just above. Or see [8] to develop this in the more general setting of two-point functions. In any case, we now have inclusions of real form families of pseudo–riemannian manifolds:

$$\begin{aligned} &\text{(weakly symmetric spaces)} \subset \text{(geodesic orbit spaces)} \subset \text{(D'Atri spaces)} \\ &\text{(naturally reductive spaces)} \subset \text{(geodesic orbit spaces)} \subset \text{(D'Atri spaces)} \end{aligned} \tag{7.1}$$

This suggests a number of open problems, one of which was noted toward the end of Section 6:

- If $(G/H, ds^2)$ is weakly symmetric and $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$, is $(G'/H', ds'^2)$ weakly symmetric?
- Can the naturally reductive weakly symmetric spaces be characterized as the weakly symmetric spaces $(G/H, ds^2)$ for which every $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$ is weakly symmetric?
- If $(G/H, ds^2)$ is a D'Atri space and $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$, is $(G'/H', ds'^2)$ a D'Atri space?
- Can the geodesic orbit spaces be characterized as the D'Atri spaces $(G/H, ds^2)$ for which every $(G'/H', ds'^2) \in \{\{(G/H, ds^2)\}\}$ is a D'Atri space?
- Which commutative spaces are weakly symmetric spaces?
- What happens if we restrict these questions to the case of spaces $(G/H, ds^2)$ for which $G$ is semisimple (or real reductive) and $H$ is reductive in $G$?
- What happens if we restrict these questions to the case of spaces $(G/H, ds^2)$ for which $G$ is of the form $N \rtimes H$ with $N$ nilpotent?

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