The Structure of Geodesic Orbit Lorentz Nilmanifolds

Yuri Nikolayevsky[1] · Joseph A. Wolf[2]

Abstract

The geodesic orbit property is useful and interesting in Riemannian geometry. It implies homogeneity and has important classes of Riemannian manifolds as special cases. Those classes include weakly symmetric Riemannian manifolds and naturally reductive Riemannian manifolds. The corresponding results for indefinite metric manifolds are much more delicate than in Riemannian signature, but in the last few years important corresponding structural results were proved for geodesic orbit Lorentz manifolds. Here, we carry out a major step in the structural analysis of geodesic orbit Lorentz nilmanifolds. Those are the geodesic orbit Lorentz manifolds $M = G/H$ such that a nilpotent analytic subgroup of G is transitive on M. Suppose that there is a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ (vector space direct sum) with $\mathfrak{n}$ nilpotent. When the metric is nondegenerate on $[\mathfrak{n}, \mathfrak{n}]$, we show that $\mathfrak{n}$ is abelian or 2-step nilpotent (this is the same result as for geodesic orbit Riemannian nilmanifolds), and when the metric is degenerate on $[\mathfrak{n}, \mathfrak{n}]$, we show that $\mathfrak{n}$ is a Lorentz double extension corresponding to a geodesic orbit Riemannian nilmanifold. In the latter case, we construct examples to show that the number of nilpotency steps is unbounded.

✉ Joseph A. Wolf
    jawolf@math.berkeley.edu

   Yuri Nikolayevsky
   Y.Nikolayevsky@latrobe.edu.au

[1] Department of Mathematical and Physical Sciences, La Trobe University, Bundoora, VIC 3086, Australia

[2] Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

 ⓐ Springer

1 Introduction

A pseudo-Riemannian manifold (M, ds^2) is called a *geodesic orbit manifold* (or a manifold with homogeneous geodesics, or simply a GO manifold), if every geodesic of M is an orbit of a 1-parameter subgroup of the full isometry group $I(M) = I(M, ds^2)$. One loses no generality if one replaces $I(M)$ by its identity component $I^0(M)$. If G is a transitive Lie subgroup of $I^0(M)$, so $(M, ds^2) = (G/H, ds^2)$ where H is an isotropy subgroup of G, and if every geodesic of M is an orbit of a 1-parameter subgroup of G, then we say that (M, ds^2) is a G-*geodesic orbit manifold*, or a G-GO manifold. Clearly every G-GO manifold is a GO manifold, but not vice versa. The class of geodesic orbit manifolds includes (but is not limited to) symmetric spaces, weakly symmetric spaces, normal and generalized normal homogeneous spaces, and naturally reductive spaces. For the current state of knowledge in the theory of Riemannian geodesic orbit manifolds, we refer the reader to [1] and its bibliography.

In this paper, we study the GO condition for pseudo-Riemannian nilmanifolds (N, ds^2), relative to subgroups $G \subset I(N)$ of the form $G = N \rtimes H$, where H is an isotropy subgroup. Most of our results apply to the case where (N, ds^2) is a Lorentz manifold.

Our results for G-GO manifolds $(M, ds^2) = (G/H, ds^2)$ require the coset space G/H to be reductive. In other words, they make use of an $\mathrm{Ad}_G(H)$–invariant decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Very few structural results are known for indefinite metric GO manifolds that are not reductive, and we always assume that G/H is reductive.

Recall that a pseudo-Riemannian *nilmanifold* is a pseudo-Riemannian manifold admitting a transitive nilpotent Lie group of isometries. In the Riemannian case, the full isometry group of a nilmanifold (N, ds^2), where N is a transitive nilpotent group of isometries, is the semidirect product $I(N) = N \rtimes H$, where H is the group of all isometric automorphisms of (N, ds^2) [20, Theorem 4.2]. In other words, N is the nilradical of $I(N)$. In the pseudo-Riemannian cases, $I(N)$ might still contain $N \rtimes H$ and yet be strictly larger. In indefinite signatures of metric, a nilmanifold is not necessarily reductive as a coset space of $I(N)$, and even when it is, N does not have to be a normal subgroup of $I(N)$. Here, the GO condition does not rescue us, for there exist 4-dimensional, Lorentz GO nilmanifolds that are reductive relative to $I(N)$, but for which N is not an ideal in $I(N)$ [8, Sect. 3]. Moreover, already in dimension 4 (the lowest dimension for homogeneous pseudo-Riemannian spaces G/H with H connected that are not reductive), every non-reductive space is a GO manifold when we make a correct choice of parameters [3, Theorem 4.1]. A complete classification of pseudo-Riemannian GO manifolds of dimension 4 is given in [2].

In Sect. 2, we recall some basic facts on reductive geodesic orbit spaces. In particular, the Geodesic Lemma (recalled as Proposition 1 below) gives an algebraic condition $\langle [T + A, T']_{\mathfrak{m}}, T \rangle = k \langle T, T' \rangle$ for a reductive pseudo-Riemannian homogeneous space $M = G/H$, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, to be GO. We also recall the notion of geodesic graph and use it in Proposition 2 for a characterization of the naturally reductive condition.

In Sect. 3, we sharpen [6, Theorem 7] to obtain a basic structure result on reductive GO Lorentz nilmanifolds $(G/H, ds^2)$. Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ with $\mathfrak{n}$ nilpotent and let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathfrak{n}$ defined by ds^2. If $\langle \cdot, \cdot \rangle|_{[\mathfrak{n},\mathfrak{n}]}$ is nondegenerate then [6,

Theorem 7] says N is either abelian, or 2-step nilpotent, or 4-step nilpotent. While there are many examples of abelian and of 2-step nilpotent, there were no examples of 4-step nilpotent. Our Theorem 2 eliminates the 4-step possibility. That is the main result of this paper. Theorem 2 in Sect. 3 is obtained as a corollary of Theorem 1; the latter is valid in higher signatures provided the derived algebra of $\mathfrak{n}$ is either abelian or Lorentz.

In Sect. 4, we recall the notion of double extension and use it to obtain a complement, Theorems, 3 to Theorem 2. While Theorem 2 requires that $\langle \cdot, \cdot \rangle|_{[\mathfrak{n}, \mathfrak{n}]}$ be nondegenerate, Theorem 3 requires that it be degenerate, and then it shows that $(G/H, ds^2)$ is a Lorentz double extension of a Riemannian GO nilmanifold.

In Sect. 5, we construct a large family of naturally reductive GO Lorentz nilmanifolds $(G/H, ds^2)$ that are double extensions of Riemannian GO nilmanifolds. There the transitive nilpotent groups are r-step nilpotent for unbounded r. Theorem 4 extracts a few of those double extension manifolds and shows that for every $d > 0$ there is a naturally reductive Lorentz nilmanifold (N, ds^2) of nilpotent step $\geq d$ and dimension $d + 4$, and a corresponding Lorentz nilmanifold (N, ds^2) of nilpotent step $\geq d$ and dimension $d + 10$, that is not naturally reductive.

The authors have no competing interests to declare that are relevant to the content of this article.

2 Preliminaries

Let $M = G/H$ be a pseudo-Riemannian homogeneous space. As usual $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H. In the Riemannian case, there is an $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-module $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as a linear space. This is not necessarily true in an arbitrary signature; if it is, the pseudo-Riemannian homogeneous space $M = G/H$ is called G-*reductive*. Note that reductivity depends on the choice of the isometry group G. Any corresponding decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is called a *reductive decomposition*.

The GO condition for reductive spaces is given in the **Geodesic Lemma**:

Proposition 1 *Let $M = G/H$ be a reductive pseudo-Riemannian homogeneous space, with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then M is a G-geodesic orbit space if and only if, for any $T \in \mathfrak{m}$, there exist $A = A(T) \in \mathfrak{h}$ and $k = k(T) \in \mathbb{R}$ such that if $T' \in \mathfrak{m}$ then*

$$\langle [T + A, T']_{\mathfrak{m}}, T \rangle = k \langle T, T' \rangle. \tag{1}$$

The subscript $_{\mathfrak{m}}$ means the $\mathfrak{m}$-component in $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

Substituting $T' = T$ one sees that $k(T) = 0$ unless T is a null vector. In particular, k is always zero in the Riemannian signature. Any map $A : \mathfrak{m} \to \mathfrak{h}$ for which (1) holds (with some function k) is called a *geodesic graph*. If a geodesic graph exists (that is, if the space is GO), it can be chosen $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-equivariant, i.e., such that $[L, A(T)] = A([L, T])$, for all $L \in \mathfrak{h}$ and all $T \in \mathfrak{m}$.

A pseudo-Riemannian homogeneous space $M = G/H$ is $(G\text{-})$*naturally reductive* if there is a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that

$$\text{if } T, T' \in \mathfrak{m} \text{ then } \langle [T', T]_{\mathfrak{m}}, T \rangle = 0. \tag{2}$$

In our nilmanifold case, if $G = N \rtimes H$, the space G/H might be naturally reductive using a choice of a complementary $\mathfrak{h}$-module $\mathfrak{m}$ different from $\mathfrak{n}$. In the case $G = N$, the natural reductivity condition says that the inner product on $\mathfrak{n}$ is invariant under the adjoint representation, so the metric on N is *bi-invariant*, in other words invariant under both left and right translations. Kostant's criterion for natural reductivity in the Riemannian signature [13, Theorem 4] is valid as well in pseudo-Riemannian case [17, Theorem 2.2].

The property of being naturally reductive depends on the choice of group G in the presentation $M = G/H$: both enlarging and reducing G may lead to gaining or losing the natural reductivity property, even in the Riemannian setting [9, § 2]. This contrasts with the GO condition, which is trivially preserved under enlarging the isometry group.

A G-naturally reductive space is always G'-geodesic orbit for any $G' \supset G$. This is seen by taking $A = 0$ and $k = 0$ in the Geodesic Lemma. The converse fails even in Riemannian case, where there are GO spaces that are not G-naturally reductive for any choice of the transitive group G ([12, Proposition 3], [14, Theorem 5.3(I)]). Also, see Theorem 4 below.

Proposition 2 below is useful for deciding whether a GO space is naturally reductive relative to the same group G. The proof is essentially the same as in [19, Corollaire 2, Lemme 10] for the affine case (although the "only if" direction there requires H to be compact), and in [14, Proposition 2.10] for the Riemannian case. For completeness, we include it below.

Proposition 2 *Let $M = G/H$ be a reductive G-GO space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then M is G-naturally reductive if and only if a geodesic graph $A : \mathfrak{m} \to \mathfrak{h}$ in the Geodesic Lemma can be chosen linear and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-equivariant.*

Proof Let $M = G/H$ be a naturally reductive G-GO space. Choose a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\langle [T', T]_{\mathfrak{m}}, T \rangle = 0$ for all $T, T' \in \mathfrak{m}$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be any other reductive decomposition. Since both $\mathfrak{m}$ and $\mathfrak{p}$ are naturally identified with the tangent space of (M, ds^2) at the base point, there is a uniquely defined $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-invariant isometry $\iota : \mathfrak{m} \to \mathfrak{p}$. Now for the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, the equation (1) holds with $A(X) = \iota^{-1}X - X$ and $k(X) = 0$, for all $X \in \mathfrak{p}$.

Conversely, given a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with an $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-equivariant linear map $A : \mathfrak{p} \to \mathfrak{h}$ such that (1) holds for $X \in \mathfrak{p}$ (forcing $k = 0$ by continuity), we define $\mathfrak{m} = \mathrm{Span}(X + A(X) \,|\, X \in \mathfrak{p})$, with the inner product such that the map $X \mapsto X + A(X)$ is an isometry. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and $\mathfrak{m}$ is an $\mathfrak{h}$-module because A is $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-equivariant. It is easy to check that $\langle [T', T]_{\mathfrak{m}}, T \rangle = 0$ for all $T, T' \in \mathfrak{m}$. $\quad\square$

In the Riemannian case, or more generally when H is compact, the existence of a linear geodesic graph implies the existence of a linear, $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-equivariant geodesic graph, and hence is equivalent to natural reductivity [5, Lemma 3].

3 Eliminating 4-Step When $[\mathfrak{n}, \mathfrak{n}]$ is Nondegenerate

Some years ago, Gordon proved [11, Theorem 2.2] that a Riemannian GO nilmanifold is at most 2-step nilpotent. More recently, Chen, Wolf and Zhang proved [6, Theorem 7] that a connected Lorentz G-geodesic orbit nilmanifold $M = G/H$, with $G = N \rtimes H$, N nilpotent and $\langle \cdot, \cdot \rangle|_{[\mathfrak{n},\mathfrak{n}]}$ nondegenerate,[1] has similar properties: $\mathfrak{n}$ is abelian, or $\mathfrak{n}$ is 2-step nilpotent, or $\mathfrak{n}$ is 4-step nilpotent. Our theorems here eliminate the 4-step possibility and go somewhat beyond the case of Lorentz signature.

Given a reductive homogeneous pseudo-Riemannian manifold $(G/H, ds^2)$, where $G = N \rtimes H$ with N nilpotent, identify $\mathfrak{n} = \mathrm{Lie}(N)$ with the tangent space to G/H at $1N$.

Theorem 1 *Let* $(M = G/H, ds^2)$ *be a connected pseudo-Riemannian G-geodesic orbit nilmanifold where $G = N \rtimes H$ with N nilpotent. Denote $\langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle|_{[\mathfrak{n},\mathfrak{n}]}$. In the following cases, N is either abelian or 2-step nilpotent:*

(a) $\langle \cdot, \cdot \rangle'$ *is of definite signature;*
(b) $\langle \cdot, \cdot \rangle'$ *is of Lorentz signature and the centralizer of $[\mathfrak{n}, \mathfrak{n}]$ is nondegenerate.*

Proof The first part of our argument is similar to part of the proof of [6, Theorem 7]. Let $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$, and denote $m = \dim \mathfrak{n}'$. We can assume that $m \geq 2$.

Suppose that $\langle \cdot, \cdot \rangle'$ is nondegenerate. Let $\mathfrak{v}$ denote the orthogonal complement to $\mathfrak{n}'$ in $\mathfrak{n}$. Then we have the $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-invariant orthogonal direct sum decomposition $\mathfrak{n} = \mathfrak{n}' \oplus \mathfrak{v}$. Let $T = X + Y$ and $T' = X' + Y'$ where $X, X' \in \mathfrak{n}'$, $Y, Y' \in \mathfrak{v}$, and T is non-null in (1). Then $k(T) = 0$ and we have $A = A(X, Y) \in \mathfrak{h}$ such that

$$\langle [A, X'], X \rangle + \langle [A, Y'], Y \rangle + \langle [X, X'] + [Y, X'] + [X, Y'] + [Y, Y'], X \rangle = 0. \quad (3)$$

Taking $Y' = Y$, $X' = 0$ we obtain, by continuity,

$$\langle [Y, X], X \rangle = 0, \quad \text{for all } Y \in \mathfrak{v}, \ X \in \mathfrak{n}'. \quad (4)$$

As $\mathfrak{v}$ generates $\mathfrak{n}$, it follows that

$$\langle [T, X], X \rangle = 0, \quad \text{for all } T \in \mathfrak{n}, \ X \in \mathfrak{n}'. \quad (5)$$

Separating the X'- and the Y'-components in (3) and using (4) and (5), we find that for all $X \in \mathfrak{n}'$ and $Y \in \mathfrak{v}$ with $X + Y$ non-null, there exists $A = A(X, Y) \in \mathfrak{h}$ such that for all $X' \in \mathfrak{n}'$, $Y' \in \mathfrak{v}$,

$$\langle [A, Y], Y' \rangle = \langle [Y, Y'], X \rangle, \quad (6)$$
$$[A + Y, X] = 0. \quad (7)$$

Assertion (a) now follows from (5): for all $T \in \mathfrak{n}$, the (nilpotent) operator $\mathrm{ad}_{\mathfrak{g}}(T)|_{\mathfrak{n}'}$ on $\mathfrak{n}'$ is skew-symmetric relative to the definite inner product $\langle \cdot, \cdot \rangle'$, and hence is zero, which implies $[\mathfrak{n}, \mathfrak{n}'] = 0$. (This follows the argument of [11, Theorem 2.2].)

---

[1] Nondegeneracy of $\langle \cdot, \cdot \rangle|_{[\mathfrak{n},\mathfrak{n}]}$ is stated in the paragraph before the statement of [6, Theorem 7] and is recalled and used in the proof, but perhaps it could have been part of the statement itself.

For (b), we suppose that $\langle \cdot, \cdot \rangle'$ is Lorentz. Denote $\mathfrak{s} := \mathfrak{so}(m-1, 1) \subset \mathfrak{gl}(\mathfrak{n}')$, the algebra of skew-symmetric endomorphisms relative to $\langle \cdot, \cdot \rangle'$. Thus, $\mathfrak{k} := \mathrm{ad}_\mathfrak{g}(\mathfrak{n})|_{\mathfrak{n}'}$ is a subalgebra of $\mathfrak{s}$ consisting of nilpotent endomorphisms. As such, using Engel's Theorem, it is triangular. Thus, it is conjugate by an inner automorphism [16, Theorem 2.1] to a subalgebra of the nilpotent part $\mathfrak{u}$ of an Iwasawa decomposition $\mathfrak{s} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{u}$. Now we may (and do) assume $\mathfrak{k} \subset \mathfrak{u}$.

Choose a basis $\{f_1, \ldots, f_m\}$ for $\mathfrak{n}'$ such that $\langle f_i, f_j \rangle = \varepsilon_i \delta_{ij}$, where $\varepsilon_1 = -1$ and $\varepsilon_i = +1$ for $i > 1$, and such that the maximal compact subalgebra $\mathfrak{t} = \mathfrak{so}(m-1)$ acts on $\mathrm{Span}(f_2, \ldots, f_m)$, and the 1-dimensional abelian subalgebra is given by $\mathfrak{a} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0_{m-2} \end{pmatrix} \mathbb{R}$. Here, 0_{m-2} denotes the $(m-2) \times (m-2)$ zero matrix. Then $\mathfrak{u}$ is the space of matrices of the form $\begin{pmatrix} 0 & 0 & u^t \\ 0 & 0 & u^t \\ u & -u & 0_{m-2} \end{pmatrix}$ where $u \in \mathbb{R}^{m-2}$. We introduce a new basis for $\mathfrak{n}'$ given by $e_1 = (f_1 + f_2)/\sqrt{2}$, $e_2 = (f_1 - f_2)/\sqrt{2}$ and $e_i = f_i$ for $i > 2$. Relative to this basis, we have

$$\langle \cdot, \cdot \rangle|_{\mathfrak{n}'} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I_{m-2} \end{pmatrix} \text{ and } \mathfrak{u} = \left\{ \begin{pmatrix} 0 & 0 & u^t \\ 0 & 0 & 0 \\ 0 & u & 0_{m-2} \end{pmatrix}, \ u \in \mathbb{R}^{m-2} \right\}.$$

As $\mathfrak{k} \subset \mathfrak{u}$, we obtain a linear map $\Phi : \mathfrak{v} \to \mathrm{Span}(e_3, \ldots, e_m)$ such that, for all $Y \in \mathfrak{v}$,

$$\mathrm{ad}_\mathfrak{g}(Y)e_1 = 0, \ \mathrm{ad}_\mathfrak{g}(Y)e_2 = \Phi Y, \text{ and } \mathrm{ad}_\mathfrak{g}(Y)e_i = \langle \Phi Y, e_i \rangle e_1 \text{ for } i > 2. \quad (8)$$

As $\mathfrak{u}$ (and hence $\mathfrak{k}$) is abelian, $[[\mathfrak{v}, \mathfrak{v}], \mathfrak{n}'] = 0$. Since $\mathfrak{v}$ generates $\mathfrak{n}$, we obtain $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}'] = 0$, which implies that $\mathfrak{n}'$ is abelian.

To complete the proof of (b), we introduce 2-forms $\omega_i \in \Lambda^2(\mathfrak{v})$ by

$$[V_1, V_2] = \sum_{i=1}^{m} \omega_i(V_1, V_2)e_i \text{ for } V_1, V_2 \in \mathfrak{v}. \quad (9)$$

As $e_1 \in \mathfrak{n}'$ and $\langle \cdot, \cdot \rangle'$ is nondegenerate, we cannot have $e_1 \perp \mathfrak{n}'$. But since $\mathfrak{n}'$ is abelian we get $\mathfrak{n}' = [\mathfrak{v}, \mathfrak{v}] + [\mathfrak{v}, \mathfrak{n}']$, and from (8) we obtain $e_1 \perp [\mathfrak{v}, \mathfrak{n}']$. Thus, $e_1 \not\perp [\mathfrak{v}, \mathfrak{v}]$, which by (9) implies $\omega_2 \neq 0$.

Using (8) and (9), the Jacobi identity gives

$$\sigma \left(\omega_2(V_1, V_2)\Phi V_3 + \sum_{i=3}^{m} \omega_i(V_1, V_2)\langle \Phi V_3, e_i \rangle e_1 \right) = 0, \quad (10)$$

where σ denotes the cyclic permutation of $V_1, V_2, V_3 \in \mathfrak{v}$. If $\mathrm{rk}\,\Phi \geq 3$, then for almost all triples $V_1, V_2, V_3 \in \mathfrak{v}$, the vectors $\Phi V_1, \Phi V_2$, and $\Phi V_3 \in \mathrm{Span}(e_3, \ldots, e_m)$ are linearly independent, so $\omega_2 = 0$ by (10). This is a contradiction, so $\mathrm{rk}\,\Phi \leq 2$.

If $\Phi = 0$, the algebra $\mathfrak{n}$ is 2-step nilpotent by (8). Now suppose $\Phi \neq 0$.

The centralizer of $\mathfrak{n}'$ in $\mathfrak{n}$ is an $\mathrm{ad}_\mathfrak{g}(\mathfrak{h})$-invariant ideal in $\mathfrak{n}$, so its intersection with $\mathfrak{v}$, which is the subspace $\mathfrak{c} = \mathrm{Ker}\,\Phi \subset \mathfrak{v}$, is also $\mathrm{ad}_\mathfrak{g}(\mathfrak{h})$-invariant. Then the subspace $\mathfrak{c}^\perp \subset \mathfrak{v}$ is $\mathrm{ad}_\mathfrak{g}(\mathfrak{h})$-invariant as well. Note that $\dim \mathfrak{c}^\perp = \mathrm{rk}\,\Phi \in \{1, 2\}$ by the above argument, and that $\langle \cdot, \cdot \rangle|_\mathfrak{c}$ is nondegenerate by our assumption. Then $\langle \cdot, \cdot \rangle|_{\mathfrak{c}^\perp}$ is also nondegenerate. As $\Phi \mathfrak{c} = 0$, (10) implies $\omega_2(\mathfrak{c}, \mathfrak{c}) = 0$, where we take $V_1, V_2 \in \mathfrak{c}$

and $V_3 \in \mathfrak{c}^\perp$. Moreover, taking in (6) $Y \in \mathfrak{c}$, $Y' \in \mathfrak{c}^\perp$, we obtain $[\mathfrak{c}, \mathfrak{c}^\perp] = 0$, and in particular, $\omega_2(\mathfrak{c}, \mathfrak{c}^\perp) = 0$.

Since $\omega_2 \neq 0$, we must have $\omega_2(\mathfrak{c}^\perp, \mathfrak{c}^\perp) \neq 0$, and so $\dim \mathfrak{c}^\perp = 2$. Let Y_1, Y_2 be a basis for $\mathfrak{c}^\perp$ such that $\langle Y_i, Y_j \rangle = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = \pm 1$ for $i = 1, 2$. As $\mathfrak{c}^\perp$ is $\mathrm{ad}_\mathfrak{g}(\mathfrak{h})$-invariant, $\mathrm{ad}_\mathfrak{g}(\mathfrak{h})|_\mathfrak{n}$ is skew-symmetric, we obtain $[A, Y_1] = \varepsilon_1 \mu(A) Y_2$ and $[A, Y_2] = -\varepsilon_2 \mu(A) Y_1$, for any $A \in \mathfrak{h}$, where μ is a linear functional on $\mathfrak{h}$. In particular, $[A, [Y_1, Y_2]] = 0$ for all $A \in \mathfrak{h}$. Taking $X = [Y_1, Y_2]$ in (7), we see that $[Y, [Y_1, Y_2]] = 0$ for all $Y \in \mathfrak{v}$ such that $Y + [Y_1, Y_2]$ is non-null. Thus, $[Y, [Y_1, Y_2]] = 0$ for all $Y \in \mathfrak{v}$. But then (8) and (9) imply $\omega_2(Y_1, Y_2) \Phi Y = 0$, so $\omega_2(Y_1, Y_2) = 0$. So $\omega_2(\mathfrak{c}^\perp, \mathfrak{c}^\perp) = 0$, which is a contradiction. $\qquad\square$

The Lorentz manifold case of Theorem 1 is of special interest, so we state it separately.

Theorem 2 *Let $(M = G/H, ds^2)$ be a connected Lorentz G-geodesic orbit nilmanifold where $G = N \rtimes H$ with N nilpotent. Let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathfrak{n}$ induced by ds^2. If $\langle \cdot, \cdot \rangle|_{[\mathfrak{n}, \mathfrak{n}]}$ is nondegenerate, then N is abelian or 2-step nilpotent.*

Proof This is an immediate consequence of assertion (b) in Theorem 1. Indeed, the centralizer of $\mathfrak{n}'$ is the direct sum of $\mathfrak{n}'$ and a subspace $\mathfrak{c}$ of $\mathfrak{v}$, and if $\langle \cdot, \cdot \rangle$ is Lorentz and its restriction to n' is Lorentz, then and its restriction to $\mathfrak{n}'$ is Lorentz, then the inner product on $\mathfrak{v}$ is definite, which implies that it is also definite on $\mathfrak{c}$, and hence the centralizer of $\mathfrak{n}'$ is nondegenerate. $\qquad\square$

If the GO condition in Theorems 1 and 2 is replaced by the natural reductivity condition, the complete description of all resulting nilmanifolds is given in [18, Theorem 3.2]. There the construction in arbitrary signature is similar to the construction for Riemannian signature.

4 The Double Extension Theorem

Given a metric Lie algebra $\mathfrak{m}_0$ with a nondegenerate inner product $\langle \cdot, \cdot \rangle_0$, say of signature (p, q), let $\mathfrak{m}_1$ be its central extension, as in the exact sequence

$$0 \to \mathbb{R}e \to \mathfrak{m}_1 \to \mathfrak{m}_0 \to 0 \text{ where } \mathfrak{m}_1 = \mathbb{R}e \oplus \mathfrak{m}_0 \text{ with } e \neq 0 \neq [e, m1] \quad (11)$$

where the arrows are Lie algebra homomorphisms. Let a Lie algebra $\mathfrak{m}_2 = \mathbb{R}f \oplus \mathfrak{m}_1$ be an extension of $\mathfrak{m}_1$ by a nonzero derivation f as follows.

$$0 \to \mathfrak{m}_1 \to \mathfrak{m}_2 \to \mathbb{R}f \to 0, \text{ Lie algebra exact Seq, } \mathrm{ad}_{\mathfrak{m}_2}(f)|_{\mathfrak{m}_1} \in \mathrm{Der}\,\mathfrak{m}_1. \quad (12)$$

Then $(\mathfrak{m}_2, \langle \cdot, \cdot \rangle)$ is a Lie algebra with a nondegenerate inner product $\langle \cdot, \cdot \rangle$ of signature $(p + 1, q + 1)$ defined by

$$\langle \cdot, \cdot \rangle|_{\mathfrak{m}_0} = \langle \cdot, \cdot \rangle_0, \ \langle e, \mathfrak{m}_0 \rangle = \langle f, \mathfrak{m}_0 \rangle = 0, \ \|e\| = \|f\| = 0, \ \langle e, f \rangle = 1. \quad (13)$$

In particular, if $\langle \cdot, \cdot \rangle_0$ is positive definite, so $\langle \cdot, \cdot \rangle$ is of Lorentz signature, then $(\mathfrak{m}_2, \langle \cdot, \cdot \rangle)$ is called the *Lorentz double extension* of $(\mathfrak{m}_0, \langle \cdot, \cdot \rangle_0)$. The double extension is a well-known tool since in [15] it has been used for constructing bi-invariant pseudo-Riemannian inner products. Our approach is closer to that of [22].

Theorem 3 *Let $(M = G/H, ds^2)$ be a connected Lorentz geodesic orbit nilmanifold, where $G = N \rtimes H$ with N nilpotent. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathfrak{n}$ defined by ds^2. Suppose that $\langle \cdot, \cdot \rangle|_{[\mathfrak{n}, \mathfrak{n}]}$ is degenerate. Then $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is a Lorentz double extension of the metric Lie algebra corresponding to a Riemannian GO nilmanifold (which necessarily is abelian or 2-step nilpotent).*

Proof Suppose the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$ is degenerate. Let $e \in \mathfrak{n}'$ be a nonzero null vector. Let $\mathfrak{v}$ denote the orthogonal complement to $\mathfrak{n}'$ in $\mathfrak{n}$, and let $\mathfrak{m}_1 = \mathfrak{n}' + \mathfrak{v}$. We have $\mathfrak{n}' \cap \mathfrak{v} = \mathbb{R}e$ and $\mathfrak{m}_1 = e^{\perp}$. Moreover, all four subspaces $\mathbb{R}e$, $\mathfrak{n}'$, $\mathfrak{v}$, and $\mathfrak{m}_1$ are $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-invariant.

The subspace $\mathfrak{m}_1$ is a degenerate hyperplane in $\mathfrak{n}$ and is an $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-invariant ideal. We choose a null vector f such that $\mathbb{R}f \oplus \mathfrak{m}_1 = \mathfrak{n}$ and $\langle f, e \rangle = 1$ (note that the choice of such f is not unique). Clearly $\mathfrak{m}_1$ is $\mathrm{ad}_{\mathfrak{g}}(f)$-invariant and $\mathrm{ad}_{\mathfrak{g}}(f)|_{\mathfrak{m}_1}$ acts on $\mathfrak{m}_1$ as a nilpotent derivation. Moreover, the restriction of $\langle \cdot, \cdot \rangle$ to $\mathrm{Span}(f, e)$ is Lorentz. Let $\mathfrak{m}_0 = (\mathrm{Span}(f, e))^{\perp}$. Define the inner product $\langle \cdot, \cdot \rangle_0$ on $\mathfrak{m}_0$ to be the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{m}_0$. Note that $\langle \cdot, \cdot \rangle_0$ is positive definite.

According to our definition, to prove that $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is a Lorentz double extension of $(\mathfrak{m}_0, \langle \cdot, \cdot \rangle_0)$, it remains to show that e lies in the center of $\mathfrak{m}_1$ (and then the Lie bracket $[\cdot, \cdot]_0$ on $\mathfrak{m}_0$ is defined by requiring that $(\mathfrak{m}_0, [\cdot, \cdot]_0)$ be isomorphic to the quotient algebra $\mathfrak{m}_1/(\mathbb{R}e)$). To see that e is central in $\mathfrak{m}_1$, take $T = X \in \mathfrak{m}_0$ and $T' = e$ in (1). Using the facts that $e \perp \mathfrak{m}_0$ and that $\mathbb{R}e$ is $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-invariant, we obtain $\langle \mathrm{ad}_{\mathfrak{g}}(e)X, X \rangle = 0$ for all $X \in \mathfrak{m}_0$. Then for $X \in \mathfrak{m}_0$, we have $\mathrm{ad}_{\mathfrak{g}}(e)X = KX + \mu(X)e$ for some endomorphism K of $\mathfrak{m}_0$ and a linear form μ on $\mathfrak{m}_0$. As K is nilpotent and skew-symmetric relative to a positive definite inner product $\langle \cdot, \cdot \rangle_0$, we get $K = 0$, and so $[X, e] = -\mu(X)e$, for all $X \in \mathfrak{m}_0$. Since $\mathrm{ad}_{\mathfrak{g}}(X)$ is nilpotent, e lies in the center of $\mathfrak{m}_1$.

To complete the proof of the theorem, we need to show that the metric Lie algebra $(\mathfrak{m}_0, \langle \cdot, \cdot \rangle_0)$ is the Lie algebra of a Riemannian GO nilmanifold.

Take a nonzero $X \in \mathfrak{m}_0$. From the Geodesic Lemma, we can find $A(X) \in \mathfrak{h}$ such that (1) with $T = X$ holds for all $T' \in \mathfrak{n}$, and in particular, for all $T' = Y \in \mathfrak{m}_0$. Note that $k(X) = 0$, as X is non-null. Define a skew-symmetric operator $D(X)$ on $(\mathfrak{m}_0, \langle \cdot, \cdot \rangle_0)$ by $\langle D(X)Y, Y' \rangle_0 = \langle [A(X), Y], Y' \rangle$ for $Y, Y' \in \mathfrak{m}_0$. As $\mathfrak{m}_1$ and $\mathbb{R}e$ are $\mathrm{ad}_{\mathfrak{g}}(A(X))$-invariant and $[e, \mathfrak{m}_1] = 0$, we find that $D(X)$ is a (skew-symmetric) derivation of the algebra $(\mathfrak{m}_0, [\cdot, \cdot]_0)$. Then (1) gives $\langle [X, Y]_0, X \rangle_0 + \langle D(X)Y, X \rangle_0 = 0$ (as $\langle e, \mathfrak{m}_1 \rangle = 0$) which completes the proof according to the Riemannian version of the Geodesic Lemma. □

5 Examples Related To Degree of Nilpotence

In this section, we use Theorem 3 to show that the set of nilpotency steps of Lorentz GO nilmanifolds is unbounded. In our examples, the group $G = N \rtimes H$ where

$H = H(N)$ is the full group of isometric automorphisms of (N, ds^2). We construct both naturally reductive examples and examples that are not naturally reductive, first constructing the naturally reductive ones and then modifying them to get examples that are not naturally reductive.

Let $d > 1$ and let S be an $s \times s$ matrix that is d-step nilpotent, that is, $S^d = 0$ but $S^{d-1} \neq 0$. For example, S could be the $(d + 1) \times (d + 1)$ matrix $(S_{i,j})$ where $S_{i,i+1} = 1$ for $1 \leq i \leq d$ and all other $S_{i,j} = 0$. Introduce the following $(2s) \times (2s)$ matrices:

$$P = \begin{pmatrix} 0_s & -S^t \\ S & S^t - S \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0_s & S^t \\ S & S^t - S \end{pmatrix}. \tag{14}$$

The matrix P is skew-symmetric and the matrix Q is nilpotent. To see the latter, compute $Q = \begin{pmatrix} I_s & I_s \\ 0_s & I_s \end{pmatrix} \begin{pmatrix} -S & 0_s \\ S & S^t \end{pmatrix} \begin{pmatrix} I_s & -I_s \\ 0_s & I_s \end{pmatrix}$. From that, Q is nilpotent and $Q^{d-1} \neq 0$.

Following the idea of the construction in Theorem 3, we start with a Riemannian metric Lie algebra $(\mathfrak{m}_0, \langle \cdot, \cdot \rangle_0)$. Here, $\mathfrak{m}_0 = \mathbb{R}^{2s}$ as a vector space, $\langle \cdot, \cdot \rangle_0$ is positive definite, and we fix an orthonormal basis $\mathcal{B}$. The Lie algebra structure of $\mathfrak{m}_0$ and a central extension $\mathfrak{m}_1 = \mathbb{R}e \oplus \mathfrak{m}_0$ (vector space direct sum), $0 \to \mathbb{R}e \to \mathfrak{m}_1 \to \mathfrak{m}_0 \to 0$, are given by

$$\begin{aligned} \langle e, \mathfrak{m}_1 \rangle &= 0 \text{ and } \langle X, Y \rangle = \langle X, Y \rangle_0 \text{ for } X, Y \in \mathfrak{m}_0 \\ [e, \mathfrak{m}_1] &= 0 \text{ and } [X, Y] = \langle PX, Y \rangle_0 \, e \text{ for } X, Y \in \mathfrak{m}_0 \end{aligned} \tag{15}$$

where P has matrix (14) relative to the basis $\mathcal{B}$ of $\mathfrak{m}_0$. Next, define the extension $\mathfrak{n}$ of $\mathfrak{m}_1$ by $\mathfrak{n} = \mathbb{R}f \oplus \mathfrak{m}_1$ (vector space direct sum), $0 \to \mathfrak{m}_1 \to \mathfrak{n} \to \mathbb{R}f \to 0$, with the Lie bracket and the inner product defined by (15) on $\mathfrak{m}_1$, and additionally, by

$$\begin{aligned} \langle f, e \rangle &= 1 \text{ and } \langle f, X \rangle = \langle f, f \rangle = 0 \text{ for } X \in \mathfrak{m}_0 \\ [f, e] &= 0 \text{ and } [f, X] = QX \text{ for } X \in \mathfrak{m}_0 \end{aligned} \tag{16}$$

with matrices relative to $\mathcal{B}$ as before. The algebra $\mathfrak{n}$ so constructed is nilpotent, and is of step at least d, as $\text{ad}_{\mathfrak{g}}(f)^{d-1} X = Q^{d-1} X \neq 0$ for some $X \in \mathfrak{m}_0$. Note $\dim \mathfrak{n} = 2s + 2$.

We claim that $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is geodesic orbit. To see this, let $T = \alpha f + X + \eta e \in \mathfrak{n}$ where $X \in \mathfrak{m}_0$ and $\alpha, \eta \in \mathbb{R}$. Define $k(T) = 0$ and $A(T) \in \mathfrak{h} = \text{Lie}(H)$ in such a way that

$$\begin{aligned} \text{ad}_{\mathfrak{g}}(A(T))e &= 0, \; \text{ad}_{\mathfrak{g}}(A(T))f = v(T), \text{ and } \text{ad}_{\mathfrak{g}}(A(T))Y \\ &= -\langle v(T), Y \rangle e \text{ for } Y \in \mathfrak{m}_0, \\ \text{where } v(T) &= (Q^t + P)X = \begin{pmatrix} 0_s & 0_s \\ 2S & 0_s \end{pmatrix} X \in \mathfrak{m}_0. \end{aligned} \tag{17}$$

It is easy to check that $\text{ad}_{\mathfrak{g}}(A(T))$ so defined is a skew-symmetric derivation of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$.

Using (15)–(17), for an arbitrary $T' = \beta f + Y + \rho e \in \mathfrak{n}$ with $Y \in \mathfrak{m}_0$ and $\beta, \rho \in \mathbb{R}$, we have

$$[T, T'] = \alpha QY - \beta QX + \langle PX, Y \rangle e \text{ and } [A(T), T'] = \beta v(T) - \langle v(T), Y \rangle e,$$

and so

$$\langle [T + A(T), T'], T \rangle = \langle (\alpha QY - \beta QX + \beta v(T)) + (\langle PX, Y \rangle - \langle v(T), Y \rangle)e, \alpha f + X + \eta e \rangle$$
$$= \beta \langle v(T) - QX, X \rangle + \alpha \langle (Q^t + P)X - v(T), Y \rangle = 0.$$

By the Geodesic Lemma, $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is GO.

In fact $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is $(G$-)naturally reductive. The geodesic graph $A(T)$ given by (17) is linear in T (and $k = 0$), so according to Proposition 2, we need to check that $A(T)$ is $\mathrm{ad}_\mathfrak{g}(\mathfrak{h})$-equivariant, where $\mathfrak{h}$ is the Lie algebra of skew-symmetric derivations of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$. Straightforward computation using (15) and (16) shows that $\mathfrak{h}$ is spanned by elements B such that

$$u := \mathrm{ad}_\mathfrak{g}(B)f \in \mathfrak{m}, \ \mathrm{ad}_\mathfrak{g}(B)e = 0 \text{ and } \mathrm{ad}_\mathfrak{g}(B)X = \Phi X - \langle u, X \rangle e \text{ for } X \in \mathfrak{m}_0,$$
$$\text{where } \Phi \in \mathfrak{so}(\mathfrak{m}_0) \text{ and } [\Phi, P] = [\Phi, Q] = 0 \text{ and } (Q^t + P)u = 0. \tag{18}$$

Then, whenever $T \in \mathfrak{n}$ and $B \in \mathfrak{h}$, we have $\mathrm{ad}_\mathfrak{g}(A)(\mathrm{ad}_\mathfrak{g}(B)T) = \mathrm{ad}_\mathfrak{g}(B)\,\mathrm{ad}_\mathfrak{g}(A(T))$ by a direct calculation, using (17), (18) and the consequence $(Q^t + P)^2 = 0$ of (14). This completes the proof that $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is $(G$-)naturally reductive.

A manifold that is not naturally reductive can be constructed by taking the direct sum of an algebra constructed above and one that is not naturally reductive. For example, let $(\mathfrak{n}_1, \langle \cdot, \cdot \rangle_1)$ be a Lorentz algebra constructed above, and let $(\mathfrak{n}_2, \langle \cdot, \cdot \rangle_2)$ be a Riemannian 2-step nilpotent Lie algebra defined by $\mathfrak{n}_2 = \mathfrak{z} \oplus \mathfrak{a}$, $\mathfrak{a} = \mathfrak{z}^\perp = \mathbb{R}^4$, with Lie bracket defined as follows. Let $\{z_1, z_2\}$ is an orthonormal basis for $\mathfrak{z}$ and $J_1, J_2 \in \mathfrak{so}(4)$ such that $J_i J_j + J_j J_i = -2\delta_{ij} I_4$ for $i, j = 1, 2$. So J_1 and J_2 lie in the same $\mathfrak{so}(3)$ factor of $\mathfrak{so}(4)$ and are orthonormal relative to the Killing form, up to scale. Then the Lie algebra structure on $\mathfrak{n}_2$ is given by

$$[\mathfrak{n}_2, \mathfrak{z}] = 0 \text{ and } [X, Y] = \langle J_1 X, Y \rangle_2 z_1 + \langle J_2 X, Y \rangle_2 z_2, \text{ for } X, Y \in \mathfrak{a}.$$

Then $(\mathfrak{n}_2, \langle \cdot, \cdot \rangle_2)$ is GO, but not naturally reductive ([12, Proposition 3], [14, Theorem 5.3(I)]). Note $\dim \mathfrak{n}_2 = 6$.

Now define the Lorentz Lie algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle) = (\mathfrak{n}_1, \langle \cdot, \cdot \rangle_1) \oplus (\mathfrak{n}_2, \langle \cdot, \cdot \rangle_2)$ as the orthogonal direct sum of $(\mathfrak{n}_1, \langle \cdot, \cdot \rangle_1)$ and $(\mathfrak{n}_2, \langle \cdot, \cdot \rangle_2)$. Let $\pi_i : \mathfrak{n} \to \mathfrak{n}_i$ denote the orthogonal projections. Let H be the full group of skew-symmetric automorphisms of the resulting nilmanifold (N, ds^2) and let $\mathfrak{h}$ its Lie algebra. Note that $\mathfrak{h}$ may a priori be bigger than the direct sum of the corresponding algebras for $(\mathfrak{n}_1, \langle \cdot, \cdot \rangle_1)$ and $(\mathfrak{n}_2, \langle \cdot, \cdot \rangle_2)$. But if $B \in \mathfrak{h}$ then $(\pi_i B)|_{\mathfrak{n}_i}$ is a skew-symmetric derivation of $(\mathfrak{n}_i, \langle \cdot, \cdot \rangle_i)$. Were $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ naturally reductive, it would admit a linear geodesic graph $A : \mathfrak{n} \to$

$\mathfrak{h}$ by Proposition 2, which would then give a linear geodesic graph $(\pi_2 A)|_{\mathfrak{n}_2}$ for $(\mathfrak{n}_2, \langle \cdot, \cdot \rangle_2)$, in contradiction with Proposition 2 and the comment after its proof.

It would be very interesting to see which of these Lorentz GO nilmanifolds are weakly symmetric.

We summarize the considerations of this section as they apply to the cases where the nilpotent matrices S are exemplified just before (14).

Theorem 4 *For every number $d > 1$, there exist both a naturally reductive GO Lorentz nilmanifold (N, ds^2) nilpotent of step $\geq d$ and dimension $2d + 4$, and a GO Lorentz d'-step nilmanifold $(N_1, ds_1^2) \times (N_2, ds_2^2)$ with $d' \geq d$ and dimension $2d + 10$.*

Note that the GO Lorentz nilmanifolds constructed in Sect. 5, in particular those cited in Theorem 4, have nilpotence bounds in sharp contrast to the bounds of Theorem 1 and the results of [6]. We now compare the constructions of this section, including the case of Theorem 4, with the nilpotence bounds of [6, Theorem 8]; there, if $\langle \cdot, \cdot \rangle|_{[\mathfrak{n}, \mathfrak{n}]}$ is degenerate and the action of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ on $\mathfrak{n}$ is completely reducible (semisimple), then N is abelian or 2-step nilpotent. The reason for the difference here is

Proposition 3 *The GO Lorentz nilmanifolds constructed in Sect. 5 have the property that the action of $\mathrm{Ad}_G(H)$ on $\mathfrak{n}$ is not completely reducible.*

Proof Fix a Lorentz nilmanifold as constructed in this section. If we assume complete reducibility, we obtain a contradiction as follows. First note from (14)–(16), that $[\mathfrak{n}, \mathfrak{n}] = Q\mathfrak{m}_0 + \mathbb{R}e$ is $\mathrm{Ad}_G(H)$-invariant. Thus, $[\mathfrak{n}, \mathfrak{n}] \cap [\mathfrak{n}, \mathfrak{n}]^\perp = \mathbb{R}e$ is $\mathrm{Ad}_G(H)$-invariant. Now $(\mathbb{R}e)^\perp = \mathfrak{m}_0 + \mathbb{R}e = \mathfrak{m}_1$ is stable under $\mathrm{Ad}_G(H)$. An invariant complement to $\mathfrak{m}_1$ in $\mathfrak{n}$ may be taken to be $\mathbb{R}f$. So now $\mathfrak{m}_0$, $\mathbb{R}e$ and $\mathbb{R}f$ are $\mathrm{Ad}_G(H)$-invariant, while (17) shows that some elements of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ map f into $\mathfrak{m}_0$, $\mathfrak{m}_0$ into $\mathbb{R}e$, and e to 0, with nonzero images. That contradicts complete reducibility of the action of $\mathrm{Ad}_G(H)$ on $\mathfrak{n}$. □

6 Remarks

Together, Theorem 4 and Proposition 3 show that the existence of a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ is crucial in Theorems 1 and 3. This gives a good indication of the difficulty of finding structural results for non-reductive GO Lorentz nilmanifolds. However, it might be worthwhile to explore two special cases: naturally reductive and weakly symmetric.

Recall that a pseudo-Riemannian manifold (M, ds^2) is *weakly symmetric* if, given $x \in M$ and a tangent vector $\xi \in M_x$, there is an isometry $s_{x,\xi}$ of (M, ds^2) such that $s_{x,\xi}(x) = x$ and $ds_{x,\xi}(\xi) = \xi$; (M, ds^2) is *symmetric* if we can always choose $s_{x,\xi}$ independent of ξ. Let (M, ds^2) be weakly symmetric and $G = I(M, ds^2)$ with $M = G/H$. In the Riemannian case [21, Theorem 13.1.1], the nilradical N of G is abelian or 2-step nilpotent. In general, if there is a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ with $\mathfrak{n} \subset \mathfrak{m}$ and the metric definite on $[\mathfrak{n}, \mathfrak{n}]$ then [7, Theorem 4.12] N is abelian or 2-step nilpotent. There N does not have to be transitive on M. This suggests that Theorem 1 might apply when N is Lorentz, or perhaps even in a general signature of metric when the inner product on $[\mathfrak{n}, \mathfrak{n}]$ is definite or Lorentz.

One can consider the possibility of a converse to Theorem 3, perhaps guided by the examples of Sect. 5. Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be a double extension of a metric Lie algebra $(\mathfrak{n}_0, \langle \cdot, \cdot \rangle_0)$ corresponding to a Riemannian GO manifold. What are the conditions for $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ to be GO? Or naturally reductive? Or weakly symmetric? And what if $(\mathfrak{n}_0, \langle \cdot, \cdot \rangle_0)$ corresponds to a Lorentz GO manifold?

References

1. Berestovskii, V.N., Nikonorov, Yu.G.: Riemannian Manifolds and Homogeneous Geodesics. Springer Monographs in Mathematics, Springer, Cham (2020)
2. Calvaruso, G., Zaeim, A.: Four-dimensional pseudo-Riemannian g.o. spaces and manifolds. J. Geom. Phys. **130**, 63–80 (2018)
3. Calvaruso, G., Fino, A., Zaeim, A.: Homogeneous geodesics of non-reductive homogeneous pseudo-Riemannian 4-manifolds. Bull. Braz. Math. Soc. **46**, 23–64 (2015)
4. Chen, Z., Wolf, J.A.: Weakly symmetric pseudo-Riemannian Nilmanifolds. J. Diff. Geom. **121**, 541–572 (2022)
5. Chen, Z., Nikolayevsky, Y., Nikonorov, Y.: Compact geodesic orbit spaces with a simple isotropy group. Ann. Glob. Anal. Geomet. **63** (2023). https://doi.org/10.1007/s10455-022-09877-7
6. Chen, Z., Wolf, J.A., Zhang, S.: On the geodesic orbit property for Lorentz manifolds. J. Geom. Anal. **32**, Bibarticle number Article (81) (2022)
7. Chen, Z., Wolf, J.A.: Pseudo-Riemannian weakly symmetric manifolds. Ann. Glob. Anal. Geom. **41**, 381–390 (2012)
8. del Barco, V., Ovando, G.: Isometric actions on pseudo-Riemannian nilmanifolds. Ann. Glob. Anal. Geom. **45**, 95–110 (2014)
9. D'Atri, J.E., Ziller, W.: Naturally reductive metrics and Einstein metrics on compact Lie groups. Mem. Am. Math. Soc. **18**, Issue (215) (1979)
10. Dušek, Z., Kowalski, O.: Light-like homogeneous geodesics and the geodesic lemma for any signature. Publ. Math. Debrecen **71**, 245–252 (2007)
11. Gordon, C.S.: Homogeneous Riemannian Manifolds Whose Geodesics are Orbits, in "Topics in Geometry, in Memory of Joseph D'Atri", Progress in Nonlinear Differential Equation, Volume number (20), pp. 155–174. Birkhäuser, Basel (1996)
12. Kaplan, A.: On the geometry of groups of Heisenberg type. Bull. Lond. Math. Soc. **15**, 35–42 (1983)
13. Kostant, B.: On differential geometry and homogeneous spaces II. Proc. N.A.S. **42**, 354–357 (1956)
14. Kowalski, O., Vanhecke, L.: Riemannian manifolds with homogeneous geodesics. Boll. Un. Math. Ital. B **5**(7), 189–246 (1991)
15. Medina, A., Revoy, P.: Algèebres de Lie et produit scalaire invariant. Ann. sci. Éc. Norm. Sup. **18**, 553–561 (1985)
16. Mostow, G.D.: On maximal subgroups of real Lie groups. Ann. Math. (2) **74**, 503–517 (1961)
17. Ovando, G.: Naturally reductive pseudo-Riemannian spaces. J. Geom. Phys. **61**, 157–171 (2011)
18. Ovando, G.: Naturally reductive pseudo-Riemannian 2-step nilpotent Lie groups. Houston J. Math **39**, 147–167 (2013)
19. Szenthe, J.: Sur la connection naturelle à torsion nulle. Acta Sci. Math. (Szeged) **38**, 383–398 (1976)
20. Wolf, J.A.: On locally symmetric spaces of non-negative curvature and certain other locally homogeneous spaces. Comm. Math. Helv. **37**, 265–295 (1963)
21. Wolf, J.A.: Harmonic Analysis on Commutative Spaces. Mathematical Surveys and Monographs, vol. 142. American Mathematical Society, New York (2007)
22. Yan, Z., Deng, S.: Double extensions on Riemannian Ricci Nilsolitons. J. Geom. Anal. **31**, 9996–10023 (2021)