LOCALIZATION AND STANDARD MODULES FOR REAL SEMISIMPLE LIE GROUPS II: IRREDUCIBILITY AND CLASSIFICATION

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Contents

1. Introduction

This paper is the continuation of [12], in which we related two constructions of representations of a connected semisimple Lie group with finite center G_0. We fix a maximal compact subgroup $K_0 \subset G_0$, which is unique up to G_0-conjugacy. We denote by K its complexification. As a matter of notation, we write $\mathfrak{g}_0$, $\mathfrak{k}_0$, for the Lie algebras of real Lie groups G_0, K_0, and $\mathfrak{g}$, $\mathfrak{k}$, for the complexified Lie algebras of real Lie groups G_0, K_0.

On the one hand, there are the admissible, finite length representations of G_0 on complete, locally convex Hausdorff topological complex vector spaces that are quasisimple in the sense of Harish-Chandra, modulo his notion of infinitesimal equivalence; on the other hand, there are Harish-Chandra modules: finitely generated modules of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ with a compatible action of K, on which $\mathcal{Z}(\mathfrak{g})$, the center of $\mathcal{U}(\mathfrak{g})$, acts by a character. The K-finite vectors of a finite length admissible epresentation (π, V) of G_0 are dense in the representation space V, and the space of C^∞ vectors is sandwiched between the space of K-finite vectors and the representation space V itself. The derivative of such representations of G_0 then sets up a bijection between representations of G_0, modulo infinitesimal

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equivalence, and Harish-Chandra modules. In that sense, to understand representations of G_0 turns into the problem of understanding Harish-Chandra modules.

In this paper, we give unified geometric proofs of various known, but widely scattered results about Harish-Chandra modules. The point of departure is the localization construction of Beilinson and Bernstein; representations of G_0 will play no role from here on.

Let X be the flag variety of $\mathfrak{g}$. For any point $x \in X$, let $\mathfrak{b}_x$ be the corresponding Borel subalgebra of $\mathfrak{g}$. Also, let $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$. We denote by $\mathfrak{h}$ the abstract Cartan algebra of $\mathfrak{g}$ (compare [12]). The dual $\mathfrak{h}^*$ is spanned by the (abstract) root system Σ of roots. It contains a set Σ^+ of positive roots, which specializes at each point $x \in X$ to the roots corresponding to the root subspaces spanning $\mathfrak{n}_x$.

To each $\lambda \in \mathfrak{h}^*$, Beilinson and Bernstein attach a twisted sheaf of differential operators $\mathcal{D}_\lambda$ on the flag variety X. As discussed in [12], the maximal ideals of the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ are parametrized by the orbits of the Weyl group W of of the root system Σ in $\mathfrak{h}^*$. Let θ be the orbit of some λ in $\mathfrak{h}^*$. Denote by $\mathcal{U}_\theta$ the quotient of $\mathcal{U}(\mathfrak{g})$ by the two-sided ideal generated by the maximal ideal I_θ in $\mathcal{Z}(\mathfrak{g})$ attached to the orbit θ. Then, we have $\Gamma(X, \mathcal{D}_\lambda) = \mathcal{U}_\theta$.

For each λ in $\mathfrak{h}^*$, we can consider the category $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ of (quasicoherent) $\mathcal{D}_\lambda$-modules on X and the category $\mathcal{M}(\mathcal{U}_\theta)$ of $\mathcal{U}_\theta$-modules. Clearly, the functor of global sections $\Gamma(X, -) : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{U}_\theta)$ has a left adjoint functor Δ_λ defined by $\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$ for any $\mathcal{U}_\theta$-module V. This is the *localization* functor of Beilinson and Bernstein [3]. Localization functors are an equivalence of the category $\mathcal{M}(\mathcal{U}_\theta)$ with the category of $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ for regular and antidominant $\lambda \in \mathfrak{h}^*$.

We can consider the derived categories $D^*(\mathcal{D}_\lambda)$ and $D^*(\mathcal{U}_\theta)$ of the categories $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ and $\mathcal{M}(\mathcal{U}_\theta)$ respectively. The derived functors $R\Gamma$ and $L\Delta_\lambda$ are adjoint functors between these categories. For regular $\lambda \in \mathfrak{h}^*$, they are equivalences of categories.

As discussed in [12], we can define analogous categories $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$ of finitely generated $\mathcal{U}_\theta$-modules and $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ of coherent $\mathcal{D}_\lambda$-modules with compatible actions of K. We call the objects of these categories Harish-Chandra modules and Harish-Chandra sheaves respectively. The above results extend formally to these categories. For example, localization functors are an equivalence of the category of Harish-Chandra modules with the category of Harish-Chandra sheaves for regular and antidominant $\lambda \in \mathfrak{h}^*$. Harish-Chandra sheaves are holonomic $\mathcal{D}_\lambda$-modules. Therefore, $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ is an artinian and noetherian category. Moreover, its irreducible objects are easily classified. As explained in [12], they are attached on to the set of *geometric data* consisting of pairs (Q, τ) where Q is a K-orbit in X and τ is a K-equivariant irreducible connection on Q compatible with the twist. The $\mathcal{D}_\lambda$-module direct image of the connection τ is the *standard Harish-Chandra sheaf* $\mathcal{I}(Q, \tau)$ attached to (Q, τ). It has the unique irreducible Harish-Chandra subsheaf $\mathcal{L}(Q, \tau)$. All irreducible Harish-Chandra sheaves in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ are isomorphic to some $\mathcal{L}(Q, \tau)$.

Now we describe im some detail the results discussed in the paper. The (derived) localizations $L\Delta_\lambda$ of $\mathcal{U}_\theta$-modules for different $\lambda \in \theta$ are related by the intertwining functors LI_w of Beilinson and Bernstein [4]. Their construction and basic results are discussed in Section 2. In Section 3. we prove a quantitative analogue of the main result in [4] which relates support of the localization of an irreducible $\mathcal{U}_\theta$-module

V for strongly antidominant λ with possible weights of the Lie algebra homology $H_0(\mathfrak{n}_x, V)$ for a dense set of $x \in X$.

Our leading principle is that information contained in the localizations $L\Delta_{w\lambda}(V)$, for some Harish-Chandra module (π, V) for specific $w \in W$ can give more obvious information about Harish-Chandra module than the localization $\Delta_\lambda(V)$ for an antidominant λ. A typical example is the main result in Section 8, which gives a necessary and sufficient condition for irreducibility of standard Harish-Chandra sheaves. Localization functors satisfy a product formula which allows a reduction to the case of reflections with respect to a simple root α. By considering the fibration of the flag variety X over the generalized flag variety X_α attached to a simple root α we can easily see that failure of the conditions for root α implies the reducibility. The general criterion for irreducibility follows from this remark and an inductive argument using intertwining functors. This irreducibility result is a $\mathcal{D}$-module analogue of the irreducibility result of Speh and Vogan in [20]. An attempt to understand this result was one of starting points of this part of our project. They remarked that the situation is much more complicated for singular infinitesimal characters. This suggested that this is naturally a result about standard Harish-Chandra sheaves and not corresponding modules. The complications at singular infinitesimal character are caused by the failure of equivalence of categories in this case.

As we already remarked, the geometric classification of irreducible Harish-Chandra sheaves is straightforward. If λ is antidominant, the category of Harish-Chandra modules is the quotient of the category of Harish-Chandra sheaves by the subcategory of all Harish-Chandra sheaves with no global sections. Therefore, irreducible Harish-Chandra modules are all nonvanishing modules $\Gamma(X, \mathcal{L}(Q, \tau))$. Hence, to have a classification of irreducible Harish-Chandra modules, we have to characterize all $\mathcal{L}(Q, \tau)$ with nonvanishing global sections. This is done in Section 9.

Finally, in Sections 11. and 12., we reprove in our setting the classical results of Harish-Chandra on asymptotic of matrix coefficients of irreducible Harish-Chandra modules. Again, assume for simplicity that G_0 is a connected semisimple Lie group with maximal compact subgroup K_0. Let $G_0 = K_0 A_0 N_0$ be the Iwasawa decomposition of G_0. Then, N_0 determines a set of positive (restricted) roots. Harish-Chandra considered the K_0-finite matrix coefficients of a Harish-Chandra module (π, V) on the corresponding negative chamber in A_0 (for details, consult [8]). The growth of these coefficients at infinity is determined by "leading exponents". In [8], it is established that these linear forms on the Lie algebra of A_0 are in the set of all weights of $H_0(\mathfrak{n}_0, V)$, where $\mathfrak{n}_0$ is the Lie algebra of N_0. By [16, Theorem II. 2.1], they correspond precisely to the "minimal" weights with respect to a natural ordering. This establishes a connection between growth conditions of K_0-finite matrix coefficients and $\mathfrak{n}_0$-homology.

The Lie algebra $\mathfrak{n}_0$ is contained in a Borel suhalgebra of $\mathfrak{g}$ which lies in the open orbit of K in the flag variety X. Therefore, to determine the "leading exponents" of (π, V), we have to understand the localizations of (π, V) supported on the full flag variety X. The main result of Section 3 implies therefore the precise estimates for possible "leading exponents" of irreducible Harish-Chandra modules. This allows to reprove the results of Harish-Chandra on classification of discrete series of G_0 [10]. First, they exist if and only if $\operatorname{rank}\mathfrak{g} = \operatorname{rank} K$. Second, they correspond (for regular and strongly antidominant λ) to standard Harish-Chandra sheaves $\mathcal{I}(Q,\tau)$ attached to closed K-orbits Q.

We also characterize tempered Harish-Chandra modules in terms of vanishing of a simple invariant which we call *Langlands invariant.* As a consequence, we see that irreducible tempered Harish-Chandra modules are global sections of specific irreducible standard Harish-Chandra sheaves $\mathcal{I}(Q,\tau)$ for strongly antidominant λ. This explains relative simplicity of tempered spectrum of G_0.

If Langlands invariant of an irreducible Harish-Chandra module is nonzero, it determines the data necessary to characterize it as a Langlands representation [13]. We shall discuss the details of this correspondence in a further publication.

2. Generalities on intertwining functors

Let θ be a Weyl group orbit in $\mathfrak{h}^*$. We consider the category $\mathcal{M}(\mathcal{U}_\theta)$ of $\mathcal{U}_\theta$-modules. For each $\lambda \in \theta$ we also consider the category $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ of (quasicoherent) $\mathcal{D}_\lambda$-modules. Assigning to a $\mathcal{D}_\lambda$-module $\mathcal{V}$ its global sections $\Gamma(X,\mathcal{V})$ defines a functor $\Gamma : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{U}_\theta)$. Its left adjoint is the *localization* functor $\Delta_\lambda : \mathcal{M}(\mathcal{U}_\theta) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ given by $\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$.

Let Σ_λ be the set of roots integral with respect to λ, i.e.,

$$\Sigma_\lambda = \{\alpha \in \Sigma \mid \alpha\check{}(\lambda) \in \mathbb{Z}\}.$$

Then the subgroup W_λ of the Weyl group W generated by the reflections with respect to the roots from Σ_λ is equal to

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \in Q(\Sigma)\},$$

where $Q(\Sigma)$ is the root lattice of Σ in $\mathfrak{h}^*$ ([7], Ch. VI, §2, Ex. 2). Let Π_λ be the set of simple roots in the root system Σ_λ attached to the set of positive roots $\Sigma^+_\lambda = \Sigma_\lambda \cap \Sigma^+$. Denote by $\ell_\lambda : W_\lambda \longrightarrow \mathbb{Z}_+$ the corresponding length function.

We say that λ is *antidominant* if $\alpha\check{}(\lambda)$ is not a strictly positive integer for any $\alpha \in \Sigma^+$. For arbitrary λ we define

$$n(\lambda) = \min\{\ell_\lambda(w) \mid \ w\lambda \text{ is antidominant}, \ w \in W_\lambda\}.$$

The following result was established in [4] and [11].

Theorem 2.1. *Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$. Then*

(i) *The right cohomological dimension of $\Gamma : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{U}_\theta)$ is $\leq n(\lambda)$.*
(ii) *The left cohomological dimension of $\Delta_\lambda : \mathcal{M}(\mathcal{U}_\theta) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is finite if and only if λ is regular.*
(iii) *If λ is regular, the left cohomological dimension of Δ_λ is $\leq n(\lambda)$.*

Consider the derived category $D(\mathcal{U}_\theta)$ of complexes of $\mathcal{U}_\theta$-modules and the derived category $D(\mathcal{D}_\lambda)$ of complexes of $\mathcal{D}_\lambda$-modules. By (i), there exists the derived functor $R\Gamma : D(\mathcal{D}_\lambda) \longrightarrow D(\mathcal{U}_\theta)$. This functor also induces functors between the corresponding full subcategories of bounded complexes. On the other hand, for arbitrary λ, there exists also the derived functor of localization functor $L\Delta_\lambda : D^-(\mathcal{U}_\theta) \longrightarrow D^-(\mathcal{D}_\lambda)$ between derived categories of complexes bounded from above.

If λ is regular, the left cohomological dimension of Δ_λ is finite by (ii), and $L\Delta_\lambda$ extends to the derived functor between $D(\mathcal{D}_\lambda)$ and $D(\mathcal{U}_\theta)$. Moreover, it maps bounded complexes into bounded complexes.

We have the following result [4].

Theorem 2.2. *Let $\lambda \in \mathfrak{h}^*$ be regular and $\theta = W \cdot \lambda$. Then $R\Gamma : D^b(\mathcal{D}_\lambda) \longrightarrow D^b(\mathcal{U}_\theta)$ and $L\Delta_\lambda : D^b(\mathcal{U}_\theta) \longrightarrow D^b(\mathcal{D}_\lambda)$ are mutually quasiinverse equivalences of categories.*

This implies, in particular, that for any two $\lambda, \mu \in \theta$, the categories $D^b(\mathcal{D}_\lambda)$ and $D^b(\mathcal{D}_\mu)$ are equivalent. This equivalence is given by the functor $L\Delta_\mu \circ R\Gamma$ from $D^b(\mathcal{D}_\lambda)$ into $D^b(\mathcal{D}_\mu)$. In this section we describe a functor, defined in geometric terms, which is (under certain conditions) isomorphic to this functor. This is the *intertwining functor* of Beilinson and Bernstein [4].

Most of the following results on the intertwining functors are due to Beilinson and Bernstein and were announced in [4], [2]. Complete details can be found in [15].

We start with some geometric remarks. Define the action of $G = \mathrm{Int}(\mathfrak{g})$ on $X \times X$ by

$$g \cdot (x, x') = (g \cdot x, g \cdot x')$$

for $g \in G$ and $(x, x') \in X \times X$. The G-orbits in $X \times X$ can be parametrized in the following way. First we introduce a relation between Borel subalgebras in $\mathfrak{g}$. Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two Borel subalgebras in $\mathfrak{g}$, $\mathfrak{n}$ and $\mathfrak{n}'$ their nilpotent radicals and N and N' the corresponding subgroups of G. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b} \cap \mathfrak{b}'$. Denote by R the root system of $(\mathfrak{g}, \mathfrak{c})$ in $\mathfrak{c}^*$ and by R^+ the set of positive roots determined by $\mathfrak{b}$. This determines a specialization of the Cartan triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ into $(\mathfrak{c}^*, R, R^+)$ [12]. On the other hand, $\mathfrak{b}'$ determines another set of positive roots in R, which corresponds via this specialization to $w(\Sigma^+)$ for some uniquely determined $w \in W$. The element $w \in W$ does not depend on the choice of $\mathfrak{c}$, and we say that $\mathfrak{b}'$ is in *relative position* w with respect to $\mathfrak{b}$.

Let

$$Z_w = \{(x, x') \in X \times X \mid \mathfrak{b}_{x'} \text{ is in the relative position } w \text{ with respect to } \mathfrak{b}_x\}$$

for $w \in W$. Then the map $w \longrightarrow Z_w$ is a bijection of W onto the set of G-orbits in $X \times X$, hence the sets Z_w, $w \in W$, are smooth subvarieties of $X \times X$.

Denote by p_1 and p_2 the projections of Z_w onto the first and second factor in the product $X \times X$, respectively. The fibrations $p_i : Z_w \longrightarrow X$, $i = 1, 2$, are locally trivial with fibres isomorphic to $\ell(w)$-dimensional affine spaces. Hence, they are are affine morphisms.

Let $\Omega_{Z_w|X}$ be the invertible $\mathcal{O}_{Z_w}$-module of top degree relative differential forms for the projection $p_1 : Z_w \longrightarrow X$. Let $\mathcal{T}_w$ be its inverse. Since the tangent space at $(x, x') \in Z_w$ to the fibre of p_1 can be identified with $\mathfrak{n}_x/(\mathfrak{n}_x \cap \mathfrak{n}_{x'})$, and $\rho - w\rho$ is the sum of roots in $\Sigma^+ \cap (-w(\Sigma^+))$, we see that

$$\mathcal{T}_w = p_1^*(\mathcal{O}(\rho - w\rho)).$$

It is easy to check that

$$(\mathcal{D}_{w\lambda})^{p_1} = (\mathcal{D}_\lambda^{p_2})^{\mathcal{T}_w}$$

([12], Appendix A). Since the morphism $p_2 : Z_w \longrightarrow X$ is a surjective submersion, the inverse image p_2^+ is an exact functor from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}_{qc}((\mathcal{D}_\lambda)^{p_2})$. Twisting by $\mathcal{T}_w$ defines an exact functor $\mathcal{V} \longrightarrow \mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V})$ from $\mathcal{M}(\mathcal{D}_\lambda)$ into $\mathcal{M}_{qc}((\mathcal{D}_{w\lambda})^{p_1})$ ([12], A.3.3.1). Therefore, we have a functor $\mathcal{V}^\cdot \longrightarrow \mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot)$ from $D^b(\mathcal{D}_\lambda)$ into $D^b((\mathcal{D}_{w\lambda})^{p_1})$. Composing it with the direct image functor $Rp_{1+} : D^b((\mathcal{D}_{w\lambda})^{p_1}) \longrightarrow D^b(\mathcal{D}_{w\lambda})$, we get the functor $J_w : D^b(\mathcal{D}_\lambda) \longrightarrow D^b(\mathcal{D}_{w\lambda})$ by the formula

$$J_w(\mathcal{V}^\cdot) = Rp_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot))$$

for any $\mathcal{V}^{\cdot} \in D^b(\mathcal{D}_\lambda)$. Let $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$. Since p_1 is an affine morphism with $\ell(w)$-dimensional fibres, it follows that $H^i(J_w(D(\mathcal{V})))$ vanishes for $i < -\ell(w)$ and $i > 0$. Moreover, the functor

$$I_w(\mathcal{V}) = R^0 p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}))$$

from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$ is right exact. This is the intertwining functor (attached to $w \in W$) between $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ and $\mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$. One knows that J_w is actually the left derived functor LI_w of I_w ([4], [15]); moreover,

Proposition 2.3. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then $LI_w = J_w : D^b(\mathcal{D}_\lambda) \longrightarrow D^b(\mathcal{D}_{w\lambda})$ is an equivalence of categories.*

We denote by $P(\Sigma)$ the weight lattice of Σ. For a weight $\nu \in P(\Sigma)$ we denote by $\mathcal{O}(\nu)$ the corresponding homogeneous invertible $\mathcal{O}_X$-module. From the construction of the intertwining functors one can easily check that they behave nicely with respect to twists by homogeneous invertible $\mathcal{O}_X$-modules:

Lemma 2.4. *Let $w \in W$, $\lambda \in \mathfrak{h}^*$ and $\nu \in P(\Sigma)$. Then*

$$LI_w(\mathcal{V}^{\cdot}(\nu)) = LI_w(\mathcal{V}^{\cdot})(w\nu)$$

for any $\mathcal{V}^{\cdot} \in D^b(\mathcal{D}_\lambda)$.

Intertwining functors satisfy a natural "product formula". To formulate it we need some additional geometric information on G-orbits in $X \times X$. Let $w, w' \in W$. Denote by p_1 and p_2 the projections of Z_w into X, and by p_1' and p_2' the corresponding projections of $Z_{w'}$ into X. Let $Z_{w'} \times_X Z_w$ be the fibre product of $Z_{w'}$ and Z_w with respect to the morphisms p_2' and p_1. Denote by $q' : Z_{w'} \times_X Z_w \longrightarrow Z_{w'}$ and $q : Z_{w'} \times_X Z_w \longrightarrow Z_w$ the corresponding projections to the first, resp. second factor. Finally, the morphisms $p_1' \circ q' : Z_{w'} \times_X Z_w \longrightarrow X$ and $p_2 \circ q : Z_{w'} \times_X Z_w \longrightarrow X$ determine a morphism $r : Z_{w'} \times_X Z_w \longrightarrow X \times X$. Therefore, we have the following commutative diagram.

$$\begin{tikzcd}
 & & X \times X & & \\
 & & Z_{w'} \times_X Z_w \arrow[u, "r"] \arrow[dl, "q'"'] \arrow[dr, "q"] & & \\
 & Z_{w'} \arrow[dl, "{p'}_1"'] \arrow[dr, "{p'}_2"'] & & Z_w \arrow[dl, "p_1"] \arrow[dr, "p_2"] & \\
X & & X & & X
\end{tikzcd}$$

All morphisms in the diagram are G-equivariant. From the construction it follows that the image of r is contained in $Z_{w'w}$, and by the G-equivariance of r it is a surjection of $Z_{w'} \times_X Z_w$ onto $Z_{w'w}$. Assume in addition that $w, w' \in W$ are such that $\ell(w'w) = \ell(w') + \ell(w)$. Then $r : Z_{w'} \times_X Z_w \longrightarrow Z_{w'w}$ is an isomorphism. Therefore, if we assume that $w, w', w'' \in W$ satisfy $w'' = w'w$ and $\ell(w'') = \ell(w') + \ell(w)$, we can identify $Z_{w''}$ and $Z_{w'} \times_X Z_w$. Under this identification the projections p_1'' and p_2'' of $Z_{w''}$ into X correspond to the maps $p_1' \circ q'$ and $p_2 \circ q$. This leads to the following result.

Proposition 2.5. *Let $w, w' \in W$ be such that $\ell(w'w) = \ell(w') + \ell(w)$. Then, for any $\lambda \in \mathfrak{h}^*$, the functors $LI_{w'} \circ LI_w$ and $LI_{w'w}$ from $D^b(\mathcal{D}_\lambda)$ into $D^b(\mathcal{D}_{w'w\lambda})$ are isomorphic; in particular the functors $I_{w'} \circ I_w$ and $I_{w'w}$ from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into $\mathcal{M}_{qc}(\mathcal{D}_{w'w\lambda})$ are isomorphic.*

Let $\alpha \in \Sigma^+$. We say that $\lambda \in \mathfrak{h}^*$ is *α-antidominant* if $\alpha\check{}(\lambda)$ is not a strictly positive integer. For any $S \subset \Sigma^+$, we say that $\lambda \in \mathfrak{h}^*$ is *S-antidominant* if it is α-antidominant for all $\alpha \in S$. Put

$$\Sigma_w^+ = \{\alpha \in \Sigma^+ \mid w\alpha \in -\Sigma^+\} = \Sigma^+ \cap (-w^{-1}(\Sigma^+))$$

for any $w \in W$. Then

$$\Sigma_{w^{-1}}^+ = -w(\Sigma_w^+),$$

and if $w, w' \in W$ are such that $\ell(w'w) = \ell(w') + \ell(w)$,

$$\Sigma_{w'w}^+ = w^{-1}(\Sigma_{w'}^+) \cup \Sigma_w^+$$

by ([7], Ch. VI, §1, no. 6, Cor. 2. of Prop. 17). In this situation, if $\lambda \in \mathfrak{h}^*$ is $\Sigma_{w'w}^+$-antidominant, then $w\lambda$ is $\Sigma_{w'}^+$-antidominant.

Since the left cohomological dimension of I_w is $\leq \ell(w)$, LI_w extends to a functor from $D(\mathcal{D}_\lambda)$ into $D(\mathcal{D}_{w\lambda})$ which is also an equivalence of categories. The next result gives one of the fundamental properties of this functor.

Theorem 2.6. *Let $w \in W$ and let $\lambda \in \mathfrak{h}^*$ be Σ_w^+-antidominant. Then the functors $LI_w \circ L\Delta_\lambda$ and $L\Delta_{w\lambda}$ from $D^-(\mathcal{U}_\theta)$ into $D^-(\mathcal{D}_{w\lambda})$ are isomorphic.*

If we also assume regularity, we get the result of Beilinson and Bernstein we mentioned before.

Theorem 2.7. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be Σ_w^+-antidominant and regular. Then LI_w is an equivalence of the category $D^b(\mathcal{D}_\lambda)$ with $D^b(\mathcal{D}_{w\lambda})$, isomorphic to $L\Delta_{w\lambda} \circ R\Gamma$.*

We can also give a more precise estimate of the left cohomological dimension of the intertwining functors.

Theorem 2.8. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then the left cohomological dimension of I_w is $\leq \operatorname{Card}(\Sigma_w^+ \cap \Sigma_\lambda)$.*

In particular, we have the following important consequence.

Corollary 2.9. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be such that $\Sigma_w^+ \cap \Sigma_\lambda = \emptyset$. Then $I_w : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$ is an equivalence of categories and $I_{w^{-1}}$ is its quasi-inverse, i.e., the compositions $I_w \circ I_{w^{-1}}$ and $I_{w^{-1}} \circ I_w$ are isomorphic to the identity functors.*

Also, for a regular λ, we see from the equivalence of derived categories 2.2 and 2.7 that $R\Gamma \circ LI_w$ is a functor isomorphic to $R\Gamma$. By a twisting argument on can actually remove this restriction, i.e., we have the following result.

Theorem 2.10. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be Σ_w^+-antidominant. Then the functors $R\Gamma \circ LI_w$ and $R\Gamma$ from $D^b(\mathcal{D}_\lambda)$ into $D^b(\mathcal{U}_\theta)$ are isomorphic.*

This theorem implies a spectral sequence, which collapses when all but one of the derived intertwining functors of a $\mathcal{D}_\lambda$-module $\mathcal{V}$ vanish, either as a consequence of 2.10, or by explicit verification:

Corollary 2.11. *Suppose $\lambda \in \mathfrak{h}^*$ is Σ_w^+-antidominant. Suppose further that $L^p I_w(\mathcal{V}) = 0$ for $p \neq -q \in -\mathbb{Z}_+$. Then*

$$H^p(X, L^{-q} I_w(\mathcal{V})) \cong H^{p-q}(X, \mathcal{V}), \quad p \in \mathbb{Z}_+,$$

as $\mathcal{U}_\theta$-modules.

Let $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ be the category of coherent $\mathcal{D}_\lambda$-modules and $D^b_{coh}(\mathcal{D}_\lambda)$ the corresponding bounded derived category. It is equivalent with the full subcategory of $D^b(\mathcal{D}_\lambda)$ consisting of complexes with coherent cohomology ([5], VI.2.11). If θ is the Weyl group orbit of λ we can also consider the bounded derived category $D^b_{fg}(\mathcal{U}_\theta)$ of finitely generated $\mathcal{U}_\theta$-modules. Again, it is equivalent with the full subcategory of $D^b(\mathcal{U}_\theta)$ consisting of complexes with finitely generated cohomology. The functor $R\Gamma$ maps complexes from $D^b_{coh}(\mathcal{D}_\lambda)$ into complexes from $D^b_{fg}(\mathcal{U}_\theta)$. If λ is regular, the localization functor $L\Delta_\lambda$ maps complexes from $D^b_{fg}(\mathcal{U}_\theta)$ into complexes from $D^b_{coh}(\mathcal{D}_\lambda)$. Hence, by 2.4 and 2.7, we see that $LI_w : D^b_{coh}(\mathcal{D}_\lambda) \longrightarrow D^b_{coh}(\mathcal{D}_{w\lambda})$ for arbitrary $w \in W$ and $\lambda \in \mathfrak{h}^*$. This is clearly an equivalence of categories. Now we want to describe the quasiinverse of this functor.

First, we recall the twisted version of the $\mathcal{D}$-module duality functor. Let $\lambda \in \mathfrak{h}^*$. It is well-known that the opposite sheaf of rings $\mathcal{D}_\lambda^\circ$ of $\mathcal{D}_\lambda$ is isomorphic to $\mathcal{D}_{-\lambda}$ ([12], A.2). Therefore, we can view the sheaf $Hom_{\mathcal{D}_\lambda}(\mathcal{V}, \mathcal{D}_\lambda)$ of right $\mathcal{D}_\lambda$-modules as a left $\mathcal{D}_{-\lambda}$-module. If $\mathcal{V}$ is a coherent $\mathcal{D}_\lambda$-module, $Hom_{\mathcal{D}_\lambda}(\mathcal{V}, \mathcal{D}_\lambda)$ is a coherent $\mathcal{D}_{-\lambda}$-module. Moreover, for any complex $\mathcal{V}^\cdot$, we have the duality functor

$$\mathbb{D} : D^b_{coh}(\mathcal{D}_\lambda) \longrightarrow D^b_{coh}(\mathcal{D}_{-\lambda})$$

given by

$$\mathbb{D}(\mathcal{V}^\cdot) = RHom_{\mathcal{D}_\lambda}(\mathcal{V}^\cdot, D(\mathcal{D}_\lambda))[\dim X].$$

One can check that this duality operation behaves well with respect to tensoring, i.e., for any weight $\nu \in P(\Sigma)$, the following diagram of functors is commutative

$$\begin{array}{ccc} D^b_{coh}(\mathcal{D}_\lambda) & \xrightarrow{\mathbb{D}} & D^b_{coh}(\mathcal{D}_{-\lambda}) \\ {\scriptstyle -(\nu)}\big\downarrow & & \big\downarrow{\scriptstyle -(-\nu)} \\ D^b_{coh}(\mathcal{D}_{\lambda+\nu}) & \xrightarrow[\mathbb{D}]{} & D^b_{coh}(\mathcal{D}_{-\lambda-\nu}) \end{array} .$$

Assume for a moment that λ is regular antidominant. Since it is equivalent to $\mathcal{M}_{fg}(\mathcal{U}_\theta)$, the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ has enough projective objects. Moreover, they are direct summands of $\mathcal{D}_\lambda^p$ for some $p \in \mathbb{Z}_+$. Hence, if $\mathcal{P}$ is a projective object in $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ and x an arbitrary point in X, the stalk $\mathcal{P}_x$ of $\mathcal{P}$ is a projective $\mathcal{D}_{\lambda,x}$-module. Since the twisting with $\mathcal{O}(\nu)$, for a weight $\nu \in P(\Sigma)$, is an equivalence of $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ with $\mathcal{M}_{coh}(\mathcal{D}_{\lambda+\nu})$, we see that the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ has enough projectives for arbitrary $\lambda \in \mathfrak{h}^*$. Moreover, if $\mathcal{P}$ is a projective object in $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$, its stalk $\mathcal{P}_x$ is a projective $\mathcal{D}_{\lambda,x}$-module for any $x \in X$. Therefore, $\mathcal{E}xt^p_{\mathcal{D}_\lambda}(\mathcal{P},\mathcal{D}_\lambda)_x = \operatorname{Ext}^p_{\mathcal{D}_{\lambda,x}}(\mathcal{P}_x,\mathcal{D}_{\lambda,x}) = 0$ for $p > 0$, the "local to global" spectral sequence

$$H^p(X, \mathcal{E}xt^q_{\mathcal{D}_\lambda}(\mathcal{P},\mathcal{D}_\lambda)) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{D}_\lambda}(\mathcal{P},\mathcal{D}_\lambda),$$

degenerates, and we conclude that

$$H^p(X, Hom_{\mathcal{D}_\lambda}(\mathcal{P},\mathcal{D}_\lambda)) = 0, \text{ for } p > 0;$$

i.e., $Hom_{\mathcal{D}_\lambda}(\mathcal{P},\mathcal{D}_\lambda)$ is acyclic for the functor of global sections Γ.

Consider the functor $\mathcal{V}^\cdot \longmapsto RHom_{\mathcal{D}_\lambda}(\mathcal{V}^\cdot, D(\mathcal{D}_\lambda))$ from $D^-_{coh}(\mathcal{D}_\lambda)$ into $D^+_{coh}(\mathcal{D}_{-\lambda})$ and the functor $R\Gamma$ from $D^+_{coh}(\mathcal{D}_{-\lambda})$ into $D^+(\mathcal{U}_\theta)$. Then the above remark implies that

$$R\Gamma(RHom_{\mathcal{D}_\lambda}(\mathcal{V}^\cdot, D(\mathcal{D}_\lambda))) = R\operatorname{Hom}_{\mathcal{D}_\lambda}(\mathcal{V}^\cdot, D(\mathcal{D}_\lambda)).$$

This yields the following result.

Lemma 2.12. *We have the isomorphism*

$$R\Gamma(\mathbb{D}(\mathcal{V}^\cdot)) = R\operatorname{Hom}_{\mathcal{D}_\lambda}(\mathcal{V}^\cdot, D(\mathcal{D}_\lambda))[\dim X]$$

of functors from $D^b_{coh}(\mathcal{D}_\lambda)$ into $D^b(\mathcal{U}_\theta)$.

Let θ be the Weyl group orbit of λ and $-\theta$ be the orbit of $-\lambda$. For regular orbit θ, the homological dimension of the ring $\mathcal{U}_\theta$ is finite. Moreover, the principal antiautomorphism of $\mathcal{U}(\mathfrak{g})$ induces an isomorphism of the ring opposite to $\mathcal{U}_\theta$ with $\mathcal{U}_{-\theta}$. We define a contravariant duality functor

$$\mathbb{D}_{alg}(V^\cdot) = R\operatorname{Hom}_{\mathcal{U}_\theta}(V^\cdot, D(\mathcal{U}_\theta))$$

from $D^b_{fg}(\mathcal{U}_\theta)$ into $D^b_{fg}(\mathcal{U}_{-\theta})$.

Let $V^\cdot$ be a complex of finitely generated $\mathcal{U}_\theta$-modules bounded from above. Then there exists a complex $F^\cdot$ bounded from above, consisting of free $\mathcal{U}_\theta$-modules of finite rank and a morphism of complexes $F^\cdot \longrightarrow V^\cdot$. Therefore,

$$\begin{aligned} R\Gamma(\mathbb{D}(L\Delta_\lambda(V^\cdot))) &= R\Gamma(\mathbb{D}(\Delta_\lambda(F^\cdot))) = R\operatorname{Hom}_{\mathcal{D}_\lambda}(\Delta_\lambda(F^\cdot), D(\mathcal{D}_\lambda))[\dim X] \\ &= \operatorname{Hom}^\cdot_{\mathcal{D}_\lambda}(\Delta_\lambda(F^\cdot), D(\mathcal{D}_\lambda))[\dim X] = \operatorname{Hom}^\cdot_{\mathcal{U}_\theta}(F^\cdot, D(\mathcal{U}_\theta))[\dim X] \\ &= R\operatorname{Hom}_{\mathcal{U}_\theta}(F^\cdot, D(\mathcal{U}_\theta))[\dim X] = \mathbb{D}_{alg}(V^\cdot)[\dim X]. \end{aligned}$$

Since $L\Delta_\lambda$ is an equivalence of $D^b_{fg}(\mathcal{U}_\theta)$ with $D^b_{coh}(\mathcal{D}_\lambda)$ we get the following result.

Lemma 2.13. *Let $\lambda \in \mathfrak{h}^*$ be regular, then the following diagram of functor commutes*

$$\begin{tikzcd} D^b_{coh}(\mathcal{D}_\lambda) \arrow[r, "\mathbb{D}"] \arrow[d, "R\Gamma"'] & D^b_{coh}(\mathcal{D}_{-\lambda}) \arrow[d, "R\Gamma"] \\ D^b_{fg}(\mathcal{U}_\theta) \arrow[r, "{\mathbb{D}_{alg}[\dim X]}"'] & D^b_{fg}(\mathcal{U}_{-\theta}) \end{tikzcd}.$$

Let α be a simple root. If λ is α-antidominant, by 2.10, we have

$$R\Gamma(\mathcal{V}^\cdot) = R\Gamma(LI_{s_\alpha}(\mathcal{V}^\cdot)).$$

Hence, we have

$$R\Gamma(\mathbb{D}(\mathcal{V}^\cdot)) = \mathbb{D}_{alg}(R\Gamma(\mathcal{V}^\cdot))[\dim X] = \mathbb{D}_{alg}(R\Gamma(LI_{s_\alpha}(\mathcal{V}^\cdot)))[\dim X] = R\Gamma(\mathbb{D}(LI_{s_\alpha}(\mathcal{V}^\cdot))).$$

Here $\mathbb{D}(\mathcal{V}^\cdot)$ is in $D^b_{coh}(\mathcal{D}_{-\lambda})$ and $\mathbb{D}(LI_{s_\alpha}(\mathcal{V}^\cdot))$ is in $D^b_{coh}(\mathcal{D}_{-s_\alpha\lambda})$. Since $-s_\alpha\lambda$ is α-antidominant, applying again 2.10, it follows that

$$R\Gamma(\mathbb{D}(\mathcal{V}^\cdot)) = R\Gamma(LI_{s_\alpha}\mathbb{D}(LI_{s_\alpha}(\mathcal{V}^\cdot))).$$

Since $\mathbb{D}(\mathcal{V}^\cdot)$ and $LI_{s_\alpha}\mathbb{D}(LI_{s_\alpha}(\mathcal{V}^\cdot))$ are in $D^b_{coh}(\mathcal{D}_{-\lambda})$ and $R\Gamma$ is an equivalence of categories, we have

$$\mathbb{D}(\mathcal{V}^\cdot) = LI_{s_\alpha}(\mathbb{D}(LI_{s_\alpha}(\mathcal{V}^\cdot))).$$

Therefore,

$$LI_{s_\alpha} \circ (\mathbb{D} \circ LI_{s_\alpha} \circ \mathbb{D}) \cong id$$

on $D^b_{coh}(\mathcal{D}_{-\lambda})$. Because all of these functors commute with twists, it follows that this relation holds for arbitrary λ.

This implies that in general

$$LI_w \circ (\mathbb{D} \circ LI_{w^{-1}} \circ \mathbb{D}) \cong id.$$

Therefore, we proved the following result.

Theorem 2.14. *The quasiinverse of the intertwining functor $LI_w : D^b_{coh}(\mathcal{D}_\lambda) \longrightarrow D^b_{coh}(\mathcal{D}_{w\lambda})$ is equal to*

$$\mathbb{D} \circ LI_{w^{-1}} \circ \mathbb{D} : D^b_{coh}(\mathcal{D}_{w\lambda}) \longrightarrow D^b_{coh}(\mathcal{D}_\lambda).$$

Finally, we want to discuss the behavior of global sections of $\mathcal{D}_\lambda$-modules for (not necessarily regular) antidominant $\lambda \in \mathfrak{h}^*$. Since the localization functor Δ_λ is the left adjoint of Γ, we have the adjunction morphisms $\Delta_\lambda \circ \Gamma \longrightarrow \mathrm{id}$ of functors on $\mathcal{M}(\mathcal{U}_\theta)$ and $\mathrm{id} \longrightarrow \Gamma \circ \Delta_\lambda$ of functors on $\mathcal{M}(\mathcal{D}_\lambda)$. By (i), Γ is exact in this situation and the functor $\Gamma \circ \Delta_\lambda$ is right exact. Moreover, by [3],

$$(\Gamma \circ \Delta_\lambda)(\mathcal{U}_\theta) = \Gamma(X, \mathcal{D}_\lambda) = \mathcal{U}_\theta.$$

Hence, from the exact sequence

$$\mathcal{U}_\theta^{(J)} \longrightarrow \mathcal{U}_\theta^{(I)} \longrightarrow V \longrightarrow 0$$

we get the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{U}_\theta^{(J)} & \longrightarrow & \mathcal{U}_\theta^{(I)} & \longrightarrow & V & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \\
\mathcal{U}_\theta^{(J)} & \longrightarrow & \mathcal{U}_\theta^{(I)} & \longrightarrow & \Gamma(X, \Delta_\lambda(V)) & \longrightarrow & 0
\end{array}.$$

We conclude that the morphism $V \longrightarrow \Gamma(X, \Delta_\lambda(V))$ is an isomorphism. Therefore, the adjunction morphism $\mathrm{id} \longrightarrow \Gamma \circ \Delta_\lambda$ is an isomorphism of functors.

Lemma 2.15. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and $\theta = W \cdot \lambda$. Then:*

(i) *for any irreducible $\mathcal{D}_\lambda$-module $\mathcal{V}$, either $\Gamma(X, \mathcal{V})$ is an irreducible $\mathcal{U}_\theta$-module or it is equal to zero;*

(ii) *for an irreducible $\mathcal{U}_\theta$-module V there exists a unique irreducible $\mathcal{D}_\lambda$-module $\mathcal{V}$ such that $V = \Gamma(X, \mathcal{V})$.*

Proof. Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module. Then the $\mathcal{D}_\lambda$-submodule of $\mathcal{V}$ generated by $\Gamma(X, \mathcal{V})$ can be either 0 or $\mathcal{V}$. Obviously, the first case corresponds to $\Gamma(X, \mathcal{V}) = 0$.

Assume now that $\Gamma(X, \mathcal{V}) \neq 0$. Then the adjunction morphism $\Delta_\lambda(\Gamma(X, \mathcal{V})) \longrightarrow \mathcal{V}$ is an epimorphism. Let $\mathcal{K}$ be the kernel of this morphism. Applying Γ to the corresponding short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \Delta_\lambda(\Gamma(X, \mathcal{V})) \longrightarrow \mathcal{V} \longrightarrow 0$$

we get

$$0 \longrightarrow \Gamma(X, \mathcal{K}) \longrightarrow \Gamma(X, \Delta_\lambda(\Gamma(X, \mathcal{V}))) \longrightarrow \Gamma(X, \mathcal{V}) \longrightarrow 0,$$

and since $\Gamma \circ \Delta_\lambda \cong \mathrm{id}$, we see that $\Gamma(X, \mathcal{K}) = 0$. Let $\mathcal{C}$ be any quasicoherent submodule of $\Delta_\lambda(\Gamma(X, \mathcal{V}))$. Then either $\mathcal{C} \subset \mathcal{K}$ and $\Gamma(X, \mathcal{C}) = 0$, or the morphism of $\mathcal{C}$ into $\mathcal{V}$ is surjective. Since Γ is exact, the natural map $\Gamma(X, \mathcal{C}) \longrightarrow \Gamma(X, \mathcal{V})$ is an isomorphism in the latter case.

Assume now that U is a nonzero quotient of $\Gamma(X, \mathcal{V})$. Then $\Delta_\lambda(U)$ is a quotient of $\Delta_\lambda(\Gamma(X, \mathcal{V}))$. Let $\mathcal{W}$ be the kernel of this epimorphism. By the preceding remark, either $\Gamma(X, \mathcal{W}) = 0$ or $\Gamma(X, \mathcal{W}) \longrightarrow \Gamma(X, \mathcal{V})$ is an isomorphism. The latter case is ruled out since $U \neq 0$, hence $\Gamma(X, \mathcal{W}) = 0$ and $U = \Gamma(X, \mathcal{V})$. Therefore, $\Gamma(X, \mathcal{V})$ is irreducible. This completes the proof of (i).

Let V be an irreducible $\mathcal{D}_\lambda$-module. Then $\Delta_\lambda(V)$ is a coherent $\mathcal{D}_\lambda$-module. Let $\mathcal{W}$ be a maximal coherent $\mathcal{D}_\lambda$-submodule and $\mathcal{V}$ the quotient of $\Delta_\lambda(V)$ by $\mathcal{W}$. Then we have the exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{W}) \longrightarrow \Gamma(X, \Delta_\lambda(V)) \longrightarrow \Gamma(X, \mathcal{V}) \longrightarrow 0.$$

Since $\Gamma(X, \Delta_\lambda(V)) = V$, either $\Gamma(X, \mathcal{W}) = V$ or $\Gamma(X, \mathcal{V}) = V$. Since $\Delta_\lambda(V)$ is, by definition, generated by its global sections, the first possibility is ruled out. It follows that $\Gamma(X, \mathcal{W}) = 0$ and $\Gamma(X, \mathcal{V}) = V$. This proves the existence part in (ii).

Let $\mathcal{S}$ be the family of all quasicoherent $\mathcal{D}_\lambda$-submodules $\mathcal{U}$ of $\Delta_\lambda(\Gamma(X, \mathcal{V}))$ ordered by inclusion. Since the functor Γ is exact, $\mathcal{S}$ has the largest element. Hence, we conclude that $\mathcal{W}$ is the largest coherent $\mathcal{D}_\lambda$-submodule and $\mathcal{V}$ is the unique irreducible quotient of $\Delta_\lambda(V)$. Let $\mathcal{U}$ be another irreducible $\mathcal{D}_\lambda$-module with $\Gamma(X, \mathcal{U}) = V$. Then, by the proof of (i), $\mathcal{U}$ is a quotient of $\Delta_\lambda(V)$. Therefore, $\mathcal{U} = \mathcal{V}$. □

This reduces the problem of classification of irreducible $\mathcal{U}_\theta$-modules to the problem of classification of irreducible $\mathcal{D}_\lambda$-modules and the problem of describing all irreducible $\mathcal{D}_\lambda$-modules with no global sections. Now we prove several simple results useful in studying the second problem (a more detailed discussion can be found in [15]).

We need some preparation. Let F be a finite-dimensional $\mathfrak{g}$-module. Then the sheaf $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$ has a natural structure of a sheaf of $\mathcal{U}(\mathfrak{g})$-modules. Fix a base point $x_0 \in X$. Let $0 = F_0 \subset F_1 \subset \cdots \subset F_m = F$ be a maximal $\mathfrak{b}_{x_0}$-invariant flag in F. Then $\mathfrak{n}_{x_0} F_i \subset F_{i-1}$ for $1 \leq i \leq m$. Therefore, $\mathfrak{b}_{x_0}/\mathfrak{n}_{x_0}$ acts naturally on F_i/F_{i-1}, and this action induces, by specialization, an action of the Cartan algebra $\mathfrak{h}$ on F_i/F_{i-1} given by a weight $\nu_i \in P(\Sigma)$. The sheaf $\mathcal{F}$ is the sheaf of local sections of the trivial homogeneous vector bundle $X \times F \longrightarrow X$. Hence, the flag induces a filtration of $\mathcal{F}$ by the sheaves of local sections $\mathcal{F}_i$ of homogeneous vector subbundles with fibres F_i, $1 \leq i \leq m$, at the base point x_0. They are locally free coherent $\mathcal{O}_X$-modules and also $\mathcal{U}(\mathfrak{g})$-modules. On the other hand, $\mathcal{F}_i/\mathcal{F}_{i-1} = \mathcal{O}(\nu_i)$ as a $\mathcal{U}(\mathfrak{g})$-module, i.e., $\mathcal{F}_i/\mathcal{F}_{i-1}$ is naturally a $\mathcal{D}_{\nu_i-\rho}$-module. Let $\mathcal{V}$ be a quasicoherent $\mathcal{D}_\lambda$-module on X. Then the $\mathcal{O}_X$-module $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}$ has a natural structure of a

$\mathcal{U}(\mathfrak{g})$-module given by

$$\xi(v \otimes s) = \xi v \otimes s + v \otimes \xi s$$

for $\xi \in \mathfrak{g}$, and local sections v and s of $\mathcal{V}$ and $\mathcal{F}$, respectively. We can define its $\mathcal{U}(\mathfrak{g})$-module filtration $\mathrm{F}(\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})$ by the submodules $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}_i$, $1 \leq i \leq m$. By the previous discussion, the corresponding graded module is $\mathrm{Gr}(\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}) = \bigoplus_{i=1}^{m} \mathcal{V}(\nu_i)$. Therefore, for any $\xi \in \mathcal{Z}(\mathfrak{g})$, the product $\prod_{1 \leq i \leq m}(\xi - \chi_{\lambda+\nu_i}(\xi))$ annihilates $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}$. Hence, $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}$ decomposes into the direct sum of its generalized $\mathcal{Z}(\mathfrak{g})$-eigensheaves.

Let $\mathcal{U}$ be a $\mathcal{U}(\mathfrak{g})$-module and $\mu \in \mathfrak{h}^*$. Denote by $\mathcal{U}_{[\mu]}$ the generalized $\mathcal{Z}(\mathfrak{g})$-eigensheaf of $\mathcal{U}$ corresponding to χ_μ. Then

$$\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F} = \bigoplus_{\nu} (\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\nu]},$$

where the sum is taken over the weights ν of F which represent the different Weyl group orbits $W \cdot (\lambda + \nu)$.

We also need to recall some standard constructions from the theory of derived categories. Let $\mathcal{A}$ be an abelian category and $D^b(\mathcal{A})$ the corresponding derived category of bounded complexes. Let $D : \mathcal{A} \longrightarrow D^b(\mathcal{A})$ be the natural functor which attaches to an object A the complex $D(A)$ such that $D(A)^n = 0$ for $n \neq 0$ and $D(A)^0 = A$. Then D is fully faithful.

Also, for any $s \in \mathbb{Z}$, we define the truncation functors $\tau_{\geq s}$ and $\tau_{\leq s}$ on $D^b(\mathcal{A})$: if $A^\cdot$ is a complex, $\tau_{\geq s}(A^\cdot)$ is a complex which is zero in degrees less than s, $\tau_{\geq s}(A^\cdot)^s = \operatorname{coker} d^{s-1}$ and $\tau_{\geq s}(A^\cdot)^q = A^q$ for $q > s$, with the differentials induced by the differentials of $A^\cdot$. On the other hand, $\tau_{\leq s}(A^\cdot)$ is a complex which is zero in degrees greater than s, $\tau_{\leq s}(A^\cdot)^s = \ker d^s$ and $\tau_{\leq s}(A^\cdot)^q = A^q$ for $q < s$, with the differentials induced by the differentials of $A^\cdot$. The natural morphisms $\tau_{\leq s}(A^\cdot) \longrightarrow A^\cdot$ and $A^\cdot \longrightarrow \tau_{\geq s}(A^\cdot)$ induce isomorphisms on cohomology in degrees $\leq s$ and $\geq s$ respectively. Moreover, for any complex $A^\cdot$ we have the distinguished triangle:

$$\begin{tikzcd}
& \tau_{\geq s+1}(A^\cdot) \arrow[dl, "{[1]}"] & \\
\tau_{\leq s}(A^\cdot) \arrow[rr] & & A^\cdot \arrow[ul]
\end{tikzcd}$$

in $D^b(\mathcal{A})$.

We return to the analysis of irreducible $\mathcal{D}_\lambda$-modules. Let $\alpha \in \Pi_\lambda$. Then, by 2.8, the left cohomological dimension of the intertwining functor I_{s_α} is ≤ 1.

Lemma 2.16. *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi_\lambda$ and $p = -\alpha\check{}(\lambda) \in \mathbb{Z}$. Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module. Then either*

(i) $I_{s_\alpha}(\mathcal{V}) = 0$ *and* $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}(p\alpha)$; *or*
(ii) $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$. *In this case, we have the exact sequence*

$$0 \longrightarrow \mathcal{C} \longrightarrow I_{s_\alpha}(\mathcal{V}) \longrightarrow \mathcal{V}(p\alpha) \longrightarrow 0$$

where $\mathcal{C}$ the largest proper coherent $\mathcal{D}_{s_\alpha\lambda}$-submodule of $I_{s_\alpha}(\mathcal{V})$. In addition, $I_{s_\alpha}(\mathcal{C}) = 0$ and $L^{-1}I_{s_\alpha}(\mathcal{C}) = \mathcal{C}(p\alpha)$.

Proof. By 2.4, we can first assume that λ is antidominant and regular. Since the left cohomological dimension of LI_{s_α} is ≤ 1, the complex $LI_{s_\alpha}(D(\mathcal{V}))$ can have nontrivial cohomology modules only in degrees -1 and 0. By the truncation construction for $s = -1$, we get the distinguished triangle

$$\begin{tikzcd} & D(I_{s_\alpha}(\mathcal{V})) \arrow[dl, "{[1]}"'] & \\ D(L^{-1}I_{s_\alpha}(\mathcal{V}))[1] \arrow[rr] & & LI_{s_\alpha}(D(\mathcal{V})) \arrow[ul] \end{tikzcd}$$

Applying to it the functor $R\Gamma$ leads to the distinguished triangle

$$\begin{tikzcd} & R\Gamma(D(I_{s_\alpha}(\mathcal{V}))) \arrow[dl, "{[1]}"'] & \\ R\Gamma(D(L^{-1}I_{s_\alpha}(\mathcal{V})))[1] \arrow[rr] & & R\Gamma(LI_{s_\alpha}(D(\mathcal{V}))) \arrow[ul] \end{tikzcd}$$

By 2.10, we conclude that that $R\Gamma(LI_{s_\alpha}(D(\mathcal{V}))) = R\Gamma(D(\mathcal{V})) = D(\Gamma(X, \mathcal{V}))$. In addition, since λ is antidominant and regular, we see that $n(s_\alpha\lambda) = 1$. By 2.1.(i), it follows that $H^p(X, \mathcal{W}) = 0$ for $p > 1$ for any quasicoherent $\mathcal{D}_{s_\alpha\lambda}$-module $\mathcal{W}$. Hence, from the long exact sequence of cohomology attached to the above distinguished triangle, we conclude that $\Gamma(X, L^{-1}I_{s_\alpha}(\mathcal{V})) = 0$, $H^1(X, I_{s_\alpha}(\mathcal{V})) = 0$ and

$$0 \longrightarrow H^1(X, L^{-1}I_{s_\alpha}(\mathcal{V})) \longrightarrow \Gamma(X, \mathcal{V}) \longrightarrow \Gamma(X, I_{s_\alpha}(\mathcal{V})) \longrightarrow 0$$

is exact. Since $\Gamma(X, \mathcal{V})$ is irreducible, either $H^1(X, L^{-1}I_{s_\alpha}(\mathcal{V})) = 0$ or $\Gamma(X, I_{s_\alpha}(\mathcal{V})) = 0$. Therefore, either $R\Gamma(D(L^{-1}I_{s_\alpha}(\mathcal{V}))) = 0$ or $R\Gamma(D(I_{s_\alpha}(\mathcal{V}))) = 0$. By the equivalence of derived categories this implies that either $I_{s_\alpha}(\mathcal{V}) = 0$ or $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$.

Again, by 2.4, we can assume that λ is antidominant and $\alpha\check{}(\lambda) = 0$. Moreover, we can assume that $\beta\check{}(\lambda)$, for $\beta \in \Sigma^+_\lambda - \{\alpha\}$, are "very large" integers. Let $\mathcal{U}$ be a $\mathcal{D}_\lambda$-module. Let μ be a "small" dominant weight and F the corresponding irreducible finite dimensional $\mathfrak{g}$-module. Let $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$ be the sheaf of sections of the corresponding trivial vector bundle. As we discussed before, this sheaf has a natural finite increasing filtration, which induces the filtration $\mathrm{F}(\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F})$ with the graded module $\mathrm{Gr}(\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F}) = \bigoplus_\nu \mathcal{U}(\nu)$. The center $\mathcal{Z}(\mathfrak{g})$ acts on $\mathcal{U}(\nu)$ with infinitesimal character $\chi_{\lambda+\nu}$. Consider the graded components $\mathcal{U}(\nu)$ on which the action is given by $\chi_{\lambda+\mu}$. In these cases, we have $w(\lambda+\nu) = \lambda+\mu$ for some $w \in W$. This implies that $w\lambda - \lambda = \mu - w\nu \in Q(\Sigma)$ and $w \in W_\lambda$. Since μ is "small" and $w\nu$ is another weight of F, $w\lambda - \lambda$ is also "small". This implies that either $w = 1$ or $w = s_\alpha$. Hence, the induced filtration of the sheaf $(\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\mu]}$ is two-step, and the corresponding graded sheaf is $\mathcal{U}(\mu) \oplus \mathcal{U}(s_\alpha\mu)$. Since μ is the highest weight of F, we see that we have the following short exact sequence

$$0 \longrightarrow \mathcal{U}(\mu) \longrightarrow (\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\mu]} \longrightarrow \mathcal{U}(s_\alpha\mu) \longrightarrow 0.$$

Under our assumptions, $\lambda + s_\alpha \mu$ is regular and antidominant. Assume that $\Gamma(X,\mathcal{U}) = 0$. Then

$$H^p(X, (\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\mu]}) = H^p(X, (\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F}))_{[\lambda+\mu]} = (H^p(X,\mathcal{U}) \otimes_{\mathbb{C}} F)_{[\lambda+\mu]} = 0.$$

Hence, from the long exact sequence of cohomology corresponding to this short exact sequence, we see that $\Gamma(X,\mathcal{U}(\mu)) = 0$ and $\Gamma(X,\mathcal{U}(s_\alpha\mu)) = H^1(X,\mathcal{U}(\mu))$. Therefore, $R\Gamma(D(\mathcal{U}(s_\alpha\mu))) = R\Gamma(D(\mathcal{U}(\mu))[1])$. On the other hand,

$$R\Gamma(D(\mathcal{U}(s_\alpha\mu))) = R\Gamma(LI_{s_\alpha}(D(\mathcal{U}(s_\alpha\mu))))$$

by 2.10. By the equivalence of derived categories, it follows that

$$LI_{s_\alpha}(D(\mathcal{U}(s_\alpha\mu))) = D(\mathcal{U}(\mu))[1].$$

By 2.4, we conclude that $LI_{s_\alpha}(D(\mathcal{U})) = D(\mathcal{U})[1]$.

When this discussion is applied to $\mathcal{V}$, we see that $\Gamma(X,\mathcal{V}) = 0$ implies (i). Conversely, if (i) holds, by 2.10,

$$D(\Gamma(X,\mathcal{V})) = R\Gamma(D(\mathcal{V})) = R\Gamma(LI_{s_\alpha}(D(\mathcal{V}))) = R\Gamma(D(\mathcal{V}))[1] = D(\Gamma(X,\mathcal{V}))[1],$$

and $\Gamma(X,\mathcal{V}) = 0$.

Therefore, if (i) does not hold, $V = \Gamma(X,\mathcal{V}) \neq 0$. By 2.15 and its proof, V is irreducible and $\mathcal{V}$ is the unique irreducible quotient of $\Delta_\lambda(V)$. Hence, by 2.6, $I_{s_\alpha}(\mathcal{V})$ is a quotient of $I_{s_\alpha}(\Delta_\lambda(V)) = \Delta_\lambda(V)$. Since $I_{s_\alpha}(\mathcal{V}) \neq 0$, $\mathcal{V}$ is the unique irreducible quotient of $I_{s_\alpha}(\mathcal{V})$. Let $\mathcal{C}$ be the largest coherent $\mathcal{D}_{s_\alpha\lambda}$-submodule of $I_{s_\alpha}(\mathcal{V})$. Then we have the exact sequence

$$0 \longrightarrow \mathcal{C} \longrightarrow I_{s_\alpha}(\mathcal{V}) \longrightarrow \mathcal{V} \longrightarrow 0.$$

By applying Γ to it we see that $\Gamma(X,\mathcal{C}) = 0$, an by the above result, $LI^{-1}(\mathcal{C}) = \mathcal{C}$. $\square$

Assume that λ is antidominant and $\alpha \in \Pi_\lambda$ such that $\alpha\check{}(\lambda) = 0$. Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module. Then, as we have shown in the preceding argument, $I_{s_\alpha}(\mathcal{V}) = 0$ implies that $\Gamma(X,\mathcal{V}) = 0$. The converse also holds:

Proposition 2.17. *Let $\lambda \in \mathfrak{h}^*$ be antidominant, $\theta = W \cdot \lambda$ and $S = \{\alpha \in \Pi_\lambda \mid \alpha\check{}(\lambda) = 0\}$. Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module. Then the following conditions are equivalent:*

(i) $\Gamma(X,\mathcal{V}) = 0$;
(ii) *there exists $\alpha \in S$ such that $I_{s_\alpha}(\mathcal{V}) = 0$.*

Proof. As we remarked above, we just need to prove that (i) implies (ii). Let $W(\lambda)$ be the stabilizer of λ in W. Then $W(\lambda)$ is generated by reflections with respect to $\Sigma(\lambda) = \{\alpha \in \Sigma \mid \alpha\check{}(\lambda) = 0\}$. The root subsystem $\Sigma(\lambda)$ is contained in Σ_λ. Since λ is antidominant, any positive root in $\Sigma(\lambda)$ is a sum of roots from S, i.e., S is a basis of $\Sigma(\lambda)$. It follows that $W(\lambda)$ is generated by reflections with respect to S. Therefore, the length function on $W(\lambda)$ is the restriction of ℓ_λ.

Assume that $I_{s_\alpha}(\mathcal{V}) \neq 0$ for all $\alpha \in S$. Let ν be a regular antidominant weight. We claim that $\Gamma(X,\mathcal{V}(w\nu)) \neq 0$ for all $w \in W(\lambda)$. The proof is by induction in $\ell_\lambda(w)$. If $\ell_\lambda(w) = 0$, $w = 1$, $\lambda + \nu$ is regular antidominant and $\Gamma(X,\mathcal{V}(\nu)) \neq 0$. Assume that the assertion holds for $v \in W(\lambda)$, $\ell_\lambda(v) < k$ for some $k > 0$. Let $\ell_\lambda(w) = k$. Then $w = s_\alpha w'$ with $\alpha \in S$ and $w' \in W(\lambda)$ such that $\ell_\lambda(w') = k - 1$.

Then, ${w'}^{-1}\alpha \in \Sigma_\lambda^+$ (see, for example, [7], Ch. VI, §1, no. 6, Cor. 1 of Prop. 17.). This implies, by the antidominance of ν,

$$-p = \alpha\check{}(\lambda + w'\nu) = \alpha\check{}(w'\nu) = ({w'}^{-1}\alpha)\check{}(\nu) \in -\mathbb{Z}_+,$$

and $\lambda + w'\nu$ is α-antidominant. By the induction assumption we have $\Gamma(X, \mathcal{V}(w'\nu)) \neq 0$, and by 2.4 and 2.15.(ii) we have the exact sequence

$$0 \longrightarrow \mathcal{C}(p\alpha) \longrightarrow I_{s_\alpha}(\mathcal{V}(w'\nu)) \longrightarrow \mathcal{V}(w\nu) \longrightarrow 0,$$

and $LI_{s_\alpha}(D(\mathcal{C})) = D(\mathcal{C}(p\alpha))[1]$. Therefore, by 2.10,

$$R\Gamma(D(\mathcal{C})) = R\Gamma(LI_{s_\alpha}(D(\mathcal{C}))) = R\Gamma(D(\mathcal{C}(p\alpha)))[1].$$

It follows that $\Gamma(X, \mathcal{C}(p\alpha)) = 0$. On the other hand, by the induction assumption and 2.10, we have

$$\Gamma(X, I_{s_\alpha}(\mathcal{V}(w'\nu))) = \Gamma(X, \mathcal{V}(w'\nu)) \neq 0,$$

so $\Gamma(X, \mathcal{V}(w\nu)) \neq 0$. This proves our earlier claim.

Let F be a finite-dimensional representation with lowest weight ν, and put $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$ as before. Assume that (i) holds. Then $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}$ satisfies

$$\Gamma(X, \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}) = \Gamma(X, \mathcal{V}) \otimes_{\mathbb{C}} F = 0,$$

hence $\Gamma(X, (\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\nu]}) = 0$. On the other hand, the filtration of $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}$, which was discussed before, induces a filtration of $(\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\nu]}$ such that the corresponding graded sheaf is a direct sum of $\mathcal{V}(\mu)$ for all weights μ of F such that $w(\lambda + \nu) = \lambda + \mu$ for some $w \in W$. This implies $w\lambda - \lambda = \mu - w\nu$, and $w \in W_\lambda$. The left side of the equality $\lambda - w^{-1}\lambda = w^{-1}\mu - \nu$ is a negative of a sum of roots from Π_λ and the right side is a sum of roots from Π. It follows that $w\lambda = \lambda$, i.e., $w \in W(\lambda)$. Let $w \in W(\lambda)$ be such that $\mathcal{V}(w\nu)$ is a submodule of $(\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\nu]}$. Then, if (ii) is violated, $\Gamma(X, \mathcal{V}(w\nu)) \neq 0$ according to the earlier claim, contradicting $\Gamma(X, (\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda+\nu]}) = 0$. □

To put 2.17 into perspective we should mention the following criterion for vanishing of intertwining functors for *simple* reflections. In this paper, we shall need only a special case, which we establish in 7.5, and which is an unpublished result of Beilinson and Bernstein.

Let $\alpha \in \Pi$, X_α the generalized flag variety of parabolic subalgebras of type α and $p_\alpha : X \longrightarrow X_\alpha$ the canonical projection. We say that a $\mathcal{D}$-module $\mathcal{V}$ *is of* X_α*-origin* if it is equal to a twist $p_\alpha^+(\mathcal{W})(\mu)$, $\mu \in P(\Sigma)$, of the inverse image $p_\alpha^+(\mathcal{W})$ for some $\mathcal{D}$-module $\mathcal{W}$ on X_α. The following result is proven in [15].

Proposition 2.18. *Let $\lambda \in \mathfrak{h}^*$ be antidominant, and $\alpha \in \Pi$ such that $\alpha\check{}(\lambda) = 0$. Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module. Then the following conditions are equivalent:*

(i) $I_{s_\alpha}(\mathcal{V}) = 0$;
(ii) $\mathcal{V}$ *is of* X_α*-origin.*

3. Supports and $\mathfrak{n}$-homology

In this section we prove some results relating the localization and $\mathfrak{n}$-homology which follow from analysis of the action of intertwining functors. They are inspired by the work of Beilinson and Bernstein on the generalization of the subrepresentation theorem of Casselman [4]. Our main result can be viewed as a quantitative version of their result.

We start with some geometric preliminaries. Let $\leq$ be the Bruhat order on W (determined by the reflections with respect to Π). Let S be a subset of the flag variety X. For $w \in W$ put

$$E_w(S) = \{x \in X \mid \mathfrak{b}_x \text{ is in relative position } v \text{ with respect to } \mathfrak{b}_y \text{ for some } v \leq w, y \in S\}.$$

Lemma 3.1. *Let S be a subset of X and $w \in W$. Then:*

(i) $\dim S \leq \dim E_w(S) \leq \dim S + \ell(w)$.
(ii) $E_w(\bar{S}) = \overline{E_w(S)}$.
(iii) *If S is a closed subset of X, $E_w(S)$ is the closure of the set*

$$\{x \in X \mid \mathfrak{b}_x \textit{ is in relative position } w \textit{ with respect to some } \mathfrak{b}_y,\ y \in S\}.$$

(iv) *If S is irreducible, $E_w(S)$ is also irreducible.*
(v) *If $w, v \in W$ are such that $\ell(wv) = \ell(w) + \ell(v)$,*

$$E_{wv}(S) = E_w(E_v(S)).$$

Proof. Let $\alpha \in \Pi$. Denote by X_α the generalized flag variety of parabolic subalgebras of type α, and by $p_\alpha : X \longrightarrow X_\alpha$ the natural projection. Then we have

$$E_{s_\alpha}(S) = p_\alpha^{-1}(p_\alpha(S)).$$

Clearly, in this case, $E_{s_\alpha}(S)$ is closed (resp. irreducible) if S is closed (resp. irreducible). Moreover, we see that

$$\dim S \leq \dim E_{s_\alpha}(S) \leq \dim S + 1.$$

Therefore, $E_{s_\alpha}(\bar{S})$ is closed. Hence, $\overline{E_{s_\alpha}(S)} \subset E_{s_\alpha}(\bar{S})$. On the other hand, since $S \subset \overline{E_{s_\alpha}(S)}$ it follows that $\bar{S} \subset \overline{E_{s_\alpha}(S)}$. If $x \in \overline{E_{s_\alpha}(S)}$, the whole fiber $p_\alpha^{-1}(p_\alpha(x))$ is contained in $\overline{E_{s_\alpha}(S)}$. This implies $E_{s_\alpha}(\bar{S}) \subset \overline{E_{s_\alpha}(S)}$. This proves (ii) for simple reflections.

Now we prove (v) by induction in the length of $w \in W$. First we claim that the formula holds if $w = s_\alpha$, $\alpha \in \Pi$. In this case, $E_{s_\alpha}(E_v(S))$ consists of all points $x \in X$ such that either $x \in E_v(S)$ or there exists $y \in E_v(S)$ such that $\mathfrak{b}_x$ is in relative position s_α with respect to $\mathfrak{b}_y$. Hence, it consists of all $x \in X$ such that there exists $y \in S$ and $\mathfrak{b}_x$ is in relative position u with respect to b_y for either $u \leq v$ or $u = s_\alpha u'$ with $u' \leq v$. In the second case, we have either $\ell(u) = \ell(u') + 1$ and $u \leq s_\alpha v$ or $\ell(u) = \ell(u') - 1$ and $u \leq u' \leq v$. Hence, $E_{s_\alpha}(E_v(S)) \subset E_{s_\alpha v}(S)$. Conversely, if $u \leq s_\alpha v$, we have either $u \leq v$ or $s_\alpha u \leq v$, hence $E_{s_\alpha}(E_v(S)) = E_{s_\alpha v}(S)$.

Assume now that w is arbitrary. Then we can find $\alpha \in \Pi$ and $w' \in W$ such that $\ell(w) = \ell(w') + 1$. Therefore, by the induction assumption,

$$E_w(E_v(S)) = E_{s_\alpha w'}(E_v(S)) = E_{s_\alpha}(E_{w'}(E_v(S))) = E_{s_\alpha}(E_{w'v}(S)),$$

which completes the proof of (v).

Now, for arbitrary $w \in W$, $\alpha \in \Pi$, and $w' \in W$ such that $\ell(w) = \ell(w') + 1$, we have $E_w(S) = E_{s_\alpha}(E_{w'}(S))$. Using the first part of the proof and an induction in $\ell(w)$, (i), (ii) and (iv) follow. In addition, we see that $E_w(S)$ is closed, if S is closed.

(iii) Let

$$V = \{x \in X \mid \mathfrak{b}_x \text{ is in relative position } w \text{ with respect to some } \mathfrak{b}_y,\ y \in S\}.$$

Then $V \subset E_w(S)$. Since $E_w(S)$ is closed, $\overline{V} \subset E_w(S)$. Let $y \in S$. Then the closure of the set of all $x \in X$ such that $\mathfrak{b}_x$ is in relative position w with respect to $\mathfrak{b}_y$ is equal to $E_w(\{x\})$. This implies

$$\overline{V} \supset \bigcup_{x \in S} E_w(\{x\}) = E_w(S).$$

□

We say that $w \in W$ is *transversal* to $S \subset X$ if

$$\dim E_w(S) = \dim S + \ell(w).$$

If w is transversal to S, $\ell(w) \leq \operatorname{codim} S$.

Lemma 3.2. *Let S be a subset of X. Then*

(i) *$w \in W$ is transversal to S if and only if it is transversal to $\bar{S}$.*

(ii) *Let $w, v \in W$ be such that $\ell(wv) = \ell(w) + \ell(v)$. Then the following statements are equivalent:*
  (a) *wv is transversal to S;*
  (b) *v is transversal to S and w is transversal to $E_v(S)$.*

Proof. (i) By 3.1.(ii) we have

$$\dim E_w(S) = \dim \overline{E_w(S)} = \dim E_w(\bar{S}),$$

and the assertion follows from the definition of transversality.

(ii) By 3.1.(i)

$$\dim E_{wv}(S) \leq \dim S + \ell(wv) = \dim S + \ell(w) + \ell(v),$$

and the equality holds if and only if wv is transversal to S. On the other hand, by 3.1.(v),

$$\dim E_{wv}(S) = \dim E_w(E_v(S)) \leq \dim E_v(S) + \ell(w) \leq \dim S + \ell(v) + \ell(w).$$

Hence, if (a) holds, the last relation is an equality, i.e.,

$$\dim E_w(E_v(S)) = \dim E_v(S) + \ell(w)$$

and

$$\dim E_v(S) = \dim S + \ell(v).$$

Hence, (b) holds.

Conversely, if (b) holds, we see immediately that wv is transversal to S. □

Lemma 3.3. *Let S be an irreducible closed subvariety of X and $w \in W$. Then there exists $v \leq w$ such that v is transversal to S and $E_v(S) = E_w(S)$.*

Proof. First we consider the case of $w = s_\alpha$, $\alpha \in \Pi$. In this case $E_{s_\alpha}(S) = p_\alpha^{-1}(p_\alpha(S))$ is irreducible and closed, and we have two possibilities:

a) s_α is transversal to S and $\dim E_{s_\alpha}(S) = \dim S + 1$, or

b) s_α is not transversal to S, $\dim E_{s_\alpha}(S) = \dim S$ and since $S \subset E_{s_\alpha}(S)$, we have $E_{s_\alpha}(S) = S$.

Now we prove the general statement by induction in $\ell(w)$. If $\ell(w) = 0$, $w = 1$ and $E_1(S) = S$, hence the assertion is obvious. Assume that $\ell(w) = k$. Then there exists $w' \in W$ and $\alpha \in \Pi$ such that $w = s_\alpha w'$ and $\ell(w) = \ell(w') + 1$. In this case, $E_w(S) = E_{s_\alpha}(E_{w'}(S))$ by 3.1.(v). By the induction assumption, there exists $v' \in W$, $v' \leq w'$ which is transversal to S and such that $E_{v'}(S) = E_{w'}(S)$.

Now, by the first part of the proof, if s_α is not transversal to $E_{w'}(S)$ we have

$$E_w(S) = E_{s_\alpha}(E_{w'}(S)) = E_{w'}(S) = E_{v'}(S).$$

Since $v' \leq w' \leq w$ the assertion follows. If s_α is transversal to $E_{w'}(S)$, we have

$$\dim E_w(S) = \dim E_{s_\alpha}(E_{w'}(S)) = \dim E_{w'}(S) + 1 = \dim S + \ell(v') + 1.$$

Put $v = s_\alpha v'$. If we have $\ell(v) = \ell(v') - 1$, $E_{v'}(S) = E_{s_\alpha}(E_v(S)) = p_\alpha^{-1}(p_\alpha(E_v(S)))$ by 3.1.(v) and

$$E_{s_\alpha}(E_{v'}(S)) = p_\alpha^{-1}(p_\alpha(p_\alpha^{-1}(p_\alpha(E_v(S))))) = E_{v'}(S),$$

contrary to transversality of s_α. Therefore, $\ell(v) = \ell(v') + 1$, $v \leq w$ and $E_v(S) = E_{s_\alpha}(E_{v'}(S))$. We conclude that $E_w(S) = E_v(S)$,

$$\dim E_v(S) = \dim E_w(S) = \dim S + \ell(v') + 1 = \dim S + \ell(v)$$

and v is transversal to S. □

As we remarked in §2, for any coherent $\mathcal{D}_\lambda$-module $\mathcal{V}$, the modules $L^p I_w(\mathcal{V})$, $p \in \mathbb{Z}$, are also coherent.

If $\mathcal{V}$ is a coherent $\mathcal{D}_\lambda$-module, the set $\{x \in X \mid \mathcal{V}_x \neq 0\}$ is closed, and thus coincides with the support $\operatorname{supp} \mathcal{V}$ of $\mathcal{V}$. We want to analyze how the action of intertwining functors changes supports of coherent $\mathcal{D}$-modules. First we point out the following simple fact which is a direct consequence of the definition of the intertwining functors and 3.1.(iii).

Lemma 3.4. *For any $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$, $p \in \mathbb{Z}$ and $w \in W$, we have*

$$\operatorname{supp} L^p I_w(\mathcal{V}) \subset E_w(\operatorname{supp} \mathcal{V}).$$

Lemma 3.5. *Let $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$ and $w \in W$ transversal to $S = \operatorname{supp} \mathcal{V}$. Assume that S is irreducible. Then*

$$\operatorname{supp} I_w(\mathcal{V}) = E_w(S),$$

and

$$\dim \operatorname{supp} I_w(\mathcal{V}) = \dim S + \ell(w).$$

Proof. We prove this result by induction in $\ell(w)$. If $\ell(w) = 1$, $w = s_\alpha$ for some $\alpha \in \Pi$. In this case, the second statement is proved in [4]. By 3.4, $\operatorname{supp} I_{s_\alpha}(\mathcal{V}) \subset E_{s_\alpha}(S)$. Also, by 3.1, both sets are closed and $E_{s_\alpha}(S)$ is irreducible. Since $\dim \operatorname{supp} I_w(\mathcal{V}) = \dim S + 1 = \dim E_{s_\alpha}(S)$ by transversality, the first statement follows.

Let $w \in W$ with $\ell(w) = k > 1$. Then $w = s_\alpha w'$ with $\alpha \in \Pi$ and $\ell(w') = k - 1$. Since w is transversal to S, w' is transversal to S and s_α is transversal to $E_{w'}(S)$ by 3.2. By the induction assumption, $\operatorname{supp} I_{w'}(\mathcal{V}) = E_{w'}(S)$. Hence, by 2.5 and 3.1.(iv), we have

$$\operatorname{supp} I_w(\mathcal{V}) = \operatorname{supp} I_{s_\alpha}(I_{w'}(\mathcal{V})) = E_{s_\alpha}(E_{w'}(S)) = E_w(S).$$

□

To any coherent $\mathcal{D}_\lambda$-module we attach two subsets of the Weyl group W:

$$S(\mathcal{V}) = \{w \in W \mid \operatorname{supp} I_w(\mathcal{V}) = X\}$$

and

$$\mathcal{E}(\mathcal{V}) = \text{ the set of minimal elements in } S(\mathcal{V}).$$

We have the following result. The statement (i) is the result of Beilinson and Bernstein we mentioned before.

Proposition 3.6. *Suppose $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$ has irreducible support. Then*

(i) *the set $S(\mathcal{V})$ is nonempty;*

(ii)

$$\mathcal{E}(\mathcal{V}) = \{w \in W \,|\, w \text{ is transversal to } \operatorname{supp} V \text{ and } \ell(w) = \operatorname{codim} \operatorname{supp} \mathcal{V}\},$$

i.e., $\mathcal{E}(\mathcal{V})$ consists of all $w \in W$ transversal to $\operatorname{supp} \mathcal{V}$ with the maximal possible length.

Proof. Assume that $w \in W$ is transversal to $\operatorname{supp} \mathcal{V}$ and $\ell(w) = \operatorname{codim} \operatorname{supp} \mathcal{V}$. Then, by 3.5, we conclude that $w \in S(\mathcal{V})$. If $v < w$, $\ell(v) < \operatorname{codim} \operatorname{supp} \mathcal{V}$, and $\dim \operatorname{supp} I_v(\mathcal{V}) < \dim X$ by 3.4. Hence, $v \notin S(\mathcal{V})$, i.e., $w \in \mathcal{E}(\mathcal{V})$.

Conversely, assume that $w \in \mathcal{E}(\mathcal{V})$. Then, by 3.4, we have $E_w(\operatorname{supp} \mathcal{V}) = X$. Since the support of $\mathcal{V}$ is irreducible, by 3.3 we can find $v \leq w$ such that v is transversal to $\operatorname{supp} \mathcal{V}$ and $E_v(\operatorname{supp} \mathcal{V}) = X$. By 3.5 this implies $v \in S(\mathcal{V})$. Since w is a minimal element in $S(\mathcal{V})$ we must have $w = v$, and w is transversal to $\operatorname{supp} \mathcal{V}$. This proves (ii).

To show (i) it is enough to show that $\mathcal{E}(\mathcal{V})$ is nonempty. Clearly, if w_0 is the longest element in W, $E_{w_0}(S) = X$. By 3.3, there exists w transversal to S such that $E_w(S) = X$, hence the assertion follows from (ii). $\square$

We recall a simple relationship between localization and $\mathfrak{n}$-homology. Let $x \in X$. Fix a Cartan subalgebra $\mathfrak{c}$ in $\mathfrak{b}_x$. Let θ be a Weyl group orbit in $\mathfrak{h}^*$. Let $V \in \mathcal{M}(\mathcal{U}_\theta)$. The $\mathfrak{n}_x$-homology modules $H_p(\mathfrak{n}_x, V)$, $p \in \mathbb{Z}_+$, have a natural structure of $\mathfrak{c}$-modules, and via the specialization we can view them as $\mathfrak{h}$-modules. According to a result of Casselman and Osborne, the modules $H_p(\mathfrak{n}_x, V)$ are annihilated by $P_\theta(\xi) = \prod_{w \in W}(\xi - (w\lambda + \rho)(\xi))$ for all $\xi \in \mathfrak{h}$. For a $\mathfrak{h}$-module U denote by $U_{(\mu)}$ the generalized weight submodule corresponding to the weight $\mu \in \mathfrak{h}^*$. Then

$$H_p(\mathfrak{n}_x, V) = \sum_{w \in W} H_p(\mathfrak{n}_x, V)_{(w\lambda+\rho)}$$

for any $p \in \mathbb{Z}_+$. Moreover, if $\lambda \in \mathfrak{h}^*$ is regular, linear forms $w\lambda + \rho$ in P_θ are all mutually different, hence the $\mathfrak{n}_x$-homology modules $H_p(\mathfrak{n}_x, V)$ are semisimple. These modules are related to localization by the following result (see, for example, [11]). For any $\mathcal{O}_X$-module $\mathcal{F}$ we denote by $T_x(\mathcal{F})$ the geometric fibre of $\mathcal{F}$.

Lemma 3.7. *Let $\lambda \in \mathfrak{h}^*$ be regular and $\theta = W \cdot \lambda$. Then for any $V \in \mathcal{M}(\mathcal{U}_\theta)$ we have the spectral sequence*

$$L^p T_x(L^q \Delta_\lambda(V)) \Longrightarrow H_{-(p+q)}(\mathfrak{n}_x, V)_{(\lambda+\rho)}.$$

This result will allow us to extract information about $\mathfrak{n}$-homology from localizations.

Unfortunately, as we remarked in 2.1.(ii), the behavior of localization functor for singular infinitesimal characters is quite bad and the corresponding relationship is much less useful. Therefore, to analyze $\mathfrak{n}$-homology in this case we shall use the translation functor technique.

Let F be a finite-dimensional $\mathfrak{g}$-module and $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$. The following lemma is implicit in [3]. We include a proof for the sake of completeness.

Lemma 3.8. *Let $\lambda \in \mathfrak{h}^*$, $\mu \in P(\Sigma)$ and $w \in W$ be such that $w\lambda$ and $-w\mu$ are antidominant. Let F be the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight $w\mu$. Then $\mathcal{V} \longrightarrow (\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}$ is a covariant functor from $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ into itself, naturally equivalent to the identity functor.*

Proof. The filtration of $\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$ described in §2 has $\mathcal{V}(-\mu+\nu)$ as its composition factors, where ν ranges over the set of all weights of F. Therefore, $\mathcal{Z}(\mathfrak{g})$ acts on them with the infinitesimal character $\chi_{\lambda-\mu+\nu}$. Assume that

$$s\lambda = \lambda - \mu + \nu$$

for some $s \in W$. Then, if we put $s' = wsw^{-1}$ and $\lambda' = w\lambda$, we have

$$s'\lambda' - \lambda' = w\nu - w\mu,$$

and since $w\mu$ and $w\nu$ are weights of F, $s'\lambda' - \lambda' \in Q(\Sigma)$. Therefore, $s' \in W_{\lambda'}$. Now, since $w\mu$ is the highest weight of F, $w\nu - w\mu$ is a sum of negative roots. On the other hand, since λ' is antidominant, $s'\lambda' - \lambda'$ is a sum of roots in $\Sigma_\lambda^+ \subset \Sigma^+$. Therefore, $s\lambda = \lambda$ and $\mu = \nu$, and the generalized eigensheaf of $\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F}$ corresponding to χ_λ is isomorphic to $\mathcal{V}$. □

Finally, we can formulate the result we need. Let $V \neq 0$ be a finitely generated $\mathcal{U}_\theta$-module. We say that $\lambda \in \theta$ is an *exponent* of V if the set

$$\{x \in X \mid H_0(\mathfrak{n}_x, V)_{(\lambda+\rho)} \neq 0\}$$

contains an open dense subset of X. Beilinson and Bernstein proved that the set of exponents of V is nonempty [4]. In particular, the set of all $x \in X$ such that $H_0(\mathfrak{n}_x, V) \neq 0$ contains an open dense subset of X.

We say that $\lambda \in \mathfrak{h}^*$ is *strongly antidominant* if $\operatorname{Re} \alpha\check{\ }(\lambda) \leq 0$ for any $\alpha \in \Sigma^+$. Clearly, a strongly antidominant λ is antidominant.

We also define a partial ordering on $\mathfrak{h}^*$ by: $\lambda \preccurlyeq \mu$ if $\mu - \lambda$ is a linear combination of simple roots in Π with coefficients with non-negative real parts. This order relation is related to the ordering on the Weyl group W by the following observation (see for example [9], 7.7.2).

Lemma 3.9. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant. Then for any $v, w \in W$, $v \leq w$ implies $v\lambda \preccurlyeq w\lambda$.*

Proof. Clearly, it is enough to show that for any $w \in W$ and $\alpha \in \Pi$ such that $\ell(s_\alpha w) = \ell(w) + 1$, we have $w\lambda \preccurlyeq s_\alpha w\lambda$. But $s_\alpha w\lambda = w\lambda - \alpha\check{\ }(w\lambda)\alpha$, hence

$$s_\alpha w\lambda - w\lambda = (w^{-1}\alpha)\check{\ }(\lambda)\alpha,$$

and it is enough to prove that $\operatorname{Re}(w^{-1}\alpha)\check{\ }(\lambda) \geq 0$. Since $w^{-1}\alpha$ is in Σ^+ (see, for example, [7], Ch. VI, §1, no. 6, Cor. 2 of Prop. 17), this follows from the strong antidominance of λ. □

The next result is the sharpening of the result of Beilinson and Bernstein we alluded to before.

Theorem 3.10. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant. Let $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda)$ be such that $S = \operatorname{supp} \mathcal{V}$ is irreducible. Put $V = \Gamma(X, \mathcal{V})$.*

(i) *If ω is an exponent of V, there exists $w \in W$ transversal to S with $\ell(w) = \operatorname{codim} S$ such that $w\lambda \preccurlyeq \omega$.*
(ii) *Assume that $\mathcal{V}$ is irreducible and $V \neq 0$. If $w \in W$ is transversal to S and $\ell(w) = \operatorname{codim} S$, then $w\lambda$ is an exponent of V.*

We first prove (i). Let μ be a regular dominant weight and F the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight μ. Let $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$. Then $\lambda - \mu$ is regular and strongly antidominant. Let $U = \Gamma(X, \mathcal{V}(-\mu))$. Then, by 3.8,

$$\mathcal{V} = (\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}.$$

This implies

$$\begin{aligned}V = \Gamma(X, \mathcal{V}) &= \Gamma(X, (\mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}) \\ &= \Gamma(X, \mathcal{V}(-\mu) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]} = (\Gamma(X, \mathcal{V}(-\mu)) \otimes_{\mathbb{C}} F)_{[\lambda]} = (U \otimes_{\mathbb{C}} F)_{[\lambda]}.\end{aligned}$$

Let ω be an exponent of V, i.e., $H_0(\mathfrak{n}_x, V)_{(\omega+\rho)} \neq 0$ for all x in some open dense subset of X. Then

$$H_0(\mathfrak{n}_x, V) = H_0(\mathfrak{n}_x, (U \otimes_{\mathbb{C}} F)_{[\lambda]}) = \bigoplus_{v \in W} H_0(\mathfrak{n}_x, U \otimes_{\mathbb{C}} F)_{(v\lambda+\rho)},$$

and

$$H_0(\mathfrak{n}_x, V)_{(\omega+\rho)} = H_0(\mathfrak{n}_x, U \otimes_{\mathbb{C}} F)_{(\omega+\rho)}.$$

Let $(F_p; 1 \leq p \leq n)$ be an increasing $\mathfrak{b}_x$-invariant maximal flag in F. It induces a filtration $(U \otimes_{\mathbb{C}} F_p; 1 \leq p \leq n)$ of the $\mathfrak{b}_x$-module $U \otimes_{\mathbb{C}} F$. The corresponding graded module is the direct sum of modules of the form $U \otimes_{\mathbb{C}} \mathbb{C}_\nu$, where ν goes over the set of weights of F. Clearly, the semisimplification of $H_0(\mathfrak{n}_x, U \otimes_{\mathbb{C}} F)$ is a submodule of the direct sum of modules $H_0(\mathfrak{n}_x, U) \otimes_{\mathbb{C}} \mathbb{C}_\nu$. Since the infinitesimal character of U is regular, $H_0(\mathfrak{n}_x, U)$ is a semisimple $\mathfrak{h}$-module. This implies that the semisimplification of $H_0(\mathfrak{n}_x, V)_{(\omega+\rho)}$ is a submodule of the direct sum of modules $H_0(\mathfrak{n}_x, U)_{(\omega-\nu+\rho)} \otimes_{\mathbb{C}} \mathbb{C}_\nu$. In particular, if $H_0(\mathfrak{n}_x, V)_{(\omega+\rho)} \neq 0$, $H_0(\mathfrak{n}_x, U)_{(\omega-\nu+\rho)} \neq 0$ for some weight ν of F. Since the set of weights is finite, we can assume that $H_0(\mathfrak{n}_x, U)_{(\omega-\nu+\rho)} \neq 0$ for all x in an open dense subset of X. On the other hand, $\omega - \nu = v(\lambda - \mu)$ for some uniquely determined $v \in W$. This implies that $v^{-1}(\omega - \nu) = \lambda - \mu$. Since $\omega = u\lambda$ for some $u \in W$, we see that

$$v^{-1}u\lambda - \lambda = -(\mu - v^{-1}\nu).$$

Since μ is the highest weight of F, the right side is the negative of a sum of positive roots. Hence $v^{-1}u \in W_\lambda$ and since λ is antidominant, we see that the left side is a sum of positive roots. It follows that both sides must be zero, $v^{-1}u$ is in the stabilizer of λ and $\omega = u\lambda = v\lambda$. Since $\lambda - \mu$ is regular, $\mathcal{V}(-\mu) = \Delta_{\lambda-\mu}(U)$. Moreover, from 3.7 we conclude that $\operatorname{supp} \Delta_{v(\lambda-\mu)}(U) = X$. Since $I_v(\mathcal{V}(-\mu)) = I_v(\Delta_{\lambda-\mu}(U)) = \Delta_{v(\lambda-\mu)}(U)$ by 2.6, we see that $v \in S(\mathcal{V}(-\mu)) = S(\mathcal{V})$. Hence, by 3.6 there exists $w \leq v$ such that w is transversal to S and $\ell(w) = \operatorname{codim} S$. But, by 3.9, this implies $w\lambda \preccurlyeq v\lambda = \omega$. This completes the proof of 3.10.(i).

To prove 3.10.(ii) we need a curious result which is a formal consequence of the equivalence of derived categories $D^b(\mathcal{U}_\theta)$ and $D^b(\mathcal{D}_\lambda)$.

Lemma 3.11. *Let $\lambda \in \mathfrak{h}^*$ be regular and $\theta = W \cdot \lambda$. Let V be a $\mathcal{U}_\theta$-module and $p = \min\{q \in \mathbb{Z} \mid L^{-q}\Delta_\lambda(V) \neq 0\}$. Assume that $H^q(X, L^{-p}\Delta_\lambda(V)) = 0$ for $q < p$. Then there exists a nontrivial morphism of V into $H^p(X, L^{-p}\Delta_\lambda(V))$.*

Proof. First a simple result about morphisms in derived categories. Let $\mathcal{A}$ be an abelian category and $D^b(\mathcal{A})$ its derived category of bounded complexes. Let $C^\cdot$ and $D^\cdot$ be two complexes in $D^b(\mathcal{A})$ and $\phi \in \operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, D^\cdot)$. Assume that

a) $H^q(C^\cdot) = 0$ for $q > 0$,
b) $H^q(D^\cdot) = 0$ for $q < 0$.

Then $\phi = 0$ if and only if $H^0(\phi) = 0$.

To prove this we use the truncation functors $\tau_{\geq s}$ and $\tau_{\leq s}$ we introduced in §2. By the hypothesis, $\tau_{\leq 0}(C^\cdot) \longrightarrow C^\cdot$ and $D^\cdot \longrightarrow \tau_{\geq 0}(D^\cdot)$ are quasiisomorphisms, and by composing them with ϕ we can assume that $C^q = 0$ for $q > 0$ and $D^q = 0$ for $q < 0$. By the definition of a morphism in derived categories, there exists a complex $B^{\cdot} \in D^b(\mathcal{A})$ and morphisms of complexes $q : B^{\cdot} \longrightarrow C^{\cdot}$, $f : B^{\cdot} \longrightarrow D^{\cdot}$, where q is a quasiisomorphism, which represent ϕ. By composing them with the truncation morphism $\tau_{\leq 0}(B^{\cdot}) \longrightarrow B^{\cdot}$, we see that we can assume in addition that $B^{\cdot}$ satisfies $B^q = 0$ for $q > 0$. But this implies that $f^q = 0$ for $q \neq 0$, $\operatorname{im} f^0 \subset \ker d^0$ and $\operatorname{im} d^{-1} \subset \ker f^0$. Hence $f^0 = 0$ is equivalent to $H^0(\phi) = 0$.

Consider now the truncation morphism

$$L\Delta_\lambda(D(V)) \longrightarrow \tau_{\geq -p}(L\Delta_\lambda(D(V))) = D(L^{-p}\Delta_\lambda(V))[p].$$

By the assumption, it is not zero. By equivalence of derived categories, it leads to a nontrivial morphism ϕ of $D(V)$ into $R\Gamma(D(L^{-p}\Delta_\lambda(V)[p]) = R\Gamma(D(L^{-p}\Delta_\lambda(V))[p]$. It induces zero morphisms between the cohomology modules of both complexes, except in degree zero where we get a morphism of V into $H^p(X, L^{-p}\Delta_\lambda(V))$. Since cohomology modules of $L^{-p}\Delta_\lambda(V)$ vanish below degree p, the complex $R\Gamma(D(L^{-p}\Delta_\lambda(V))[p]$ satisfies the condition (b). Hence, by the preceding result, the morphism $H^0(\phi)$ of V into $R\Gamma(D(L^{-p}\Delta_\lambda(V)))[p]^0 = H^p(X, L^{-p}\Delta_\lambda(V))$ is nonzero. □

Now we can prove 3.10.(ii). If $\mathcal{V}$ is irreducible, $\mathcal{V}(-\mu)$ is also irreducible and their support S is irreducible. Hence, U is irreducible by the equivalence of categories. Since w is transversal to S and $\ell(w) = \operatorname{codim} S$, by 3.5 we see that $\operatorname{supp} \Delta_{w(\lambda-\mu)}(U) = X$. Put $\mathcal{U} = \Delta_{w(\lambda-\mu)}(U)$. Since U is irreducible, by applying 3.11 with $p = 0$, we get $U \subset \Gamma(X, \mathcal{U})$.

Assume that $s \in U$ is a global section of $\mathcal{U}$ which vanishes on the open dense subset in X. Then it generates a submodule of global sections supported in the complement of this open set. This submodule must be either equal to U or to zero. The first possibility would imply that the localization $\Delta_{w(\lambda-\mu)}(U)$ is also supported in the complement of this open set, contradicting our assumption. Therefore this submodule is equal to zero, i.e., $s = 0$. This implies that the support of any nonzero global section in U is equal to X. Let F be the irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight μ. Then, as before, by 3.8,

$$\mathcal{U}(w\mu) = (\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}.$$

Hence, we see

$$\begin{aligned}\Gamma(X, \mathcal{U}(w\mu)) &= \Gamma(X, (\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}) \\ &= \Gamma(X, \mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]} = (\Gamma(X, \mathcal{U}) \otimes_{\mathbb{C}} F)_{[\lambda]} \supset (U \otimes_{\mathbb{C}} F)_{[\lambda]} = V.\end{aligned}$$

Moreover, the support of any nonzero global section of $\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{F} = \mathcal{U} \otimes_{\mathbb{C}} F$ which comes from $U \otimes_{\mathbb{C}} F$ is equal to X, and the support of any nonzero global section of its subsheaf $\mathcal{U}(w\mu)$ which belongs to $(U \otimes_{\mathbb{C}} F)_{[\lambda]} = V$ is also equal to X. Since $\mathcal{U}(w\mu)$ is coherent, there exists an open dense subset O in X such that $\mathcal{U}(w\mu)|O$ is a locally free $\mathcal{O}_O$-module ([5], VII.9.3). Therefore, on this set, a section vanishes if and only if its values (i.e., its images in geometric fibres) vanish everywhere. Hence, there exists an open dense subset O' of O, such that for $x \in O'$, some sections from V do not vanish at x. On the other hand, for any $x \in O'$, the global sections in $\mathfrak{n}_x V$ vanish at that point. Therefore, for $x \in O'$, the geometric fibre map $\mathcal{U}(w\mu) \longmapsto T_x(\mathcal{U}(w\mu))$ induces a nonzero map of V into $T_x(\mathcal{U}(w\mu))$, which factors through $H_0(\mathfrak{n}_x, V)$, and this factor map is a morphism of $\mathfrak{b}_x$-modules. It follows that $H_0(\mathfrak{n}_x, V)_{(w\lambda+\rho)} \neq 0$ for $x \in O'$, i.e., $w\lambda$ is an exponent of V. This completes the proof of 3.10.(ii).

4. Calculations for $\mathfrak{sl}(2, \mathbb{C})$

In this section we discuss the simplest case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. In this case the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$ can be identified with $\mathrm{PSL}(2, \mathbb{C})$, and we can identify the flag variety X of $\mathfrak{g}$ with the one-dimensional projective space $\mathbb{P}^1$. If we denote by $[x_0, x_1]$ the projective coordinates of $x \in \mathbb{P}^1$, the corresponding Borel subalgebra $\mathfrak{b}_x$ is the Lie subalgebra of $\mathfrak{sl}(2, \mathbb{C})$ which leaves the line x invariant.

First we want to classify all possible Harish-Chandra pairs $(\mathfrak{g}, K)$ with $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. We say that two Harish-Chandra pairs $(\mathfrak{g}, K)$ and $(\mathfrak{g}, K')$ are *conjugate* if there exists an isomorphism $\psi : K \longrightarrow K'$ and an inner automorphism β of $\mathfrak{g}$ such that $\beta \circ \varphi = \varphi' \circ \psi$.

Let B be the Borel subgroup of $\mathrm{PSL}(2, \mathbb{C})$ corresponding to $[1, 0]$, N its unipotent radical and T the one-dimensional torus which stabilizes both $0 = [1, 0]$ and $\infty = [0, 1]$.

Lemma 4.1. *Up to conjugacy, the only connected algebraic groups K such that $(\mathfrak{g}, K)$ is a Harish-Chandra pair are:*

(i) *N with $\varphi =$ identity,*
(ii) *finite coverings of T with $\varphi =$ covering map,*
(iii) *finite coverings of B with $\varphi =$ covering map,*
(iv) $\mathrm{PSL}(2, \mathbb{C})$ *with $\varphi =$ identity,*
(v) $\mathrm{SL}(2, \mathbb{C})$ *with $\varphi =$ covering map.*

Proof. Clearly, $\dim K > 0$, since otherwise there would be infinitely many K-orbits. Therefore, if the Lie algebra $\mathfrak{k}$ of K is one-dimensional, the elements of $\mathfrak{k}$ are either all nilpotent, or they are all semisimple. This implies that $\mathfrak{k}$ is conjugate either to the Lie algebra of N or the Lie algebra of T. Since N is simply connected, either (i) or (ii) holds.

If $\dim \mathfrak{k} = 2$, $\mathfrak{k}$ must be solvable, hence a Borel subalgebra. This implies (iii). Finally, if $\dim \mathfrak{k} = 3$, φ must be surjective, hence (iv) and (v) follows from the fact that $\mathrm{SL}(2, \mathbb{C})$ is simply connected and its center is $\mathbb{Z}_2$. □

Let $(\mathfrak{g}, K)$ and $(\mathfrak{g}, K')$ be two Harish-Chandra pairs and $\iota : K' \longrightarrow K$ a morphism of algebraic groups with the property that $\varphi \circ \iota = \varphi'$. Then we have a natural functor from the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ into $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K')$. If the groups K and K' are connected, this functor is fully faithful. To see this, one can argue as follows. The corresponding statement for the categories $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$ and $\mathcal{M}(\mathcal{U}_\theta, K')$ is clear. Therefore, by the equivalence of categories, it holds also for $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ and $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K')$ if $\lambda \in \mathfrak{h}^*$ is antidominant and regular. By twisting, this statement holds for arbitrary $\lambda \in \mathfrak{h}^*$. Hence we can view $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K')$ as a full subcategory of $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$. In particular, in the case of a connected group K, the general situation can be reduced to (i) and (ii).

We need to determine the structure of standard Harish-Chandra sheaves in these cases. We start with (i).

First we want to construct a suitable trivializations of $\mathcal{D}_\lambda$ on the open cover of $\mathbb{P}^1$ consisting of $\mathbb{P}^1 - \{0\}$ and $\mathbb{P}^1 - \{\infty\}$. We denote by $\alpha \in \mathfrak{h}^*$ the positive root of $\mathfrak{g}$ and put $\rho = \frac{1}{2}\alpha$ and $t = \alpha\check{}(\lambda)$. Denote by $\bar{N}$ the unipotent radical of the Borel subgroup of $\mathrm{PSL}(2,\mathbb{C})$ which stabilizes $\infty = [0,1]$ in $\mathbb{P}^1$. Then the subgroups N and $\bar{N}$ correspond to the subgroups

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{C} \right\}$$

and

$$\left\{ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \,\middle|\, y \in \mathbb{C} \right\}$$

of $\mathrm{SL}(2,\mathbb{C})$. Both are normalized by the image in $\mathrm{PSL}(2,\mathbb{C})$ of the torus

$$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \,\middle|\, t \in \mathbb{C}^* \right\}.$$

Let $\{E, F, H\}$ denote the standard basis of $\mathfrak{sl}(2,\mathbb{C})$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the commutation relations

$$[H,E] = 2E \qquad [H,F] = -2F \qquad [E,F] = H.$$

Also, E spans the Lie algebra of N, F spans the Lie algebra of $\bar{N}$ and H spans the Lie algebra of T. If we specialize at 0, H corresponds to the dual root $\alpha\check{}$, but if we specialize at ∞, H corresponds to the negative of $\alpha\check{}$.

First we discuss $\mathbb{P}^1 - \{\infty\}$. We define on it the usual coordinate z by $z([1,x_1]) = x_1$. In this way one identifies $\mathbb{P}^1 - \{\infty\}$ with the complex plane $\mathbb{C}$, which is an $\bar{N}$-orbit. The matrix

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

moves 0 into x, and this map is an isomorphism of $\bar{N}$ onto $\mathbb{C}$. Also, if ∂ denotes differentiation with respect to z considered as a vector field on $\mathbb{C}$, then F corresponds to ∂ under the above isomorphism. Now H and E are represented by first order differential operators on $\mathbb{C}$, i.e.,

$$H = a\partial + b \quad \text{and} \quad E = c\partial + d$$

where a, b, c, d are polynomials. Clearly,

$$[H,F] = [a\partial + b, \partial] = -a'\partial - b'$$

which implies $a = 2z + a_0$ and $b = b_0$ where a_0 and b_0 are constants. On the other hand, in the geometric fibre of $\mathcal{D}_\lambda$ at 0, $H - (t+1)$ maps into 0, which implies $a_0 = 0$ and $b_0 = t+1$. It remains to determine E. We have

$$[E,F] = [c\partial + d, \partial] = -c'\partial - d',$$

which implies $c = -z^2 + c_0$ and $d = -(t+1)z + d_0$; and

$$\begin{aligned} [H,E] = [2z\partial+(t+1), -z^2\partial+c_0\partial-(t+1)z+d_0] &= -2[z\partial, z^2\partial]+2c_0[z\partial,\partial]-2(t+1)z \\ &= -2z^2\partial - 2c_0\partial - 2(t+1)z = 2(-z^2\partial - c_0\partial - (t+1)z), \end{aligned}$$

which implies $c_0 = 0$ and $d_0 = 0$. Therefore, in our coordinate system the basis of $\mathfrak{g}$ is given by

$$E = -z^2\partial - (t+1)z, \qquad F = \partial, \qquad H = 2z\partial + (t+1).$$

Consider now $\mathbb{P}^1 - \{0\}$. Let

$$w = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then $w \in \mathrm{SL}(2,\mathbb{C})$, $w^{-1} = -w$ and $w[x_0, x_1] = [x_1, x_0]$ for any $[x_0, x_1] \in \mathbb{P}^1$. In particular, the automorphism μ of $\mathbb{P}^1$ induced by w maps $\mathbb{P}^1 - \{\infty\}$ onto $\mathbb{P}^1 - \{0\}$. Since $\mathcal{D}_\lambda$ is homogeneous, $\mu^*(\mathcal{D}_\lambda) \cong \mathcal{D}_\lambda$ and $\mu^*(\xi) = \mathrm{Ad}(w)\xi$ for any $\xi \in \mathfrak{g}$. In particular, $\mu^*(E) = F$, $\mu^*(F) = E$ and $\mu^*(H) = -H$. The natural coordinate is $\zeta([x_0, 1]) = x_0$ which identifies $\mathbb{P}^1 - \{0\}$ with the complex plane $\mathbb{C}$. Since $\zeta(\mu([x_0, x_1])) = \zeta([x_1, x_0]) = x_1 = z([x_0, x_1])$, it follows that in this coordinate system we have

$$E = \partial, \qquad F = -\zeta^2\partial - (t+1)\zeta, \qquad H = -2\zeta\partial - (t+1).$$

On $\mathbb{C}^*$ these two coordinate systems are related by $\zeta = \frac{1}{z}$. This implies $\partial_\zeta = -z^2\partial_z$, i.e., on $\mathbb{C}^*$ the second trivialization gives

$$E = -z^2\partial, \quad F = \partial - \frac{1+t}{z}, \quad H = 2z\partial - (t+1).$$

Therefore, the first and the second trivialization on $\mathbb{C}^*$ are related by the automorphism of $\mathcal{D}_{\mathbb{C}^*}$ induced by

$$\partial \longrightarrow \partial - \frac{1+t}{z} = z^{1+t}\,\partial\, z^{-(1+t)}.$$

The N-orbits are $0 = [1, 0]$ and its complement $X^* = \mathbb{P}^1 - \{0\}$. Since the group N is unipotent, the representation which induces the connection at 0 is trivial. This implies that the standard Harish-Chandra sheaf $\mathcal{I}(\{0\}, \lambda)$ is isomorphic to the $\mathcal{D}$-module of truncated Laurent series at 0. Its generator z^{-1} is annihilated by E, and H acts on it by multiplication by $t-1$. Also, the module is spanned by $F^n z^{-1} = (-1)^n n! z^{-(n+1)}$. This implies that the global sections of $\mathcal{I}(\{0\}, \lambda)$ are isomorphic to the Verma module $M((t-1)\rho + \rho) = M(t\rho) = M(\lambda)$.

To see what happens with the standard Harish-Chandra sheaf on the open N-orbit we first remark that $\mathcal{I}(X^*, \lambda)|X^* = \mathcal{O}_{X^*}$ in our second trivialization. Since the irreducibility of $\mathcal{D}_\lambda$-modules is a local property, to analyze the reducibility of $\mathcal{I}(X^*, \lambda)$ it is enough to consider the restriction to $\mathbb{P}^1 - \{\infty\}$ (since the restriction to $\mathbb{P}^1 - \{0\}$ is obviously irreducible). Using the relation between our trivializations, we see that we can view $\mathcal{I}(X^*, \lambda)|\mathbb{P}^1 - \{\infty\}$ as the $\mathcal{D}_{\mathbb{C}}$-module which is the direct image of the module on $\mathbb{C}^*$ generated by z^{1+t}. This module is irreducible if and only if $t \notin \mathbb{Z}$. If $t \in \mathbb{Z}$, $\lambda \in P(\Sigma)$ and $\mathcal{I}(X^*, \lambda)$ contains the invertible homogeneous $\mathcal{O}_X$-module $\mathcal{O}(\lambda+\rho)$ as its unique irreducible $\mathcal{D}_\lambda$-submodule, i. e. we have the exact sequence

$$0 \longrightarrow \mathcal{O}(\lambda + \rho) \longrightarrow \mathcal{I}(X^*, \lambda) \longrightarrow \mathcal{I}(\{0\}, \lambda) \longrightarrow 0.$$

To calculate $\Gamma(X, \mathcal{I}(X^*, \lambda))$ we remark first that (with respect to the trivialization on X^*) constant functions on X^* are annihilated by E, and H acts on them by multiplication with $-(t+1)$. Moreover, $F\zeta^n = -(n+t+1)\zeta^{n+1}$, for $n \in \mathbb{Z}_+$, which implies $\Gamma(X, \mathcal{I}(X^*, \lambda))$ is generated by 1 if t is not a negative integer. Therefore, if $\alpha\check{}(\lambda)$ is not a negative integer, $\Gamma(X, \mathcal{I}(X^*, \lambda))$ is the Verma module $M(-(t+1)\rho + \rho) = M(-t\rho) = M(-\lambda)$.

If $t = -k$, k a strictly positive integer, by the equivalence of categories, $\Gamma(X, \mathcal{I}(X^*, \lambda))$ is reducible, and it contains, as the unique $\mathfrak{g}$-submodule, the finite-dimensional irreducible $\mathfrak{g}$-module with lowest weight $\lambda + \rho$. The quotient of $\Gamma(X, \mathcal{I}(X^*, \lambda))$ by this submodule is isomorphic to the Verma module $M(\lambda)$.

Lemma 4.2. *Let $\lambda \in \mathfrak{h}^*$. Then:*

(i) *$\mathcal{I}(\{0\}, \lambda)$ is an irreducible $\mathcal{D}_\lambda$-module;*

(ii) *$\Gamma(X, \mathcal{I}(\{0\}, \lambda)) = M(\lambda)$ and $H^i(X, \mathcal{I}(\{0\}, \lambda)) = 0$ for $i > 0$.*

(iii) *If $\alpha\check{}(\lambda)$ is not an integer, $\mathcal{I}(X^*, \lambda)$ is an irreducible $\mathcal{D}_\lambda$-module. If $\alpha\check{}(\lambda)$ is an integer, we have the exact sequence*

$$0 \longrightarrow \mathcal{O}(\lambda + \rho) \longrightarrow \mathcal{I}(X^*, \lambda) \longrightarrow \mathcal{I}(\{0\}, \lambda) \longrightarrow 0$$

of $\mathcal{D}_\lambda$-modules.

(iv) *If $\alpha\check{}(\lambda)$ is not a strictly negative integer, we have $\Gamma(X, \mathcal{I}(X^*, \lambda)) = M(-\lambda)$.*

(v) *If $\alpha\check{}(\lambda)$ is a strictly negative integer, we have an exact sequence of $\mathfrak{g}$-modules*

$$0 \longrightarrow F_{\lambda+\rho} \longrightarrow \Gamma(X, \mathcal{I}(X^*, \lambda)) \longrightarrow M(\lambda) \longrightarrow 0$$

where $F_{\lambda+\rho}$ is the finite-dimensional $\mathfrak{g}$-module with lowest weight $\lambda + \rho$.

(vi) *$H^i(X, \mathcal{I}(X^*, \lambda)) = 0$ for $i > 0$.*

This enables us to calculate the action of the intertwining functor $I = I_{s_\alpha}$.

Lemma 4.3. *Let $\lambda \in \mathfrak{h}^*$.*

(i) *If $\alpha\check{}(\lambda)$ is not an integer,*

$$I(\mathcal{I}(\{0\}, \lambda)) = \mathcal{I}(X^*, -\lambda) \text{ and } I(\mathcal{I}(X^*, \lambda)) = \mathcal{I}(\{0\}, -\lambda).$$

(ii) *If $\alpha\check{}(\lambda)$ is an integer,*

$$I(\mathcal{O}(\lambda + \rho)) = 0 \text{ and } L^{-1}I(\mathcal{O}(\lambda + \rho)) = \mathcal{O}(-\lambda + \rho),$$

$$I(\mathcal{I}(\{0\}, \lambda)) = \mathcal{I}(X^*, -\lambda) \text{ and } L^{-1}I(\mathcal{I}(\{0\}, \lambda)) = 0;$$

and

$$I(\mathcal{I}(X^*, \lambda)) = \mathcal{I}(X^*, -\lambda) \text{ and } L^{-1}I(\mathcal{I}(X^*, \lambda)) = \mathcal{O}(-\lambda + \rho).$$

Proof. (i) If $\alpha\check{}(\lambda)$ is not an integer, λ and $-\lambda$ are antidominant, hence the assertion follows from 4.2 and the equivalence of categories.

(ii) To prove the first statement, by 2.4 we can assume that λ is antidominant and regular. Since in this situation

$$\Gamma(X, \mathcal{O}(\lambda + \rho)) = F_{\lambda+\rho} = H^1(X, \mathcal{O}(-\lambda + \rho))$$

by the Borel-Weil-Bott theorem, the assertion follows from the equivalence of derived categories and 2.10.

The second statement follows from 4.2, 2.10 and the equivalence of derived categories. Finally, to get the third statement we use the short exact sequence of 4.2.(iii). It implies the long exact sequence

$$\begin{aligned} 0 \longrightarrow L^{-1}I(\mathcal{O}(\lambda + \rho)) \longrightarrow L^{-1}I(\mathcal{I}(X^*, \lambda)) \longrightarrow L^{-1}I(\mathcal{I}(\{0\}, \lambda) \\ \longrightarrow I(\mathcal{O}(\lambda + \rho)) \longrightarrow I(\mathcal{I}(X^*, \lambda)) \longrightarrow I(\mathcal{I}(\{0\}, \lambda)) \longrightarrow 0. \end{aligned}$$

If we apply the first statement, the assertion follows. □

Before turning to the second basic case (ii), we digress to consider a possibly non-connected group K (compare [12], Appendix B). First, let $(\mathfrak{g}, K)$ be a Harish-Chandra pair such that the unipotent radical of K is nontrivial. In this case, by 4.1, the identity component of K is up to conjugacy either equal to N or to a covering of B. Therefore, K has exactly two orbits in X. By conjugating, we can assume that $X^* = \mathbb{P}^1 - \{0\}$ is the open orbit.

Lemma 4.4. *Let τ be an irreducible K-homogeneous connection on X^* compatible with $\lambda + \rho \in \mathfrak{h}^*$. Then $\mathcal{I}(X^*, \tau)$ is an irreducible $(\mathcal{D}_\lambda, K)$-module if and only if $\alpha\check{}(\lambda) \notin \mathbb{Z}$.*

Proof. If we view $\mathcal{I}(X^*, \tau)$ as a $\mathcal{D}_\lambda$-module, it is a direct sum of finitely many copies of $\mathcal{I}(X^*, \lambda)$. If $\alpha\check{}(\lambda) \notin \mathbb{Z}$, $\mathcal{I}(X^*, \lambda)$ is irreducible by 4.2.(iii), hence $\mathcal{I}(X^*, \tau)$ has no quotients supported in 0. Therefore, $\mathcal{L}(X^*, \tau)$ must be equal to $\mathcal{I}(X^*, \tau)$, i.e., $\mathcal{I}(X^*, \tau)$ is irreducible.

Assume now that $\alpha\check{}(\lambda) \in \mathbb{Z}$. Then $\mathcal{I}(X^*, \lambda)$ contains $\mathcal{O}(\lambda + \rho)$ as a $\mathcal{D}_\lambda$-submodule. Hence, the $\mathcal{D}_\lambda$-module $\mathcal{I}(X^*, \tau)$ contains the largest $\mathcal{D}_\lambda$-submodule $\mathcal{V}$ which is a connection. It is equal to the direct sum of the submodules $\mathcal{O}(\lambda + \rho)$ for various copies of $\mathcal{I}(X^*, \lambda)$. The quotient of $\mathcal{I}(X^*, \tau)$ by $\mathcal{V}$ is nontrivial and supported in 0. Clearly, the K-action maps this connection into itself, i.e., it is a $(\mathcal{D}_\lambda, K)$-submodule. Therefore, $\mathcal{V} = \mathcal{L}(X^*, \tau)$ and $\mathcal{I}(X^*, \tau)$ is reducible. □

Now suppose the connected component K_0 of K is a cover of $\mathrm{PSL}(2, \mathbb{C})$. In this case, K acts transitively on X. If $K = K_0$, the standard modules are $\mathcal{O}(\lambda + \rho)$, $\lambda \in P(\Sigma)$, and the action of the intertwining functor I is given by 4.3.(ii). In general, we have the following result.

Lemma 4.5. *Let τ be a K-homogeneous connection on X compatible with $\lambda + \rho \in \mathfrak{h}^*$. Then $p = -\alpha\check{}(\lambda) \in \mathbb{Z}$ and*

$$LI(D(\tau)) = D(\tau(p\alpha))[1].$$

Proof. Since τ must be a direct sum of K_0-homogeneous invertible $\mathcal{O}_X$-modules we conclude that $p \in \mathbb{Z}$ and τ, as a K_0-homogeneous connection, is a direct sum of copies of $\mathcal{O}(\lambda + \rho)$.

Let $C = \ker \varphi$. Then C is a normal subgroup of K. On the other hand, since K_0 is connected, it centralizes C. Therefore, the map $C \times K_0 \longrightarrow K$ given by $(c, k) \longmapsto ck$ is a surjective homomorphism. Its kernel is $C_0 = C \cap K_0$ imbedded by the map $c \longmapsto (c, c^{-1})$ into $C \times K_0$. The subgroup C_0 is the kernel of the restriction of φ to the identity component K_0 of K. This map is a covering map and K_0 is either $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{PSL}(2, \mathbb{C})$. Therefore, C_0 is either trivial or $\mathbb{Z}_2$. By construction, C_0 is a normal subgroup of K, hence it must be a central subgroup. Hence, by using the map $C \times K_0 \longrightarrow K$ we can always reduce the situation to the case of $K = C \times K_0$. In this situation the result follows immediately as in 4.3.(ii). □

Now the case (ii). Then K is an n-fold covering of the torus T in $\mathrm{PSL}(2, \mathbb{C})$ and φ is the covering map. We realize K as $\mathbb{C}^*$ and take $\varphi(\zeta)([x_0, x_1]) = [x_0, \zeta^n x_1]$. Let $\zeta\partial_\zeta$ be a basis vector of the Lie algebra $\mathfrak{k}$ of K. Then the differential of φ maps $\zeta\partial_\zeta$ into $\frac{1}{2}nH$. The K-orbits in this case are $\{0\}$, $\{\infty\}$ and $\mathbb{C}^*$, the stabilizers of $\{0\}$ and $\{\infty\}$ are equal to K, and the stabilizer of any point in $\mathbb{C}^*$ is the group M of n^{th} roots of 1. The irreducible representations of K are $\omega_k : \zeta \longmapsto \zeta^k$ for $k \in \mathbb{Z}$.

The only "new" standard Harish-Chandra sheaves arise on the open orbit $\mathbb{C}^*$. Let η_0 be the trivial representation of M, η_1 the identity representation of M, and $\eta_k = (\eta_1)^k$, $2 \le k \le n - 1$, the remaining irreducible representations of the cyclic group M. To analyze these $\mathcal{D}_\lambda$-modules it is convenient to introduce a trivialization of $\mathcal{D}_\lambda$ on $\mathbb{C}^* = \mathbb{P}^1 - \{0, \infty\}$ such that H corresponds to the differential operator

$2z\partial$ on $\mathbb{C}^*$. We obtain this trivialization by restricting the original z-trivialization to $\mathbb{C}^*$ and twisting it by the automorphism

$$\partial \longmapsto \partial - \frac{1+t}{2z} = z^{\frac{1+t}{2}}\, \partial\, z^{-\frac{1+t}{2}}.$$

This gives a trivialization of $\mathcal{D}_\lambda | \mathbb{C}^*$ which satisfies

$$E = -z^2\partial - \frac{1+t}{2} z, \qquad F = \partial - \frac{1+t}{2}, \qquad H = 2z\partial.$$

Denote by τ_k the K-equivariant connection on $\mathbb{C}^*$ corresponding to the representation η_k of M, and by $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)$ the corresponding standard Harish-Chandra sheaf in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$. The global sections of τ_k on $\mathbb{C}^*$ form the linear space spanned by functions $z^{p+\frac{k}{n}}$, $p \in \mathbb{Z}$. Therefore, the function $z^{p+\frac{k}{n}}$, $p \in \mathbb{Z}$, is an eigenvector of H for eigenvalue $2(p+\frac{k}{n})$ and K acts on it via representation ω_{np+k}. To analyze the irreducibility of the standard $\mathcal{D}_\lambda$-module $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)$ we have to study its behavior at 0 and ∞. By the preceding discussion, if we use the z-trivialization of $\mathcal{D}_\lambda$ on $\mathbb{C}^*$, $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)$ looks like the $\mathcal{D}_{\mathbb{C}}$-module which is the direct image of the $\mathcal{D}_{\mathbb{C}^*}$-module generated by $z^{\frac{k}{n}-\frac{1+t}{2}}$. This module is reducible if and only if it contains constant functions, i.e., if and only if $\frac{k}{n} - \frac{1+t}{2}$ is an integer. On the other hand, $\mu^*(\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)) = \mathcal{I}(\mathbb{C}^*, \eta_{n-k}, \lambda)$, hence $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)|\mathbb{P}^1 - \{0\}$ is reducible if and only if $\frac{n-k}{n} - \frac{1+t}{2}$ is an integer, i.e., if and only if $\frac{k}{n} + \frac{1+t}{2}$ is an integer. Therefore, $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)$ is irreducible if and only if neither $\frac{k}{n} - \frac{1+t}{2}$ nor $\frac{k}{n} + \frac{1+t}{2}$ is an integer.

We can summarize this as follows.

Lemma 4.6. *Let K be the n-fold covering of T, $k \in \{0, 1, \dots, n-1\}$ and $\lambda \in \mathfrak{h}^*$. Then the following conditions are equivalent:*

(i) $\alpha\check{}(\lambda) \notin \left\{\frac{2k}{n}, -\frac{2k}{n}\right\} + 2\mathbb{Z} + 1$*;*

(ii) *the standard module $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)$ is irreducible.*

In the following, we shall refer to

$$\alpha\check{}(\lambda) \notin \left\{\frac{2k}{n}, -\frac{2k}{n}\right\} + 2\mathbb{Z} + 1$$

as the *parity condition*.

If a standard module $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)$ is reducible, it has irreducible quotients supported in $\{0, \infty\}$. All such irreducible modules are obtained in this way:

Corollary 4.7. *Every standard module supported in a closed K-orbit is isomorphic to a quotient of a unique standard module attached the open orbit $\mathbb{C}^*$.*

Proof. For simplicity, assume that a standard module is supported in $\{0\}$. An irreducible K-homogeneous connection on $\{0\}$ compatible with $\lambda + \rho$ is just an irreducible representation of K with differential equal to $\lambda + \rho$ (under the specialization at 0). If $\omega_k : \zeta \longmapsto \zeta^k$ is this irreducible representation of K, the compatibility implies that $k = \frac{1}{2}n(t+1)$. Hence, for each λ there is at most one standard module supported in $\{0\}$. Since η_k is the restriction ω_k to M, from the discussion preceding 4.6 we see that the standard module $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)$ is reducible and has an irreducible quotient supported at $\{0\}$. This irreducible module must be isomorphic to our standard module. □

The global sections of $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)$ are the $\mathcal{U}_\theta$-module spanned by $e_p = z^{p+\frac{k}{n}}$, $p \in \mathbb{Z}$, and the action of $\mathfrak{g}$ is given by

$$\begin{aligned}
Ee_p &= -\left(p + \frac{k}{n} + \frac{1}{2}(1+t)\right) e_{p+1}, \\
Fe_p &= \left(p + \frac{k}{n} - \frac{1}{2}(1+t)\right) e_{p-1}, \\
He_p &= 2\left(p + \frac{k}{n}\right) e_p.
\end{aligned}$$

This implies that this $\mathcal{U}_\theta$-module is irreducible if the parity condition holds.

Clearly, this condition is symmetric under the change $t \longmapsto -t$. If it is satisfied, we can define rational functions α_p, $p \in \mathbb{Z}$, such that

$$\alpha_{p+1} = \frac{p + \frac{k}{n} + \frac{1}{2}(1+t)}{p + \frac{k}{n} + \frac{1}{2}(1-t)} \alpha_p$$

and change the basis by $f_p = \alpha_p e_p$, $p \in \mathbb{Z}$. This leads to

$$\begin{aligned}
Ef_p &= \alpha_p Ee_p = -\frac{\alpha_p}{\alpha_{p+1}}\left(p + \frac{k}{n} + \frac{1}{2}(1+t)\right) f_{p+1} = -\left(p + \frac{k}{n} + \frac{1}{2}(1-t)\right) f_{p+1}, \\
Ff_p &= \alpha_p Fe_p = \frac{\alpha_p}{\alpha_{p-1}}\left(p + \frac{k}{n} - \frac{1}{2}(1+t)\right) f_{p-1} = \left(p + \frac{k}{n} - \frac{1}{2}(1-t)\right) f_{p-1}, \\
Hf_p &= 2\left(p + \frac{k}{n}\right) f_p.
\end{aligned}$$

It follows that $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \eta_k, \lambda))$ and $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \eta_k, -\lambda))$ are isomorphic as $\mathcal{U}_\theta$-modules. Also, since $\mathbb{C}^*$ is an affine variety,

$$H^i(X, \mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)) = H^i(\mathbb{C}^*, \tau_k) = 0$$

for $i > 0$, and the same statement is true for $\mathcal{I}(\mathbb{C}^*, \eta_k, -\lambda)$. Therefore,

$$R\Gamma(D(\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda))) = R\Gamma(D(\mathcal{I}(\mathbb{C}^*, \eta_k, -\lambda))).$$

For regular antidominant λ satisfying the parity condition this implies, via the equivalence of derived categories,

$$LI(D(\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda))) = D(\mathcal{I}(\mathbb{C}^*, \eta_k, -\lambda)).$$

Therefore, by translation, this holds for arbitrary λ satisfying the parity condition.

Lemma 4.8. *Let K be the n-fold covering of T, $k \in \{0, 1, \dots, n-1\}$ and $\lambda \in \mathfrak{h}^*$. Assume also that λ and k satisfy the parity condition. Then*

$$LI(D(\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda))) = D(\mathcal{I}(\mathbb{C}^*, \eta_k, -\lambda)).$$

Now we want to extend the last three results to the case of non-connected K. Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair such that the identity component K_0 of K is the n-fold covering of the torus T. Then the image $\varphi(K)$ of K in $\mathrm{PSL}(2,\mathbb{C})$ is a subgroup of the normalizer $N(T)$ of the torus T. Since T is in the image and T has index two in $N(T)$, we have two possibilities:

(a) $\varphi(K) = T$;
(b) $\varphi(K) = N(T)$.

Let $K_1 = \varphi^{-1}(T)$. Then, in the case (a), $K_1 = K$; and in the case (b), K_1 has index two in K. Since K_1 acts trivially on the Lie algebra of K, K_0 is a central subgroup of K_1. Moreover, K_1 is the centralizer of K_0, since in the case (b) K does not centralize K_0.

By dimension reasons, the K_0-orbit $\mathbb{C}^*$ is also a K-orbit. Let S be the stabilizer in K of $1 \in \mathbb{C}^*$, $S_1 = S \cap K_1$ and $S_0 = S \cap K_0$. Since S_1 also stabilizes 0 and ∞, it acts trivially on X. The orbit $\mathbb{C}^*$ is connected, hence the map $K_0 \times S \longrightarrow K$ given by $(k, s) \longmapsto ks$, is surjective. Therefore, in case (b), S_1 is a proper subgroup of S. Any representative of the nontrivial element in S/S_1 acts on $\mathbb{C}^*$ as the inversion $z \longmapsto z^{-1}$, hence S stabilizes only 1 and -1 in $\mathbb{C}^*$.

Lemma 4.9. *The restriction of any irreducible algebraic representation of S to S_0 is a direct sum of copies of η_k or a direct sum of copies of $\eta_k \oplus \eta_{n-k}$ for some $0 \le k \le n-1$.*

Proof. In the case (a) the assertion is obvious since S_0 is a central subgroup of S.

In the case (b) S_1 is a subgroup of index two in S, hence the restriction of an irreducible representation of S to S_1 is either irreducible or a direct sum of two irreducible representations conjugated by the action of S/S_1. In the first case the restriction to S_0 is a direct sum of copies of η_k for some $k \in \mathbb{Z}$. In the second case, the representation restricted to S_0 is a direct sum of two isotypic components of the same dimension corresponding to two irreducible representations conjugated by the action of S/S_1. Since the nontrivial element of S/S_1 acts as $k \longmapsto k^{-1}$ on S_0, the orbit of η_k is equal to $\{\eta_k, \eta_{n-k}\}$ and the isotypic components correspond to these representations. □

Since the parity condition is symmetric with respect to $k \longmapsto n-k$, we see that we can say that the pair (ω, λ), where ω is a finite-dimensional algebraic representation of S and $\lambda \in \mathfrak{h}^*$, satisfies the *parity condition* if $\omega|S_0$ contains only representations η_k, $0 \le k \le n-1$, such that the pairs (k, λ) satisfy the parity condition. If ω is irreducible, by 4.9 it is enough that one irreducible component η_k of $\omega|S_0$ is such that the pair (k, λ) satisfies the parity condition.

The next result generalizes 4.6 to this setting.

Proposition 4.10. *Let ω be an irreducible representation of S, τ the corresponding connection on $\mathbb{C}^*$ and $\lambda \in \mathfrak{h}^*$. The following conditions are equivalent:*

(i) *the pair (ω, λ) satisfies the parity condition;*
(ii) *the standard $(\mathcal{D}_\lambda, K)$-module $\mathcal{I}(\mathbb{C}^*, \tau, \lambda)$ is irreducible.*

Proof. The $\mathcal{D}_\lambda$-module $\mathcal{I}(\mathbb{C}^*, \tau, \lambda)$ is the direct sum of $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)$, where η_k goes over all irreducible components of $\omega|S_0$. Let $\mathcal{L}(\mathbb{C}^*, \tau, \lambda)$ be the unique irreducible $(\mathcal{D}_\lambda, K)$-submodule of $\mathcal{I}(\mathbb{C}^*, \tau, \lambda)$. Since $\mathcal{L}(\mathbb{C}^*, \tau, \lambda)|\mathbb{C}^*$ is τ and as a K_0-homogeneous connection τ corresponds to $\omega|S_0$, we see that the $\mathcal{D}_\lambda$-module $\mathcal{L}(\mathbb{C}^*, \tau, \lambda)$ must contain the direct sum $\mathcal{V}$ of all $\mathcal{L}(\mathbb{C}^*, \eta_k, \lambda)$, where η_k ranges over all irreducible components of $\omega|S_0$. On the other hand, the action of K maps the irreducible $\mathcal{D}_\lambda$-module $\mathcal{L}(\mathbb{C}^*, \eta_k, \lambda)$ into a submodule of $\mathcal{V}$. Therefore, $\mathcal{V}$ is a $(\mathcal{D}_\lambda, K)$-submodule of $\mathcal{I}(\mathbb{C}^*, \tau, \lambda)$, and must contain $\mathcal{L}(\mathbb{C}^*, \tau, \lambda)$. It follows that $\mathcal{V} = \mathcal{L}(\mathbb{C}^*, \tau, \lambda)$. Therefore, $\mathcal{I}(\mathbb{C}^*, \tau, \lambda)$ is irreducible $(\mathcal{D}_\lambda, K)$-module if and only if all $\mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)$, where η_k ranges over all irreducible components of $\omega|S_0$, are irreducible $\mathcal{D}_\lambda$-modules. By 4.6 and 4.9, this implies our assertion. □

Let C be a closed K-orbit in X, i.e., either $\{0\}$, or $\{\infty\}$ or the union of these two points. The next result generalizes 4.7.

Lemma 4.11. *Every standard module attached to C is isomorphic to a quotient of a standard module on the open orbit $\mathbb{C}^*$.*

Proof. By twisting we can assume that λ is regular and dominant. In the case (a), K is a quotient of the direct product $K_0 \times S$. Therefore, we can assume that $K = K_0 \times D$ for some finite group D and that $\varphi|\{1\} \times D = 1$. The orbit C consists of just one point and we can assume that $C = \{0\}$. An irreducible K-homogeneous connection on C compatible with $\lambda+\rho$ is just an irreducible representation of K with differential equal to a direct sum of copies of $\lambda + \rho$ (under the specialization at 0). Such representation is an exterior tensor product $\omega \boxtimes \delta$ of irreducible representations ω of K_0 and δ of D. If $\omega = \omega_k$, the compatibility implies that $k = \frac{1}{2}n(t + 1)$. If we denote by $\mathcal{I}(C, \omega_k)$ the standard $(\mathfrak{g}, K_0)$-module on C determined by ω_k, we have $\Gamma(X, \mathcal{I}(C, \omega)) = \Gamma(X, \mathcal{I}(C, \omega_k)) \boxtimes \delta$ where $\mathfrak{g}$ acts only on the first factor in the tensor product. On the other hand, $\eta_k \boxtimes \delta$ is then an irreducible representation of the stabilizer of 1 in K and determines an irreducible K-homogeneous connection τ on $\mathbb{C}^*$. Its global sections are

$$\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau, \lambda)) = \Gamma(X, \mathcal{I}(\mathbb{C}^*, \eta_k, \lambda)) \boxtimes \delta,$$

and the assertion follows from 4.7 and the equivalence of categories.

In the case (b), we have $C = \{0, \infty\}$. As we remarked in Appendix B of [12], in this situation

$$\Gamma(X, \mathcal{I}(C, \omega)) = \operatorname{Ind}_{K_1}^{K}(\Gamma(X, \mathcal{I}(\{0\}, \omega|\{0\})))$$

for any irreducible K-homogeneous connection ω on C. On the other hand, by the first part of the proof, $\Gamma(X, \mathcal{I}(\{0\}, \omega|\{0\}))$ is a quotient of $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau, \lambda))$ for some irreducible K_1-homogeneous connection τ on $\mathbb{C}^*$. This connection corresponds to some irreducible representation γ of the stabilizer S_1 of 1 in K_1. Let $\tilde{\gamma} = \operatorname{Ind}_{K_1}^{K}(\gamma)$. Then $\tilde{\gamma}$ is either irreducible or the sum of two irreducible representations γ_+ and γ_-. Denote by $\tilde{\tau}$, resp. τ_+ and τ_-, the corresponding irreducible K-homogeneous connections on $\mathbb{C}^*$. One can check that

$$\operatorname{Ind}_{K_1}^{K}(\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau, \lambda))) = \Gamma(X, \mathcal{I}(\mathbb{C}^*, \tilde{\tau}, \lambda))$$

in the first case, and

$$\operatorname{Ind}_{K_1}^{K}(\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau, \lambda))) = \Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_+, \lambda)) \oplus \Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_-, \lambda))$$

in the second case. Therefore, $\Gamma(X, \mathcal{I}(C, \omega))$ is a quotient of either $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tilde{\tau}, \lambda))$ or $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_+, \lambda)) \oplus \Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_-, \lambda))$. The assertion again follows from the equivalence of categories. □

Now we generalize 4.8. A K-homogeneous connection τ on $\mathbb{C}^*$ is determined by the representation ω of the stabilizer S in the geometric fiber $T_1(\tau)$. On the other hand, S also stabilizes the point -1. Therefore, there exists a unique K-homogeneous connection $\tilde{\tau}$ on $\mathbb{C}^*$ determined by ω considered as the representation of S in the geometric fiber $T_{-1}(\tilde{\tau})$. Since K_0 is transitive on $\mathbb{C}^*$ and K_1 is the centralizer of K_0, it follows that $\tau \cong \tilde{\tau}$ as K_1-homogeneous connections.

Proposition 4.12. *Let ω is an irreducible representation of S and $\lambda \in \mathfrak{h}^*$. Assume that the pair (ω, λ) satisfies the parity condition. Then*

$$LI(D(\mathcal{I}(\mathbb{C}^*, \tau, \lambda))) = D(\mathcal{I}(\mathbb{C}^*, \tilde{\tau}, -\lambda)).$$

Proof. If (ω, λ) satisfies the parity condition, all η_k appearing in $\omega|S_0$ satisfy this condition too. Therefore, $I(\mathcal{I}(\mathbb{C}^*, \tau, \lambda))$ is as a $\mathcal{D}_{-\lambda}$-module equal to a direct sum of finitely many $\mathcal{I}(\mathbb{C}^*, \eta_k, -\lambda)$ for η_k contained in $\omega|S_0$, and the higher derived intertwining functors vanish on $\mathcal{I}(\mathbb{C}^*, \tau, \lambda)$. Moreover,

$$I(\mathcal{I}(\mathbb{C}^*, \tau, \lambda)) = \mathcal{I}(\mathbb{C}^*, \tau', -\lambda),$$

where τ' is the K-equivariant connection which is the restriction of $I(\mathcal{I}(\mathbb{C}^*, \tau, \lambda))$ to $\mathbb{C}^*$. By translation we can assume that λ is antidominant and regular. Then by 2.10 we have

$$\Gamma(\mathbb{C}^*, \tau) = \Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau, \lambda)) = \Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau', -\lambda)) = \Gamma(\mathbb{C}^*, \tau'),$$

as K-modules.

Assume first that we are in the case (a). In this situation K is a central extension of K_0. Therefore the map $K_0 \times S \longrightarrow K$ given by $(k, s) \longmapsto ks$ is a surjective homomorphism. This implies that any irreducible representation of K can be viewed as an irreducible representation of $K_0 \times S$. Since K_0 is commutative, the restriction of this representation to S is irreducible. By Frobenius reciprocity, the preceding formula implies that the representations of S determining τ and τ' are equivalent. Hence, in this case $\tau \cong \tilde{\tau}$.

Assume now that we are in the case (b). In this case K_1 is a normal subgroup of index two in K. Thus we can define a character δ of K which is 1 on K_1 and -1 outside K_1. If π is an irreducible algebraic representation of K, $\pi \otimes \delta$ is an irreducible algebraic representation of K. There are two possibilities for π.

(i) $\pi_1 = \pi|K_1$ is irreducible. In this case, we can induce $\pi|K_1$ to K. The induced representation $\operatorname{Ind}(\pi_1)$ contains exactly one copy of π by Frobenius reciprocity. Since $\dim \operatorname{Ind}(\pi_1) = 2 \dim \pi$, $\operatorname{Ind}(\pi_1)$ is reducible and it is a sum of two irreducible representations of K. Let ν be the other irreducible component of $\operatorname{Ind}(\pi_1)$. Then $\nu|K_1 = \pi_1$ by Frobenius reciprocity. Therefore, $\nu|K_1 = \pi|K_1$. Since the character of $\operatorname{Ind}(\pi_1)$ vanishes outside K_1, $\operatorname{tr} \nu(k) = -\operatorname{tr} \pi(k)$ outside K_1. Therefore, $\nu \cong \pi \otimes \delta$. On the other hand, $\nu \ncong \pi$, since $\operatorname{Ind}(\pi_1)$ contains only one copy of π. Therefore, in this case there exists exactly two irreducible representations extending π_1 to K, the representation π and $\pi \otimes \delta$. Since $\pi|K_0$ is an isotypic K_0-module, and K/K_1 conjugates all nontrivial characters of K_0 into their inverses, we see that the restriction of π to K_0 is trivial.

(ii) $\pi_1 = \pi|K_1$ is reducible. In this case, π_1 consists of two irreducible representations ν_+ and ν_- of K_1 conjugated by the action of K/K_1. By Frobenius reciprocity, π is contained in $\operatorname{Ind}(\nu_+)$ and $\operatorname{Ind}(\nu_-)$, but $\nu_+ \ncong \nu_-$. Since $\dim \pi = \dim \operatorname{Ind}(\nu_+) = \dim \operatorname{Ind}(\nu_-)$, we conclude that $\pi \cong \operatorname{Ind}(\nu_+) \cong \operatorname{Ind}(\nu_-)$. This implies that the character of π vanishes outside K_1 and $\pi \cong \pi \otimes \delta$.

Assume that $\Gamma(\mathbb{C}^*, \tau)$ contains at least one irreducible component π of the type (i). In this case, $\pi|K_1$ is irreducible, hence as in (a) we conclude that the restriction of π to S_1 is irreducible. This implies that the restriction of π to S is irreducible. By Frobenius reciprocity, the representation ω defining τ is equivalent to $\pi|S$. Since the same argument applies to $\Gamma(\mathbb{C}^*, \tau')$, we conclude $\tau \cong \tau'$. On the other hand, again by Frobenius reciprocity, we see that the representation of S in $T_{-1}(\tau)$ is also equivalent to $\pi|S$, and $\tilde{\tau} \cong \tau \cong \tau'$. Also, since $\pi|K_0$ is trivial, $\omega|S_0$ is trivial in this case.

It remains to treat the case when all irreducible representations of K in $\Gamma(\mathbb{C}^*, \tau)$ are of type (ii). Then $\pi|S \cong \operatorname{Ind}(\nu_+|S_1) \cong \operatorname{Ind}(\nu_-|S_1)$. If this is an irreducible representation of S for some π in $\Gamma(\mathbb{C}^*, \tau)$, the preceding argument applies again and $\tau \cong \tilde{\tau}$. It remains to analyze the situation when $\pi|S$ is reducible for all π in $\Gamma(\mathbb{C}^*, \tau)$. This implies that $\pi|S$ contains two irreducible subrepresentations σ_+ and σ_-. By Frobenius reciprocity, $\sigma_+|S_1 \cong \sigma_-|S_1 \cong \nu_+|S_1 \cong \nu_-|S_1$, and σ_+ and σ_- are not equivalent. As before, we conclude that $\sigma_- \cong \sigma_+ \otimes \iota$ where $\iota = \delta|S$. Therefore, the representation ω determining τ is either σ_+ or $\sigma_+ \otimes \iota$. Since $K/K_1 = S/S_1$ conjugates $\nu_+|S_1$ and $\nu_-|S_1$, if $\nu_+|S_0$ is a direct sum of copies of η_k, $\nu_-|S_0$ is a direct sum of copies of η_{n-k}. Therefore, we also have $\eta_k \cong \eta_{n-k}$. This is possible only if $\nu_+|S_0$ is either trivial or its kernel is of order two in S_0.

Since $K_0 \times S_1 \longrightarrow K_1$ is a surjective homomorphism, if $\nu_+|S_0 \cong 1$, there exists an irreducible representation γ of K_1 such that $\gamma|K_0 = 1$ and $\gamma|S_1 = \nu_+|S_1$. Since K/K_1 conjugates $\nu_+|S_1$ and $\nu_-|S_1$ and they are equivalent, we conclude that the conjugate of γ is equivalent to γ. By Frobenius reciprocity, there exists an irreducible representation of K contained in $\Gamma(X, \tau)$ which, restricted to K_1, contains γ. By the preceding discussion, this representation must be of the type (i) and we have a contradiction. Therefore, $\ker(\nu_+|S_0)$ is of index two in S_0. Since this is a normal subgroup of K, we can divide S_0 by it and assume that $S_0 \cong \mathbb{Z}_2$. In this case, K_0 is a two-fold cover of T. Also, there exists an element k_o of K which maps into the image of w in $\mathrm{PSL}(2, \mathbb{C})$. It acts as $z \longmapsto z^{-1}$ on $\mathbb{C}^*$, and therefore lies in S. Since T acts with no fixed points on $\mathbb{C}^*$, it follows that $\varphi(K_0) \cap B_1$ is trivial. This implies that $\varphi(k_o)$ is the only nontrivial element of $\varphi(K) \cap B_1$. If we consider $\mathcal{I}(\mathbb{C}^*, \tau, \lambda)$ as $(\mathcal{D}_\lambda, K_1)$-module, from the preceding argument we conclude that the restrictions of each isotypic K_1-submodule of $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau, \lambda))$ to S_1 are mutually equivalent and irreducible. Therefore, we can assume that they are all isomorphic to some irreducible S_1-module V. Hence we see that the global sections are spanned by $e_p \otimes v$, $p \in \mathbb{Z}$ and $v \in V$. Since $\varphi(S_1) = 1$, the actions of E, F and H are

$$\begin{aligned}
E(e_p \otimes v) &= -\left(p + 1 + \frac{t}{2}\right) e_{p+1} \otimes v, \\
F(e_p \otimes v) &= \left(p - \frac{t}{2}\right) e_{p-1} \otimes v, \\
H(e_p \otimes v) &= (2p+1) e_p \otimes v,
\end{aligned}$$

for all $p \in \mathbb{Z}$. Let R be the linear transformation which describes the action of k_0 on $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau, \lambda))$. Then,

$$R(e_p \otimes v) = e_{-p-1} \otimes Q_p v$$

for some linear transformation Q_p on V. By a direct calculation,

$$\begin{aligned}
R^{-1}ER(e_p \otimes v) &= F(e_p \otimes Q_{p-1}^{-1} Q_p v) \\
R^{-1}FR(e_p \otimes v) &= E(e_p \otimes Q_{p+1}^{-1} Q_p v) \\
R^{-1}HR(e_p \otimes v) &= -H(e_p \otimes v)
\end{aligned}$$

for any $p \in \mathbb{Z}$. Since $\mathrm{Ad}(w)(E) = F$, $\mathrm{Ad}(w)F = E$ and $\mathrm{Ad}(w)(H) = -H$, we see that $Q_p = Q$ for all $p \in \mathbb{Z}$. This implies that k_0 acts as Q on the geometric fibre $T_1(\tau) \cong V$, and as $-Q$ on $T_{-1}(\tau) \cong V$.

If we change the basis $\{e_p \,|\, p \in \mathbb{Z}\}$ to the basis $\{f_p \mid p \in \mathbb{Z}\}$ as before, we get

$$T(f_p \otimes v) = \alpha(e_{-p-1} \otimes Qv) = \frac{\alpha_p}{\alpha_{-p-1}}(f_{-p-1} \otimes Qv).$$

On the other hand, for $p \in \mathbb{N}$, we have

$$\frac{\alpha_p}{\alpha_{-p-1}} = \frac{\alpha_p}{\alpha_{p-1}} \cdot \frac{\alpha_{p-1}}{\alpha_{-p}} \cdot \frac{\alpha_{-p}}{\alpha_{-p-1}} = \frac{(p+\frac{t}{2})}{(p-\frac{t}{2})} \cdot \frac{(-p+\frac{t}{2})}{(-p-\frac{t}{2})} \cdot \frac{\alpha_{p-1}}{\alpha_{-p}} = \frac{\alpha_{p-1}}{\alpha_{-p}} = \ldots = \frac{\alpha_0}{\alpha_{-1}} = -1.$$

This implies

$$T(f_p \otimes v) = -(f_{-p-1} \otimes Qv)$$

for any $p \in \mathbb{Z}$. Hence the actions of k_0 on the fibre of τ and τ' at 1 differ in sign. This implies $\tau' \cong \tilde{\tau}$ in this case. □

Corollary 4.13. *Assume that the pair (ω, λ) satisfies the parity condition and that $p = -\alpha\check{}(\lambda) \in \mathbb{Z}$. Then*

$$LI(D(\mathcal{I}(\mathbb{C}^*, \tau, \lambda))) = D(\mathcal{I}(\mathbb{C}^*, \tau, \lambda)(p\alpha)).$$

Proof. In the case (a), $\varphi(K) = T$. Since T acts with no fixed points on $\mathbb{C}^*$, it follows that $\varphi(K) \cap B_1$ is trivial. Hence, the representation of the stabilizer $S = \varphi^{-1}(\varphi(K) \cap B_1)$ corresponding to the K-homogeneous $\mathcal{O}_{\mathbb{C}^*}$-connection $i^*(\mathcal{O}(p\alpha))$ is trivial. This proves that $\tilde{\tau} \cong \tau \cong \tau \otimes_{\mathcal{O}_{\mathbb{C}^*}} i^*(\mathcal{O}(p\alpha))$ in this case.

In the case (b), the element k_o of K maps into the image of w in $\mathrm{PSL}(2, \mathbb{C})$. It acts as $z \longmapsto z^{-1}$ on $\mathbb{C}^*$ and therefore lies in S. As in the preceding argument, this implies that $\varphi(k_o)$ is the only nontrivial element of $\varphi(K) \cap B_1$. Its square maps into the identity element of $\mathrm{PSL}(2, \mathbb{C})$, hence it acts as -1 in the one-dimensional representation of S attached to the K-homogeneous $\mathcal{O}_{\mathbb{C}^*}$-connection $i^*(\mathcal{O}(\alpha))$. If p is even, the representation of the stabilizer attached to $i^*(\mathcal{O}(p\alpha))$ is trivial and $i^*(\mathcal{O}(p\alpha)) \cong \mathcal{O}_{\mathbb{C}^*}$. Since the parity condition holds, $2k \neq n$ in this situation and $\ker \omega | S_0$ is not of index two in S_0. Therefore, as we have seen in the preceding argument, $\tau \cong \tilde{\tau}$ and the assertion holds in this case. If p is odd, by the parity condition $k \neq 0$. Hence, either ω is induced from an irreducible representation of S_1 or $\ker \omega | S_0$ is of index two in S_0. In the first case, the representations of the stabilizer S at 1 and -1 attached to τ are equivalent and $\omega \cong \omega \otimes \iota$. Hence, $\tilde{\tau} \cong \tau \otimes_{\mathcal{O}_{\mathbb{C}^*}} i^*(\mathcal{O}(p\alpha))$ in this case. In the second case, the representation of S at 1 corresponding to $\tilde{\tau}$ is $\omega \otimes \iota$, hence $\tilde{\tau} \cong \tau \otimes_{\mathcal{O}_{\mathbb{C}^*}} i^*(\mathcal{O}(p\alpha))$ again. □

Finally, we want to make an observation about the action of the intertwining functor I on irreducible Harish-Chandra sheaves. In particular, we want to establish an analogue of 2.18 in this case. First, by 2.8, $L^{-1}I \neq 0$ implies that $\alpha\check{}(\lambda) \in \mathbb{Z}$.

Lemma 4.14. *Let $p = -\alpha\check{}(\lambda) \in \mathbb{Z}$. Let $\mathcal{L}(Q, \tau)$ be an irreducible Harish-Chandra sheaf. Then the following conditions are equivalent:*

- (i) $I(\mathcal{L}(Q, \tau)) = 0$;
- (ii) *either*
  - (a) *K contains a conjugate of N and Q is the open orbit in X; or*
  - (b) *the identity component of K covers a conjugate of T, Q is the open orbit in X and the parity condition fails for τ.*

Proof. Assume that K contains a conjugate of N and Q is not the open orbit. Then Q is a point and $I(\mathcal{L}(Q, \tau)) \neq 0$ by 4.3.(ii). If the identity component of K covers T and Q is not the open orbit, Q is either a point or a pair of points, hence the same argument applies. If Q is the open orbit and the parity condition holds for τ, $\mathcal{L}(Q,\tau)=\mathcal{I}(Q,\tau)$ by 4.10, and $I(\mathcal{L}(Q,\tau))\neq 0$ by 4.12. Therefore, (i) implies (ii).

Assume that (ii) holds. By 2.4, we can assume that $\lambda=-\rho$. First, assume that K contains N. Then, by replacing K by its identity component we see that $\mathcal{I}(Q,\tau)$ is isomorphic to a finite direct sum of $\mathcal{I}(X^*,-\rho)$. By 4.2.(iii), each of these modules contains a copy of $\mathcal{O}_X$ as the unique irreducible submodule, we see that $\mathcal{I}(Q,\tau)$ contains a connection $\mathcal{C}$ which is the direct sum of the same number of copies of $\mathcal{O}_X$. The connection $\mathcal{C}$ is clearly K-homogeneous, and the quotient of $\mathcal{I}(Q,\tau)$ by $\mathcal{C}$ is supported in the complement of Q. Hence, it is equal to $\mathcal{L}(Q,\tau)$. By 4.3.(ii), we see that $I(\mathcal{L}(Q,\tau))=0$.

Assume now that (ii) holds and identity component of K covers T. Again, by replacing K by its identity component we can assume that $\mathcal{I}(Q,\tau)$ is, as a $\mathcal{D}_\lambda$-module, a finite direct sum of $\mathcal{I}(\mathbb{C}^*,\eta_k,\lambda)$ (for possibly different k). Moreover, the failure of the parity condition implies that k must be equal to 0. It follows that $\mathcal{I}(Q,\tau)$ is a direct sum of finitely many copies of $\mathcal{I}(\mathbb{C}^*,\eta_0,-\rho)$ as a $\mathcal{D}_X$-module. By 4.6, each of these modules contains a copy of $\mathcal{O}_X$ as the unique irreducible submodule, we see that $\mathcal{I}(Q,\tau)$ contains a connection $\mathcal{C}$ which is the direct sum of the same number of copies of $\mathcal{O}_X$. The connection $\mathcal{C}$ is clearly K-homogeneous, and the quotient of $\mathcal{I}(Q,\tau)$ by $\mathcal{C}$ is supported in the complement of Q. Hence, it is equal to $\mathcal{L}(Q,\tau)$. The assertion $I(\mathcal{L}(Q,\tau))=0$ again follows by applying 4.3.(ii). □

5. Some results on root systems with involution

In this section we prove some technical lemmas about root systems with involution. Let V be a vector space over $\mathbb{Q}$ and Σ a (restricted) root system in V. We assume that V is equipped with a natural inner product $(.\,,.)$ invariant under the action of $\mathrm{Aut}(\Sigma)$. Let σ be an involution on Σ, i.e., an automorphism of the root system Σ such that $\sigma^2=1$. A root $\alpha\in\Sigma$ is called *imaginary* if $\sigma\alpha=\alpha$, *real* if $\sigma\alpha=-\alpha$ and *complex* otherwise. If $\mathfrak{g}$ is the complexified Lie algebra of a real semisimple Lie group $\mathfrak{g}_0$, σ a Cartan involution on $\mathfrak{g}$ and $\mathfrak{c}$ the complexification of a σ-stable Cartan subalgebra $\mathfrak{c}_0$ of $\mathfrak{g}$, the vector space V over $\mathbb{Q}$ spanned by the roots of $(\mathfrak{g},\mathfrak{c})$ in $\mathfrak{c}^*$ is a root system with involution induced by the Cartan involution σ, and the notions of imaginary, real and complex roots agree with the usual ones.

Denote by Σ_I the set of imaginary roots, $\Sigma_\mathbb{R}$ the set of real roots and $\Sigma_\mathbb{C}$ the set of complex roots in Σ. Let Σ^+ be a set of positive roots in Σ. We say that Σ^+ is of *Langlands type* if for any positive complex root α the root $\sigma\alpha$ is negative; and that Σ^+ is of *Zuckerman type* if for any positive complex root α the root $\sigma\alpha$ is positive.

If (Σ,σ) is a root system with involution, $(\Sigma,-\sigma)$ is also a root system with involution. The sets of complex roots are the same in both cases; and real, respectively imaginary, roots for (Σ,σ) are imaginary, respectively real, roots for $(\Sigma,-\sigma)$. Thus, replacing the involution σ with $-\sigma$ switches the two types of sets of positive roots: a set of Langlands type, respectively of Zuckerman type, for σ is a set of Zuckerman type, respectively of Langlands type, for $-\sigma$.

Lemma 5.1. *The root system Σ admits sets of positive roots of Langlands type and of Zuckerman type.*

Proof. Let $V=V_+\oplus V_-$ be the decomposition of V into the σ-eigenspaces with eigenvalues 1 and -1. Define a lexicographical ordering on V with respect to a basis of V which consists of a basis of V_+ followed by a basis of V_-. Let Σ^+ be the corresponding set of positive roots. Then $\sigma\alpha$ is a positive root for any positive root α which is not real. Therefore, Σ^+ is of Zuckerman type. The existence of sets of positive roots of Langlands type follows by replacing σ with $-\sigma$. □

Now we want to refine the argument of the preceding lemma. Let $\gg_+$ be an order relation on V_+, and $\gg_-$ be an order relation on V_-, compatible with the vector space structures on V_+ and V_- respectively. Then we can define an order relation $\gg_{+,-}$ on $V = V_+ \oplus V_-$ as $(v, w) \gg_{+,-} (v', w')$ if and only if $v - v' \gg_+ 0$ if $v \neq v'$, and $w - w' \gg_- 0$ if $v = v'$. Analogously, we can define an order relation $\gg_{-,+}$ on V by reversing the roles of V_+ and V_-.

Lemma 5.2. *Let Σ^+ be a set of positive roots in V and $\lambda \in V$ such that $(\alpha, \lambda) \leq 0$ for all $\alpha \in \Sigma^+$. Then there exists a set of positive roots $\Sigma^{+,L}$ of Langlands type such that:*

(L1) $(\alpha, \lambda) \leq 0$ *for all imaginary roots in $\Sigma^{+,L}$;*
(L2) $(\alpha, \lambda - \sigma\lambda) \leq 0$ *for all nonimaginary roots in $\Sigma^{+,L}$.*
(I) $\Sigma^+ \cap (-\Sigma^{+,L})$ *consists of complex roots satisfying $\sigma\alpha \in \Sigma^+$.*

Proof. By continuity we may assume that λ is regular and $\lambda - \sigma\lambda$ is not orthogonal to any nonimaginary roots. Then we can define an ordering on V_- by $\mu \gg_- 0$ if $(\mu, \lambda - \sigma\lambda) \leq 0$ and an ordering $\gg_+$ on V_+ compatible with $\Sigma^+ \cap \Sigma_I$. This gives the ordering $\gg_{-,+}$ on V. Since $\lambda - \sigma\lambda$ is not orthogonal to any nonimaginary root α, they are either positive or negative. On the other hand, the order relation on imaginary roots is given by $\gg_+$. Thus any root is either positive or negative with respect to $\gg_{-,+}$, hence the set of all roots $\alpha \gg_{-,+} 0$ is a set of positive roots. Clearly it satisfies the conditions (L1) and (L2) of the lemma, and it is of Langlands type. In addition, if α is a positive real root with respect to this ordering,

$$2(\alpha, \lambda) = (\alpha, \lambda - \sigma\lambda) < 0.$$

Hence, $\alpha \in \Sigma^+$. This implies that roots in $\Sigma^+ \cap (-\Sigma^{+,L})$ are complex. Moreover, if α belongs to $\Sigma^+ \cap (-\Sigma^{+,L})$, $\sigma\alpha \in \Sigma^{+,L}$ and

$$0 > (\sigma\alpha, \lambda - \sigma\lambda) = (\alpha, \sigma\lambda - \lambda),$$

which implies

$$(\sigma\alpha, \lambda) = (\alpha, \sigma\lambda - \lambda) + (\alpha, \lambda) < 0$$

and $\sigma\alpha \in \Sigma^+$. □

Let Σ^+ be a set of positive roots in Σ. Put

$$D(\Sigma^+) = \{\alpha \in \Sigma^+ \mid \sigma\alpha \in \Sigma^+ \text{ and } \sigma\alpha \neq \alpha\}.$$

Proposition 5.3. *Let Σ^+ be a set of positive roots in Σ.*

(i) *There exists a set of positive roots of Langlands type $\Sigma^{+,L}$ such that*

$$\Sigma^+ \cap (-\Sigma^{+,L}) \subset D(\Sigma^+).$$

(ii) *Let $\Sigma^{+,L}$ be a set of positive root of Langlands type such that $S = \Sigma^+ \cap (-\Sigma^{+,L}) \subset D(\Sigma^+)$. Then*

$$S \cap \sigma S = \emptyset \quad \textit{and} \quad S \cup \sigma S = D(\Sigma^+).$$

(iii) *Let $\Sigma^{+,\prime}$ be another set of positive roots in Σ such that $S = \Sigma^+ \cap (-\Sigma^{+,\prime})$ satisfies*

$$S \cap \sigma S = \emptyset \quad \text{and} \quad S \cup \sigma S = D(\Sigma^+).$$

Then $\Sigma^{+,\prime}$ is a set of positive roots of Langlands type.

Proof. Suppose $\lambda \in V$ satisfies $(\alpha, \lambda) < 0$ for all $\alpha \in \Sigma^+$. Then (i) follows from 5.2.

(ii) Let $\alpha \in S \cap \sigma S$. Then

$$-\alpha, -\sigma\alpha \in -S \subset \Sigma^{+,L}.$$

Since $\Sigma^{+,L}$ is of Langlands type, this would imply that α is an imaginary root contradicting $\alpha \in D(\Sigma^+)$. Therefore, $S \cap \sigma S$ is empty.

Let $\alpha \in D(\Sigma^+)$. Then $\alpha \in \Sigma^{+,L}$ or $-\alpha \in \Sigma^{+,L}$. In the first case, $\sigma\alpha \in -\Sigma^{+,L}$ and $\sigma\alpha \in \Sigma^+ \cap (-\Sigma^{+,L}) = S$. In the second case, $\alpha \in \Sigma^+ \cap (-\Sigma^{+,L}) = S$. Therefore, $\alpha \in S \cup \sigma S$.

(iii) Let α be a complex root in $\Sigma^{+,\prime}$.

Assume first that $\alpha \in -\Sigma^+$. Then $-\alpha \in S \subset D(\Sigma^+)$. This implies that $-\sigma\alpha \in D(\Sigma^+) \subset \Sigma^+$ and $-\sigma\alpha \notin S$. Therefore $-\sigma\alpha \notin -\Sigma^{+,\prime}$, i.e., $\sigma\alpha \in -\Sigma^{+,\prime}$.

Assume now that $\alpha \in D(\Sigma^+)$. Since $\alpha \in \Sigma^{+,\prime}$, $\alpha \notin S$. Hence $\sigma\alpha \in S$, i.e., $\sigma\alpha \in -\Sigma^{+,\prime}$.

Finally, assume that $\alpha \in \Sigma^+$ and $\alpha \notin D(\Sigma^+)$. In this case, $\sigma\alpha \in -\Sigma^+$, i.e., $-\sigma\alpha \in \Sigma^+$. If $\sigma\alpha \in \Sigma^{+,\prime}$, $-\sigma\alpha \in S \subset D(\Sigma^+)$ and $-\alpha \in D(\Sigma^+)$ contradicting $\alpha \in \Sigma^+$. Therefore, $\sigma\alpha \in -\Sigma^{+,\prime}$.

Consequently $\Sigma^{+,\prime}$ is a set of positive roots of Langlands type. □

Since the Weyl group W of Σ acts transitively on the sets of positive roots, this result can be rephrased as follows.

Corollary 5.4. *Let Σ^+ be a set of positive roots in Σ. There exists $w \in W$ such that*

$$\Sigma^+_w \cap \sigma(\Sigma^+_w) = \emptyset \text{ and } \Sigma^+_w \cup \sigma(\Sigma^+_w) = D(\Sigma^+).$$

In particular, if $D(\Sigma^+) \neq \emptyset$, it must contain a simple root.

For any such $w \in W$, $w^{-1}(\Sigma^+)$ is a set of positive roots of Langlands type.

Any set of positive roots of Langlands type $\Sigma^{+,L}$ satisfying

$$S = \Sigma^+ \cap (-\Sigma^{+,L}) \subset D(\Sigma^+)$$

defines a section S of the σ-orbits in $D(\Sigma^+)$. Such sections are not completely arbitrary. Actually, they are all contained in a smaller subset of $D(\Sigma^+)$.

To analyze these sections in more detail we first have to study the case of root systems of rank 2. If Σ is a root system of rank 2 with nonempty $D(\Sigma^+)$, the involution σ must be different from ± 1. On the other hand, if $\alpha \in D(\Sigma^+)$, $\alpha, \sigma\alpha, -\alpha, -\sigma\alpha$ are complex roots. Moreover, if α and $\sigma\alpha$ are not strongly orthogonal, Σ contains at least a pair of either imaginary or real roots. This implies that $\operatorname{Card} D(\Sigma^+)$ is either 2 or 4.

Assume first that $\operatorname{Card} D(\Sigma^+) = 2$. Therefore, by 5.3, there exists a set of positive roots of Langlands type $\Sigma^{+,L}$ in Σ such that $S = \Sigma^+ \cap (-\Sigma^{+,L})$ consists of only one root in $D(\Sigma^+)$. We can assume that $S = \{\alpha\}$. Let w be the element of the Weyl group of Σ with the property that $w(\Sigma^{+,L}) = \Sigma^+$. Then $S = \Sigma^+_w$. Since $\ell(w) = \operatorname{Card} \Sigma^+_w$, we see that $\alpha \in \Pi$ and $w = s_\alpha$. Therefore, $S \subset D(\Sigma^+) \cap \Pi$. The only ambiguity about S is in the case when $D(\Sigma^+) = \Pi$. This is possible only if both simple roots are of the same length, i.e., we have the following cases:

(i) Σ is of type $A_1 \times A_1$ and $\Pi = \{\alpha, \sigma\alpha\}$;
(ii) Σ is of type A_2 and $\Pi = \{\alpha, \sigma\alpha\}$.

Assume that $D(\Sigma^+) = 4$. Then Σ must contain at least eight complex roots, and if it contains only eight roots all pairs $\alpha, \sigma\alpha$ must be strongly orthogonal. This implies that Σ must be of type G_2, and $D(\Sigma^+)$ consists of a pair of short roots and a pair of long roots. The remaining four roots are two pairs of mutually orthogonal roots: a pair of real roots and a pair of imaginary roots. By 5.3, there exists a set of positive roots of Langlands type $\Sigma^{+,L}$ in Σ such that $S = \Sigma^+ \cap (-\Sigma^{+,L})$ consists of two roots in $D(\Sigma^+)$. Let w be the element of the Weyl group of Σ with the property that $w(\Sigma^{+,L}) = \Sigma^+$. Then $S = \Sigma^+_w$, and we see that S contains a simple root. We can assume that this root is α. Moreover, if β is the other simple root, $S = \{\alpha, s_\alpha\beta\}$ and $w = s_\alpha s_\beta$. If β is complex, $\beta \in D(\Sigma^+)$ and $\sigma(\Sigma^+) = \Sigma^+$, contradicting the existence of a positive real root. The same is true if β is an imaginary root. Therefore β is a real root. It follows that S is the uniquely determined subset of $D(\Sigma^+)$ which consists of the one complex simple root α and the root which is the reflection of the other simple root β with respect to α.

This proves:

Lemma 5.5. *Let Σ be a root system of rank 2 and $\Sigma^+ \subset \Sigma$ a set of positive roots. Let*

$$C = \{\alpha \in D(\Sigma^+) \mid \alpha \textit{ is a minimal element of } \{\alpha, \sigma\alpha\}\}.$$

If $w \in W$ is such that

$$\Sigma^+_w \cap \sigma(\Sigma^+_w) = \emptyset \textit{ and } \Sigma^+_w \cup \sigma(\Sigma^+_w) = D(\Sigma^+),$$

we have $\Sigma^+_w \subset C$. Moreover, $\Sigma^+_w = C$ except if $D(\Sigma^+) = \Pi$, i.e., except in the following cases:

(i) *Σ is of type $A_1 \times A_1$, $\Pi = \{\alpha, \sigma\alpha\}$;*
(ii) *Σ is of type A_2 and $\Pi = \{\alpha, \sigma\alpha\}$.*

Now we can discuss the general case. For each α in $D(\Sigma^+)$ we denote by Σ_α the smallest closed root subsystem containing α and $\sigma\alpha$. Clearly Σ_α is σ-invariant, hence the restriction of σ to the vector subspace V_α of V spanned by Σ_α defines an involution σ_α on the root system Σ_α. Thus $(\Sigma_\alpha, \sigma_\alpha)$ is a root system with involution of rank 2. We can define an ordering on Σ_α by $\Sigma^+_\alpha = \Sigma_\alpha \cap \Sigma^+$. Denote by Π_α the corresponding set of simple roots in Σ_α. If $\Sigma^{+,L}$ is a set of positive roots of Langlands type in Σ, $\Sigma^{+,L}_\alpha$ is a set of positive roots of Langlands type in Σ_α. Define

$$C(\Sigma^+) = \{\alpha \in D(\Sigma^+) \mid \alpha \text{ is minimal in } \{\alpha, \sigma\alpha\} \text{ with respect to } \Sigma^+_\alpha\}.$$

Now 5.5 combined with the preceding discussion implies:

Proposition 5.6. *Let $w \in W$ be such that*

$$\Sigma^+_w \cap \sigma(\Sigma^+_w) = \emptyset \textit{ and } \Sigma^+_w \cup \sigma(\Sigma^+_w) = D(\Sigma^+).$$

Then we have

$$\Sigma^+_w \subset C(\Sigma^+).$$

Moreover, $\{\alpha, \sigma\alpha\} \subset C(\Sigma^+)$ if and only if

(i) *Σ_α is of type $A_1 \times A_1$, $\Pi_\alpha = \{\alpha, \sigma\alpha\}$;*
(ii) *Σ_α is of type A_2 and $\Pi_\alpha = \{\alpha, \sigma\alpha\}$.*

The next result is a converse of 5.2.

Lemma 5.7. *Let $\Sigma^{+,L}$ be a set of positive roots of Langlands type in V and $\lambda \in V$ such that (L1) and (L2) hold. Then there exists a set of positive roots Σ^+ such that*

(AD) $(\alpha, \lambda) \leq 0$ *for all roots in* Σ^+*;*

(I) $\Sigma^+ \cap (-\Sigma^{+,L})$ *consists of complex roots satisfying* $\sigma\alpha \in \Sigma^+$*.*

Proof. Again, by continuity we can assume that λ is regular and $\lambda - \sigma\lambda$ is not orthogonal to any nonimaginary root. Then the set of all roots α satisfying $(\alpha, \lambda) \leq 0$ is a set of positive roots in Σ. Also, it contains all imaginary and real roots from $\Sigma^{+,L}$. Now (I) follows as in the proof of 5.2. □

We shall also need:

Lemma 5.8. *Let Σ^+ be a set of positive roots and $\lambda \in V$ such that*

(V1) $(\alpha, \lambda + \sigma\lambda) \leq 0$ *for all roots in* $\alpha \in \Sigma^+$ *such that* $\sigma\alpha \in \Sigma^+$*;*

(V2) $(\alpha, \lambda - \sigma\lambda) \geq 0$ *for all roots* $\alpha \in \Sigma^+$ *such that* $-\sigma\alpha \in \Sigma^+$*.*

Then there exists a set of positive roots $\Sigma^{+,L}$ of Langlands type such that

(DL1) $(\alpha, \lambda) \leq 0$ *for all imaginary roots in* $\Sigma^{+,L}$*;*

(DL2) $(\alpha, \lambda - \sigma\lambda) \geq 0$ *for all nonimaginary roots in* $\Sigma^{+,L}$*;*

(I) *all* $\alpha \in \Sigma^+ \cap (-\Sigma^{+,L})$ *are complex and satisfy:*

(I1) $\sigma\alpha \in \Sigma^+$*; and*

(I2) $(\alpha, \lambda) \leq 0$.

Proof. By continuity, we may assume that λ is regular, $\lambda + \sigma\lambda$ is not orthogonal to imaginary roots, and $\lambda - \sigma\lambda$ is not orthogonal to nonimaginary roots. Then we can define an order relation $\gg_+$ on V_+ by $\mu \gg_+ 0$ if $(\mu, \lambda + \sigma\lambda) \leq 0$ and an order relation $\gg_-$ on V_- by $\mu \gg_- 0$ if $(\mu, \lambda - \sigma\lambda) \geq 0$. Together they define the order relation $\gg_{-,+}$ on V. As before $\gg_{-,+}$ determines a set of positive roots $\Sigma^{+,L}$ of Langlands type. It satisfies the condition (DL2). Moreover, since

$$2(\alpha, \lambda) = (\alpha, \lambda + \sigma\lambda)$$

for any imaginary root α, we see that (DL1) holds. Since λ is regular,

$$\Sigma_I \cap \Sigma^+ = \Sigma_I \cap \Sigma^{+,L}.$$

Analogously, for any real root $\alpha \in \Sigma^{+,L}$,

$$2(\alpha, \lambda) = (\alpha, \lambda - \sigma\lambda) \geq 0$$

and $\alpha \in \Sigma^+$. Conversely, if α is a real root in Σ^+, it follows that $(\alpha, \lambda - \sigma\lambda) \geq 0$ and $\alpha \in \Sigma^{+,L}$. Therefore,

$$\Sigma_{\mathbb{R}} \cap \Sigma^+ = \Sigma_{\mathbb{R}} \cap \Sigma^{+,L}.$$

Hence, the roots in $\Sigma^+ \cap (-\Sigma^{+,L})$ are complex. Moreover, if $\alpha \in \Sigma^+ \cap (-\Sigma^{+,L})$ and $-\sigma\alpha \in \Sigma^+$, it would follow from (V2) that $(\alpha, \lambda - \sigma\lambda) \geq 0$, and from the definition of $\Sigma^{+,L}$ that $(\alpha, \lambda - \sigma\lambda) \leq 0$ which is impossible since α is not orthogonal to $\lambda - \sigma\lambda$. Therefore, $\sigma\alpha \in \Sigma^+$ for any $\alpha \in \Sigma^+ \cap (-\Sigma^{+,L})$. Finally

$$2(\alpha, \lambda) = (\alpha, \lambda + \sigma\lambda) + (\alpha, \lambda - \sigma\lambda) \leq 0,$$

because of (V1) and (DL2). □

Lemma 5.9. *Let $\Sigma^{+,L}$ be a set of positive roots of Langlands type and $\lambda \in V$ such that (DL1) and (DL2) hold. Then there exists a set of positive roots $\Sigma^{+,Z}$ of Zuckerman type such that*

(Z1) $(\alpha, \lambda) \geq 0$ *for all real roots in* $\Sigma^{+,Z}$*;*

(Z2) $(\alpha, \lambda + \sigma\lambda) \leq 0$ *for all nonreal roots in* $\Sigma^{+,Z}$.

(I) $\Sigma^{+,Z} \cap (-\Sigma^{+,L})$ *consists of complex roots and* $(\alpha, \lambda) \leq 0$ *for* $\alpha \in \Sigma^{+,Z} \cap (-\Sigma^{+,L})$.

Proof. To prove this statement argue as in the preceding argument, but replace the order $\gg_{-,+}$ with $\gg_{+,-}$. $\square$

Finally, we shall need the following simple result.

Lemma 5.10. *Let* Σ^+ *be a set of positive roots of Langlands type. Then:*

(i) *The set* $P = \Sigma_I \cup \Sigma^+$ *is a parabolic set of roots in* Σ.

(ii) *There exists* $v \in V_-$ *such that* $P = \{\alpha \in \Sigma \mid (\alpha, v) \geq 0\}$.

Proof. (i) Let $\alpha \in \Sigma_I$ and $\beta \in \Sigma^+ - \Sigma_I$ be such that $\alpha + \beta$ is a root. We have to show that $\alpha + \beta$ is positive. This is evident if α is positive. On the other hand, if α is negative, the root β is either complex or real, hence $\sigma\beta \in -\Sigma^+$. Assume that $\alpha + \beta \in -\Sigma^+$. Since $\alpha + \beta$ is not imaginary, $\alpha + \sigma\beta = \sigma(\alpha + \beta) \in \Sigma^+$, and $\sigma\beta = (\alpha + \sigma\beta) - \alpha \in \Sigma^+$, contradicting the preceding statement.

(ii) Let $u \in V$ be such that $\Sigma^+ = \{\alpha \in \Sigma \mid (\alpha, u) > 0\}$. Since Σ^+ is of Langlands type, for any positive nonimaginary root α we have $(\alpha, \sigma u) < 0$. If we put $v = u - \sigma u$, we have $v \in V_-$ and $(\alpha, v) > 0$ for any positive nonimaginary root α. Therefore, $(\alpha, v) \geq 0$ for any $\alpha \in P$. On the contrary, if $\alpha \notin P$, $-\alpha$ is in $\Sigma^+ - \Sigma_I$, hence $(\alpha, v) < 0$. $\square$

6. K-orbits in the flag variety

A K-orbit in X can be viewed as a K-conjugacy class of Borel subalgebras in $\mathfrak{g}$. The following result is due to Matsuki [14].

Lemma 6.1. *Let* $\mathfrak{b}$ *be a Borel subalgebra of* $\mathfrak{g}$ *and* N *the unipotent radical of the Borel subgroup* B *of* $G = \mathrm{Int}(\mathfrak{g})$ *corresponding to* $\mathfrak{b}$. *Then the algebra* $\mathfrak{b}$ *contains a* σ*-stable Cartan subalgebra* $\mathfrak{h}_0$. *All such Cartan subalgebras are conjugate by* $K \cap N$.

Let Q be a K-orbit in X and $x \in Q$. Then, by 6.1, Q determines a K-conjugacy class of σ-stable Cartan subalgebras in $\mathfrak{g}$. Therefore, we have a map from the set of K-orbits in X onto the set of K-conjugacy classes of σ-stable Cartan subalgebras in $\mathfrak{g}$; in particular the latter set is finite. Let $\mathfrak{c}$ be a σ-stable Cartan subalgebra in $\mathfrak{g}$, and let R be the root system of $(\mathfrak{g}, \mathfrak{c})$ in $\mathfrak{c}^*$. Any choice of positive roots R^+ in R determines a Borel subalgebra, spanned by $\mathfrak{c}$ and the root subspaces corresponding to the roots in R^+, and thus determines a K-orbit in X. Assume that two such choices of positive roots define Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$ lying in the same K-orbit in X. Choose $k \in K$ such that $\mathrm{Ad}\, k(\mathfrak{b}') = \mathfrak{b}$. Then $\mathrm{Ad}\, k(\mathfrak{c})$ is a σ-stable Cartan subalgebra which is contained in $\mathfrak{b}$. By 6.1, there is $u \in K \cap N$ such that $\mathrm{Ad}\, k(\mathfrak{c}) = \mathrm{Ad}\, u(\mathfrak{c})$, i.e., $k' = u^{-1}k \in K$ lies in the normalizer $N_K(\mathfrak{c})$ of $\mathfrak{c}$ in K, and

$$\mathfrak{b}' = \mathrm{Ad}(k^{-1})(\mathfrak{b}) = \mathrm{Ad}({k'}^{-1}u^{-1})(\mathfrak{b}) = \mathrm{Ad}({k'}^{-1})(\mathfrak{b}).$$

Therefore, the K-orbits in X which map into the K-conjugacy class of $\mathfrak{c}$ are parametrized by the conjugacy classes of positive root systems in R with respect to $N_K(\mathfrak{c})$. To summarize:

Observation 6.2. (i) *Each* K*-orbit in* X *is attached to a unique* K*-conjugacy class of* σ*-stable Cartan subalgebras.*

(ii) *Let $\mathfrak{c}$ be a σ-stable Cartan subalgebra. Then the K-orbits corresponding to the K-conjugacy class of $\mathfrak{c}$ are parametrized bijectively by the $N_K(\mathfrak{c})$-orbits of sets of positive roots R^+ for $(\mathfrak{g}, \mathfrak{c})$.*

Let Q be a K-orbit in X, x a point of Q, and $\mathfrak{c}$ a σ-stable Cartan subalgebra contained in $\mathfrak{b}_x$. Then σ induces an involution on the root system R in $\mathfrak{c}^*$. Let R^+ be the set of positive roots determined by $\mathfrak{b}_x$. The specialization map from the Cartan triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ into the triple $(\mathfrak{c}^*, R, R^+)$ pulls back σ to an involution of Σ. From the construction, one sees that this involution on Σ depends only on the orbit Q, so we denote it by σ_Q. Let $\mathfrak{h} = \mathfrak{t}_Q \oplus \mathfrak{a}_Q$ be the decomposition of $\mathfrak{h}$ into σ_Q-eigenspaces for the eigenvalue 1 and -1. Under the specialization map this corresponds to the decomposition $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ of $\mathfrak{c}$ into σ-eigenspaces for the eigenvalue 1 and -1. As we discussed in §5, we can divide the roots in (Σ, σ_Q) into imaginary, real and complex roots. This division depends on the orbit Q, hence we have

$$\begin{aligned}\Sigma_{Q,I} &= Q\text{-imaginary roots},\\ \Sigma_{Q,\mathbb{R}} &= Q\text{-real roots},\\ \Sigma_{Q,\mathbb{C}} &= Q\text{-complex roots}.\end{aligned}$$

Via specialization, these roots correspond to imaginary, real and complex roots in the root system R in $\mathfrak{c}^*$.

Put

$$D_+(Q) = \{\alpha \in \Sigma^+ \mid \sigma_Q\alpha \in \Sigma^+,\ \sigma_Q\alpha \neq \alpha\};$$

then $D_+(Q)$ is σ_Q-invariant and consists of Q-complex roots. Each σ_Q-orbit in $D_+(Q)$ consists of two roots, hence $d(Q) = \operatorname{Card} D_+(Q)$ is even. The complement of the set $D_+(Q)$ in the set of all positive Q-complex roots is

$$D_-(Q) = \{\alpha \in \Sigma^+ \mid -\sigma_Q\alpha \in \Sigma^+, \sigma_Q\alpha \neq -\alpha\}.$$

In addition, for imaginary $\alpha \in R$, $\sigma\alpha = \alpha$ and the root subspace $\mathfrak{g}_\alpha$ is σ-invariant. Therefore, σ acts on it either as 1 or as -1. In the first case $\mathfrak{g}_\alpha \subset \mathfrak{k}$ and α is a *compact imaginary* root, in the second case $\mathfrak{g}_\alpha \not\subset \mathfrak{k}$ and α is a *noncompact imaginary* root. We denote by R_{CI} and R_{NI} the sets of compact, resp. noncompact, imaginary roots in R. Also, we denote the corresponding sets of roots in Σ by $\Sigma_{Q,CI}$ and $\Sigma_{Q,NI}$.

Lemma 6.3. (i) *The Lie algebra $\mathfrak{k}$ is the direct sum of $\mathfrak{t}$, the root subspaces $\mathfrak{g}_\alpha$ for compact imaginary roots α, and the σ-eigenspaces of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha}$ for the eigenvalue* 1 *for real and complex roots α.*

(ii) *The Lie algebra $\mathfrak{k} \cap \mathfrak{b}_x$ is spanned by $\mathfrak{t}$, $\mathfrak{g}_\alpha$ for positive compact imaginary roots α, and the σ-eigenspaces of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha}$ for the eigenvalue* 1 *for complex roots $\alpha \in R^+$ with $\sigma\alpha \in R^+$.*

Proof. Evident. □

Lemma 6.4. *Let Q be a K-orbit in X. Then*

$$\dim Q = \frac{1}{2}(\operatorname{Card}\Sigma_{Q,CI} + \operatorname{Card}\Sigma_{Q,\mathbb{R}} + \operatorname{Card}\Sigma_{Q,\mathbb{C}} - d(Q)).$$

Proof. The tangent space to Q at x can be identified with $\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{b}_x)$. By 6.3,

$$\begin{aligned}\dim Q &= \dim\mathfrak{k} - \dim(\mathfrak{k} \cap \mathfrak{b}_x)\\ &= \operatorname{Card}\Sigma_{Q,CI} + \frac{1}{2}(\operatorname{Card}\Sigma_{Q,\mathbb{R}} + \operatorname{Card}\Sigma_{Q,\mathbb{C}}) - \frac{1}{2}\operatorname{Card}\Sigma_{Q,CI} - \frac{1}{2}d(Q).\end{aligned}$$

□

By 6.4, since $D_+(Q)$ consists of at most half of all Q-complex roots, the dimension of K-orbits attached to $\mathfrak{c}$ lies between

$$\frac{1}{2}(\operatorname{Card}\Sigma_{Q,CI} + \operatorname{Card}\Sigma_{Q,\mathbb{R}} + \frac{1}{2}\operatorname{Card}\Sigma_{Q,\mathbb{C}})$$

and

$$\frac{1}{2}(\operatorname{Card}\Sigma_{Q,CI} + \operatorname{Card}\Sigma_{Q,\mathbb{R}} + \operatorname{Card}\Sigma_{Q,\mathbb{C}}).$$

The first, minimal, value is attained if Q corresponds to a set of positive roots R^+ of Zuckerman type. We call such orbits *Zuckerman orbits* attached to $\mathfrak{c}$. The second, maximal, value is attained on the K-orbits corresponding to sets of positive roots of Langlands type. We call those orbits *Langlands orbits* attached to $\mathfrak{c}$. As we have shown in 5.1, there exist both Langlands and Zuckerman orbits attached to $\mathfrak{c}$.

The following simple observation will play a critical role later. Let $\alpha \in \Pi$ and X_α be the generalized flag variety of $\mathfrak{g}$ of parabolic subalgebras of type α. Denote by p_α the natural projection of X onto X_α, which maps a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ into the parabolic subalgebra of type α containing $\mathfrak{b}$. Let Q be a K-orbit in X and $V = p_\alpha^{-1}(p_\alpha(Q))$. Then V is an union of finitely many K-orbits. Let $x \in Q$ and $y = p_\alpha(x)$. Let P_y be the parabolic subgroup of G of type α which stabilizes y and $\mathfrak{p}_y$ its Lie algebra. Let U be the unipotent radical of P_y. Then the quotient of P_y/U by its center is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$. Denote by τ the corresponding homomorphism of P_y into $\mathrm{PSL}(2,\mathbb{C})$. The differential of τ defines an isomorphism of the fiber $p_\alpha^{-1}(y)$, i.e., the set of Borel subalgebras of $\mathfrak{g}$ contained in $\mathfrak{p}_y$, with the flag variety $X_o = \mathbb{P}^1$ of $\mathfrak{sl}(2,\mathbb{C})$. Also, $\varphi^{-1}(\varphi(K) \cap \ker\tau)$ is a normal subgroup of the closed subgroup $\varphi^{-1}(\varphi(K) \cap P_y)$ of K. Therefore, we have a natural homomorphism φ_o of the group

$$K_o = \varphi^{-1}(\varphi(K) \cap P_y)/\varphi^{-1}(\varphi(K) \cap \ker\tau)$$

into $\mathrm{PSL}(2,\mathbb{C})$.

Lemma 6.5. (i) $(\mathfrak{sl}(2,\mathbb{C}), K_o)$ *is a Harish-Chandra pair.*

(ii) *The identification of X_o and the fiber $p_\alpha^{-1}(y)$ identifies K_o-orbits in X_o with the intersections of K-orbits in V with $p_\alpha^{-1}(y)$.*

(iii) *If α is a compact Q-imaginary root, the identity component of K_o is a covering of* $\mathrm{PSL}(2,\mathbb{C})$. *The orbit Q is equal to V.*

(iv) *If α is a noncompact Q-imaginary root or a Q-real root, the identity component of K_o is a one dimensional torus. In the first case,* $\dim Q = \dim V - 1$, *in the second* $\dim Q = \dim V$. *The variety V is a union of two or three K-orbits.*

(v) *If α is a Q-complex root, the unipotent radical of K_o is one dimensional. The variety V is a union of two K-orbits,* $\dim Q = \dim V$ *if* $\sigma_Q\alpha \notin \Sigma^+$, *and* $\dim Q = \dim V - 1$ *if* $\sigma_Q\alpha \in \Sigma^+$. *In the second case,* $p_\alpha : Q \longrightarrow p_\alpha(Q)$ *is an isomorphism.*

Proof. (ii) Let Q_o be a K_o-orbit in $X_o \cong p_\alpha^{-1}(y)$. Then Q_o is contained in a K-orbit O. Let $x' \in O \cap X_o$. Then there exists $k \in K$ such that $k \cdot x' \in Q_o$. Moreover, $k \cdot y = k \cdot p_\alpha(x') = p_\alpha(k \cdot x') = y$ implies that $k \in \varphi^{-1}(\varphi(K) \cap P_y)$, which yields $x' \in Q_o$. Therefore, $Q_o = O \cap X_o$.

(i) follows from (ii).

To prove the remaining statements, we calculate the Lie algebra of K_o. Using the notation of 6.3, we see that $\mathfrak{k} \cap \mathfrak{p}_y$ is spanned by $\mathfrak{t}$, $\mathfrak{g}_\beta$ for positive compact roots

β, σ-eigenspaces of $\mathfrak{g}_\beta \oplus \mathfrak{g}_{\sigma\beta}$ for the eigenvalue 1 for complex roots $\beta \in R^+$ with $\sigma\beta \in R^+$, and either $\mathfrak{g}_{-\alpha}$ if α is compact imaginary, or σ-eigenspaces of $\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\sigma\alpha}$ for the eigenvalue 1 if α is a complex root such that $\sigma\alpha \notin R^+$.

If α is compact imaginary in R, $\mathfrak{k} \cap \mathfrak{p}_y$ has a Levi factor that contains $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Therefore, the Lie algebra of K_o in this case must be $\mathfrak{sl}(2, \mathbb{C})$. This completes the proof of (iii).

If α is noncompact imaginary or real, the Lie algebra of K_o is the image of $\mathfrak{t}$ under the differential of τ. By (i), K_o must be at least one-dimensional and from 4.1 we conclude that its identity component is an one-dimensional torus. An application of 4.1 and (ii) completes the proof of (iv).

If α is complex, the Lie algebra of K_o is solvable and contains the image under the differential of τ of either $\mathfrak{g}_\alpha$ if $\sigma\alpha \in R^+$, or $\mathfrak{g}_{-\alpha}$ if $\sigma\alpha \notin R^+$. Therefore, the unipotent radical of K_o is one dimensional by (i) and 4.1, and K_o acts on X_o with two orbits. By (ii), this implies that V contains two K-orbits. Applying 4.1 again we see that in the first case K_o stabilizes x, $Q \cap X_o = \{x\}$, and $\varphi^{-1}(\varphi(K) \cap P_y) = \varphi^{-1}(\varphi(K) \cap B_x)$; in the second case K_o does not stabilize x, and Q is the open orbit in V. □

Let w be transversal to a K-orbit Q. Then $E_w(Q)$ is K-invariant. Since it is irreducible by 3.1.(iv), and the number of K-orbits is finite, there exists a unique K-orbit Q_w of maximal dimension in $E_w(Q)$. The next result reduces the analysis of elements of W transversal to a K-orbits to simple reflections.

Lemma 6.6. *Let $w, v \in W$ be such that $\ell(wv) = \ell(w) + \ell(v)$, and Q a K-orbit in X. Then the following conditions are equivalent:*

(i) *wv is transversal to Q;*
(ii) *v is transversal to Q and w is transversal to Q_v.*

If these conditions are satisfied, $Q_{wv} = (Q_v)_w$.

Proof. Assume that wv is transversal to Q. Then, by 3.2, v is transversal to Q and w is transversal to $E_v(Q)$. Since Q_v is dense in $E_v(Q)$, $E_v(Q) \subset \overline{Q_v}$. Hence, by 3.1.(ii) and 3.1.(v),

$$E_{wv}(Q) = E_w(E_v(Q)) \subset E_w(\overline{Q_v}) = \overline{E_w(Q_v)}.$$

This implies

$$\dim Q + \ell(wv) = \dim E_{wv}(Q) \leq \dim E_w(Q_v) \leq \dim Q_v + \ell(w) \leq \dim Q + \ell(v) + \ell(w),$$

hence the inequalities must be equalities. Therefore, w is transversal to Q_v. In addition this implies that the K-orbit Q_{wv} is open in $E_w(Q_v)$, i.e., $Q_{wv} = (Q_v)_w$.

If v is transversal to Q and w is transversal to Q_v, by 3.2.(ii) and 3.1.(ii), v is transversal to $\bar{Q}$ and w is transversal to $\overline{Q_v} = \overline{E_v(Q)} = E_v(\bar{Q})$. By 3.2.(ii), it follows that wv is transversal to Q. □

The case of simple reflections is treated in the following result.

Lemma 6.7. *Let Q be a K-orbit and α a simple root. Then s_α is transversal to Q if and only if α is either noncompact Q-imaginary or Q-complex satisfying $\sigma_Q\alpha \in \Sigma^+$.*

Proof. By our definition, $E_{s_\alpha}(Q) = p_\alpha^{-1}(p_\alpha(Q))$, hence s_α is transversal to Q if and only if Q is of codimension one in $p_\alpha^{-1}(p_\alpha(Q))$. By 6.5.(iii) and (iv), if α is compact Q-imaginary of Q-real, $\dim p_\alpha^{-1}(p_\alpha(Q)) = \dim Q$, hence s_α is not transversal to Q. On the other hand, by 6.5.(iv), if α is noncompact Q-imaginary, s_α is transversal to Q. If α is Q-complex, s_α is transversal to Q if and only if $\sigma_Q\alpha$ is a positive root by 6.5.(v). □

Assume first that $\alpha \in \Pi$ is noncompact Q-imaginary. Then $E_{s_\alpha}(Q)$ is the union of two or three K-orbits by 6.5.(iv). Fix $x \in Q$. Let $\mathfrak{c}$ be a σ-stable Cartan subalgebra in $\mathfrak{b}_x$. Then roots α and $-\alpha$ via specialization determine root subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ of $\mathfrak{g}$. Let $\mathfrak{s}_\alpha$ be the subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{c}$. Then $\mathfrak{s}_\alpha$ is σ-stable, since σ acts as -1 on $\mathfrak{g}_\alpha$. Let σ_α be the restriction of σ to $\mathfrak{s}_\alpha$. Therefore, $\mathfrak{l}_\alpha = \mathfrak{s}_\alpha + \mathfrak{c}$ is a σ-stable Levi factor of the parabolic subalgebra $\mathfrak{p}$ of type α which contains $\mathfrak{b}_x$. This parabolic subalgebra corresponds to the point $p_\alpha(x)$ in the generalized flag variety X_α. Let $\xi_\alpha \in \mathfrak{g}_\alpha$ and $\xi_{-\alpha} \in \mathfrak{g}_{-\alpha}$. Then $\xi_\alpha - \xi_{-\alpha}$ is a semisimple element in $\mathfrak{s}_\alpha$ and

$$\sigma(\xi_\alpha - \xi_{-\alpha}) = -(\xi_\alpha - \xi_{-\alpha}).$$

Therefore, the kernel of the root α in $\mathfrak{c}$ and the line spanned by $\xi_\alpha - \xi_{-\alpha}$ span another σ-stable Cartan subalgebra in $\mathfrak{g}$, which we denote by $\mathfrak{d}$. The σ-invariant vectors in $\mathfrak{d}$ are the subspace of codimension 1 in the σ-invariants of $\mathfrak{c}$. Therefore, $\mathfrak{c}$ and $\mathfrak{d}$ are not K-conjugate. Since $\mathfrak{d} \subset \mathfrak{p}$, there exists a Borel subalgebra $\mathfrak{b}_{x'}$ containing $\mathfrak{d}$ which lies inside $\mathfrak{p}$. The point x' lies in a K-orbit which projects onto $p_\alpha(Q)$ in X_α. In the notation of 6.5, the fiber over $y = p_\alpha(x)$ can be viewed as the flag variety X_o of $\mathfrak{sl}(2, \mathbb{C})$. Since K_o is an one dimensional torus by the discussion in 6.5, by the results of §4 it follows that $\mathfrak{c} \cap \mathfrak{s}_\alpha$ is the only σ_α-stable Cartan subalgebra in $\mathfrak{s}_\alpha$ on which σ_α acts as identity. The representative of the other class of σ_α-stable Cartan subalgebras is $\mathfrak{d} \cap \mathfrak{s}_\alpha$. The involution σ_α acts on it as -1, and it corresponds to the open orbit in X_o. Hence the K-orbit of x' is open in $p_\alpha^{-1}(p_\alpha(Q))$, i.e., this orbit is Q_{s_α}.

From the construction it is clear that the involutions σ_Q and $\sigma_{Q_{s_\alpha}}$ agree on $\ker \alpha$. On the other hand, on the complementary line spanned by $\alpha\check{}$, σ_Q acts as 1 and $\sigma_{Q_{s_\alpha}}$ as -1. Therefore,

$$\sigma_{Q_{s_\alpha}} = s_\alpha \circ \sigma_Q = \sigma_Q \circ s_\alpha.$$

It follows that α is a Q_α-real root.

Hence, we established the following fact.

Lemma 6.8. *Let $\alpha \in \Pi$ be a noncompact Q-imaginary root. Then*

(i) $\sigma_{Q_{s_\alpha}} = s_\alpha \circ \sigma_Q = \sigma_Q \circ s_\alpha$;
(ii) *α is Q_{s_α}-real.*

Now we want to discuss elements of W transversal to K-orbits and which are products of complex simple reflections only. Let $w \in W$ and Z_w be the subvariety of $X \times X$ consisting of pairs of Borel subalgebras in relative position w. Denote by p_1, p_2, the projections of Z_w onto the first, resp. second, factor in $X \times X$. As we mentioned in §2, $p_2 : Z_w \longrightarrow X$ is a locally trivial fibration with fibres isomorphic to $\mathbb{C}^{\ell(w)}$. Therefore, for any K-orbit Q in X, $p_2^{-1}(Q)$ is a smooth K-invariant subvariety of Z_w. Recall the notation established in §2.

Lemma 6.9. *Let Q be a K-orbit in X attached to a σ-stable Cartan subalgebra $\mathfrak{c}$ and a set of positive roots R^+ in $\mathfrak{c}^*$. Let $w \in W$. Assume that $\Sigma_w^+ \subset D_+(Q)$ and $\Sigma_w^+ \cap \sigma_Q(\Sigma_w^+) = \emptyset$. Then:*

(i) *$p_2^{-1}(Q)$ is a K-orbit in Z_w;*

(ii) *the projection* p_1 *induces an isomorphism of* $p_2^{-1}(Q)$ *onto the* K*-orbit* $p_1(p_2^{-1}(Q))$ *in* X*, which is attached to* $\mathfrak{c}$ *and the set of positive roots in* R *corresponding to* $w^{-1}(\Sigma^+)$ *under the specialization determined by* Q*.*

By 3.1.(iii), the K-orbit $p_1(p_2^{-1}(Q))$ is dense in $E_w(Q)$. Hence,

$$\dim E_w(Q) = \dim p_1(p_2^{-1}(Q)) = \dim Q + \ell(w),$$

and w is transversal to Q. It follows that $Q_w = p_1(p_2^{-1}(Q))$.

We prove this statement by induction on $\ell(w)$. Assume first that $\ell(w) = 1$, i.e., $w = s_\alpha$ for some simple root α. Then $s_\alpha(\Sigma^+) = (\Sigma^+ - \{\alpha\}) \cup \{-\alpha\}$, hence $\Sigma^+_{s_\alpha} = \{\alpha\}$ and the only condition is that $\alpha \in D_+(Q)$. Let $x \in Q$ and $y = p_\alpha(x)$ as before. The fiber of $p_2 : Z_{s_\alpha} \longrightarrow X$ at x consists of all pairs $(x', x) \in X \times X$ such that $\mathfrak{b}'_x$ and $\mathfrak{b}_x$ are in relative position s_α. This is equivalent to $x' \neq x$ and $p_\alpha(x') = p_\alpha(x)$. To prove (i), it is enough to show that the stabilizer $\varphi^{-1}(\varphi(K) \cap B_x)$ of x in K acts transitively on this fiber. Since α is Q-complex and $\sigma_Q\alpha \in \Sigma^+$, by 6.5.(v), $K_o = \varphi^{-1}(\varphi(K) \cap B_x) = \varphi^{-1}(\varphi(K) \cap P_y)$, and this group acts transitively on $\{x' \in X \mid p_\alpha(x') = y, x' \neq x\}$.

Let $(x, x'), (x, x'') \in p_2^{-1}(Q)$. Then $p_\alpha(x') = p_\alpha(x) = p_\alpha(x'')$, so $x' = x''$ since $p_\alpha : Q \longrightarrow p_\alpha(Q)$ is a bijection by 6.5.(v). This proves (ii) in this situation.

Now we can prove the result for an arbitrary w by induction on $\ell(w)$. Assume that the statement holds for all $w' \in W$ such that $\ell(w') < k$, and that w satisfies $\ell(w) = k$. Let $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}$ be a reduced expression of w. Denote $w' = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_{k-1}}$. Then $\ell(w') = k - 1$ and $w = w' s_{\alpha_k}$. Moreover, as we remarked in §2, we see that $\Sigma^+_w = s_{\alpha_k}(\Sigma^+_{w'}) \cup \{\alpha_k\}$, and this union is disjoint. So $\alpha_k \in D_+(Q)$, and by the first part of the proof, 6.9 holds for s_{α_k}. Hence, the K-orbit $Q_{s_{\alpha_k}}$ is attached to $\mathfrak{c}$ and the set of positive roots $s_{\alpha_k}(R^+)$. The specializations of $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ to $(\mathfrak{c}^*, R, R^+)$ and $(\mathfrak{c}^*, R, s_{\alpha_k}(R^+))$ differ by s_{α_k}. Therefore,

$$\sigma_{Q_{s_{\alpha_k}}} = s_{\alpha_k} \circ \sigma_Q \circ s_{\alpha_k}.$$

Since $\sigma_Q(\Sigma^+_w) \cap \Sigma^+_w = \emptyset$, we have

$$\emptyset = \sigma_Q(s_{\alpha_k}(\Sigma^+_{w'})) \cap s_{\alpha_k}(\Sigma^+_{w'}) = s_{\alpha_k}(\sigma_{Q_{s_{\alpha_k}}}(\Sigma^+_{w'}) \cap \Sigma^+_{w'}),$$

and $\sigma_{Q_{s_{\alpha_k}}}(\Sigma^+_{w'}) \cap \Sigma^+_{w'} = \emptyset$. Before we complete the proof of 6.9 we need to describe $D_+(Q_{s_{\alpha_k}})$.

Lemma 6.10. *Let* $\alpha \in D_+(Q)$ *be a simple root. Then*

$$s_\alpha(D_+(Q_{s_\alpha})) = D_+(Q) - \{\alpha, \sigma_Q\alpha\}.$$

Proof. Let $\beta \in D_+(Q)$, different from α and $\sigma_Q\alpha$. Then $\sigma_Q\beta \neq \alpha$, hence $s_\alpha(\beta) \in \Sigma^+$ and $s_\alpha(\sigma_Q\beta) \in \Sigma^+$. It follows that $s_\alpha(\beta)$ and $\sigma_{Q_{s_\alpha}}(s_\alpha(\beta)) = (s_\alpha\sigma_Q)(\beta)$ are contained in Σ^+. Therefore, $D_+(Q) - \{\alpha, \sigma_Q\alpha\} \subset s_\alpha(D_+(Q_{s_\alpha}))$.

Clearly,

$$\sigma_{Q_{s_\alpha}}(\alpha) = (s_\alpha\sigma_Q s_\alpha)(\alpha) = -s_\alpha(\sigma_Q\alpha).$$

Since $\sigma_Q\alpha \in \Sigma^+$ is different from α, it follows that $s_\alpha(\sigma_Q\alpha) \in \Sigma^+$ and $\sigma_{Q_{s_\alpha}}(\alpha) \in -\Sigma^+$. Therefore, $\alpha \notin D_+(Q_{s_\alpha})$. Let $\beta \in s_\alpha(D_+(Q_{s_\alpha}))$. Since $\alpha \notin D_+(Q_{s_\alpha})$, $\beta \in \Sigma^+$. Also $s_\alpha(\beta) \in D_+(Q_{s_\alpha})$, i.e., $s_\alpha(\beta) \in \Sigma^+$ and $\sigma_{Q_{s_\alpha}}(s_\alpha(\beta)) = (s_\alpha\sigma_Q)(\beta) \in \Sigma^+$. Assume that $(s_\alpha s_Q)(\beta) = \alpha$. This would imply that $\beta = -\sigma_Q\alpha \in -\Sigma^+$ what contradicts the preceding statement. Therefore, $(s_\alpha\sigma_Q)(\beta) \neq \alpha$ and $\sigma_Q(\beta) \in \Sigma^+$. This implies that $\beta \in D_+(Q)$. Since $D_+(Q_{s_\alpha})$ is a set of positive roots,

$s_\alpha(D_+(Q_{s_\alpha}))$ cannot contain α. If $\sigma_{Q_{s_\alpha}}\alpha$ would be in $s_\alpha(D_+(Q_{s_\alpha}))$, this would imply that

$$-\sigma_{Q_{s_\alpha}}(\alpha) = -(s_\alpha \sigma_Q s_\alpha)(\alpha) = s_\alpha(\sigma_Q \alpha) \in D_+(Q_{s_\alpha})$$

and $-\alpha = -\sigma_{Q_{s_\alpha}}(\sigma_{Q_{s_\alpha}}(\alpha)) \in D_+(Q_{s_\alpha})$, which is again impossible. $\square$

We now resume the proof of 6.9. Since $\sigma_Q(\Sigma_w^+) \cap \Sigma_w^+ = \emptyset$, $\sigma_Q \alpha_k \notin s_{\alpha_k}(\Sigma_{w'}^+)$. By 6.10,

$$s_{\alpha_k}(\Sigma_{w'}^+) \subset \Sigma_w^+ - \{\alpha_k, \sigma_Q\alpha_k\} \subset D_+(Q) - \{\alpha_k, \sigma_Q \alpha_k\} \subset s_{\alpha_k}(D_+(Q_{s_{\alpha_k}})),$$

and $\Sigma_{w'}^+ \subset D_+(Q_{s_{\alpha_k}})$. Therefore w' satisfies the conditions of 6.9 with respect to the K-orbit $Q_{s_{\alpha_k}}$.

Now the induction step. Let ${p'}_1$, ${p'}_2$ be the projections of $Z_{w'}$ onto the first, resp. second, factor in $X \times X$. Denote the corresponding projections for $Z_{s_{\alpha_k}}$ by ${p''}_1$ and ${p''}_2$. Since $\ell(w) = \ell(w') + 1 = \ell(w') + \ell(s_{\alpha_k})$, as we remarked in §2, the natural map from the fibered product $r : Z_{w'} \times_X Z_{s_{\alpha_k}} \longrightarrow Z_w$, given by $r((x, x'), (x', x'')) = (x, x'')$, is an isomorphism of varieties. It maps ${p'}_2^{-1}(Q_{s_{\alpha_k}}) \times_{Q_{s_{\alpha_k}}} {p''}_2^{-1}(Q)$ onto $p_2^{-1}(Q)$. By the first step of the proof, the projection of ${p'}_2^{-1}(Q_{s_{\alpha_k}}) \times_{Q_{s_{\alpha_k}}} {p''}_2^{-1}(Q)$ onto ${p'}_2^{-1}(Q_{s_{\alpha_k}})$ is a K-equivariant bijection, hence ${p'}_2^{-1}(Q_{s_{\alpha_k}}) \times_{Q_{s_{\alpha_k}}} {p''}_2^{-1}(Q)$ is a K-orbit. This implies that $p_2^{-1}(Q)$ is a K-orbit. Its projection $p_1(p_2^{-1}(Q))$ is equal to the projection ${p'}_1({p'}_2^{-1}(Q_{s_{\alpha_k}}))$, i.e., to the K-orbit $(Q_{s_{\alpha_k}})_{w'} = Q_w$, and the projection map is an isomorphism of $p_2^{-1}(Q)$ onto Q_w. This ends the proof of 6.9.

Another consequence of this inductive analysis gives the following proposition, which is a generalization of 6.10. First we remark that

$$\sigma_{Q_w} = w \circ \sigma_Q \circ w^{-1}.$$

This is evident if w is a simple reflection. On the other hand,

$$\sigma_{Q_w} = \sigma_{(Q_{s_{\alpha_k}})_{w'}} = w' \circ \sigma_{Q_{s_{\alpha_k}}} \circ {w'}^{-1} = w' s_{\alpha_k} \circ \sigma_Q \circ s_{\alpha_k} {w'}^{-1} = w \circ \sigma_Q \circ w^{-1}.$$

Proposition 6.11. *Let Q be a K-orbit and $w \in W$. Assume that $\Sigma_w^+ \subset D_+(Q)$ and $\Sigma_w^+ \cap \sigma_Q(\Sigma_w^+) = \emptyset$. Then*

$$\sigma_{Q_w} = w \circ \sigma_Q \circ w^{-1},$$
$$w^{-1} D_+(Q_w) = D_+(Q) - (\Sigma_w^+ \cup \sigma_Q(\Sigma_w^+)).$$

and

$$D_-(Q_w) = wD_-(Q) \cup \Sigma_{w^{-1}}^+ \cup \left(-\sigma_{Q_w}(\Sigma_{w^{-1}}^+)\right).$$

Proof. We prove this statement by induction in $\ell(w)$. If $\ell(w) = 1$ this is the statement of 6.10. Assume that $\ell(w) = k$, with $k > 1$. Let $w' \in W$ be such that $\ell(w') = k - 1$ and $w = w' s_\alpha$. Then, as we remarked in §2, $\Sigma_w^+ = s_\alpha(\Sigma_{w'}^+) \cup \{\alpha\}$, and this union is disjoint. As we checked in the preceding argument, w' satisfies the conditions of the proposition with respect to the orbit Q_{s_α}, hence by the induction assumption we have

$${w'}^{-1} D_+(Q_w) = {w'}^{-1} D_+((Q_{s_\alpha})_{w'}) = D_+(Q_{s_\alpha}) - (\Sigma_{w'}^+ \cup \sigma_Q \Sigma_{w'}^+).$$

This implies that

$$\begin{aligned} w^{-1}D_+(Q_w) &= s_\alpha {w'}^{-1} D_+(Q_w) \\ &= s_\alpha D_+(Q_{s_\alpha}) - (s_\alpha \Sigma^+_{w'} \cup \sigma_Q s_\alpha \Sigma^+_{w'}) = D_+(Q) - (\Sigma^+_w \cup \sigma_Q \Sigma^+_w). \end{aligned}$$

On the other hand, since $w^{-1}D_+(Q_w) \subset \Sigma^+ \cap w^{-1}(\Sigma^+)$, we see that $D_+(Q_w) \subset \Sigma^+ \cap w(\Sigma^+)$. It follows that

$$\begin{aligned} D_-(Q_w) &= w(\Sigma_{Q,\mathbb{C}}) \cap \Sigma^+ - D_+(Q_w) \\ &= \big(w(\Sigma_{Q,\mathbb{C}}) \cap \Sigma^+ \cap w(\Sigma^+) - D_+(Q_w)\big) \cup \big(w(\Sigma_{Q,\mathbb{C}}) \cap \Sigma^+_{w^{-1}}\big) \\ &= w\big(\Sigma_{Q,\mathbb{C}} \cap w^{-1}(\Sigma^+) \cap \Sigma^+ - w^{-1}D_+(Q_w)\big) \cup w\big(\Sigma_{Q,\mathbb{C}} \cap (-\Sigma^+_w)\big) \\ &= w\big(\Sigma_{Q,\mathbb{C}} \cap \Sigma^+ - (w^{-1}D_+(Q_w) \cup \Sigma^+_w)\big) \cup w(-\Sigma^+_w) \\ &= wD_-(Q) \cup \sigma_Q(\Sigma^+_w) \cup \Sigma^+_{w^{-1}} \\ &= wD_-(Q) \cup \Sigma^+_{w^{-1}} \cup \big(-\sigma_{Q_w}(\Sigma^+_{w^{-1}})\big). \end{aligned}$$

□

In particular, if Q is a Zuckerman orbit, we get the following result.

Corollary 6.12. *Let Q be a Zuckerman orbit and $w \in W$ such that Σ^+_w consists of Q-complex roots and $\Sigma^+_w \cap \sigma_Q(\Sigma^+_w) = \emptyset$. Then*

$$D_-(Q_w) = \Sigma^+_{w^{-1}} \cup \big(-\sigma_{Q_w}\big(\Sigma^+_{w^{-1}}\big)\big).$$

This finally leads to the following statement.

Proposition 6.13. *Let Q_1 be an arbitrary K-orbit in X and $w \in W$. Then the following conditions are equivalent:*

(i) *there exist a Zuckerman orbit Q attached to the same conjugacy class of Cartan subalgebras such that Σ^+_w consists of Q-complex roots, $\Sigma^+_w \cap \sigma_Q(\Sigma^+_w) = \emptyset$ and $Q_1 = Q_w$.*
(ii) *$\Sigma^+_{w^{-1}} \cap \big(-\sigma_{Q_1}(\Sigma^+_{w^{-1}})\big) = \emptyset$ and $D_-(Q_1) = \Sigma^+_{w^{-1}} \cup \big(-\sigma_{Q_1}(\Sigma^+_{w^{-1}})\big)$.*

Let Θ be a subset of the set of simple roots Π. Let X_Θ be the variety of parabolic subalgebras of $\mathfrak{g}$ of type Θ. For a point y in X_Θ we denote by $\mathfrak{p}_y$ the corresponding parabolic subalgebra of $\mathfrak{g}$. Let $X_{\Theta,\sigma}$ be the subset of all $y \in X_\Theta$ such that $\mathfrak{p}_y$ and $\sigma(\mathfrak{p}_y)$ have a common Levi subalgebra. Then $X_{\Theta,\sigma}$ is a union of K-orbits.

Proposition 6.14. *Let Q be one of the K-orbits in $X_{\Theta,\sigma}$. Then Q is affinely imbedded in X_Θ.*

If $\Theta = \emptyset$, X_Θ coincides with X. In this case, every K-orbit is affinely imbedded. The proof of this result for involutive Harish-Chandra pairs in ([12], 4.1) (due to Beilinson and Bernstein), applies to the present situation. We leave it to the reader to make the necessary modifications.

Now consider the case when Θ consists of only one simple root α. To simplify the notation assume that our orbit in X_α is the projection $p_\alpha(Q)$ of an orbit Q in X. Then $p_\alpha(Q)$ is in $X_{\alpha,\sigma}$ if and only if the set $\{\alpha, -\alpha\}$ is σ_Q-invariant, i.e., if α is either Q-imaginary or Q-real. Thus we obtain:

Corollary 6.15. *Let Q be a K-orbit in X and $\alpha \in \Pi$. Assume that α is either Q-imaginary or Q-real. Then $p_\alpha(Q)$ is affinely imbedded in X_α.*

We shall also need the following simple (and well-known) remark.

Lemma 6.16. (i) *A K-orbit in the flag variety X is closed if and only if it consists of σ-stable Borel subalgebras.*

(ii) *The K-orbit of any σ-stable parabolic subalgebra in a generalized flag variety X_Θ is closed.*

Proof. Let $\Theta \subset \Pi$ and equip $X_\Theta \times X_\Theta$ with the G-action given by

$$g(x, y) = (gx, \sigma(g)y),$$

for $g \in G$ and $X, y \in X_\Theta$. Let $(x, x) \in \Delta$. If P_x is the parabolic subgroup which stabilizes $x \in X_\Theta$, the stabilizer of (x, x) equals $P_x \cap \sigma(P_x)$. Therefore, if the Lie algebra $\mathfrak{p}_x$ of P_x is σ-stable, the stabilizer of (x, x) is P_x, and the G-orbit of (x, x) is closed. Let C be the connected component containing (x, x) of the intersection of this orbit with the diagonal Δ. We have just seen that C is closed. Via the correspondence set up in the proof of 4.1 in [12], C corresponds to the K-orbit of x under the diagonal imbedding of X_Θ in $X_\Theta \times X_\Theta$. This proves (ii) and one implication in (i).

Let Q be a closed K-orbit, and $x \in Q$. Then the stabilizer of x in K is a solvable parabolic subgroup, i.e., it is a Borel subgroup of K. Therefore, by 6.3 and 6.4,

$$\dim Q = \frac{1}{2}(\dim \mathfrak{k} - \dim \mathfrak{t}) = \frac{1}{2}(\operatorname{Card} \Sigma_{Q,CI} + \frac{1}{2}(\operatorname{Card} \Sigma_{Q,\mathbb{C}} + \operatorname{Card} \Sigma_{Q,\mathbb{R}}))$$

and

$$\dim Q = \frac{1}{2}(\operatorname{Card} \Sigma_{Q,CI} + \operatorname{Card} \Sigma_{Q,\mathbb{R}} + \operatorname{Card} \Sigma_{Q,\mathbb{C}} - d(Q)).$$

This implies

$$\operatorname{Card} \Sigma_{Q,\mathbb{R}} + \operatorname{Card} \Sigma_{Q,\mathbb{C}} = 2d(Q).$$

Since $D_+(Q)$ consists of at most half of all Q-complex roots, we see that there are no Q-real roots, and all positive Q-complex root lie in $D_+(Q)$. This implies that all Borel subalgebras $\mathfrak{b}_x$, $x \in Q$, are σ-stable. □

We shall also need some information on Weyl group elements transversal to Langlands orbits. Let Q be a Langlands orbit in X. Then, by 5.10, the set $P = \Sigma_{Q,I} \cup \Sigma^+$ is a parabolic set of roots in Σ. It determines a set of simple roots Θ. Since $P \cap (-P) = \Sigma_{Q,I}$, Θ consists of Q-imaginary roots. Let W_Θ be the subgroup of W generated by reflections with respect to roots in Θ.

Lemma 6.17. $\sigma_Q(P) = -P$.

Proof. We have

$$\sigma_Q(P) = \sigma_Q(\Sigma_{Q,I}) \cup \sigma_Q(\Sigma^+ - \Sigma_{Q,I}) = \Sigma_{Q,I} \cup \sigma_Q(\Sigma^+ - \Sigma_{Q,I}) = (-\Sigma_{Q,I}) \cup \sigma_Q(\Sigma^+ - \Sigma_{Q,I}).$$

Let $\alpha \in \Sigma^+ - \Sigma_{Q,I}$. If α is Q-real, $\sigma_Q(\alpha) = -\alpha$ and $\sigma_Q(\alpha)$ is a negative root. If α is Q-complex, $\sigma_Q(\alpha)$ is also a negative root, since Q is a Langlands orbit. Therefore, $\sigma_Q(\Sigma^+ - \Sigma_{Q,I}) \subset -\Sigma^+$, and

$$\sigma_Q(P) = (-\Sigma_{Q,I}) \cup \sigma_Q(\Sigma^+ - \Sigma_{Q,I}) \subset (-\Sigma_{Q,I}) \cup (-\Sigma^+) = -P.$$

□

As before, let X_Θ be the generalized flag variety of parabolic subalgebras of type Θ. Denote by p_Θ the canonical projection of X onto X_Θ.

Lemma 6.18. *$p_\Theta(Q)$ is the open K-orbit in X_Θ.*

Proof. Let $y \in X_\Theta$ and denote by $\mathfrak{p}_y$ the corresponding parabolic subalgebra of $\mathfrak{g}$. Then the tangent space to X_Θ at y can be identified with $\mathfrak{g}/\mathfrak{p}_y$, and the tangent space to the K-orbit through y with $\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{p}_y)$. Hence, the K-orbit through y is open in X_Θ if and only if $\mathfrak{k} + \mathfrak{p}_y = \mathfrak{g}$.

Assume that $y = p_\Theta(x)$, $x \in Q$. Then, by 6.17, $\mathfrak{g} = \mathfrak{p}_y + \sigma(\mathfrak{p}_y)$. Hence, any $\xi \in \mathfrak{g}$ can be represented as $\xi = \xi_1 + \sigma(\xi_2)$ with $\xi_1, \xi_2 \in \mathfrak{p}_y$. This implies

$$\xi = \xi_1 - \xi_2 + (\xi_2 + \sigma(\xi_2)) \in \mathfrak{k} + \mathfrak{p}_y;$$

i.e., $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}_y$. $\square$

Let Q' be another K-orbit in X which contains Q in its closure. Then, since $p_\Theta(Q)$ is open in X_Θ, the projection of Q' to X_Θ must be equal to $p_\Theta(Q)$. Let $x' \in Q'$ be such that $p_\Theta(x') = y = p_\Theta(x)$. Let $\mathfrak{c}'$ be a σ-stable Cartan subalgebra in $\mathfrak{b}_{x'}$. By 6.17, $\mathfrak{l}_y = \sigma(\mathfrak{p}_y) \cap \mathfrak{p}_y$ is the σ-stable Levi factor of $\mathfrak{p}_y$. Hence, it contains $\mathfrak{c}'$. Since $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$, and $\mathfrak{l}_y$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, we conclude that $\mathfrak{a} \subset \mathfrak{c}'$, i.e., $\mathfrak{a} \subset \mathfrak{a}'$. This implies $\mathfrak{a}_Q \subset \mathfrak{a}_{Q'}$ and

$$\sigma_{Q'}|\mathfrak{a}_Q = -1.$$

Hence, for any $\alpha \in \Sigma$ the restrictions of $\sigma_{Q'}\alpha$ and $-\alpha$ to $\mathfrak{a}_Q$ agree. By 5.10.(ii), if $\alpha \in P$ we see that $-\sigma_{Q'}\alpha \in -P$. Hence, $\sigma_{Q'}(P) = -P$.

Therefore, we proved the following strengthening of 6.17.

Lemma 6.19. *Let Q be a Langlands orbit in X and Q' another K-orbit in X such that Q is contained in the closure of Q'. Then:*
- (i) $\sigma_{Q'}(P) = -P$;
- (ii) $\mathfrak{a}_Q \subset \mathfrak{a}_{Q'}$.

Corollary 6.20. $\sigma_{Q'}(\Sigma^+ - \Sigma_{Q,I}) = -(\Sigma^+ - \Sigma_{Q,I})$.

Proof. By 6.19, we have

$$\Sigma_{Q,I} = P \cap (-P) = P \cap \sigma_{Q'}(P).$$

Let $\alpha \in \Sigma^+ - \Sigma_{Q,I}$. Since α is not Q-imaginary, by the preceding relation $\sigma_{Q'}\alpha \notin P$. Hence $\sigma_{Q'}\alpha \in -P$. This implies $\sigma_{Q'}\alpha \in -(\Sigma^+ - \Sigma_{Q,I})$. $\square$

Proposition 6.21. *Let $w \in W$ be transversal to a Langlands orbit Q. Then $w \in W_\Theta$ for the set Θ of all simple Q-imaginary roots.*

Proof. Let $w = s_\alpha w'$, $\ell(w') = \ell(w) - 1$ with $\alpha \in \Pi$. Then, by 6.6, w' is transversal to Q and s_α is transversal to $Q_{w'}$. Assume that $\alpha \notin \Theta$. By the definition of $Q_{w'}$, $Q \subset \overline{Q_{w'}}$. This implies

$$\sigma_{Q_{w'}}(\Sigma^+ - \Sigma_{Q,I}) = -(\Sigma^+ - \Sigma_{Q,I}),$$

and $\sigma_{Q_{w'}} \in -\Sigma^+$. But this contradicts the transversality of s_α to $Q_{w'}$, by 6.7. Hence, $\alpha \in \Theta$. By the induction in length the statement follows. $\square$

Finally, we analyze the structure of the stabilizers in K of points in X. Let Q be a K-orbit in the flag variety X. Let $x \in Q$ and $\mathfrak{b}_x$ the corresponding Borel subalgebra. Denote by B_x the corresponding subgroup of $G = \mathrm{Int}(\mathfrak{g})$. Fix a σ-stable Cartan subalgebra $\mathfrak{c}$ in $\mathfrak{b}_x$ and let C be the corresponding torus in $\mathrm{Int}(\mathfrak{g})$. Let S_x be the stabilizer of x in K, i.e.,

$$S_x = \varphi^{-1}(\varphi(K) \cap B_x).$$

Then the Lie algebra $\mathfrak{s}_x = \mathfrak{k} \cap \mathfrak{b}_x$ is a semidirect product of $\mathfrak{t} = \{\xi \in \mathfrak{c} \mid \sigma(\xi) = \xi\}$ with the nilpotent radical $\mathfrak{u}_x = \{\eta \in \mathfrak{n}_x \mid \sigma(\eta) = \eta\}$ of $\mathfrak{s}_x$. Let U_x be the unipotent subgroup of K corresponding to $\mathfrak{u}_x$; it is the unipotent radical of S_x. Put $T = \varphi^{-1}(\varphi(K) \cap C)$. Then we have:

Lemma 6.22. *The stabilizer S_x is the semidirect product of T with U_x.*

Proof. Let $s \in \varphi(S_x)$. Then $s \in \sigma(B_x) \cap B_x$, and this group is a semidirect product of C with $\sigma(N_x) \cap N_x$. This implies that we have a unique representation $s = cn$ with $c \in C$ and $n \in \sigma(N_x) \cap N_x$. Therefore, $s = \sigma(s) = \sigma(c)\sigma(n)$ implies that $c = \sigma(c)$ and $n = \sigma(n)$, i.e., $c \in \varphi(K) \cap C$ and $n \in \phi(U_x)$. This implies $S_x = T \cdot U_x$. Since T is a reductive subgroup of S_x, it is contained in a Levi factor T' of S_x. This in turn implies that the natural map $T \longrightarrow S_x/U_x \cong T'$ is surjective, i.e., $T = T'$. □

Let

$$F = \{\exp(\xi) \mid \sigma(\exp(\xi)) = \exp(\xi),\ \xi \in \mathfrak{a}\}.$$

Then for any $s \in F$, we have $s = \sigma(s) = \exp(\sigma(\xi)) = \exp(-\xi) = s^{-1}$, i.e., $s^2 = 1$. Hence F is a direct product of several copies of $\mathbb{Z}_2$. Let $s \in F$ and α a Q-complex root. Then the character e^α of C satisfies

$$e^\alpha(s) = \exp(\alpha(\xi)) = \exp(\sigma\alpha(\sigma(\xi))) = \exp(-\sigma\alpha(\xi)) = e^{-\sigma\alpha}(s).$$

Therefore, $e^\alpha(s) = e^{-\sigma\alpha}(s) = \pm 1$. Denote by A a set of representatives of the $(-\sigma_Q)$-orbits in $D_-(Q)$. Then

$$\delta_Q(t) = \prod_{\alpha \in A} e^\alpha(t),\ t \in F,$$

is a character of F independent of the choice of A.

Let α be a Q-real root. Denote by $\mathfrak{s}_\alpha$ the three-dimensional simple algebra spanned by $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Let S_α be the connected subgroup of $G = \mathrm{Int}(\mathfrak{g})$ with Lie algebra $\mathfrak{s}_\alpha$; it is isomorphic either to $\mathrm{SL}(2,\mathbb{C})$ or to $\mathrm{PSL}(2,\mathbb{C})$. Denote by H_α the element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{a}$ such that $\alpha(H_\alpha) = 2$. Then H_α is the dual root in $\mathfrak{c}$, and $\beta(H_\alpha) \in \mathbb{Z}$ for any $\beta \in \Sigma$. This implies $m_\alpha = \exp(\pi i H_\alpha)$ satisfies $m_\alpha^2 = 1$ in G. Moreover, $\sigma(m_\alpha) = \exp(-\pi i H_\alpha) = m_\alpha^{-1} = m_\alpha$, and $m_\alpha \in F$. Clearly $m_\alpha = 1$ if $S_\alpha \cong \mathrm{PSL}(2,\mathbb{C})$, and $m_\alpha \neq 1$ if $S_\alpha \cong \mathrm{SL}(2,\mathbb{C})$, and in this latter case m_α corresponds to the negative of the identity matrix in $\mathrm{SL}(2,\mathbb{C})$.

Lemma 6.23. *Let $\alpha \in \Pi$ be Q-real. Then $\delta_Q(m_\alpha) = 1$.*

Proof. Let $\beta \in D_-(Q)$. Then $s_\alpha\beta \in \Sigma^+$ and $s_\alpha\sigma_Q\beta = \sigma_Q s_\alpha\beta$. Hence, $s_\alpha\beta \in \Sigma_{Q,\mathbb{C}}$ and $-\sigma_Q s_\alpha\beta \in \Sigma^+$, i.e., $s_\alpha\beta \in D_-(Q)$. Clearly,

$$e^{s_\alpha\beta}(m_\alpha) = e^{\beta - \alpha\check{}(\beta)\alpha}(m_\alpha) = e^\beta(m_\alpha) e^\alpha(m_\alpha)^{\alpha\check{}(\beta)} = e^\beta(m_\alpha).$$

On the other hand, if $s_\alpha\beta = \beta$ we see that $\alpha\check{}(\beta) = 0$. Therefore $\beta(H_\alpha) = 0$ and $e^\beta(m_\alpha) = 1$. Hence the expression for $\delta_Q(m_\alpha)$ contains either both $e^\beta(m_\alpha)$ and $e^{s_\alpha\beta}(m_\alpha)$ if $s_\alpha\beta \neq \beta$, or $e^\beta(m_\alpha) = 1$ if $s_\alpha\beta = \beta$. □

Let $\mathfrak{k}_\alpha = \mathfrak{s}_\alpha \cap \mathfrak{k}$; it is the Lie algebra of a one dimensional torus K_α in K. Its image $\varphi(K_\alpha)$ in G is a torus in S_α. Therefore, $m_\alpha \in \varphi(K_\alpha)$. The composition of $\varphi : K_\alpha \longrightarrow S_\alpha$ and the covering projection $S_\alpha \longrightarrow \mathrm{Int}(\mathfrak{s}_\alpha)$ is an n-fold covering map between two one dimensional tori. We shall need to know an explicit lifting of m_α to K_α. If we identify K_α with $\mathbb{C}^*$, the kernel of this map is isomorphic to

$\{e^{\frac{2\pi p}{n}} \mid 0 \leq p \leq n-1\}$. Let n_α correspond to $e^{\frac{2\pi}{n}}$ under this isomorphism (there are two possible choices for n_α and they are inverses of each other). Then φ maps n_α to m_α, hence n_α lies in T. We have shown:

Observation 6.24. *$n_\alpha \in T$, $\varphi(n_\alpha) = m_\alpha$.*

7. Intertwining functors and standard Harish-Chandra sheaves

First we want to describe a simple necessary condition on $\lambda \in \mathfrak{h}^*$ for the existence of a K-homogeneous connection τ on a K-orbit Q in X. We introduce a real structure in $\mathfrak{h}$ by putting $\mathfrak{h}_{\mathbb{R}}$ to be the real span of all dual roots $\alpha\check{}$, $\alpha \in \Sigma$. For any $\lambda \in \mathfrak{h}^*$ we denote by $\operatorname{Re}\lambda$ the complex linear form on $\mathfrak{h}$ which satisfies $(\operatorname{Re}\lambda)(\xi) = \operatorname{Re}\lambda(\xi)$ for $\xi \in \mathfrak{h}_{\mathbb{R}}$, and by $\operatorname{Im}\lambda$ the complex linear form on $\mathfrak{h}$ which satisfies $(\operatorname{Im}\lambda)(\xi) = \operatorname{Im}\lambda(\xi)$ for $\xi \in \mathfrak{h}_{\mathbb{R}}$.

If K is a subgroup of a covering $\tilde{G}$ of $\operatorname{Int}(\mathfrak{g})$ with Lie algebra $\mathfrak{k}$, we say that the Harish-Chandra pair is *linear*.

Lemma 7.1. *Let $\lambda \in \mathfrak{h}^*$, Q an arbitrary K-orbit in X and τ a K-homogeneous connection on Q compatible with $\lambda + \rho$. Then*

(i) *$\alpha\check{}(\lambda + \sigma_Q\lambda) \in \mathbb{Q}$ for any $\alpha \in \Sigma$. In particular, $\operatorname{Im}\lambda$ vanishes on $\mathfrak{t}_Q$.*

(ii) *If, in addition, $(\mathfrak{g}, K)$ is a linear Harish-Chandra pair, $\alpha\check{}(\lambda + \sigma_Q\lambda) \in \mathbb{Z}$ for any $\alpha \in \Sigma$. Hence, $\alpha \in \Sigma_\lambda$ if and only if $\sigma_Q\alpha \in \Sigma_\lambda$.*

Proof. Let $x \in Q$ and $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ a σ-stable Cartan subalgebra of $\mathfrak{b}_x$. Then $\mathfrak{t} \subset \mathfrak{k}$ and it defines a closed subgroup in K. The image $\varphi(T)$ of T in $\operatorname{Int}(\mathfrak{g})$ is contained in the Cartan subgroup C of $\operatorname{Int}(\mathfrak{g})$. Let r be the order of the kernel of the homomorphism of T into C. Since τ is compatible with $\lambda + \rho$, there exists a character ω of T with differential equal to the restriction to $\mathfrak{t}$ of the specialization of $\lambda + \rho$. Then ω^r is a character of T which factors through $\varphi(T)$. It defines a character μ of $\varphi(T)$ with differential equal to the restriction to $\mathfrak{t}$ of the specialization of $r(\lambda + \rho)$. This in turn implies that $c \longmapsto \mu(c\sigma(c))$ is a character of C with the differential equal to the specialization of $r(\lambda + \sigma_Q\lambda + \rho + \sigma_Q\rho)$. Therefore, $r(\lambda + \sigma_Q\lambda)$ is a weight.

To prove (ii), without loss of generality, we can assume that $\tilde{G}$ is simply connected. Let $\tilde{B}_x$ denote the Borel subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{b}_x$ and $\tilde{C}$ the complex torus in $\tilde{G}$ with Lie algebra $\mathfrak{c}$. Let $c \in C$. Then $c\sigma(c) \in S_x$. Since the exponential map from $\mathfrak{c}$ onto C is surjective, any $c \in C$ is of the form $c = \exp(\xi)$ for some $\xi \in \mathfrak{c}$. This implies $c\sigma(c) = \exp(\xi + \sigma(\xi)) \in S_x$. In particular $c\sigma(c)$ lies in the connected component of S_x.

Let ω denote the representation of the stabilizer S_x induced by the connection τ on Q, which is compatible with $\lambda + \rho$. Then

$$\omega(c\sigma(c)) = \exp((\lambda + \rho)(\xi + \sigma(\xi))) = \exp(\lambda + \sigma_Q\lambda + \rho + \sigma_Q\rho)(\xi)).$$

On the other hand, if $\mu(\xi) \in 2\pi i\mathbb{Z}$ for any $\mu \in P(\Sigma)$, $c = \exp(\xi)$ is equal to the identity in $\tilde{G}$, and $\exp(\lambda + \sigma_Q\lambda)(\xi) = 1$. This implies $\lambda + \sigma_Q\lambda \in P(\Sigma)$. Therefore, for any root $\alpha \in \Sigma$, we have $\alpha\check{}(\lambda) + (\sigma_Q\alpha)\check{}(\lambda) \in \mathbb{Z}$, and $\alpha \in \Sigma_\lambda$ is equivalent to $\sigma_Q\alpha \in \Sigma_\lambda$. □

Let Q be a K-orbit in X and $i_Q : Q \longrightarrow X$ the natural inclusion. Assume that Q is not a Langlands orbit, i.e., the set $D_+(Q)$ is not empty. Let $w \in W$ satisfy the conditions of 6.9, i.e., $\Sigma_w^+ \subset D_+(Q)$ and $\Sigma_w^+ \cap \sigma_Q(\Sigma_w^+) = \emptyset$. There we defined the K-orbit $Q_w = p_1(p_2^{-1}(Q))$. Since $p_1 : p_2^{-1}(Q) \longrightarrow Q_w$ is an isomorphism, p_2

composed with the inverse of this map induces a natural projection of Q_w onto Q. This fibration is locally trivial and its fibres are isomorphic to $\mathbb{C}^{\ell(w)}$.

Let x be a point of Q, $\mathfrak{c}$ a σ-stable Cartan subalgebra contained in $\mathfrak{b}_x$, and R^+ the set of positive roots in the root system R of $(\mathfrak{g}, \mathfrak{c})$ corresponding to the orbit Q. Fix $\lambda \in \mathfrak{h}^*$. The homogeneous twisted sheaf of differential operators $\mathcal{D}_\lambda$ on X induces a homogeneous twisted sheaf of differential operators $(\mathcal{D}_\lambda)^{i_Q}$ on the orbit Q. Let τ be a K-homogeneous connection on Q compatible with $(\mathcal{D}_\lambda)^{i_Q}$. This means that the differential of the corresponding representation of the stabilizer $K \cap B_x$ of x is a direct sum of copies of the one dimensional representation of $\mathfrak{k} \cap \mathfrak{b}_x$ given by the specialization of $\lambda + \rho$. Let q_2 be the restriction of p_2 to $p_2^{-1}(Q)$. Then we have the following commutative diagram:

$$\begin{tikzcd} p_2^{-1}(Q) \arrow[r, "j"] \arrow[d, "q_2"'] & Z_w \arrow[d, "p_2"'] \\ Q \arrow[r, "i_Q"] & X \end{tikzcd}$$

Since the orbit map i_Q is an affine immersion and p_2 is a locally trivial fibration, we conclude that j is also an affine immersion. Therefore, by base change ([5], VI.8.4) we see that:

$$p_2^*(\mathcal{I}(Q,\tau)) = p_2^*(R^0 i_{Q+}(\tau)) = R^0 j_+(q_2^*(\tau)).$$

Let $\mathcal{T}_w$ be the inverse of the invertible $\mathcal{O}_{Z_w}$-module of top degree relative differential forms for the projection $p_1 : Z_w \longrightarrow X$. Then

$$\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^*(\mathcal{I}(Q,\tau)) = \mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} R^0 j_+(q_2^*(\tau)) = R^0 j_+(j^*(\mathcal{T}_w) \otimes_{\mathcal{O}_{p_2^{-1}(Q)}} q_2^*(\tau)).$$

Therefore,

$$\begin{aligned} R^q p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} R^0 j_+(q_2^*(\tau))) &= R^q p_{1+}(R^0 j_+(j^*(\mathcal{T}_w) \otimes_{\mathcal{O}_{p_2^{-1}(Q)}} q_2^*(\tau))) \\ &= R^q (p_1 \circ j)_+(j^*(\mathcal{T}_w) \otimes_{\mathcal{O}_{p_2^{-1}(Q)}} q_2^*(\tau)). \end{aligned}$$

The map $p_1 \circ j$ induces an isomorphism q_1 of $p_2^{-1}(Q)$ onto Q_w, so

$$R^q p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} R^0 j_+(q_2^*(\tau))) = R^q i_{Q_w+}(R^0 q_{1+}(j^*(\mathcal{T}_w) \otimes_{\mathcal{O}_{p_2^{-1}(Q)}} q_2^*(\tau))).$$

Since the orbit map i_{Q_w} is an affine immersion, these expressions vanish for $q \neq 0$. Hence, if we let τ_w denote the K-homogeneous connection $q_{1+}(j^*(\mathcal{T}_w) \otimes_{\mathcal{O}_{p_2^{-1}(Q)}} q_2^*(\tau))$, we see that

$$R^0 p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^*(\mathcal{I}(Q,\tau))) = \mathcal{I}(Q_w, \tau_w).$$

By the definition of the intertwining functors, this gives

$$L^0 I_w(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q_w, \tau_w)$$

and

$$L^q I_w(\mathcal{I}(Q,\tau)) = 0$$

for $q \neq 0$. To describe τ_w more explicitly, we let q_w denote the natural projection of Q_w onto Q which we described previously. Then

$$\tau_w = q_{1+}(j^*(\mathcal{T}_w) \otimes_{\mathcal{O}_{p_2^{-1}(Q)}} q_2^*(\tau)) = q_{1+}(j^*(\mathcal{T}_w)) \otimes_{\mathcal{O}_{Q_w}} q_w^*(\tau).$$

Since $\mathcal{T}_w = p_1^*(\mathcal{O}(\rho - w\rho))$, we also conclude that $q_{1+}(j^*(\mathcal{T}_w)) = i_{Q_w}^*(\mathcal{O}(\rho - w\rho))$. Therefore, we finally get

$$\tau_w = q_w^*(\tau) \otimes_{\mathcal{O}_{Q_w}} i_{Q_w}^*(\mathcal{O}(\rho - w\rho)).$$

Let $x \in Q_w$. Then the stabilizer of x in K is a quotient of the stabilizer of $q_w(x) \in Q$ by a unipotent normal subgroup. Therefore, the quotient map induces a bijection between irreducible algebraic representations of these stabilizers. This implies that $\tau \longrightarrow q_w^*(\tau)$ is a bijection between irreducible K-homogeneous connections on Q compatible with $\lambda + \rho$ and irreducible K-homogeneous connections on Q_w compatible with $w\lambda + w\rho$. We have proved:

Lemma 7.2. *Let Q be an arbitrary K-orbit in X. Suppose $w \in W$ satisfies $\Sigma_w^+ \subset D_+(Q)$ and $\Sigma_w^+ \cap \sigma_Q(\Sigma_w^+) = \emptyset$. Then:*

(i) *the map $\tau \longmapsto \tau_w$ is a bijection between irreducible K-homogeneous connections on Q compatible with $\lambda + \rho$ and irreducible K-homogeneous connections on Q_w compatible with $w\lambda + \rho$;*

(ii) *for any standard Harish-Chandra module $\mathcal{I}(Q, \tau)$, we have*

$$LI_w(D(\mathcal{I}(Q,\tau))) = D(\mathcal{I}(Q_w, \tau_w)).$$

Let $(\mathfrak{g}, K)$ be an arbitrary Harish-Chandra pair. Let Θ be a subset of the set of simple roots Π. Then it defines the generalized flag variety X_Θ of all parabolic subalgebras of type Θ. Let $p_\Theta : X \longrightarrow X_\Theta$ be the natural projection.

Let O be a K-orbit of K in X_Θ. Then $V = p_\Theta^{-1}(O)$ is a smooth subvariety of X and a union of K-orbits. Denote by j the natural immersion of V into X. Then $\mathcal{D}_\lambda$ defines a K-equivariant twisted sheaf of differential operators $\mathcal{D}_\lambda^j$ on V. Let $\mathcal{M}_{coh}(\mathcal{D}_\lambda^j, K)$ be the category of K-equivariant coherent $\mathcal{D}_\lambda^j$-modules.

Let $o \in O$ and $X_o = p_\Theta^{-1}(o)$ the fiber over o. Denote by $s : X_o \longrightarrow V$ the natural immersion of the fiber X_o into V. Then $j \circ s$ is the natural immersion of X_o into X. Let P_o be the stabilizer of o in $G = \operatorname{Int}(\mathfrak{g})$, and U_o its unipotent radical. Let $L_o = G_o/U_o$ and G_o the quotient of L_o by its center. Let τ be the natural homomorphism of P_o into G_o. Then its differential defines a surjective morphism of the Lie algebra $\mathfrak{p}_o$ onto $\mathfrak{g}_o$. This map induces an identification of the fiber X_o with the flag variety of $\mathfrak{g}_o$, which maps any Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ contained in $\mathfrak{p}_o$ into $\mathfrak{b}/(\mathfrak{b} \cap \ker \tau)$. These maps induce a canonical isomorphism of the Cartan algebra $\mathfrak{h}$ of $\mathfrak{g}$ with the product of the center of the Lie algebra $\mathfrak{l}_o$ of L_o with the Cartan algebra $\mathfrak{h}_o$ of $\mathfrak{g}_o$. Therefore, we get a natural splitting of $\mathfrak{h}^*$ into the subspace spanned by roots in Θ and the complement $\mathfrak{h}^*(\Theta) = \{\mu \in \mathfrak{h}^* \mid \alpha\check{\ }(\mu) = 0 \text{ for } \alpha \in \Theta\}$, and $\mathfrak{h}_o^*$ can be identified with the first subspace. The root system Σ_o of $\mathfrak{g}_o$ can be identified with the root subsystem Σ_Θ of Σ generated by Θ, and Σ_o^+ with $\Sigma_\Theta \cap \Sigma^+$. Let r be the projection of $\mathfrak{h}^*$ onto $\mathfrak{h}_o^*$ along $\mathfrak{h}^*(\Theta)$. Let ρ_o be the half-sum of roots in Σ_o^+. Then

$$\alpha\check{\ }(\rho) = 2 = \alpha\check{\ }(\rho_o)$$

for any $\alpha \in \Theta$, hence $r(\rho) = \rho_o$. This implies

$$(\mathcal{D}_\lambda^j)^s = (\mathcal{D}_\lambda)^{j \circ s} = (\mathcal{D}_{X, \lambda+\rho})^{j \circ s} = \mathcal{D}_{X_o, r(\lambda+\rho)} = \mathcal{D}_{\lambda_o}^o,$$

where we put $\lambda_o = r(\lambda)$ and we let $\mathcal{D}_\mu^o$ denote the homogeneous twisted sheaf on X_o attached to $\mu \in \mathfrak{h}_o^*$.

The subgroup $\varphi^{-1}(\varphi(K) \cap P_o)$ acts on X_o, and this action factors through $K_o = \varphi^{-1}(\varphi(K) \cap P_o)/\varphi^{-1}(\varphi(K) \cap \ker \tau)$. The pair $(\mathfrak{g}_o, K_o)$ is a Harish-Chandra pair, since K_o-orbits in X_o are exactly the intersections of K-orbits with X_o. Since the K-orbits are affinely imbedded in X by the result in Appendix A, it follows that K_o-orbits are affinely imbedded in X_o.

Consequently, the inverse image functor s^+ is an additive functor from the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda^j, K)$ into the category $\mathcal{M}_{coh}(\mathcal{D}_{\lambda_o}^o, K_o)$.

Lemma 7.3. *The functor $s^+ : \mathcal{M}_{coh}(\mathcal{D}_\lambda^j, K) \longrightarrow \mathcal{M}_{coh}(\mathcal{D}_{\lambda_o}^o, K_o)$ is exact. It is an equivalence of categories.*

This is certainly a known fact (compare [6], 3.10).

Consider now the special case when Θ consists of one simple root α. Then $\mathfrak{g}_o \cong \mathfrak{sl}(2, \mathbb{C})$, $\Sigma_o = \{\alpha, -\alpha\}$ and $\Sigma_o^+ = \{\alpha\}$. In this case, we have $\mathfrak{g}_o \cong \mathfrak{sl}(2, \mathbb{C})$ and the connected component of K_o is one of the groups listed in 4.1. By 6.5, we have the following possibilities:

(a) If α is compact Q-imaginary, the identity component of K_o is isomorphic to either $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{PSL}(2, \mathbb{C})$.
(b) If α is Q-complex the unipotent radical of K_o is nontrivial.
(c) If α is either noncompact Q-imaginary or Q-real, the identity component of K_o is a one dimensional torus.

In the case (c), we generalize the definition of the SL_2-parity condition from §4. By 6.5, if α is Q-real, the K_o-orbit $Q_o = Q \cap X_o$ is open in X_o. Hence, for any irreducible K-homogeneous connection τ, the K_o-homogeneous connection τ_o can be viewed as a homogeneous connection on $\mathbb{C}^* \subset \mathbb{P}^1$. If this connection satisfies the parity condition from §4, we say that τ satisfies the SL_2*-parity condition* with respect to the simple Q-real root α.

As in §2, let I denote the intertwining functor I_{s_α} for $\mathfrak{g}_o \cong \mathfrak{sl}(2, \mathbb{C})$.

Lemma 7.4. *Let $\lambda \in \mathfrak{h}^*$. For any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ we have*

$$R(j \circ s)^!(LI_{s_\alpha}(D(\mathcal{V}))) = LI(R(j \circ s)^!(D(\mathcal{V}))).$$

Proof. Put $s' = j \circ s$. The morphism $s' \times s' : X_o \times X_o \longrightarrow X \times X$ is an identification of $X_o \times X_o$ with its image in $X \times X$. The intersection of $X_o \times X_o$ with Z_α consists of pairs $(\mathfrak{b}_x, \mathfrak{b}_{x'})$, $\mathfrak{b}_x, \mathfrak{b}_{x'} \subset \mathfrak{p}_o$, which are in relative position s_α. Since $\mathfrak{p}_\alpha$ is a parabolic subalgebra of type α, any two Borel subalgebras of $\mathfrak{g}$ contained in it are either in relative position s_α, or they are equal. This implies that the inverse image $(s' \times s')^{-1}(Z_{s_\alpha})$ is the complement Z_o of the diagonal in $X_o \times X_o$. Denote by $\bar{s}$ the isomorphism of Z_o onto $(X_o \times X_o) \cap Z_{s_\alpha}$. Then we have the commutative diagram

$$\begin{array}{ccc} Z_o & \xrightarrow{\bar{s}} & Z_{s_\alpha} \\ {\scriptstyle p_{01}}\downarrow & & {\scriptstyle p_1}\downarrow \\ X_o & \xrightarrow{s'} & X \end{array},$$

and by base change ([5], VI.8.4),

$$Rs'^! \circ Rp_{1+} = Rp_{01+} \circ R\bar{s}^!.$$

On the other hand,

$$R\bar{s}^!(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} p_2^+(\mathcal{V})) = \mathcal{T}_{0s_\alpha} \otimes_{\mathcal{O}_{Z_0}} R\bar{s}^!(p_2^+(\mathcal{V})),$$

and since

$$\begin{array}{ccc} Z_0 & \xrightarrow{\bar{s}} & Z_{s_\alpha} \\ {\scriptstyle p_{02}}\downarrow & & {\scriptstyle p_2}\downarrow \\ X_0 & \xrightarrow{s'} & X \end{array}$$

is also commutative,

$$R\bar{s}^!(p_2^+(\mathcal{V})) = R(p_2\circ\bar{s})^!(\mathcal{V})[-\dim X+1] = R(s'\circ p_{02})^!(\mathcal{V})[-\dim X+1] = p_{02}^+(Rs^!(\mathcal{V})),$$

which finally implies the assertion. □

In some cases, the lemma reduces the calculation of the action of the intertwining functor LI_{s_α} on a standard module to an SL_2-calculation.

Let Q be a K-orbit and $V = p_\alpha^{-1}(p_\alpha(Q))$. Assume that $\mathcal{V}$ is a Harish-Chandra sheaf supported in $\bar{V}$. Then, by Kashiwara's theorem, $R^p j^!(\mathcal{V}) = 0$ for $p \neq 0$, and $j^!(\mathcal{V})$ is in $\mathcal{M}_{coh}(D_\lambda^j, K)$. By 3.1 and 3.4, the support of $L^p I_{s_\alpha}$ is also contained in $\bar{V}$. Therefore, the same applies to these Harish-Chandra sheaves. Hence, by the preceding lemma, we see that

$$s^+(j^!(L^q I_{s_\alpha}(\mathcal{V}))) = L^q I(s^+(j^!(\mathcal{V})))$$

for any $q \in \mathbb{Z}$. Assume that $\mathcal{V}$ is irreducible Harish-Chandra sheaf with support $\bar{Q}$, i.e., $\mathcal{V} = \mathcal{L}(Q, \tau)$. Then, since the restriction of an irreducible $\mathcal{D}$-module is either irreducible or 0, we see that the restriction of $\mathcal{L}(Q, \tau)$ to the complement of ∂V is irreducible. By Kashiwara's equivalence of categories, $j^!(\mathcal{L}(Q, \tau))$ is an irreducible object in $\mathcal{M}_{coh}(\mathcal{D}_\lambda^j, K)$. Moreover, by 7.3, $s^+(j^!(\mathcal{L}(Q, \tau)))$ is an irreducible object in $\mathcal{M}_{coh}(\mathcal{D}_{\lambda_o}, K_o)$. Hence, it is equal to $\mathcal{L}(Q_o, \tau_o)$, where $Q_o = Q \cap S$ and τ_o is the restriction of τ to Q_o. It follows that

$$s^+(j^!(L^q I_{s_\alpha}(\mathcal{L}(Q, \tau)))) = L^q I(\mathcal{L}(Q_o, \tau_o))$$

for any $q \in \mathbb{Z}$. Assume that $\alpha\check{}(\lambda) \in \mathbb{Z}$. Then, by 2.16, either $I_{s_\alpha}(\mathcal{L}(Q, \tau)) = 0$ (and therefore $I(\mathcal{L}(Q_o, \tau_o)) = 0$) or $L^{-1} I_{s_\alpha}(\mathcal{L}(Q, \tau)) = 0$ (and $L^{-1} I(\mathcal{L}(Q_o, \tau_o)) = 0$). This leads immediately to the following generalization of 4.14. It is an unpublished result of Beilinson and Bernstein, which is a special case of 2.18.

Lemma 7.5. *Let $\alpha \in \Pi$ and $\alpha\check{}(\lambda) \in \mathbb{Z}$. Then $I_{s_\alpha}(\mathcal{L}(Q, \tau)) = 0$ if and only if either*

(i) *α is compact Q-imaginary root; or*
(ii) *α is a Q-complex root such that $-\sigma_Q\alpha$ is positive; or*
(iii) *α is a Q-real root which doesn't satisfy the SL_2-parity condition.*

Proof. Consider the cases (a), (b) and (c) we discussed before. If (a) holds, α is compact Q-imaginary and $I_{s_\alpha}(\mathcal{L}(Q, \tau)) = 0$ by 4.14. If (b) holds, α is Q-complex. If $\sigma_Q(\alpha)$ is a positive root, Q_o is a point and $I_{s_\alpha}(\mathcal{L}(Q, \tau)) \neq 0$ by 4.14. If $\sigma_Q(\alpha)$ is a negative root, Q_o is open in X_o and $I_{s_\alpha}(\mathcal{L}(Q, \tau)) = 0$. If (c) holds, α is either noncompact Q-imaginary or Q-real. In the first case, Q_o is either one or two points, and $I_{s_\alpha}(\mathcal{L}(Q, \tau)) \neq 0$ by 4.14. In the second case, Q_o is the open orbit in X_o. By 4.14, $I_{s_\alpha}(\mathcal{L}(Q, \tau)) = 0$ holds if and only if the SL_2-parity condition fails for τ. □

Assume now that V is affinely imbedded in X. Since the fibration $p_\alpha : X \longrightarrow X_\alpha$ is locally trivial, this is the case if $p_\alpha(Q)$ is affinely imbedded in X_α. As we have seen in 6.15, $p_\alpha(Q)$ is affinely imbedded in X_α if the root α is either Q-imaginary or Q-real. Then $p_2^{-1}(V)$ is a smooth subvariety of Z_{s_α}. Moreover, since V is affinely imbedded and $p_2 : Z_{s_\alpha} \longrightarrow X$ is locally trivial, it is also affinely imbedded in Z_{s_α}. Let q_2 be the restriction of p_2 to $p_2^{-1}(V)$. Then we have the following commutative diagram:

$$\begin{array}{ccc} p_2^{-1}(V) & \xrightarrow{k} & Z_{s_\alpha} \\ {\scriptstyle q_2}\big\downarrow & & {\scriptstyle p_2}\big\downarrow \\ V & \xrightarrow{j} & X \end{array}.$$

Since k is an affine immersion, and p_2 and q_2 are submersions, from base change ([5], VI.8.4) we see:

$$p_2^*(j_+(\mathcal{V})) = k_+(q_2^*(\mathcal{V})),$$

for any $\mathcal{D}_\lambda^j$-module $\mathcal{V}$. Also,

$$\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} p_2^*(j_+(\mathcal{V})) = \mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} k_+(q_2^*(\mathcal{V})) = k_+(k^*(\mathcal{T}_{s_\alpha}) \otimes_{\mathcal{O}_{p_2^{-1}(V)}} q_2^*(\mathcal{V})).$$

Therefore,

$$\begin{aligned} R^q p_{1+}(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} p_2^*(j_+(\mathcal{V}))) &= R^q p_{1+}(k_+(k^*(\mathcal{T}_{s_\alpha}) \otimes_{\mathcal{O}_{p_2^{-1}(V)}} q_2^*(\mathcal{V}))) \\ &= R^q(p_1 \circ k)_+(k^*(\mathcal{T}_{s_\alpha}) \otimes_{\mathcal{O}_{p_2^{-1}(V)}} q_2^*(\mathcal{V})). \end{aligned}$$

Since $V = p_\alpha^{-1}(p_\alpha(Q))$, we see that $p_2^{-1}(V) = p_1^{-1}(V)$. Hence, if we denote by q_1 the restriction of p_1 to $p_2^{-1}(V)$, we get the commutative diagram

$$\begin{array}{ccc} p_2^{-1}(V) & \xrightarrow{k} & Z_{s_\alpha} \\ {\scriptstyle q_1}\big\downarrow & & {\scriptstyle p_1}\big\downarrow \\ V & \xrightarrow{j} & X \end{array}.$$

From it we conclude that

$$R^q(p_1 \circ k)_+ = R^q(j \circ q_1)_+ = j_+ \circ R^q q_{1+}.$$

This implies

$$L^q I_{s_\alpha}(j_+(\mathcal{V})) = j_+(R^q q_{1+}(k^*(\mathcal{T}_{s_\alpha}) \otimes_{\mathcal{O}_{p_2^{-1}(V)}} q_2^*(\mathcal{V}))),$$

Hence, by Kashiwara's theorem, we have

$$L^q I_{s_\alpha}(j_+(\mathcal{V})) = j_+(R^0 j^!(L^q I_{s_\alpha}(j_+(\mathcal{V}))))$$

for all $q \in \mathbb{Z}$.

On the other hand, if $\mathcal{V}$ is a coherent $(\mathcal{D}_\lambda^j, K)$-module, by 7.3, $R^0 j^!(L^q I_{s_\alpha}(j_+(\mathcal{V})))$ is completely determined by its restriction to X_o, i.e., by

$$s^+(R^0 j^!(L^q I_{s_\alpha}(j_+(\mathcal{V}))) = R^{\dim p_\alpha(Q)}(j \circ s)^!(L^q I_{s_\alpha}(j_+(\mathcal{V}))).$$

By 7.4 and base change, we have

$$R(j \circ s)^!(LI_{s_\alpha}(D(j_+(\mathcal{V})))) = LI(R(j \circ s)^!(D(j_+(\mathcal{V})))) = LI(D(s_+(\mathcal{V})))[-\dim p_\alpha(Q)].$$

Hence

$$s^+(R^0 j^!(L^q I_{s_\alpha}(j_+(\mathcal{V}))) = L^q I(s^+(\mathcal{V})).$$

Clearly, since Q is a K-orbit in V, we have $\mathcal{I}(Q,\tau) = i_{Q+}(\tau)) = j_+(l_+(\tau))$, where $l : Q \longrightarrow V$ is the natural inclusion. Hence the preceding identity, combined with 4.5 and 4.12, leads to the following two propositions.

Proposition 7.6. *Let $\lambda \in \mathfrak{h}^*$, Q be a K-orbit and τ a connection on Q compatible with λ. Let $\alpha \in \Pi$ a compact Q-imaginary root. Then $p = -\alpha\check{}(\lambda)$ is an integer, and*

$$LI_{s_\alpha}(D(\mathcal{I}(Q,\tau))) = D(\mathcal{I}(Q,\tau)(p\alpha))[1].$$

Proof. In this case, by 6.5.(iii), we have $Q = p_\alpha^{-1}(p_\alpha(Q))$. Then, by Kashiwara's equivalence of categories, $R^0 j^!(\mathcal{I}(Q,\tau)) = \tau$ and $R^q j^!(\mathcal{I}(Q,\tau)) = 0$ for $q \neq 0$. This implies

$$R^p(j \circ s)^!(\mathcal{I}(Q,\tau)) = R^p s^!(R^0 j^!(\mathcal{I}(Q,\tau))) = R^p s^!(\tau).$$

Hence $R^p(j \circ s)^!(\mathcal{I}(Q,\tau)) = 0$ for $p \neq \dim Q - 1 = \dim p_\alpha(Q)$, and

$$R^{\dim Q-1}(j \circ s)^!(\mathcal{I}(Q,\tau)) = R^{\dim Q-1}s^!(\tau) = s^*(\tau) = \tau_o,$$

where τ_o is the restriction of τ to X_o. By 4.5 and the preceding calculations, we see that

$$R^q(j \circ s)^!(L^q I_{s_\alpha}(\mathcal{I}(Q,\tau))) = 0$$

for $q \neq -1$, and

$$\begin{aligned} R^{\dim Q-1}(j \circ s)^!(L^{-1} I_{s_\alpha}(\mathcal{I}(Q,\tau))) &= L^{-1}I(\tau_o) = \tau_o(p\alpha) \\ &= R^{\dim Q-1}(j \circ s)^!(\mathcal{I}(Q,\tau))(p\alpha) = R^{\dim Q-1}(j \circ s)^!(\mathcal{I}(Q,\tau)(p\alpha)). \end{aligned}$$

This implies our statement. □

Consider now the case of a Q-real root α. In this situation, by 6.5.(iv), Q is the open orbit in $p_\alpha^{-1}(p_\alpha(Q))$. The restriction to $Q_o = X_o \cap Q$ of the K-equivariant connection τ on Q defines a K_o-equivariant connection τ_o on $\mathbb{C}^*$. We say that τ on Q satisfies the SL_2-*parity condition with respect to* α if $R^{\dim Q-1}s^!(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q_o,\tau_o)$ satisfies the SL_2-parity condition.

Moreover, since α is a Q-real root, the twisted sheaves $\mathcal{D}_\lambda^{i_Q}$ and $\mathcal{D}_{s_\alpha\lambda}^{i_Q}$ correspond to the same invariant linear form on $\mathfrak{k} \cap \mathfrak{b}_x$, i.e., they are naturally isomorphic. Since the stabilizer S_x of $x \in Q$ in K maps into the stabilizer S of 1 in $X_o \cong \mathbb{P}^1$, we see that there is a point $\tilde{x}$ in Q which corresponds to -1 in $X_o \cong \mathbb{P}^1$, such that $S_{\tilde{x}} = S_x$. Let τ be a K-homogeneous connection on Q corresponding to the representation ω of S_x in the geometric fibre $T_x(\tau)$. Then there exists a unique K-homogeneous connection τ_{s_α} on Q such that ω is the representation of $S_{\tilde{x}} = S_x$ in $T_{\tilde{x}}(\tau_{s_\alpha})$. It can be interpreted as a K-homogeneous $\mathcal{D}_{s_\alpha\lambda}^{i_Q}$-connection on Q.

Proposition 7.7. *Let Q be a K-orbit in X, $\alpha \in \Pi$ a Q-real root and $\lambda \in \mathfrak{h}^*$. Assume that τ satisfies the* SL_2*-parity condition with respect to α. Then*

$$LI_{s_\alpha}(D(\mathcal{I}(Q,\tau))) = D(\mathcal{I}(Q,\tau_{s_\alpha})).$$

Proof. As in the preceding proof we first see, by base change, that $R^p(j \circ s)^!(\mathcal{I}(Q,\tau)) = 0$ for $p \neq \dim Q - 1$ and

$$R^{\dim Q-1}(j \circ s)^!(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q_o,\tau_o,\lambda_o)$$

where $Q_o = Q \cap X_o$ is the open orbit in X_o and τ_o is the restriction of τ to Q_o. On the other hand, by the calculation preceding 7.4 and 4.12,

$$R^{\dim Q-1}(j \circ s)^!(L^q I_{s_\alpha}(\mathcal{I}(Q,\tau)) = L^q I(\mathcal{I}(Q_o,\tau_o,\lambda)) = 0$$

if $q \neq 0$, and

$$R^{\dim Q-1}(j\circ s)^!(I_{s_\alpha}(\mathcal{I}(Q,\tau)) = I(\mathcal{I}(Q_o,\tau_o,\lambda)) \\ = \mathcal{I}(Q_o,\tilde{\tau}_o,-\lambda) = R^{\dim Q-1}(j\circ s)^!(\mathcal{I}(Q,\tau_{s_\alpha})).$$

As in the preceding argument, this implies our assertion. □

In addition, if $p=-\alpha\check{}(\lambda)\in\mathbb{Z}$, we see from 4.13 that $\tau_{s_\alpha}\cong\tau\otimes_{\mathcal{O}_Q} i_Q^*(\mathcal{O}(p\alpha))$. Hence we have the following result.

Corollary 7.8. *Let Q be a K-orbit in X, $\alpha\in\Pi$ a Q-real root and $\lambda\in\mathfrak{h}^*$. Assume that $p=-\alpha\check{}(\lambda)\in\mathbb{Z}$, and that τ satisfies the* SL$_2$*-parity condition with respect to α. Then*

$$LI_{s_\alpha}(D(\mathcal{I}(Q,\tau))) = D(\mathcal{I}(Q,\tau)(p\alpha)).$$

Finally, we have to introduce the notion of the SL$_2$*-parity condition* with respect to an arbitrary Q-real root α. Let $x\in Q$ and ω the representation of the stabilizer S_x of x in K in the geometric fibre $T_x(\tau)$. Then, as we explained at the end of §6, to α we attach an element $n_\alpha\in S_x$ of x in K. We say that τ satisfies the SL$_2$-parity condition with respect to α if the spectrum of the linear transformation $\omega(n_\alpha)$ does not contain $-e^{\pm i\pi\alpha\check{}(\lambda)}\delta_Q(\varphi(n_\alpha))$. Since n_α is determined up to inversion, this condition does not depend on the choice of n_α. By 6.23, if $\alpha\in\Pi$ it agrees with the previous defined parity condition. Moreover, for $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$ this condition agrees with the one in §4.

The next result describes how the parity condition behaves under the action of intertwining functors.

Lemma 7.9. (i) *Let τ and τ_w be the connections on Q and Q_w respectively, as in 7.2, and α a Q-real root. Then $w\alpha$ is a Q_w-real root and the following conditions are equivalent:*

(a) *τ satisfies the* SL$_2$*-parity condition with respect to α;*
(b) *τ_w satisfies the* SL$_2$*-parity condition with respect to $w\alpha$.*

(ii) *Let α be a Q-real root and τ and τ_{s_α} the K-homogeneous connections on Q as in 7.7. Let β be a Q-real root. Then $s_\alpha\beta$ is a Q-real root and the following conditions are equivalent:*

(a) *τ satisfies the* SL$_2$*-parity condition with respect to β;*
(b) *τ_{s_α} satisfies the* SL$_2$*-parity condition with respect to $s_\alpha\beta$.*

Proof. (i) Let $x\in Q$ and $\mathfrak{c}$ a σ-stable Cartan subalgebra contained in $\mathfrak{b}_x$. Denote by C the torus with Lie algebra $\mathfrak{c}$ in $G=\mathrm{Int}(\mathfrak{g})$. Then there exists $x'\in Q_w$ such that $\mathfrak{c}\subset\mathfrak{b}_{x'}$. Therefore, by 6.22, the stabilizers S_x and $S_{x'}$ of x and x' in K have a common Levi factor $T=\varphi^{-1}(\varphi(K)\cap C)$. Let ω and ω' be the representations of S_x and $S_{x'}$ respectively in geometric fibers of τ and τ_w. Let s and s' be the specializations determined by the Cartan subalgebra $\mathfrak{c}$ in $\mathfrak{b}_x$ and $\mathfrak{b}_{x'}$ respectively. Then $s'=s\circ w$. Since $\sigma_{Q_w}=w\circ\sigma_Q\circ w^{-1}$ by 6.11, we see that $w\alpha$ is a Q_w-real root if and only if α is a Q-real root. Moreover, the elements n_β, $\beta\in\Sigma_{Q_w,\mathbb{R}}$, and n'_γ, $\gamma\in\Sigma_{Q,\mathbb{R}}$, of T attached to these two specializations satisfy $n_\alpha=n'_{w\alpha}$ for $\alpha\in\Sigma_{Q,\mathbb{R}}$. From 6.11 we see that

$$\delta_{Q_w}=\prod_{\beta\in\Sigma^+_{w^{-1}}} e^\beta \prod_{\beta\in A} e^{w\beta},$$

where A is a set of representatives of the $(-\sigma_Q)$-orbits in $D_-(Q)$, and where the characters e^α are defined via the specialization s'. Since $\rho - w\rho$ is the sum of roots in $\Sigma^+_{w^{-1}}$, we have

$$\delta_{Q_w}(\varphi(n'_{w\alpha})) = e^{\rho - w\rho}(\varphi(n'_{w\alpha}))\delta_Q(\varphi(n_\alpha)),$$

and finally

$$\omega'(n'_{w\alpha})\delta_{Q_w}(\varphi(n'_{w\alpha})) = \omega(n'_{w\alpha})e^{\rho-w\rho}(\varphi(n'_{w\alpha}))\delta_{Q_w}(\varphi(n'_{w\alpha})) = \omega(n_\alpha)\delta_Q(\varphi(n_\alpha)).$$

This implies (i), since τ is a $\mathcal{D}_\lambda^{i_Q}$-connection and τ_w is a $\mathcal{D}_{w\lambda}^{i_{Q_w}}$-connection.

(ii) Clearly we have $\sigma_Q(s_\alpha\beta) = -\beta + \alpha\check{}(\beta)\alpha = -s_\alpha\beta$, and $s_\alpha\beta$ is a Q-real root. Let $x \in Q$ and ω the representation of the stabilizer S_x in the geometric fibre $T_x(\tau)$. Let $\mathfrak{c}$ be a σ-stable Cartan subalgebra in $\mathfrak{b}_x$. Then there exists a unique point $\tilde{x} \in Q$ different from x such that $p_\alpha(\tilde{x}) = p_\alpha(x)$ and $\mathfrak{b}_x \supset \mathfrak{c}$. The stabilizer $S_{\tilde{x}}$ is equal to S_x. The specializations s and $\tilde{s}$ attached to the Cartan subalgebra $\mathfrak{c}$ at these two points differ by the reflection s_α, hence the elements n_γ and $\tilde{n}_\gamma$, $\gamma \in \Sigma_{Q,\mathbb{R}}$, of the stabilizer attached to these two specializations satisfy $n_\gamma = \tilde{n}_{s_\alpha\gamma}$. Since the representations ω and $\tilde{\omega}$ of the stabilizer $S_x = S_{\tilde{x}}$, attached to τ at the points x and $\tilde{x}$ respectively, are conjugate, we see that the spectrum of $\omega(n_\beta)$ is equal to the spectrum of $\tilde{\omega}(n_{s_\alpha\beta}) = \tilde{\omega}(\tilde{n}_\beta)$. The representation of the stabilizer attached to τ_{s_α} at x is $\tilde{\omega}$, and the assertion follows since τ is a $\mathcal{D}_\lambda^{i_Q}$-connection and τ_{s_α} is a $\mathcal{D}_{s_\alpha\lambda}^{i_Q}$-connection. □

Finally, we want to analyze the structure of the standard module $\mathcal{I}(Q,\tau)$ in the situation when α is a Q-real simple root, $\alpha\check{}(\lambda) \in \mathbb{Z}$ and the SL_2-parity condition fails for τ with respect to α. Then $\mathcal{I}(Q,\tau) = j_+(l_+(\tau))$ with $l_+(\tau)$. Clearly, $l_+(\tau)$ is reducible by 7.3, since $s^+(l_+(\tau)) = \mathcal{I}(Q_o,\tau_o)$ is reducible by 4.10. Let $\mathcal{K}$ be its unique irreducible submodule corresponding under the restriction s^+ to $\mathcal{L}(Q_o,\tau_o)$. Then, by 7.3, we have the following exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow l_+(\tau) \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathcal{Q}$ is the direct sum of irreducible standard $(\mathcal{D}_\lambda^j, K)$-modules on V attached to the K-orbits in $V - Q$. Since j_+ is exact, this short exact sequence leads to the short exact sequence

$$0 \longrightarrow j_+(\mathcal{K}) \longrightarrow \mathcal{I}(Q,\tau) \longrightarrow j_+(\mathcal{Q}) \longrightarrow 0,$$

where $j_+(\mathcal{Q})$ is a direct sum of standard modules $\mathcal{I}(Q',\tau')$ for some K-orbits Q' in $V - Q$ and irreducible K-homogeneous connections τ' on Q'. By 4.14 and a previous discussion, we also have $I_{s_\alpha}(j_+(\mathcal{K})) = 0$. This establishes the following result.

Lemma 7.10. *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi$, Q a K-orbit in X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda + \rho$. Assume that α is Q-real, $\alpha\check{}(\lambda) \in \mathbb{Z}$ and the SL_2-parity condition fails for τ with respect to α. Then the standard Harish-Chandra sheaf $\mathcal{I}(Q,\tau)$ contains a Harish-Chandra subsheaf $\mathcal{C}$ such that*

(i) *$I_{s_\alpha}(\mathcal{C}) = 0$;*
(ii) *the quotient $\mathcal{I}(Q,\tau)/\mathcal{C}$ is a direct sum of standard Harish-Chandra sheaves on the K-orbits in $p_\alpha^{-1}(p_\alpha(Q)) - Q$.*

Finally, the same discussion, combined with 4.11, leads to the following result.

Lemma 7.11. *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi$, Q a K-orbit in X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda+\rho$. Assume that α is Q-imaginary root. Then the orbit Q is closed in $p_\alpha^{-1}(p_\alpha(Q))$. Let Q' be the open orbit in $p_\alpha^{-1}(p_\alpha(Q))$. Then there exists an irreducible K-homogeneous connection τ' on Q' such that $\mathcal{I}(Q,\tau)$ is a quotient of $\mathcal{I}(Q',\tau')$.*

In addition, if $\alpha\check{}(\lambda) \in \mathbb{Z}$, the kernel of the quotient map $\mathcal{I}(Q',\tau') \longrightarrow \mathcal{I}(Q,\tau)$ contains the Harish-Chandra sheaf $\mathcal{C}$ described in 7.10.

8. Irreducibility of standard Harish-Chandra sheaves

In this section we prove a necessary and sufficient condition for irreducibility of standard Harish-Chandra sheaves.

We start with a necessary condition for irreducibility. We use the notation from the preceding section. Let $\alpha \in \Pi$ and Q be a K-orbit. Denote $V = p_\alpha^{-1}(p_\alpha(Q))$ and $X_o = p_\alpha^{-1}(p_\alpha(x))$ for some $x \in Q$. Let τ be an irreducible K-homogeneous connection on Q compatible with $\lambda+\rho$ such that the standard Harish-Chandra sheaf $\mathcal{I}(Q,\tau)$ is irreducible. Clearly, V is a smooth subvariety of X and Q is affinely imbedded in V. Then we have the following diagram:

$$\begin{array}{ccc} Q & \xrightarrow{i} & V \\ \Big\| & & \Big\downarrow{\scriptstyle j} \\ Q & \xrightarrow{i_Q} & X \end{array} .$$

Therefore, by the base change

$$j^!(\mathcal{I}(Q,\tau)) = i_+(\tau)$$

and this is an irreducible $(\mathcal{D}_\lambda^j, K)$-module. Moreover, by 7.3, the restriction $s^+(i_+(\tau))$ is irreducible $(\mathcal{D}_{\lambda_o}^o, K_o)$-module. If we denote by $\tau_o = s^+(\tau)$ the restriction of τ to $Q_o = Q \cap X_o$, by the base change calculation, we get $s^+(i_+(\tau)) = \mathcal{I}(Q_o,\tau_o)$, i.e., it is a standard module on X_o. The following result follows immediately from 6.5, 4.4 and 4.10.

Lemma 8.1. *Let $\alpha \in \Pi$ and $\lambda \in \mathfrak{h}^*$. Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q such that $\mathcal{I}(Q,\tau)$ is irreducible. Then:*

(i) *if α is Q-complex and $\sigma_Q\alpha \notin \Sigma^+$, we have $\alpha\check{}(\lambda) \notin \mathbb{Z}$;*
(ii) *if α is Q-real, τ satisfies the SL_2-parity condition with respect to α.*

We shall use this result and intertwining functors to study the irreducibility of standard Harish-Chandra sheaves. We start with a discussion of a special case.

Assume that Q_o is the open orbit of K in X. Then it is the Langlands orbit attached to the conjugacy class of maximally split σ-stable Cartan subalgebras of $\mathfrak{g}$. We say that the pair $(\mathfrak{g}, K)$ is *split* if it satisfies the additional assumption:

(sp) there exists a σ-stable Cartan subalgebra in $\mathfrak{g}$ on which σ acts as -1.

This implies that all roots in Σ are Q-real.

Proposition 8.2. *Let $(\mathfrak{g}, K)$ be a split Harish-Chandra pair. Let $\mathcal{I}(Q_o,\tau)$ be a standard Harish-Chandra sheaf on Q_o. Then the following conditions are equivalent:*

(i) *τ satisfies the SL_2-parity condition for all $\alpha \in \Sigma$;*
(ii) *$\mathcal{I}(Q_o,\tau)$ is an irreducible Harish-Chandra sheaf.*

Proof. Assume first that $\alpha \in \Pi$ and that it satisfies the SL_2-parity condition. Then, by 7.7, we have

$$I_{s_\alpha}(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q,\tau').$$

There are two possibilities:

(a) $\alpha\check{}(\lambda) \notin \mathbb{Z}$. In this case I_{s_α} is an equivalence of categories. Hence $\mathcal{I}(Q,\tau)$ is irreducible if and only if $\mathcal{I}(Q,\tau')$ is irreducible.

(b) $p = -\alpha\check{}(\lambda) \in \mathbb{Z}$. Then, by 7.8, we have

$$I_{s_\alpha}(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q,\tau') = \mathcal{I}(Q,\tau)(p\alpha).$$

Hence, again $\mathcal{I}(Q,\tau)$ is irreducible if and only if $\mathcal{I}(Q,\tau')$ is irreducible.

By 7.9, this enables us to reduce the question of the SL_2-parity condition for an arbitrary root α to the case of *simple* root α. But in this case, the the irreducibility implies that the SL_2-parity condition holds by 8.1.(ii).

It remains to show the converse. Assume that τ satisfies the SL_2-parity condition for any root α. Then by the above discussion, for any $w \in W$ we have

$$I_w(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q,\tau_w),$$

and $\mathcal{I}(Q,\tau)$ is irreducible if and only if $\mathcal{I}(Q,\tau_w)$ is irreducible. Assume that $\mathcal{I}(Q,\tau)$ is reducible. Let $\mathcal{B}$ be an irreducible quotient of $\mathcal{I}(Q,\tau)$. Then its support is irreducible and, by 3.1 and 3.5, there exists $w \in W$, with the following property: supp $I_w(\mathcal{B})$ is irreducible and $\dim \operatorname{supp} I_w(\mathcal{B}) = \dim X - 1$. Since I_w is right exact, we conclude that $I_w(\mathcal{B})$ is a quotient of $\mathcal{I}(Q,\tau_w)$.

Therefore, again by 7.9, it is enough to show that the SL_2-parity condition implies that there are no quotients of $\mathcal{I}(Q,\tau)$ with irreducible support of dimension $\dim X - 1$. Assume that $\mathcal{A}$ is such quotient and that its support is the closure of an orbit Q' with $\dim Q' = \dim X - 1$. Then there exists a simple root $\alpha \in \Pi$ which is “transversal” to Q', i.e., if $p_\alpha : X \longrightarrow X_\alpha$ is the natural projection of X onto the variety X_α of all parabolic subalgebras of type α, $\dim p_\alpha^{-1}(p_\alpha(Q')) = \dim Q' + 1 = \dim X$. This implies that the projection of Q' into X_α is the open and dense orbit of K in X_α. The fiber over an arbitrary point in this orbit is isomorphic to the flag variety $X_o \cong \mathbb{P}^1$ of $\mathfrak{sl}(2,\mathbb{C})$. Let $s : \mathbb{P}^1 \longrightarrow X$ be the corresponding map. Since α is a Q-real root, the identity component of K_o is a one dimensional torus by 6.5.(iv). By base change, $s^+(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q_o,\tau_o)$ is a standard Harish-Chandra sheaf on X_o corresponding to the restriction τ_o of τ to the open orbit Q_o of K_o, and it has a nontrivial quotient supported in $\{0\} \cup \{\infty\}$. Since the SL_2-parity condition holds for α this is impossible by 4.10. □

We shall use 8.2 to prove a necessary and sufficient criterion for the irreducibility of standard Harish-Chandra sheaves on Zuckerman orbits. We start with the following observation. Let O be a closed orbit of K in X_Θ. Then $V = p_\Theta^{-1}(O)$ is a closed smooth subvariety of X and a union of K-orbits. Let $\mathcal{M}_{coh}^{\leq O}(\mathcal{D}_\lambda, K)$ be the full subcategory of $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ consisting of modules supported in V. The direct image functor j_+ is an equivalence of the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda^j, K)$ with $\mathcal{M}_{coh}^{\leq O}(\mathcal{D}_\lambda, K)$. Its inverse is $j^!$. This, in combination with 7.3, leads to the following result.

Lemma 8.3. *The functor $R^{\dim O}(j\circ s)^!$ is an equivalence of the category $\mathcal{M}_{coh}^{\leq O}(\mathcal{D}_\lambda, K)$ with $\mathcal{M}_{coh}(\mathcal{D}_{\lambda_o}^o, K_o)$.*

Let Q be a Zuckerman orbit. Then Σ^+ is a set of positive roots of Zuckerman type for (Σ, σ_Q). The set $P_Q = \Sigma_{Q,\mathbb{R}} \cup \Sigma^+$ is a σ_Q-stable parabolic set of roots by

5.10. Let $\Theta \subset \Pi$ be the corresponding set of simple roots, and X_Θ the generalized flag variety of all parabolic subalgebras of type Θ in $\mathfrak{g}$. Let $O = p_\alpha(Q)$. By 6.16.(ii), the orbit O is closed in X_Θ.

The fibre X_o over y is identified with the flag variety of $\mathfrak{g}_o$. Since $\mathfrak{p}_y$ is σ-stable, the Lie algebra of $\varphi^{-1}(\varphi(K) \cap P_y)$ is equal to $\mathfrak{k} \cap \mathfrak{p}_y$. Let $\mathfrak{c}$ be a σ-stable Cartan subalgebra in $\mathfrak{b}_x$, R the root system of $(\mathfrak{g}, \mathfrak{c})$ in $\mathfrak{c}^*$, and R^+ the set of positive roots determined by $\mathfrak{b}_x$. Then $\mathfrak{p}_y$ is spanned by the Borel subalgebra $\mathfrak{b}_x$ and the root subspaces $\mathfrak{g}_\alpha$ for all real roots $\alpha \in R$. Clearly $\mathfrak{c}$ and $\mathfrak{g}_\alpha$, for all real roots $\alpha \in R$, span a σ-stable Levi factor $\mathfrak{l}_o$ of $\mathfrak{p}_y$, and $\mathfrak{g}_o$ is canonically isomorphic to $[\mathfrak{l}_o, \mathfrak{l}_o]$. The center of $\mathfrak{l}_o$ is the intersection of kernels of all real roots in $\mathfrak{c}^*$. The involution σ induces an involution σ_o on $\mathfrak{g}_o \cong [\mathfrak{l}_o, \mathfrak{l}_o]$. Therefore, the Lie algebra $\mathfrak{k}_o$ of K_o can be identified with $\mathfrak{k} \cap [\mathfrak{l}_o, \mathfrak{l}_o]$ which is the set of fixed points of σ_o. The intersection of $\mathfrak{c}$ and $[\mathfrak{l}_o, \mathfrak{l}_o]$ determines a Cartan subalgebra of $\mathfrak{g}_o$ on which σ_o acts as -1. Thus $(\mathfrak{g}_o, K_o)$ is a split Harish-Chandra pair.

Proposition 8.4. *Let Q be a Zuckerman orbit in X, λ an element of $\mathfrak{h}^*$, and τ a K-homogeneous connection on Q compatible with λ. Then the following conditions are equivalent:*

(i) *τ satisfies the SL_2-parity condition for all Q-real roots $\alpha \in \Sigma$;*
(ii) *the standard module $\mathcal{I}(Q, \tau)$ is irreducible.*

Proof. Let $Q_o = Q \cap X_o$. Then Q_o is a K_o-orbit in X_o of a Borel subalgebra of $\mathfrak{g}_o$ which contains a Cartan subalgebra on which σ_o acts as -1. This implies that all roots in Σ_o are Q_o-real, and that Q_o is open in X_o. By base change,

$$R^{\dim O}(j \circ s)^!(\mathcal{I}(Q, \tau)) = \mathcal{I}(Q_o, \tau_o),$$

where τ_o is the restriction of τ to Q_o. Since all irreducible composition factors of $\mathcal{I}(Q, \tau)$ lie in $\mathcal{M}_{coh}^{\leq O}(\mathcal{D}_\lambda, K)$, by 8.3, $\mathcal{I}(Q, \tau)$ is irreducible if and only if $\mathcal{I}(Q_o, \tau_o)$ is irreducible. Now τ satisfies the SL_2-parity condition for all Q-real roots if and only if τ_o satisfies the the SL_2-parity condition for all Q_o-real roots, i.e., for all roots in Σ_o, so the assertion follows from 8.2. □

Next, we prove a result which reduces the problem of irreducibility of standard Harish-Chandra sheaves to the special case of Zuckerman orbits.

Lemma 8.5. *Let Q be a Zuckerman orbit, $\lambda \in \mathfrak{h}^*$, and $w \in W$. Suppose Σ_w^+ consists of Q-complex roots, and $\Sigma_w^+ \cap \sigma_Q(\Sigma_w^+) = \emptyset$. Then the following conditions are equivalent:*

(i) *$\Sigma_w^+ \cap \Sigma_\lambda = \emptyset$ and $\mathcal{I}(Q, \tau)$ is irreducible $\mathcal{D}_\lambda$-module;*
(ii) *$\mathcal{I}(Q_w, \tau_w)$ is irreducible $\mathcal{D}_{w\lambda}$-module.*

Proof. First we remark that in this case $D_+(Q)$ consists of all positive Q-complex roots. If $\Sigma_w^+ \cap \Sigma_\lambda = \emptyset$, by 2.9, the intertwining functor $I_w : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$ is an equivalence of categories and $I_{w^{-1}}$ its inverse. By 7.2, we have

$$I_w(\mathcal{I}(Q, \tau)) = \mathcal{I}(Q_w, \tau_w).$$

Therefore $\mathcal{I}(Q, \tau)$ is irreducible if and only if $\mathcal{I}(Q_w, \tau_w)$ is an irreducible $\mathcal{D}_{w\lambda}$-module.

Now we shall prove, by induction on $\ell(w)$, that $\Sigma_w^+ \cap \Sigma_\lambda \neq \emptyset$ only if $\mathcal{I}(Q_w, \tau_w)$ is a reducible $\mathcal{D}_{w\lambda}$-module. If $\ell(w) = 0$, $w = 1$ and $\Sigma_w^+ = \emptyset$, so the assertion is obvious. Thus we assume the statement holds for all $w' \in W$ with $\ell(w') < k$. Let

$\ell(w) = k$. Then $w = s_\alpha w'$ for some $\alpha \in \Pi$ and $w' \in W$ with $\ell(w') = k-1$. As we remarked in §2, $\Sigma_w^+ = \{w'^{-1}(\alpha)\} \cup \Sigma_{w'}^+$. Therefore, $\Sigma_{w'}^+$ consists of Q-complex roots and $\Sigma_{w'}^+ \cap \sigma_Q(\Sigma_{w'}^+) = \emptyset$. By 2.5 and 7.2, we have

$$\mathcal{I}(Q_w, \tau_w) = I_w(\mathcal{I}(Q,\tau)) = I_{s_\alpha}(I_{w'}(\mathcal{I}(Q,\tau))) = I_{s_\alpha}(\mathcal{I}(Q_{w'}, \tau_{w'})),$$

and $L^{-1} I_{s_\alpha}(\mathcal{I}(Q_{w'}, \tau_{w'})) = 0$.

If $w'^{-1}(\alpha) \notin \Sigma_\lambda$, i.e., $\alpha \notin \Sigma_{w'\lambda}$, we have $\Sigma_w^+ \cap \Sigma_\lambda = \Sigma_{w'}^+ \cap \Sigma_\lambda$, and by the induction assumption $\mathcal{I}(Q_{w'}, \tau_{w'})$ is a reducible $\mathcal{D}_{w'\lambda}$-module if $\Sigma_{w'}^+ \cap \Sigma_\lambda \neq \emptyset$. Since, by 2.9, in this case $I_{s_\alpha} : \mathcal{M}_{qc}(\mathcal{D}_{w'\lambda}) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{w\lambda})$ is an equivalence of categories, $\mathcal{I}(Q_w, \tau_w)$ is a reducible $\mathcal{D}_{w\lambda}$-module if $\Sigma_w^+ \cap \Sigma_\lambda \neq \emptyset$.

If $\alpha \in \Sigma_{w'\lambda}$, $p = -\alpha\check{}(w'\lambda) = \alpha\check{}(w\lambda)$ is an integer, and $-w^{-1}\alpha \in \Sigma_w^+$. This implies that $-w^{-1}\alpha$ is a positive Q-complex root, and $-\sigma_Q(w^{-1}\alpha) \notin \Sigma_w^+$. Since $\sigma_{Q_w} = w \circ \sigma_Q \circ w^{-1}$ by 6.11, we see that α is a Q_w-complex root and $\sigma_{Q_w}\alpha \notin \Sigma^+$. By 8.1.(i) this implies that $\mathcal{I}(Q,\tau)$ is reducible. □

We shall need the following auxiliary result. Let $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ and σ an involution on $\mathfrak{g}$ given by $\sigma(A) = J\,A\,J^{-1}$, $A \in \mathfrak{g}$, with

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Then σ is a Cartan involution for the real form $\mathfrak{su}(2,1)$ of $\mathfrak{sl}(3,\mathbb{C})$. Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair such that $\mathfrak{k}$ is the Lie algebra of fixed points of σ. For simplicity assume that K covers the subgroup of $\mathrm{SL}(3,\mathbb{C})$ consisting of all matrices of the form

$$\begin{pmatrix} \begin{matrix} & A & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \det A^{-1} \end{pmatrix} ,$$

where A is an arbitrary 2×2 matrix. Write ψ for the projection of K into $\mathrm{SL}(3,\mathbb{C})$. Let $\mathfrak{c}$ be the σ-stable Cartan subalgebra spanned by

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

and C the corresponding Cartan subgroup of $\mathrm{SL}(3,\mathbb{C})$. Let R be the root system of $(\mathfrak{g},\mathfrak{c})$ in $\mathfrak{c}^*$. Then R contains a unique real root α such that the dual root H_α is equal to H. The only other real root is $-\alpha$, and the remaining roots are complex. Recall the meaning of m_α and n_α, which were defined at the end of §6. Note that

$$m_\alpha = \exp(i\pi H_\alpha) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \exp(i\pi T).$$

Lemma 8.6. (i) *The subgroup $S = \psi^{-1}(\psi(K) \cap C)$ of K is isomorphic to $\mathbb{C}^*$.*

(ii) *We can choose $n_\alpha = \exp(i\pi T)$.*

(iii) *Let ω be a character of S and $\mu \in \mathfrak{c}^*$ such that the differential of ω agrees with the restriction of μ to the subspace of $\mathfrak{c}$ spanned by T. If β is a complex root such that $\beta\check{}(\mu) \in \mathbb{Z}$, we have*

$$\omega(n_\alpha) = e^{\pm i\pi\alpha\check{}(\mu)}.$$

Proof. (i) First we claim that $\psi(K)\cap C$ is the one-parameter subgroup in $\mathrm{SL}(3,\mathbb{C})$ determined by T. Since the exponential map $\exp : \mathfrak{c} \longrightarrow C$ is surjective, any $k \in \psi(K)\cap C$ has the form $k = \exp(aT + bH)$. From $k = \sigma(k)$ we get $k = \exp(aT - bH)$ and $k^{-1} = \exp(-aT + bH)$. This implies $1 = \exp(2bH)$ and $b \in i\pi\mathbb{Z}$. Since $m_\alpha = \exp(i\pi H) = \exp(i\pi T)$, the assertion follows.

We can identify $\psi(K)$ with $\mathrm{GL}(2,\mathbb{C})$. Then $\pi(A,z) = zA$ defines a homomorphism of $\mathrm{SL}(2,\mathbb{C})\times\mathbb{C}^*$ into $\psi(K)$, which is a two-fold covering. The nontrivial element of the kernel of π is $(-I,-1)$. Since $\mathrm{SL}(2,\mathbb{C})$ is simply connected, the fundamental group of $\psi(K)$ is $\mathbb{Z}$. For even $n = 2k$, the n-fold covering of $\psi(K)$ factors through $\mathrm{SL}(2,\mathbb{C})\times\mathbb{C}^*$. Without any loss of generality we can assume that K is the n-fold cover of $\psi(K)$. Hence $K \cong \mathrm{SL}(2,\mathbb{C})\times\mathbb{C}^*$ and $\psi(A,z) = z^k A$. This implies that $\ker\psi$ consists of all elements of the form (I,ζ) and $(-I, e^{i\frac{\pi}{k}}\,\zeta)$, where ζ is an arbitrary k-th root of unity. Since

$$T = \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ,$$

and since the first matrix lies in the image of the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$ and the second in the center of $\mathfrak{k}$, we see that the first component of $\exp(zT) \in K = \mathrm{SL}(2,\mathbb{C})\times\mathbb{C}^*$ is equal to

$$\exp\begin{pmatrix} \frac{3z}{2} & 0 \\ 0 & -\frac{3z}{2} \end{pmatrix} = \begin{pmatrix} e^{\frac{3z}{2}} & 0 \\ 0 & e^{-\frac{3z}{2}} \end{pmatrix}$$

for any $z \in \mathbb{C}^*$, hence the second is equal to $e^{-\frac{z}{n}}$. If $z = 2\pi i q$, $q \in \mathbb{Z}$, $\exp(2q\pi i T)$ is one of the elements of $\ker\psi$, and all of them are obtained in this way. This completes the proof of (i).

(ii) The matrix

$$T + X_\alpha + X_{-\alpha} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

lies in the image of $\mathfrak{sl}(2,\mathbb{C})$. Therefore,

$$\exp(i\pi(T + X_\alpha + X_{-\alpha})) = \exp\begin{pmatrix} 2\pi i & 0 & 0 \\ 0 & -2\pi i & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1$$

in the image of $\mathrm{SL}(2,\mathbb{C})$, and this identity persists in K. Hence, in K,

$$\exp(-i\pi(X_\alpha + X_{-\alpha})) = \exp(i\pi T).$$

Since $\mathfrak{k}_\alpha$ is spanned by $X_\alpha + X_{-\alpha}$, and since

$$\exp(t(X_\alpha + X_{-\alpha})) = \exp\begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -t \end{pmatrix} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix},$$

we see that

$$\exp(t(X_\alpha + X_{-\alpha})) \neq m_\alpha$$

for $t \notin i\pi(2\mathbb{Z}+1)$. This implies that $n_\alpha = \exp(i\pi T)$ is a possible choice for n_α.

(iii) We may assume that β is a complex root in $\mathfrak{c}$ such that $\alpha = \beta - \sigma\beta$. Then $\alpha\check{}(\beta) = 1$ and

$$e^\beta(m_\alpha) = e^{i\pi\alpha\check{}(\beta)} = -1.$$

On the other hand,

$$e^{\beta}(m_{\alpha}) = e^{i\pi\beta(T)},$$

so $\beta(T)$ is an odd integer and T is a weight in the dual root system. Since $\exp(2\pi i T) = 1$, T lies in the dual root lattice. Analogously, we see that sT, $0 < s < 1$, does not lie in the dual root lattice. This implies $\beta(T) = \pm 1$ and

$$T = \pm(\beta\check{} + (\sigma\beta)\check{}).$$

Hence

$$\omega(n_{\alpha}) = e^{i\pi\mu(T)} = e^{i\pi(\beta\check{}(\mu)+(\sigma\beta)\check{}(\mu))} = e^{-i\pi\alpha\check{}(\mu)}e^{2\pi i\beta\check{}(\mu)} = e^{-i\pi\alpha\check{}(\mu)}$$

if $T = \beta\check{} + (\sigma\beta)\check{}$, and

$$\omega(n_{\alpha}) = e^{i\pi\mu(T)} = e^{-i\pi(\beta\check{}(\mu)+(\sigma\beta)\check{}(\mu))} = e^{i\pi\alpha\check{}(\mu)}e^{-2\pi i\beta\check{}(\mu)} = e^{i\pi\alpha\check{}(\mu)}$$

otherwise. □

Now we prove the irreducibility criterion in the general situation. Let Q be K-orbit in X, σ_Q the induced involution on the root system Σ. As explained in §5, the root system with involution $(\Sigma, -\sigma_Q)$ determines a subset $C(\Sigma^+) \subset D(\Sigma^+)$ of Σ^+, which we now denote by $C_-(Q)$.

Theorem 8.7. *Let Q be a K-orbit in X, λ an element of $\mathfrak{h}^*$, and τ an irreducible K-homogeneous connection on Q compatible with $\lambda+\rho$. Then the following conditions are equivalent:*

(i) *$C_-(Q) \cap \Sigma_{\lambda} = \emptyset$, and τ satisfies the* SL_2*-parity condition with respect to every Q-real root in Σ;*

(ii) *the standard $\mathcal{D}_{\lambda}$-module $\mathcal{I}(Q,\tau)$ is irreducible.*

Proof. By 6.13 there exist a Zuckerman orbit Q_1 and $w \in W$, such that $\Sigma^+_{w^{-1}}$ consists of Q_1-complex roots, $\Sigma_{w^{-1}} \cap (-\sigma_{Q_1}(\Sigma_{w^{-1}})) = \emptyset$, $Q = (Q_1)_{w^{-1}}$, $\Sigma_w \cap (-\sigma_Q(\Sigma_w)) = \emptyset$, and

$$D_-(Q) = \Sigma_w \cup (-\sigma_Q(\Sigma_w)).$$

Also there exists an irreducible K-homogeneous connection τ' on Q_1 such that $\tau = \tau'_{w^{-1}}$. Then, by 8.5, the standard $\mathcal{D}_{\lambda}$-module $\mathcal{I}(Q,\tau)$ is irreducible if and only if $\mathcal{I}(Q_1,\tau')$ is an irreducible $\mathcal{D}_{w\lambda}$-module and $\Sigma_{w^{-1}} \cap \Sigma_{w\lambda} = \emptyset$.

By 8.4, $\mathcal{I}(Q_1,\tau')$ is irreducible if and only if τ' satisfies the SL_2-parity condition for every Q_1-real root in Σ. By 7.9.(i), this is equivalent to the SL_2-parity condition for τ and every Q-real root in Σ.

It remains to analyze the condition $\Sigma_{w^{-1}} \cap \Sigma_{w\lambda} = \emptyset$, which is equivalent also to $\Sigma_w \cap \Sigma_{\lambda} = \emptyset$. We have to show that $C_-(Q) \cap \Sigma_{\lambda} = \emptyset$ is equivalent to $\Sigma^+_w \cap \Sigma_{\lambda} = \emptyset$ when the SL_2-parity condition is satisfied for all Q-real roots in Σ.

By 5.6 we have $\Sigma^+_w \subset C_-(Q)$, hence $C_-(Q) \cap \Sigma_{\lambda} = \emptyset$ implies $\Sigma^+_w \cap \Sigma_{\lambda} = \emptyset$. We still must prove the opposite implication when the SL_2-parity condition is satisfied for τ and all Q-real roots in Σ.

By 5.6, it is enough to establish the following statement:

(*) Assume the SL_2-parity condition is satisfied for τ and all Q-real roots in Σ. Let $\alpha \in \Sigma_{Q,\mathbb{C}}$ be such that either
   (a) closed root subsystem Σ_{α} of Σ generated by α and $\sigma_Q\alpha$ is of type $A_1 \times A_1$, or
   (b) the closed root subsystem Σ_{α} of Σ generated by α and $\sigma_Q\alpha$ is of type A_2, and $\alpha - \sigma_Q\alpha$ is a Q-real root.

Then either $\{\alpha, \sigma_Q\alpha\} \subset \Sigma_\lambda$ or $\{\alpha, \sigma_Q\alpha\} \cap \Sigma_\lambda = \emptyset$.

By 7.9, this statement is equivalent to the analogous statement for the connection τ' on Q_1. Therefore, in proving (*) we can assume without any loss of generality that Q is a Zuckerman orbit.

Let $x \in Q$ and $\mathfrak{c}$ a σ-stable Cartan subalgebra in $\mathfrak{b}_x$. Then, by the specialization corresponding to Q, the root subsystem Σ_α of Σ determines the semisimple Lie subalgebra generated by the root subspaces $\mathfrak{g}_\beta$, $\beta \in \Sigma_\alpha$, which we denote by $\mathfrak{g}_\circ$. By its construction $\mathfrak{g}_\circ$ is σ-invariant. Let $\sigma_\circ$ be the involution on $\mathfrak{g}_\circ$ induced by σ. Its fixed point set is the subalgebra $\mathfrak{k}_\circ = \mathfrak{k} \cap \mathfrak{g}_\circ$. Let $N(\mathfrak{g}_\circ)$ be the connected component of the normalizer of $\mathfrak{g}_\circ$ in K, $C(\mathfrak{g}_\circ)$ the connected component of the centralizer of $\mathfrak{g}_\circ$ in K, and put $K_\circ = N(\mathfrak{g}_\circ)/C(\mathfrak{g}_\circ)$. Then $K_\circ$ acts on $\mathfrak{g}_\circ$, and the differential of this action defines an isomorphism of the Lie algebra of $K_\circ$ with $\mathfrak{k}_\circ$. Therefore $(\mathfrak{g}_\circ, K_\circ)$ is a Harish-Chandra pair. The subalgebra $\mathfrak{c}_\circ = \mathfrak{g}_\circ \cap \mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}_\circ$, which lies in the Borel subalgebra $\mathfrak{b}_\circ = \mathfrak{g}_\circ \cap \mathfrak{b}_x$. If we let $\mathfrak{h}_\circ$ denote the Cartan algebra of $\mathfrak{g}_\circ$, we get a natural injection $\mathfrak{h}_\circ \longrightarrow \mathfrak{c}_\circ \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{h}$. This map induces a restriction map $\mathfrak{h}^* \longrightarrow \mathfrak{h}_\circ^*$. The kernel of this map equals $\{\mu \in \mathfrak{h}^* \mid \alpha\check{}(\mu) = (\sigma\alpha)\check{}(\mu) = 0\}$. The restriction map identifies the root subsystem Σ_α with the root system of $\mathfrak{g}_\circ$ in $\mathfrak{h}_\circ^*$ and maps $\Sigma_\alpha^+ = \Sigma_\alpha \cap \Sigma^+$ into a set of positive roots $\Sigma_\circ^+$. The set of simple roots $\Pi_\circ$ determined by $\Sigma_\circ^+$ corresponds to Π_α under this identification. In addition, if we let $\lambda_\circ$ denote the restriction of $\lambda \in \mathfrak{h}^*$ to $\mathfrak{h}_\circ$, we see that $\beta \in \Sigma_\lambda$ is equivalent to $\beta \in (\Sigma_\circ)_{\lambda_\circ}$ for any $\beta \in \Sigma_\alpha$.

In the case (a), $\mathfrak{g}_\circ \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$, and $\sigma_\circ$ acts as $\sigma_\circ(\xi,\eta) = (\eta,\xi)$ for $\xi, \eta \in \mathfrak{sl}(2,\mathbb{C})$. Thus $\mathfrak{k}_\circ$ is the diagonal in $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$. This implies that the group $K_\circ$ is a covering of the group $\mathrm{PSL}(2,\mathbb{C})$, i.e., it is either $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{C})$. In the first case, $K_\circ$ is the diagonal subgroup of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, and in the second the diagonal subgroup of $\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$. The Harish-Chandra pair $(\mathfrak{g}_\circ, K_\circ)$ is therefore linear in this case. By 7.1.(ii), $(\Sigma_\circ)_{\lambda_\circ}$ is either empty or equal to $\Sigma_\circ$. This implies that either $\Sigma_\alpha \cap \Sigma_\lambda = \emptyset$ or $\Sigma_\alpha \subset \Sigma_\lambda$. This proves (*) in this case.

In the case (b), $\mathfrak{g}_\circ = \mathfrak{sl}(3,\mathbb{C})$ and the Harish-Chandra pair $(\mathfrak{g}_\circ, K_\circ)$ is the one described before 8.6 (by passing to a finite cover of $K_\circ$ if necessary). Let $\beta = \alpha - \sigma\alpha$ be the unique positive Q-real root in Σ_α^+. Suppose $\{\alpha, \sigma_Q\alpha\} \cap \Sigma_\lambda \neq \emptyset$. Then, without any loss of generality, we can assume that $\alpha\check{}(\lambda) \in \mathbb{Z}$.

Assume first $\beta \in \Pi$. Then $\beta\check{}(\rho) = 1$ and $e^{i\pi\beta\check{}(\rho)} = -1$. If ω is the representation of the stabilizer S_x, by 8.6.(iii) we see that either $\omega(n_\beta)e^{i\pi\beta\check{}(\lambda+\rho)}$ or $\omega(n_\beta)e^{-i\pi\beta\check{}(\lambda+\rho)}$ has an eigenvalue equal to 1. This implies that $\omega(n_\beta)$ has an eigenvalue equal to $-e^{\pm i\pi\beta\check{}(\lambda)}$. Hence the SL_2-parity condition fails for β, contrary to our assumption. It follows that $\alpha\check{}(\lambda) \notin \mathbb{Z}$ and we have a contradiction.

Assume now that β is not simple. Since Q is a Zuckerman orbit, by 5.10 we see that the root system of Q-real roots has the set of all simple Q-real roots as a basis. Therefore, $\beta = w^{-1}\gamma$, where w is a product of simple reflections with respect to Q-real roots, and γ a simple Q-real root satisfying $\gamma = w\alpha - \sigma_Q w\alpha$. By 7.9, the SL_2-parity conditions hold for a connection τ' on Q which is compatible with $w\lambda + \rho$. Therefore, by the preceding part of the proof, we conclude $(w\alpha)\check{}(w\lambda) \notin \mathbb{Z}$. This in turn implies that $\alpha\check{}(\lambda) \notin \mathbb{Z}$, and we have a contradiction again. Therefore $\{\alpha, \sigma_Q\alpha\} \cap \Sigma_\lambda = \emptyset$, and (*) holds also in this case. $\square$

The preceding argument implies that we can replace $C_-(Q)$ by a smaller subset which does not contain any element of the pair $\{\alpha, -\sigma_Q\alpha\}$ if Σ_α is of type A_2, since integrality with respect to one of these roots automatically implies that the SL_2-parity condition fails for the Q-real root $\beta = \alpha - \sigma_Q\alpha$. Since the integrality of λ with respect to a Q-complex root is easier to check than the parity condition, it seems natural to leave this redundant condition.

The next corollary is the $\mathcal{D}$-module version of a result of B. Speh and D. Vogan ([20]). It can be deduced directly from 8.2, 8.4 and 8.5, skipping a considerable amount of combinatorics related to $C_-(Q)$.

Corollary 8.8. *Let $(\mathfrak{g}, K)$ be a linear Harish-Chandra pair. Let Q be a K-orbit, $\lambda \in \mathfrak{h}^*$. Then the following conditions are equivalent:*

(i) *$D_-(Q) \cap \Sigma_\lambda = \emptyset$, and τ satisfies the* SL_2*-parity condition for every Q-real root in Σ;*

(ii) *$\mathcal{I}(Q, \tau)$ is an irreducible $\mathcal{D}_\lambda$-module.*

Proof. Using the notation from the preceding proof, by 7.1.(ii), it follows that the condition $\Sigma_w^+ \cap \Sigma_\lambda = \emptyset$ is equivalent to

$$D_-(Q) \cap \Sigma_\lambda = (\Sigma_w^+ \cup (-\sigma_Q \Sigma_w^+)) \cap \Sigma_\lambda = \emptyset$$

for linear Harish-Chandra pairs. This, in conjunction with the preceding proof, completes the argument. □

9. Geometric classification of irreducible Harish-Chandra modules

In this section we describe the geometric classification of irreducible Harish-Chandra modules due to Beilinson and Bernstein [3].

Let V be an irreducible Harish-Chandra module. We can view V as an irreducible object in the category $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$. Clearly, the real parts of the elements of θ form a Weyl group orbit $\operatorname{Re}\theta$ and contain a unique strongly antidominant element. If we fix a strongly antidominant $\lambda \in \theta$, $\operatorname{Re}\lambda$ is independent of the choice of λ. By 2.15.(ii), there exists a unique irreducible $\mathcal{D}_\lambda$-module $\mathcal{V}$ such that $\Gamma(X, \mathcal{V}) = V$. Since this $\mathcal{D}_\lambda$-module must be a Harish-Chandra sheaf, it is of the form $\mathcal{L}(Q, \tau)$ for some K-orbit Q in X and irreducible K-homogeneous connection τ on Q. Hence, there is a unique pair (Q, τ) such that $\Gamma(X, \mathcal{L}(Q, \tau)) = V$. Therefore, if σ_Q is the involution determined by Q, we can define

$$\lambda_Q = \frac{1}{2}(\lambda - \sigma_Q\lambda)$$

and

$$\lambda^Q = \frac{1}{2}(\lambda + \sigma_Q\lambda).$$

Clearly, $\lambda = \lambda_Q + \lambda^Q$. Moreover, by 7.1.(i), we have $\alpha\check{}(\lambda_Q) \in \mathbb{R}$, i.e., λ_Q is a real linear form on $\mathfrak{h}^*$. In addition,

$$\lambda^Q + \operatorname{Re}\lambda_Q = \operatorname{Re}\lambda$$

is an invariant which depends only on θ.

If λ is in addition regular, the above correspondence gives a parametrization of equivalence classes of irreducible Harish-Chandra modules by all pairs (Q, τ). On the other hand, if λ is not regular, some of pairs (Q, τ) correspond to irreducible Harish-Chandra sheaves $\mathcal{L}(Q, \tau)$ with $\Gamma(X, \mathcal{L}(Q, \tau)) = 0$. Therefore, to give a precise formulation of this classification of irreducible Harish-Chandra modules, we have to determine a necessary and sufficient condition for nonvanishing of global sections of irreducible Harish-Chandra sheaves $\mathcal{L}(Q,\tau)$.

Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant and Q a K-orbit in X.

If α is Q-imaginary root, its dual root $\alpha\check{}$ vanishes on λ_Q, and $\alpha\check{}(\lambda)$ is real.

Let

$$\Sigma_0 = \{\alpha \in \Sigma \mid \operatorname{Re} \alpha\check{}(\lambda) = 0\}.$$

Put $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$ and $\Pi_0 = \Pi \cap \Sigma_0$. Since λ is strongly antidominant, Π_0 is the basis of the root system Σ_0 determined by the set of positive roots Σ_0^+. Let W_0 be the Weyl group of Σ_0.

Let $\Sigma_1 = \Sigma_0 \cap \sigma_Q(\Sigma_0)$; equivalently, Σ_1 is the largest root subsystem of Σ_0 invariant under σ_Q. Let $\Sigma_1^+ = \Sigma_1 \cap \Sigma^+$, and Π_1 the corresponding basis of the root system Σ_1. Clearly, $\Pi_0 \cap \Sigma_1 \subset \Pi_1$, but this inclusion is strict in general.

If $\alpha \in \Pi_0$, there are the following possibilities:

(i) α is Q-imaginary root and $\alpha\check{}(\lambda) = 0$;
(ii) α is Q-complex and $\sigma_Q\alpha$ is positive;
(iii) α is Q-complex, $-\sigma_Q\alpha$ is positive;
(iv) α is Q-real.

Simple roots in Π_0 of type (i) and (iv) are automatically in Π_1. The roots in $\Pi_0 - \Pi_1$ must be of type (ii) or (iii).

Let

$$\Sigma_2 = \{\alpha \in \Sigma_1 \mid \alpha\check{}(\lambda) = 0\}.$$

If $\alpha \in \Sigma_1$, by a previous remark, we have

$$\alpha\check{}(\lambda^Q) = \frac{1}{2}(\operatorname{Re} \alpha\check{}(\lambda) + \operatorname{Re}(\sigma_Q\alpha)\check{}(\lambda)) = 0.$$

Therefore, it follows that for $\alpha \in \Sigma_2$, we have $(\sigma_Q\alpha)\check{}(\lambda) = 0$ and $\sigma_Q\alpha \in \Sigma_2$. Hence, Σ_2 is also σ_Q-invariant. Let $\Sigma_2^+ = \Sigma_2 \cap \Sigma^+$, and Π_2 the corresponding basis of the root system Σ_2. Again, $\Pi_0 \cap \Sigma_2 \subset \Pi_1 \cap \Sigma_2 \subset \Pi_2$, but these inclusions are strict in general.

The next theorem gives the simple necessary and sufficient condition for $\Gamma(X, \mathcal{L}(Q,\tau)) \neq 0$, that was alluded to before. In effect, this completes the classification of irreducible Harish-Chandra modules.

Theorem 9.1. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant. Let Q be a K-orbit in X and τ a K-homogeneous irreducible connection on Q compatible with $\lambda + \rho$. Then the following conditions are equivalent:*

(i) $\Gamma(X, \mathcal{L}(Q,\tau)) \neq 0$*;*
(ii) *the following conditions hold for the pair (Q,τ):*
  (a) *the set Π_2 contains no compact Q-imaginary roots;*
  (b) *for any Q-complex root $\alpha \in \Sigma^+$ with $\alpha\check{}(\lambda) = 0$, the root $\sigma_Q\alpha$ is also positive;*
  (c) *for any Q-real $\alpha \in \Sigma$ with $\alpha\check{}(\lambda) = 0$, τ must satisfy the SL_2-parity condition with respect to α.*

The proof is based on the following lemma.

Lemma 9.2. *Let $D_-(Q) \cap \Pi_0 = \emptyset$. Then*

(i) $\Pi_1 \subset \Pi_0$*;*
(ii) $D_-(Q) \cap \Sigma_0^+ = \emptyset$.

Proof. (i) Let $\Pi' = \Pi_0 \cap \Sigma_1$. Let $\beta \in \Sigma_1^+$. Then $\beta = \sum_{\alpha\in\Pi_0} n_\alpha \alpha$, $n_\alpha \in \mathbb{Z}_+$, and

$$\sigma_Q\beta = \sum_{\alpha\in\Pi'} n_\alpha \sigma_Q\alpha + \sum_{\alpha\in\Pi_0-\Pi'} n_\alpha \sigma_Q \alpha.$$

Since Σ_1 is σ_Q-invariant, $\sigma_Q\beta$ and $\sigma_Q\alpha$, $\alpha\in\Pi'$, are in Σ_1. This implies that

$$\sum_{\alpha\in\Pi_0-\Pi'} n_\alpha\sigma_Q\alpha \in Q(\Sigma_1)\subset Q(\Sigma_0).$$

Hence, with respect to the canonical inner product on $\mathfrak{h}_0^*$, we have

$$0 = \left(\sum_{\alpha\in\Pi_0-\Pi'} n_\alpha \sigma_Q\alpha \middle| \operatorname{Re}\lambda \right) = \sum_{\alpha\in\Pi_0-\Pi'} n_\alpha(\sigma_Q\alpha \mid \operatorname{Re}\lambda).$$

Since Π_0 does not contain roots of type (iii), roots $\alpha\in\Pi_0-\Pi'$ are of type (ii). Hence, the $\sigma_Q\alpha$ are positive roots and $(\sigma_Q\alpha\mid\operatorname{Re}\lambda)\le 0$. On the other hand, $\alpha\notin\Sigma_1$ leads to $(\sigma_Q\alpha|\operatorname{Re}\lambda)\neq 0$. Therefore, $(\sigma_Q\alpha\mid\operatorname{Re}\lambda)<0$ for $\alpha\in\Pi_0-\Pi'$, and $n_\alpha=0$ for these roots. It follows that $\beta=\sum_{\alpha\in\Pi'} n_\alpha\alpha$, i.e., Π' is a basis of Σ_1. Hence, $\Pi_1=\Pi'\subset\Pi_0$.

(ii) Let $\beta\in D_-(Q)\cap\Sigma_0^+$. Since β is a positive root in Σ_0, we have $\beta = \sum_{\alpha\in\Pi_0} m_\alpha\alpha$ with $m_\alpha\in\mathbb{Z}_+$. By our assumption Π_0 consists of simple roots of type (i), (ii) and (iv) only. Therefore, $\Pi_0=\Pi_0'\cup\Pi_0''$ where Π_0' contains the simple roots of type (i) and (iv) and Π_0'' contains the simple roots of type (ii). Since $\Pi_0'\subset\Pi_1$, we have $\operatorname{Re}(\sigma_Q\alpha)\check{}(\lambda)=0$ for $\alpha\in\Pi_0'$. On the other hand, $\sigma_Q\alpha$ are positive roots for $\alpha\in\Pi_0''$, hence $\operatorname{Re}(\sigma_Q\alpha)\check{}(\lambda)\le 0$. Since $\sigma_Q\beta$ is a negative root, it follows that

$$\begin{aligned}
0 \le (\sigma_Q\beta|\operatorname{Re}\lambda) &= \sum_{\alpha\in\Pi_0} m_\alpha(\sigma_Q\alpha|\operatorname{Re}\lambda)\\
&= \sum_{\alpha\in\Pi_0'} m_\alpha(\sigma_Q\alpha|\operatorname{Re}\lambda) + \sum_{\alpha\in\Pi_0''} m_\alpha(\sigma_Q\alpha|\operatorname{Re}\lambda) = \sum_{\alpha\in\Pi_0''} m_\alpha(\sigma_Q\alpha|\operatorname{Re}\lambda)\le 0.
\end{aligned}$$

Hence, $\operatorname{Re}(\sigma_Q\beta)\check{}(\lambda)=0$, i.e., we have $\sigma_Q\beta\in\Sigma_0$. Therefore, we proved that $\beta\in\Sigma_1^+$, i.e., we have

$$D_-(Q)\cap\Sigma_0^+ = D_-(Q)\cap\Sigma_1^+.$$

Since Σ_1 is σ_Q-invariant, by 5.4, we see that $\Pi_1\cap D_-(Q)=\emptyset$ implies $\Sigma_1\cap D_-(Q)=\emptyset$. Therefore, we have $D_-(Q)\cap\Sigma_0^+=\emptyset$. □

Now we can prove 9.1. Let $\alpha\in\Pi_0$ be such that $\alpha\check{}(\lambda)\neq 0$. Then $\alpha\check{}(\lambda)$ is purely imaginary and $s_\alpha\lambda$ is also strongly antidominant. Hence, $I_{s_\alpha}:\mathcal{M}_{qc}(\mathcal{D}_\lambda)\longrightarrow\mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$ is an equivalence of categories by 2.9. Therefore,

$$I_{s_\alpha}(\mathcal{L}(Q,\tau))=\mathcal{L}(Q',\tau'),$$

for some K-orbit Q' and an irreducible K-homogeneous connection on Q' compatible with $s_\alpha\lambda+\rho$. Also, by 2.10, we have

$$\Gamma(X,\mathcal{L}(Q,\tau))=\Gamma(X,\mathcal{L}(Q',\tau')).$$

Therefore, the conditions (i) for $\mathcal{L}(Q,\tau)$ and $\mathcal{L}(Q',\tau')$ are equivalent.

We claim that the conditions (ii) for $\mathcal{L}(Q,\tau)$ and $\mathcal{L}(Q',\tau')$ are also equivalent. Clearly, α is either Q-complex or Q-real.

Assume first that α is Q-complex. By 6.5.(v), the set $p_\alpha^{-1}(p_\alpha(Q))$ is union of two K-orbits Q and Q''. Since $I_{s_\alpha}:\mathcal{M}_{qc}(\mathcal{D}_\lambda)\longrightarrow\mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$ and $I_{s_\alpha}:\mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})\longrightarrow$

$\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ are equivalences of categories, by 7.2, there exists a connection τ'' on Q'' compatible with $s_\alpha\lambda + \rho$ such that $I_{s_\alpha}(\mathcal{I}(Q'', \tau'')) = \mathcal{I}(Q, \tau)$. It follows that $I_{s_\alpha}(\mathcal{L}(Q, \tau)) = \mathcal{L}(Q'', \tau'')$. Therefore, $Q' = Q''$ and $\tau' = \tau''$. Since the situation is completely symmetric, without any lack of generality, by possible switching of the roles of the pairs (Q, τ) and (Q', τ'), we can assume that $\dim Q' = \dim Q - 1$. Since $\sigma_{Q'} = s_\alpha \circ \sigma_Q \circ s_\alpha$ by 6.11, we see that s_α maps compact Q-imaginary, noncompact Q-imaginary, Q-complex and Q-real roots into compact Q'-imaginary, noncompact Q'-imaginary, Q'-complex and Q'-real roots respectively. In addition, Σ_0 is s_α-invariant. Hence, s_α maps Σ_1 into ${\Sigma_1}'$ and Σ_2 into ${\Sigma_2}'$. Since α is not in Σ_2, s_α maps Σ_2^+ into $({\Sigma_2}')^+$. Therefore, s_α maps Π_2 into Π_2'. Therefore, the conditions (ii)(a) for $\mathcal{L}(Q, \tau)$ and $\mathcal{L}(Q', \tau')$ are equivalent. By 7.9.(i), the conditions (ii)(c) for $\mathcal{L}(Q, \tau)$ and $\mathcal{L}(Q', \tau')$ are also equivalent. By 6.11, $D_-(Q) = s_\alpha(D_-(Q')) \cup \{\alpha, -\sigma_Q\alpha\}$. Therefore, $D_-(Q) \cap \Sigma_0^+$ consists of $s_\alpha(D_-(Q') \cap \Sigma_0^+)$, α and possibly $-\sigma_Q\alpha$ (if it is in Σ_0^+). Since $\operatorname{Im} \alpha\check{}(\lambda) \neq 0$ and $\alpha\check{}(\lambda^Q)$ is real, we see that

$$\operatorname{Im}(\sigma_Q\alpha)\check{}(\lambda) = -\operatorname{Im} \alpha\check{}(\lambda) \neq 0.$$

Hence, the conditions (ii)(b) for $\mathcal{L}(Q, \tau)$ and $\mathcal{L}(Q', \tau')$ are also equivalent.

If α is Q-real, $I_{s_\alpha}(\mathcal{I}(Q, \tau)) = \mathcal{I}(Q, \tau_{s_\alpha})$ by 7.7. Hence, $I_{s_\alpha}(\mathcal{L}(Q, \tau)) = \mathcal{L}(Q, \tau_{s_\alpha})$ and $Q' = Q$ and $\tau' = \tau_{s_\alpha}$ in this case. Since s_α commutes with σ_Q, it follows that it maps Σ_1 into Σ_1' and Σ_2 into Σ_2'. Since α is not in Σ_2, s_α maps Σ_2^+ into $({\Sigma_2}')^+$. Therefore, s_α maps Π_2 into Π_2'. Clearly s_α acts trivially on Q-imaginary roots, and the conditions (ii)(a) for $\mathcal{L}(Q, \tau)$ and $\mathcal{L}(Q, \tau')$ are identical. Moreover, s_α permutes Q-real roots in this case and, by 7.9.(ii), the conditions (ii)(c) for $\mathcal{L}(Q, \tau)$ and $\mathcal{L}(Q, \tau')$ are equivalent. Also, s_α permutes positive Q-complex roots, hence the conditions (ii)(b) for $\mathcal{L}(Q, \tau)$ and $\mathcal{L}(Q, \tau')$ are equivalent.

This completes the proof of our claim.

First we establish the implication (ii)$\Rightarrow$(i). Assume that $\Gamma(X, \mathcal{L}(Q, \tau)) = 0$. By 2.17, there exists $w \in W_0$ such that $I_w(\mathcal{L}(Q, \tau)) = 0$. We prove that (ii) does not hold by induction in $\ell(w)$. First, assume that $\ell(w) = 1$. Then, $w = s_\alpha$, $\alpha \in \Pi_0$, and the assertion follows from 7.5.

Assume that $\ell(w) = p > 1$. Then there exists $\alpha \in \Pi_0$ and $w' \in W_0$ such that $w = w's_\alpha$ and $\ell(w') = p - 1$. If $I_{s_\alpha}(\mathcal{L}(Q, \tau)) = 0$, we are done by the previous step. Therefore, we can assume that $I_{s_\alpha}(\mathcal{L}(Q, \tau)) \neq 0$. There are two possibilities:

(a) $\alpha\check{}(\lambda) = 0$;
(b) $\alpha\check{}(\lambda) \neq 0$.

Assume first that (a) holds. Then, by 2.16.(ii) we see that $\mathcal{L}(Q, \tau)$ is the unique irreducible quotient of $I_{s_\alpha}(\mathcal{L}(Q, \tau))$. Therefore, since $I_{w'}$ is right exact, $I_{w'}(\mathcal{L}(Q, \tau))$ is a quotient of

$$I_{w'}(I_{s_\alpha}(\mathcal{L}(Q, \tau))) = I_w(\mathcal{L}(Q, \tau)) = 0,$$

i.e., $I_{w'}(\mathcal{L}(Q, \tau)) = 0$. By the induction assumption, (ii) cannot hold for $\mathcal{L}(Q, \tau)$.

Assume now that (b) holds. In this case, by the previous discussion, $I_{s_\alpha}(\mathcal{L}(Q, \tau)) = \mathcal{L}(Q', \tau')$ and $\Gamma(X, \mathcal{L}(Q', \tau')) = \Gamma(X, \mathcal{L}(Q, \tau)) = 0$. Therefore, $I_{w'}(\mathcal{L}(Q', \tau')) = 0$ and by the induction assumption the condition (ii) fails for $\mathcal{L}(Q', \tau')$. The preceding discussion now implies that (ii) also fails for $\mathcal{L}(Q, \tau)$. This completes the proof of the implication (ii)$\Rightarrow$(i).

Now we prove (i)$\Rightarrow$(ii). Assume that $\Gamma(X, \mathcal{L}(Q, \tau)) \neq 0$. The first step in the reduction to the case $D_-(Q) \cap \Sigma_0^+ = \emptyset$. The proof is by downward induction on $\operatorname{Card}(D_-(Q))$. Assume that $D_-(Q) \cap \Sigma_0^+$ is not empty. By 9.2, there exists $\alpha \in \Pi_0$

such that α is Q-complex and $-\sigma_Q\alpha$ is positive. By 2.17 and 7.5.(ii), $\alpha\check{}(\lambda) = 0$ is impossible. Therefore, $\alpha\check{}(\lambda) \neq 0$ holds. By the preceding discussion, in this case $I_{s_\alpha}(\mathcal{L}(Q,\tau)) = \mathcal{L}(Q',\tau')$ and $\Gamma(X,\mathcal{L}(Q',\tau')) = \Gamma(X,\mathcal{L}(Q,\tau)) \neq 0$. Moreover, the conditions (ii) for $\mathcal{L}(Q,\tau)$ and $\mathcal{L}(Q',\tau')$ are equivalent. On the other hand, by 6.10, we have $\operatorname{Card} D_-(Q') = \operatorname{Card} D_-(Q) - 2$, and in finitely many steps we are reduced to the situation where $D_-(Q) \cap \Sigma_0^+$ is empty.

In this situation the condition (ii)(b) is vacuous. Now we prove that (ii)(c) holds. For a simple Q-real root $\alpha \in \Pi_1$ there are two possibilities:

(a) $\alpha\check{}(\lambda) \neq 0$;
(b) $\alpha\check{}(\lambda) = 0$;

If (a) holds, as before, we conclude that $I_{s_\alpha}(\mathcal{L}(Q,\tau)) = \mathcal{L}(Q,\tau')$ and $\Gamma(X,\mathcal{L}(Q,\tau)) = \Gamma(X,\mathcal{L}(Q,\tau'))$. Also, conditions (ii) for $\mathcal{L}(Q,\tau)$ and $\mathcal{L}(Q,\tau')$ are equivalent. Therefore, we can replace (Q,τ) with (Q,τ').

If (b) holds, by 7.5, τ must satisfy the SL_2-parity condition with respect to α. In this case, by 7.8, we have $I_{s_\alpha}(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q,\tau)$, i.e., $\tau' = \tau$. Moreover, by 7.9.(ii), the SL_2-parity condition for τ is satisfied for Q-real root β if and only if it is satisfied for the Q-real root $s_\alpha\beta$.

By definition, Σ_1 is σ_Q-invariant. Also, by 9.2, $\Sigma_1 \cap D_-(Q) = \emptyset$. Hence, Σ_1^+ is of Zuckerman type in Σ_1 with respect to the induced involution. By 5.10.(i), Q-real roots in Π_1 form a basis of the root system of all Q-real roots in Σ_1. By applying consecutive reflections with respect to simple Q-real roots in Π_1, we see that the SL_2-parity condition holds for all Q-real roots in Σ_1. Hence the condition (ii)(c) holds for $\mathcal{L}(Q,\tau)$.

It remains to show that (ii)(a) holds. This is an immediate consequence of the following lemma.

Lemma 9.3. *Assume that the pair (Q,τ) satisfies $\Pi_1 \subset \Pi_0$ and the conditions (ii)(b) and (ii)(c) from 9.1. Then the following conditions are equivalent:*

(i) *the set Π_2 contains no compact Q-imaginary roots;*
(ii) $\Gamma(X,\mathcal{L}(Q,\tau)) \neq 0$*;*
(iii) $\Gamma(X,\mathcal{I}(Q,\tau)) \neq 0$*.*

Proof. We already established that (i) implies (ii). That (ii) implies (iii) is obvious. Therefore, we have to show that (iii) implies (i).

The root system Σ_2 can be characterized as

$$\Sigma_2 = \{\alpha \in \Sigma_1 \mid \operatorname{Im} \alpha\check{}(\lambda) = 0\}.$$

If λ satisfies the condition $\operatorname{Im}\alpha\check{}(\lambda) \leq 0$ for all $\alpha \in \Pi_1$, we have $\Pi_2 \subset \Pi_1 \subset \Pi_0 \subset \Pi$. Hence, if $\alpha \in \Pi_2$ is a compact Q-imaginary root, α is a simple root. And in this case, by 7.6, we have $I_{s_\alpha}(\mathcal{I}(Q,\tau)) = 0$. Since λ is antidominant, by 2.10, this implies by that

$$\Gamma(X,\mathcal{I}(Q,\tau)) = \Gamma(X, I_{s_\alpha}(\mathcal{I}(Q,\tau))) = 0.$$

Hence, we have a contradiction and (i) holds.

Assume that the above condition on λ doesn't hold. Then, there exist an element w of minimal length in the Weyl group W_1 generated by the reflections corresponding to roots in Σ_1 such that $w\lambda$ satisfies this property. Put $k = \ell(w)$. We prove that (i) holds by induction in k. Let $w = w's_\alpha$, with $w' \in W_1$, $\alpha \in \Pi_1$, satisfying $\ell(w') = k-1$. Then, by the minimality of $\ell(w)$, we have $s_\alpha \operatorname{Im}\lambda \neq \operatorname{Im}\lambda$. Therefore, $\operatorname{Im}\alpha\check{}(\lambda) \neq 0$ and α is either Q-complex or a Q-real. By a previous argument,

I_{s_α} is an equivalence of categories, there exists a K-orbit Q' and an irreducible K-homogeneous connection τ' compatible with $s_\alpha\lambda+\rho$ such that $I_{s_\alpha}(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q',\tau')$. Then, by 2.10, we have $\Gamma(X,\mathcal{I}(Q',\tau')) = \Gamma(X,\mathcal{I}(Q,\tau)) \neq 0$.

Since Σ_0 is determined by $\operatorname{Re}\lambda$, it doesn't change if we replace λ by $s_\alpha\lambda$. Therefore, Π_0 is the same for (Q,τ) and (Q',τ'). If α is Q-complex, by 6.11, we have

$$\sigma_{Q'} = s_\alpha \circ \sigma_Q \circ s_\alpha = \sigma_Q \circ s_{\sigma_Q\alpha} \circ s_\alpha.$$

Since $\alpha \in \Sigma_1$, we have $\alpha \in \Sigma_0$ and $\sigma_Q\alpha \in \Sigma_0$. It follows that

$$\Sigma_1' = \Sigma_0 \cap \sigma_{Q'}(\Sigma_0) = \Sigma_0 \cap \sigma_Q(\Sigma_0) = \Sigma_1,$$

and $\Pi_1' = \Pi_1$.

If α is Q-real, we know from a previous discussion that $Q' = Q$, hence $\Pi_1' = \Pi_1$ in this case too. Therefore, the conditions of the lemma are satisfied for (Q',τ').

Now, $w'(s_\alpha\lambda)$ satisfies the above condition and $\ell(w') = k-1$. Hence, (i) holds for $\mathcal{I}(Q',\tau')$ by the induction assumption. We already established that the condition (i) holds for (Q,τ) if and only if it holds for (Q',τ'). Therefore, (i) holds for $\mathcal{I}(Q,\tau)$. □

Let V be an irreducible Harish-Chandra module in $\mathcal{M}_{fg}(\mathcal{U}_\theta,K)$. In general, the Weyl group orbit θ contains several strongly antidominant elements. For different strongly antidominant λ in θ, $V \cong \Gamma(X,\mathcal{L}(Q,\tau))$ for different pairs (Q,τ), as one can easily check in simple examples (like the discussions of $\mathrm{SL}(2,\mathbb{R})$ in the introduction of [12] and $\mathrm{SL}(2,\mathbb{C})$ at the end of [19]). Still, the K-conjugacy class of σ-stable Cartan subalgebras attached to K-orbits Q is uniquely determined by V:

Proposition 9.4. *Let V be an irreducible Harish-Chandra module in $\mathcal{M}_{fg}(\mathcal{U}_\theta,K)$. Let $\lambda,\lambda' \in \theta$ be strongly antidominant, Q, Q' be K-orbits in X and τ, τ' irreducible K-homogeneous connections on Q, resp. Q', compatible with $\lambda+\rho$, resp. $\lambda'+\rho$, such that*

$$V \cong \Gamma(X,\mathcal{L}(Q,\tau)) \cong \Gamma(X,\mathcal{L}(Q',\tau')).$$

Then:

(i) *the orbits Q and Q' are attached to the same K-conjugacy class of σ-stable Cartan subalgebras in $\mathfrak{g}$;*
(ii) $\operatorname{Re}\lambda_Q = \operatorname{Re}\lambda'_{Q'}$;
(iii) $\lambda^Q = (\lambda')^{Q'}$.

Proof. Fix an antidominant λ in θ. Then, by 2.15.(ii), there exists a unique pair (Q,τ) consisting of a K-orbit Q and an irreducible homogeneous connection τ on Q compatible with $\lambda+\rho$ such that $V = \Gamma(X,\mathcal{L}(Q,\tau))$.

Let $W_0 = W(\operatorname{Re}\lambda)$ be the stabilizer of $\operatorname{Re}\lambda$ in W. Then W_0 is generated by reflections with respect to the roots $\alpha \in \Pi$ orthogonal to $\operatorname{Re}\lambda$. Clearly, $w\lambda$ is strongly antidominant if and only if $w \in W_0$. Consider the set $\mathcal{S}$ of pairs (Q',λ') of K-orbits Q' and strongly antidominant λ' such that there exists an irreducible K-homogeneous connection τ' on Q' compatible with $\lambda'+\rho$ satisfying $\Gamma(X,\mathcal{L}(Q',\tau')) \cong \Gamma(X,\mathcal{L}(Q,\tau))$. Fix such pair (Q',λ'). Let w be the shortest element in W_0 such that $\lambda' = w\lambda$. We prove the statements (i) and (ii) by induction in $\ell(w)$. If $\ell(w)=0$, by 2.15.(ii), we see that $Q = Q'$. Let $\ell(w) > 0$. Then $w = s_\alpha w'$, where $w' \in W_0$, $\ell(w) = \ell(w')+1$, and $\alpha \in \Pi$ such that $\alpha\check{}(\operatorname{Re}\lambda) = 0$. Then, $\alpha\check{}(\lambda') \neq 0$ by the minimality of w; and $\alpha\check{}(\lambda')$ is purely imaginary. Therefore, by 7.1.(i), α cannot be a Q'-imaginary root.

Moreover, by 2.9, I_{s_α} is an equivalence of categories. Hence $I_{s_\alpha}(\mathcal{L}(Q',\tau')) = \mathcal{L}(Q'',\tau'')$ for some pair (Q'',τ''), where τ'' is an irreducible K-homogeneous connection on Q'' compatible with $\lambda''+\rho = s_\alpha\lambda'+\rho = w'\lambda+\rho$. By 2.10, we have

$$\Gamma(X,\mathcal{L}(Q'',\tau'')) = \Gamma(X, I_{s_\alpha}(\mathcal{L}(Q',\tau'))) = \Gamma(X,\mathcal{L}(Q',\tau')) \cong \Gamma(X,\mathcal{L}(Q,\tau))$$

Hence, by the induction assumption, Q'' corresponds to the same conjugacy class of σ-stable Cartan subalgebras as Q and $\operatorname{Re}\lambda''_{Q''} = \operatorname{Re}\lambda_Q$.

We already remarked that α is either Q'-real or Q'-complex. In the first case, by 7.7, we have $I_{s_\alpha}(\mathcal{I}(Q',\tau')) = \mathcal{I}(Q',\tau_{s_\alpha})$ for some irreducible K-homogeneous connection τ_{s_α} on Q' compatible with $\lambda''+\rho$. Therefore, $I_{s_\alpha}(\mathcal{L}(Q',\tau')) = \mathcal{L}(Q',\tau_{s_\alpha})$. It follows that $Q''=Q'$. Since $\operatorname{Re}\lambda'' = \operatorname{Re}\lambda'$, we have

$$\operatorname{Re}\lambda_Q = \operatorname{Re}\lambda''_{Q''} = \operatorname{Re}\frac{1}{2}(\lambda''-\sigma_{Q''}\lambda'') = \operatorname{Re}\frac{1}{2}(\lambda'-\sigma_{Q'}\lambda') = \operatorname{Re}\lambda'_{Q'}.$$

In the second case, by an analogous argument using 7.2, we see that Q'' and Q' correspond to the same K-conjugacy class of σ-stable Cartan subalgebras, and $\sigma_{Q''} = s_\alpha\circ\sigma_{Q'}\circ s_\alpha$ by 6.11. Therefore,

$$\begin{aligned}\operatorname{Re}\lambda_Q = \operatorname{Re}\lambda''_{Q''} = \operatorname{Re}\frac{1}{2}(\lambda''-\sigma_{Q''}\lambda'') = \operatorname{Re}\frac{1}{2}(\lambda''-s_\alpha\sigma_{Q'}s_\alpha\lambda'') \\ = \operatorname{Re}\frac{1}{2}(\lambda'-\sigma_{Q'}\lambda') = \operatorname{Re}\lambda'_{Q'}.\end{aligned}$$

On the other hand, as we remarked before,

$$\operatorname{Re}\lambda = \lambda^Q + \operatorname{Re}\lambda_Q$$

depends only of θ. Hence, we have

$$\lambda^Q + \operatorname{Re}\lambda_Q = (\lambda')^{Q'} + \operatorname{Re}\lambda'_{Q'}$$

and, finally, $\lambda^Q = (\lambda')^{Q'}$. □

Therefore, the invariants $\operatorname{Re}\lambda_Q$ and λ^Q do not depend on $\mathcal{L}(Q,\tau)$ but only on the Harish-Chandra module $V=\Gamma(X,\mathcal{L}(Q,\tau))$. Hence, can define

$$\kappa_V = \operatorname{Re}\lambda_Q \text{ and } \kappa^V = \lambda^Q$$

and call it them the *Langlands invariant* κ_V and the *Vogan-Zuckerman invariant* κ^V of V.

In 11.7 we are going to show that an irreducible Harish-Chandra module V is tempered if and only if $\kappa_V=0$. If $\kappa^V=0$, we say that V is *quasispherical.* modules.

10. Decomposition of global sections of standard Harish-Chandra sheaves

Let θ be a Weyl group orbit in $\mathfrak{h}^*$ and $\lambda\in\theta$ strongly antidominant. Let Q be a K-orbit in the flag variety X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda+\rho$. Let $\mathcal{I}(Q,\tau)$ be the standard Harish-Chandra sheaf attached to (Q,τ) and $\mathcal{L}(Q,\tau)$ its unique irreducible Harish-Chandra subsheaf. In 9.1 we established a necessary and sufficient criterion for $\Gamma(X,\mathcal{L}(Q,\tau))=0$. In this section we want to prove some preliminary results on the structure of Harish-Chandra modules $\Gamma(X,\mathcal{I}(Q,\tau))$. We start with the easy case.

Lemma 10.1. *Let θ be a Weyl group orbit in $\mathfrak{h}^*$, and $\lambda \in \theta$ antidominant. Let Q be a K-orbit in the flag variety X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda + \rho$. Assume that $\Gamma(X, \mathcal{L}(Q, \tau)) \neq 0$. Then $\Gamma(X, \mathcal{L}(Q, \tau))$ is the unique irreducible Harish-Chandra submodule in $\Gamma(X, \mathcal{I}(Q, \tau))$.*

Proof. Let $V = \Gamma(X, \mathcal{L}(Q, \tau))$. Then, by 2.15.(i), V is an irreducible Harish-Chandra module. Hence, it is an irreducible Harish-Chandra submodule of $\Gamma(X, \mathcal{I}(Q, \tau))$. Assume that U is another irreducible Harish-Chandra submodule of $\Gamma(X, \mathcal{I}(Q, \tau))$. Then the adjointness of Δ_λ and $\Gamma(X, -)$ implies that we have a nontrivial $\mathcal{D}_\lambda$-module morphism ϕ of $\Delta_\lambda(U)$ into $\mathcal{I}(Q, \tau)$. It follows that the image $\operatorname{im} \phi$ of $\Delta_\lambda(U)$ is a Harish-Chandra subsheaf of $\mathcal{I}(Q, \tau)$ which contains $\mathcal{L}(Q, \tau)$. Therefore, $\Gamma(X, \mathcal{J})$ is a Harish-Chandra submodule of $\Gamma(X, \mathcal{I}(Q, \tau))$ which contains V as a composition factor. On the other hand, it must also be a quotient of $\Gamma(X, \Delta_\lambda(U)) = U$, and we have $U = V$. □

In particular, if $\Gamma(X, \mathcal{L}(Q, \tau)) \neq 0$, $\Gamma(X, \mathcal{I}(Q, \tau))$ is an indecomposable Harish-Chandra module.

Now we want to consider the general case. The main result is the following theorem.

Theorem 10.2. *Let θ be a Weyl group orbit in $\mathfrak{h}^*$, and $\lambda \in \theta$ strongly antidominant. Let Q be a K-orbit in the flag variety X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda + \rho$. Assume that $\Gamma(X, \mathcal{I}(Q, \tau)) \neq 0$. Then, there exist*

(a) *a unique family $(Q_1, Q_2, \dots, Q_p)$ of K-orbits in X;*
(b) *a unique family of K-homogeneous irreducible connections τ_i on Q_i, $1 \leq i \leq p$, compatible with $\lambda + \rho$;*

such that

(i) *$V_i = \Gamma(X, \mathcal{L}(Q_i, \tau_i)) \neq 0$ for $1 \leq i \leq p$;*
(ii)
$$\Gamma(X, \mathcal{I}(Q, \tau)) = \bigoplus_{i=1}^{p} \Gamma(X, \mathcal{I}(Q_i, \tau_i))$$
is the (unique) decomposition of $\Gamma(X, \mathcal{I}(Q, \tau))$ into a direct sum of indecomposable Harish-Chandra modules.

Then Q_i, $1 \leq i \leq p$, are in the closure of Q.

The Langlands invariants and the Vogan-Zuckerman invariants of irreducible Harish-Chandra modules V_i, $1 \leq i \leq p$, are given by

$$\kappa_{V_i} = \operatorname{Re} \lambda_Q \text{ and } \kappa^{V_i} = \lambda^Q \text{ for } 1 \leq i \leq p.$$

If the pair (Q, τ) satisfies the condition (ii)(c) *from 9.1 we have $p = 1$.*

If the pair (Q, τ) satisfies the conditions (ii)(b) *and* (ii)(c) *from 9.1, the condition* (ii)(a) *from 9.1 is also satisfied.*

Proof. Since the Harish-Chandra modules $\Gamma(X, \mathcal{I}(Q_i, \tau_i))$ are indecomposable if (i) holds, the decomposition of $\Gamma(X, \mathcal{I}(Q, \tau))$ is just the decomposition into indecomposable direct summands. Hence, the modules $\Gamma(X, \mathcal{I}(Q_i, \tau_i))$ are uniquely determined. Moreover, each indecomposable direct summand $\Gamma(X, \mathcal{I}(Q_i, \tau_i))$ has a unique irreducible submodule $\Gamma(X, \mathcal{L}(Q_i, \tau_i))$ by 10.1. Therefore, by 2.15, the irreducible $\mathcal{D}_\lambda$-modules $\mathcal{L}(Q_i, \tau_i)$ are uniquely determined. This proves the uniqueness in (a) and (b).

Clearly, the composition factors of the Harish-Chandra sheaf $\mathcal{I}(Q,\tau)$ are of the form $\mathcal{L}(Q',\tau')$ for some K-orbits $Q' \subset \bar{Q}$ and irreducible K-homogeneous connections τ' on Q'. Since λ is antidominant, $\Gamma(X,-)$ is exact and, by 2.15, the composition factors of $\Gamma(X,\mathcal{I}(Q,\tau))$ are exactly such $\Gamma(X,\mathcal{L}(Q',\tau')) \neq 0$. Therefore, V_i, $1 \leq i \leq p$, are also of the form $\Gamma(X,\mathcal{L}(Q',\tau'))$. Applying 2.15 again, we see that Q_i must be among Q', and $Q_i \subset \bar{Q}$.

It remains to prove the existence of the decomposition and its last two properties. We use a reduction argument similar to the proof of 9.1. We use freely the notation and results from this proof.

First we recall some results from the proof of 9.1. Let α be a root from Π_0, i.e., a simple root such that $\operatorname{Re}\alpha\check{}(\lambda) = 0$. Assume that $\alpha\check{}(\lambda) \neq 0$. Then $I_{s_\alpha} : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$ is an equivalence of categories. Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda+\rho$. By 7.1.(i), α is either Q-complex or Q-real. If α is Q-complex, there exists a K-orbit Q' such that $Q \cup Q' = p_\alpha^{-1}(p_\alpha(Q))$ and an irreducible K-homogeneous connection τ' on Q' compatible with $s_\alpha\lambda+\rho$, such that $I_{s_\alpha}(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q',\tau')$. If α is Q-real, $I_{s_\alpha}(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q',\tau')$ for $Q' = Q$ and an irreducible K-homogeneous connection τ' on Q' compatible with $s_\alpha\lambda+\rho$. In addition, $I_{s_\alpha}(\mathcal{L}(Q,\tau)) = \mathcal{L}(Q',\tau')$ in both cases, and the conditions (ii)(a), (ii)(b) and (ii)(c) from 9.1 for the pairs (Q,τ) and (Q',τ') are equivalent.

Now we prove a reduction argument. Let Q be a K-orbit in X and τ and irreducible K-homogeneous connection on Q-compatible with $\lambda+\rho$. Then, since both λ and $s_\alpha\lambda$ are antidominant, we have

$$\Gamma(X,\mathcal{I}(Q,\tau)) = \Gamma(X, I_{s_\alpha}(\mathcal{I}(Q',\tau'))) = \Gamma(X,\mathcal{I}(Q',\tau')).$$

Assume that the assertion of the theorem holds for $\mathcal{I}(Q',\tau')$, i.e., we have the decomposition

$$\Gamma(X,\mathcal{I}(Q',\tau')) = \bigoplus_{i=1}^{p} \Gamma(X,\mathcal{I}(Q'_i,\tau'_i))$$

for some K-orbits Q'_i and irreducible K-homogeneous connections τ'_i on Q'_i compatible with $s_\alpha\lambda+\rho$. Clearly, $I_{s_\alpha} : \mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda}) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ is also an equivalence of categories. Therefore, as before, we get

$$\Gamma(X,\mathcal{I}(Q,\tau)) = \bigoplus_{i=1}^{p} \Gamma(X, I_{s_\alpha}(\mathcal{I}(Q'_i,\tau'_i))).$$

But, as we remarked before, $I_{s_\alpha}(\mathcal{I}(Q'_i,\tau'_i)) = \mathcal{I}(Q_i,\tau_i)$ for some K-orbits Q_i in X and irreducible K-homogeneous connections τ_i on Q_i compatible with $\lambda+\rho$. In addition,

$$V_i = \Gamma(X,\mathcal{L}(Q_i,\tau_i)) = \Gamma(X, I_{s_\alpha}(\mathcal{L}(Q'_i,\tau'_i))) = \Gamma(X,\mathcal{L}(Q'_i,\tau'_i)) = V'_i,$$

and we obtained a decomposition of $\Gamma(X,\mathcal{I}(Q,\tau))$ with the required properties. Clearly, $\kappa_{V_i} = \kappa_{V'_i}$ and $\kappa^{V_i} = \kappa^{V'_i}$ for $1 \leq i \leq p$. On the other hand, we have $\operatorname{Re}\lambda_Q = \operatorname{Re}(s_\alpha\lambda)_{Q'}$ and $\lambda^Q = (s_\alpha\lambda)^{Q'}$ as in the proof of 9.4. Therefore, the assertion of the theorem holds for $\mathcal{I}(Q,\tau)$.

Now we prove the existence by induction in $\dim Q$. If $\dim Q$ is minimal possible, Q is closed and $\mathcal{I}(Q,\tau)$ is irreducible. Hence, by 9.1, the statement follows immediately.

Assume that $\dim Q$ is not minimal and that the assertions of the theorem hold for all standard Harish-Chandra sheaves attached to the orbits of lower dimension. If $\Gamma(X, \mathcal{L}(Q,\tau)) \neq 0$, then the statement follows from 10.1. Hence, we can assume that $\Gamma(X, \mathcal{L}(Q,\tau)) = 0$. Then, there exists a root in Σ which fails to satisfy one of the conditions in 9.1.(ii).

Assume first that $D_-(Q) \cap \Pi_0 \neq \emptyset$. Let $\alpha \in D_-(Q) \cap \Pi_0$. Then α is a Q-complex simple root and $\operatorname{Re}\alpha(\lambda) = 0$.

Assume first that $\alpha(\lambda) \neq 0$. As in the proof of 9.1, we conclude that $I_{s_\alpha}(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q', \tau')$ for the other K-orbit Q' in $p_\alpha^{-1}(p_\alpha(Q))$ and an irreducible connection τ' on Q' compatible with $s_\alpha\lambda + \rho$. Moreover, by 6.5.(v), $\dim Q' = \dim Q - 1$. By the reduction statement, we see that the assertions of the theorem for $\mathcal{I}(Q,\tau)$ follow from the induction assumption for $\mathcal{I}(Q', \tau')$.

Assume now that $\alpha\check{}(\lambda) = 0$. In this case, by 7.2, there exists an irreducible K-homogeneous connection τ' on Q' compatible with $\lambda+\rho$ and such that $I_{s_\alpha}(\mathcal{I}(Q',\tau')) = \mathcal{I}(Q,\tau)$. Hence,

$$\Gamma(X, \mathcal{I}(Q,\tau)) = \Gamma(X, I_{s_\alpha}(\mathcal{I}(Q',\tau'))) = \Gamma(X, \mathcal{I}(Q',\tau')) = \bigoplus_{i=1}^{p} \Gamma(X, \mathcal{I}(Q_i', \tau_i'))$$

and by the induction assumption applied to $\mathcal{I}(Q',\tau')$. Therefore, if we put $Q_i = Q_i'$ and $\tau_i = \tau_i'$ we get the existence of the decomposition having the properties (i) and (ii). Moreover, as in the proof of 9.4, we see that $\lambda^Q = \lambda^{Q'}$ and $\operatorname{Re}\lambda_Q = \operatorname{Re}\lambda_{Q'}$. Finally, as we remarked in the proof of 9.1, s_α maps Q'-real roots into Q'-real roots and by 7.9, τ satisfies the SL_2-parity condition with respect to a Q-real root β if and only if τ' satisfies the SL_2-parity condition with respect to a Q'-real root $s_\alpha\beta$. Therefore, the theorem holds for $\mathcal{I}(Q,\tau)$ by the induction assumption.

As in the proof of 9.1, by downward induction on $\operatorname{Card} D_-(Q)$, we reduce this to the case $D_-(Q) \cap \Sigma_0^+ = \emptyset$. In this situation, the condition (ii)(b) from 9.1 becomes vacuous.

Assume that the connection τ satisfies the SL_2-parity condition with respect to all Q-real roots β such that $\beta\check{}(\lambda) = 0$. Since we assumed that $\Gamma(X, \mathcal{L}(Q,\tau)) = 0$, we have a contradiction with the assumption that $\Gamma(X, \mathcal{I}(Q,\tau)) \neq 0$ by 9.3.

Therefore, for $\mathcal{I}(Q,\tau)$ there exists a Q-real root $\beta \in \Sigma_0$ such that τ fails the SL_2-parity condition with respect to β. In this situation, as we remarked in the proof of 9.1, the Q-real roots in Π_0 form a basis of all the root system of all Q-real roots in Σ_0. Let α be such Q-real simple root. Then we have either $\operatorname{Im}\alpha\check{}(\lambda) \neq 0$ or $\alpha\check{}(\lambda) = 0$.

Assume first that $\operatorname{Im}\alpha\check{}(\lambda) \neq 0$. Then, $I_\alpha(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q, \tau_{s_\alpha})$ by 7.7. By the reduction result, the theorem holds for $\mathcal{I}(Q,\tau)$ if and only if it holds for $\mathcal{I}(Q, \tau_{s_\alpha})$.

Assume now that $\alpha\check{}(\lambda) = 0$ and the SL_2-parity condition holds for τ with respect α. Then by 7.8 and 7.9, we see that for any Q-real root β, τ satisfies the SL_2-parity condition with respect to β if and only if it also satisfies the SL_2-parity condition with respect to $s_\alpha\beta$.

Therefore, there exists $w \in W$ which is a product of reflections with respect to Q-real roots in Π_0, an irreducible K-homogeneous connection τ' on Q compatible with $w\lambda+\rho$ and such that τ' fails the SL_2-parity condition with respect to $w\beta \in \Pi_0$, and the theorem holds for $\Gamma(X, \mathcal{I}(Q,\tau))$ if and only if it holds for $\Gamma(X, \mathcal{I}(Q,\tau'))$.

Hence, we can assume that β is a Q-real simple root and the connection τ fails the SL_2-parity condition with respect to β. By 7.10, there exists a Harish-Chandra subsheaf $\mathcal{C}$ such that

$$O \longrightarrow \mathcal{C} \longrightarrow \mathcal{I}(Q,\tau) \longrightarrow \mathcal{Q} \longrightarrow 0,$$

$I_{s_\beta}(\mathcal{C}) = 0$ and $\mathcal{Q}$ is a direct sum of standard Harish-Chandra sheaves attached to K-orbits in $p_\beta^{-1}(p_\beta(Q)) - Q$. Hence, by 2.10, $\Gamma(X,\mathcal{C}) = \Gamma(X, I_{s_\beta}(\mathcal{C})) = 0$. Therefore,

$$\Gamma(X, \mathcal{I}(Q,\tau)) = \Gamma(X, \mathcal{Q}).$$

Since $\dim(p_\beta^{-1}(p_\beta(Q)) - Q) < \dim(Q)$, by the induction assumption

$$\Gamma(X, \mathcal{I}(Q,\tau)) = \bigoplus_{i=1}^{p} \Gamma(X, \mathcal{I}(Q_i,\tau_i))$$

for some K-orbits Q_i and irreducible K-homogeneous connections τ_i on Q_i. Let Q' be an orbit in $p_\beta^{-1}(p_\beta(Q)) - Q$. Then, by 6.5 and 6.8, $\sigma_{Q'} = s_\beta \circ \sigma_Q = \sigma_Q \circ s_\beta$. Therefore, $\operatorname{Re}\lambda_Q = \operatorname{Re}\lambda_{Q'}$ and $\lambda^{Q'} = \lambda^Q$. Hence, by the induction assumption, $\kappa_{V_i} = \operatorname{Re}\lambda_Q$ and $\kappa^{V_i} = \lambda^Q$. □

11. $\mathfrak{n}$-homology of Harish-Chandra modules

In this section we specialize the results of §3 to Harish-Chandra modules.

The open K-orbit $Q_o \subset X$ is clearly a Langlands orbit. By 6.4, all Q_o-imaginary roots are compact, and by 5.10, the set $P = \Sigma_I \cup \Sigma^+$ is a parabolic set of roots. Therefore, for an arbitrary $x \in Q_o$, P determines a parabolic subalgebra $\mathfrak{p}_x \supset \mathfrak{b}_x$. Let $\mathfrak{u}_x$ be the nilpotent radical of $\mathfrak{p}_x$. For any σ-stable Cartan subalgebra $\mathfrak{c}$ in $\mathfrak{b}_x$, let $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ be the decomposition into the σ-eigenspaces with eigenvalues 1 and -1 respectively. Then the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ is a σ-stable Levi factor of $\mathfrak{p}_x$. Since all Q_o-imaginary roots are compact, it is the direct product of the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$ with $\mathfrak{a}$. Let M be the centralizer of $\mathfrak{a}$ in K. Then M is a reductive subgroup of K with Lie algebra $\mathfrak{m}$.

Let V be a Harish-Chandra module in $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$ for some W-orbit θ in $\mathfrak{h}^*$. Then $H_0(\mathfrak{u}_x, V)$ is an algebraic M-module and an $\mathfrak{a}$-module. By the specialization we can view it as an $\mathfrak{a}_{Q_o}$-module. The $\mathfrak{h}$-module $H_0(\mathfrak{n}_x, V)$ is a quotient of $H_0(\mathfrak{u}_x, V)$, and the natural projection is a morphism of $\mathfrak{a}_{Q_o}$-modules. It can be viewed as the module of lowest weight vectors of $H_0(\mathfrak{u}_x, V)$. Since $H_0(\mathfrak{n}_x, V)$ is finite-dimensional, $H_0(\mathfrak{u}_x, V)$ must be finite-dimensional too.

A nonzero restriction of a root from Σ to $\mathfrak{a}_{Q_o}$ is called a *restricted root*. It is well-known [1], that the set Σ_o of all restricted roots is a root system in $\mathfrak{a}_{Q_o}$, the *restricted root system* of the involutive Harish-Chandra pair $(\mathfrak{g}, K)$. We define an ordering on this root system by choosing Σ_o^+ to be the set consisting of all nonzero restrictions of roots from Σ^+. Denote by Π_o the corresponding set of simple restricted roots. This is the set of distinct non-zero restrictions of elements of Π. Let $\mathcal{C}$ be the real cone in $\mathfrak{a}_{Q_o}^*$ consisting of restrictions of all $\lambda \in \mathfrak{h}^*$ such that $0 \preccurlyeq \lambda$. In other words, this is the cone consisting of all linear combinations of elements of Π_o with coefficients with nonnegative real part. We call $\mathcal{C}$ the *tempered cone*. We denote the corresponding ordering on the vector space $\mathfrak{a}_{Q_o}$ by $\ll$.

Let $\delta = \rho|\mathfrak{a}_{Q_o}$. We say that a linear form $\mu \in \mathfrak{a}_{Q_o}^*$ is a *restricted exponent* of V if $H_0(\mathfrak{u}_x, V)_{(\mu+\delta)} \neq 0$. The set of restricted exponents is independent of the choice of $x \in Q_o$.

In §3 we introduced the notion of an exponent of a finitely generated $\mathcal{U}_\theta$-module. If $\lambda \in \mathfrak{h}^*$ is an exponent of V, $H_0(\mathfrak{n}_x, V)_{(\lambda+\rho)} \neq 0$ for all x in some open dense subset of K. If V is a Harish-Chandra module, by K-equivariance, this set must include the open K-orbit Q_o. This implies the following result:

Lemma 11.1. *The set of restricted exponents of $V \in \mathcal{M}_{fg}(\mathcal{U}_\theta, K)$ is equal to the set of all restrictions of the exponents of V to $\mathfrak{a}_{Q_o}$.*

A Harish-Chandra module V is *tempered* if all of its restricted exponents lie in the tempered cone $\mathcal{C}$. A tempered Harish-Chandra module is *square-integrable* if all of its restricted exponents lie in the interior of the tempered cone $\mathcal{C}$.

Remark 11.2. Let G_0 be a connected semisimple Lie group with finite center and K_0 its maximal compact subgroup. Denote by $\mathfrak{g}$ the complexified Lie algebra of G_0 and by K the complexification of K_0. Let σ be the corresponding Cartan involution of $\mathfrak{g}$. Then our category $\mathcal{M}_{coh}(\mathcal{U}_\theta, K)$ is the "classical" category of Harish-Chandra modules with infinitesimal character corresponding to θ. In this situation, the notions of tempered and square-integrable representations were introduced by Harish-Chandra in terms of growth of K_0-finite matrix coefficients on G_0. By the results of ([8], [16]) these two definitions are equivalent.

Now we use the results of §3 to obtain information on restricted exponents of global sections of Harish-Chandra sheaves with irreducible support. Recall the notation Q_w, for K-orbits Q, introduced in §6.

Lemma 11.3. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant, Q a K-orbit in X and $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ with $\operatorname{supp}(\mathcal{V}) = \bar{Q}$. Let $w \in W$ be transversal to Q. Then:*

(i) $w(\mathfrak{a}_Q) \subset \mathfrak{a}_{Q_w}$.

There exists a set Φ_w of mutually orthogonal Q_w-real roots in Σ^+ with the following properties:

(ii) *the roots in Φ_w vanish on $w(\mathfrak{a}_Q)$ and their dual roots span a complement of $w(\mathfrak{a}_Q)$ in $\mathfrak{a}_{Q_w}$;*

(iii) $\alpha\check{}(w\lambda) \geq 0$ *for all* $\alpha \in \Phi_w$.

Proof. We proceed by induction in $\ell(w)$. If $\ell(w) = 0$, $w = 1$ and $\Phi_w = \emptyset$. Thus we may assume that $\ell(w) > 1$. In this case, $w = s_\alpha w'$, where $\alpha \in \Pi$ and $w' \in W$ with $\ell(w') = \ell(w) - 1$. If w is transversal to Q, w' is transversal to Q and s_α is transversal to $Q_{w'}$ by 6.6. Assume that α is $Q_{w'}$-complex. Then $\sigma_{Q_w} = s_\alpha \circ \sigma_{Q_{w'}} \circ s_\alpha$ by 6.11. Hence,

$$s_\alpha(\mathfrak{a}_{Q_{w'}}) = \mathfrak{a}_{Q_w}$$

and, by the induction assumption,

$$w(\mathfrak{a}_Q) = s_\alpha(w'(\mathfrak{a}_Q)) \subset s_\alpha(\mathfrak{a}_{Q_{w'}}) = \mathfrak{a}_{Q_w}.$$

Also if β is a $Q_{w'}$-real root, $s_\alpha\beta$ is a Q_w-real root. Since s_α permutes positive roots different from α, $\Phi_w = s_\alpha(\Phi_{w'})$ consists of positive Q_w-real roots. The roots of $\Phi_{w'}$ vanish on $w'(\mathfrak{a}_Q)$, hence the roots of Φ_w vanish on $w(\mathfrak{a}_Q) = s_\alpha w'(\mathfrak{a}_Q)$. Also, by induction assumption, the dual roots of the roots in $\Phi_{w'}$ span a complement to $w'(\mathfrak{a}_Q)$ in $\mathfrak{a}_{Q_{w'}}$. Hence, the dual roots of the roots in Φ_w span a complement to $w(\mathfrak{a}_Q) = s_\alpha(w'(\mathfrak{a}_Q))$ in $\mathfrak{a}_{Q_w} = s_\alpha(\mathfrak{a}_{Q_{w'}})$. Moreover, for $\beta \in \Phi_w$,

$$\beta\check{}(w\lambda) = \beta\check{}(s_\alpha w'\lambda) = (s_\alpha\beta)\check{}(w'\lambda) \geq 0,$$

by the induction assumption.

Assume now that α is noncompact $Q_{w'}$-imaginary. Then, by 6.8, $\sigma_{Q_w} = s_\alpha \circ \sigma_{Q_{w'}}$ and α vanishes on $\mathfrak{a}_{Q_{w'}}$. Hence, the roots in $\Phi_{w'}$ are Q_w-real and Q_w-real root α is orthogonal to them. Put $\Phi_w = \Phi_{w'} \cup \{\alpha\}$. Since $\mathfrak{a}_{Q_w}$ is the direct sum of $\mathfrak{a}_{Q_{w'}}$ and the line spanned by $\alpha\check{}$, we see by the induction assumption that

$$w(\mathfrak{a}_Q) = s_\alpha(w'(\mathfrak{a}_Q)) \subset s_\alpha(\mathfrak{a}_{Q_{w'}}) = \mathfrak{a}_{Q_{w'}} \subset \mathfrak{a}_{Q_w},$$

the root α vanishes on $w(\mathfrak{a}_Q)$ and the dual roots of the roots in Φ_w span a complement of $w(\mathfrak{a}_Q)$ in $\mathfrak{a}_{Q_w}$. Finally, for $\beta \in \Phi_{w'}$, since β is orthogonal to α,

$$\beta(w\lambda) = \beta(s_\alpha w'\lambda) = (s_\alpha\beta)\check{}(w'\lambda) = \beta\check{}(w'\lambda) \geq 0,$$

by the induction assumption. On the other hand, since $\operatorname{supp}(\mathcal{V}) = \bar{Q}$ and w' is transversal to Q, by 3.2.(i) and 3.5, $\operatorname{supp} I_{w'}(\mathcal{V}) = \overline{Q_{w'}}$. Hence, $I_{w'}(\mathcal{V})$ must contain an irreducible composition factor isomorphic to $\mathcal{L}(Q_{w'}, \tau)$ for some irreducible K-homogeneous connection τ on $Q_{w'}$. Therefore, $\alpha\check{}(w'\lambda) \in \mathbb{R}$ by 7.1. Since λ is strongly antidominant and $w' \leq w$, we have $w'\lambda \preccurlyeq w\lambda = s_\alpha w'\lambda = w'\lambda - \alpha\check{}(w'\lambda)\alpha$ by 3.9. Hence, $0 \geq \alpha\check{}(w'\lambda) = -\alpha\check{}(w\lambda)$, i.e., $\alpha\check{}(w\lambda) \geq 0$. $\square$

Let λ be strongly antidominant, Q a K-orbit and $\mathcal{V} \in \mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ with $\operatorname{supp}(\mathcal{V}) = \bar{Q}$. Let $w \in W$ be transversal to Q of maximal possible length. Denote by $\mathfrak{d}_w$ the subspace of $\mathfrak{a}_{Q_o}$ spanned by roots dual to Φ_w. Then we have the direct sum decomposition

$$\mathfrak{a}_{Q_o} = w(\mathfrak{a}_Q) \oplus \mathfrak{d}_w$$

and

$$\mathfrak{h} = \mathfrak{t}_{Q_o} \oplus w(\mathfrak{a}_Q) \oplus \mathfrak{d}_w.$$

On the other hand,

$$\mathfrak{h} = \mathfrak{t}_Q \oplus \mathfrak{a}_Q,$$

Hence, $w(\mathfrak{t}_Q) = \mathfrak{t}_{Q_o} \oplus \mathfrak{d}_w$. Let $\lambda_Q = \frac{1}{2}(\lambda - \sigma_Q\lambda)$ be the linear form on $\mathfrak{h}$ which was introduced in the last section. Then, $w\lambda_Q$ vanishes on $\mathfrak{t}_{Q_o} \oplus \mathfrak{d}_w$, and we can view it as a linear form on $\mathfrak{a}_{Q_o}$. On $w(\mathfrak{a}_Q)$ it agrees with $w\lambda$. On the other hand, the restriction of $w\lambda$ to $\mathfrak{d}_w$ is equal to $\frac{1}{2}\sum_{\alpha\in\Phi_w} \alpha\check{}(w\lambda)\alpha$. This implies that the restriction of $w\lambda$ to $\mathfrak{a}_{Q_o}$ is equal to the sum of the restriction of $w\lambda_Q$ to $\mathfrak{a}_{Q_o}$ and $\frac{1}{2}\sum_{\alpha\in\Phi_w} \alpha\check{}(w\lambda)\alpha$. Therefore, by 11.3.(iii),

$$w\lambda|\mathfrak{a}_{Q_o} \gg w\lambda_Q.$$

Let ν be the unique element of the Weyl group orbit of $\operatorname{Re}\lambda_Q$ which lies in the closure of the negative Weyl chamber in $\mathfrak{h}^*$.

Lemma 11.4. (i) *The linear form $\nu \in \mathfrak{h}^*$ vanishes on $\mathfrak{t}_{Q_o}$, i.e. it can be viewed as an element of $\mathfrak{a}^*_{Q_o}$.*

(ii) *Let Ψ be the subset of Π_o consisting of all roots orthogonal to ν. There exists $w \in W$ transversal to Q of maximal possible length such that $w \operatorname{Re}\lambda_Q = \nu$. For any such w*

$$w \operatorname{Re}\lambda|\mathfrak{a}_{Q_o} = \nu + \sum_{\beta\in\Psi} c_\beta \beta,$$

where $c_\beta \geq 0$.

Proof. By the preceding discussion, (ii) implies (i). By applying 5.2 to $\operatorname{Re}\lambda$ we see that there exists $v \in W$ such that $\alpha\check{}(v \operatorname{Re}\lambda_Q) \leq 0$ for $\alpha \in \Sigma^+$ and $\Sigma^+_v \subset D_+(Q)$. By definition, this implies $\nu = v \operatorname{Re}\lambda_Q$. Also, v is transversal to Q by 6.9, and Q_v is a Langlands orbit attached to the same conjugacy class of σ-stable Cartan subalgebras as Q by 5.2 and 6.11. Let u be an element of W transversal to Q_v of maximal length. Then, by 6.6, $w = uv$ is transversal to Q of maximal length. By 6.21, $u \in W_\Theta$, where Θ is the set of all Q_v-imaginary simple roots. Since λ_Q vanishes on $\mathfrak{t}_Q$, $\nu = v \operatorname{Re} \lambda_Q$ vanishes on $\mathfrak{t}_{Q_v} = v\mathfrak{t}_Q$. This implies $u\nu = \nu$, i.e., $\nu = w \operatorname{Re} \lambda_Q$.

Assume that $w \in W$ is any element transversal to Q of maximal possible length such that $\nu = w \operatorname{Re} \lambda_Q$. By the preceding discussion, $w\lambda_Q$ vanishes on $\mathfrak{t}_{Q_o} \oplus \mathfrak{d}_w$ and the roots in Φ_w vanish on $w(\mathfrak{a}_Q)$. Hence the roots in Φ_w are orthogonal to ν. Moreover,

$$w \operatorname{Re} \lambda|\mathfrak{a}_{Q_o} = \nu + \frac{1}{2} \sum_{\alpha \in \Phi_w} \alpha\check{\ }(w\lambda)\alpha.$$

Let $\Sigma_{o,\Psi}$ be the root subsystem of Σ_o generated by Ψ. Since ν lies in the closure of the negative (restricted) Weyl chamber, $\Sigma_{o,\Psi}$ is the set of all restricted roots orthogonal to ν. On the other hand, Φ_w consists of Q_o-real roots, what yields $\Phi_w \subset \Sigma_{o,\Psi} \cap \Sigma_o^+$. Hence

$$w \operatorname{Re} \lambda|\mathfrak{a}_{Q_o} = \nu + \sum_{\beta \in \Psi} c_\beta \beta,$$

where $c_\beta \geq 0$. □

Since $\nu \preccurlyeq w'\lambda_Q$ for all $w' \in W$, by 11.4.(i) and a preceding inequality, we have

$$\nu \ll w\lambda|\mathfrak{a}_{Q_o}$$

for any w transversal to Q of maximal possible length. By 3.10.(i), it follows that if ω is a restricted exponent of $V = \Gamma(X, \mathcal{V})$, $\nu \ll \omega$. This implies the following result.

Proposition 11.5. *Let $\lambda \in \theta$ be strongly antidominant, Q a K-orbit in X and $\mathcal{V}$ a Harish-Chandra sheaf in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ with* $\operatorname{supp} \mathcal{V} = \bar{Q}$. *Then:*

- (i) *if* $\operatorname{Re} \lambda|\mathfrak{a}_Q = 0$, *the Harish-Chandra module $\Gamma(X, \mathcal{V})$ is either tempered or zero;*
- (ii) *if $\mathfrak{a}_Q = 0$ and λ is regular, the Harish-Chandra module $\Gamma(X, \mathcal{V})$ is square-integrable.*

Proof. (i) follows immediately from the preceding discussion, since $\operatorname{Re} \lambda|\mathfrak{a}_Q = 0$ implies $\operatorname{Re} \lambda_Q = 0$ and $\nu = 0$.

(ii) In this case, by 3.10.(i), for any restricted exponent ω there exists $w \in W$ transversal to Q of maximal possible length such that $w\lambda|\mathfrak{a}_{Q_o} \ll \omega$. By the preceding discussion, this implies that $\omega \gg \frac{1}{2} \sum_{\alpha \in \Phi_w} \alpha\check{\ }(w\lambda)\alpha$. Since in our situation Φ_w consists of positive roots which span $\mathfrak{a}_{Q_o}^*$ and the coefficients are strictly positive by regularity and 11.3.(iii), we conclude that ω is in the interior of the tempered cone $\mathcal{C}$. □

Conversely,

Proposition 11.6. *Let $\lambda \in \theta$ be strongly antidominant, Q a K-orbit in X and $\mathcal{V}$ an irreducible Harish-Chandra sheaf in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ with* $\operatorname{supp} \mathcal{V} = \bar{Q}$ *such that $V = \Gamma(X, \mathcal{V}) \neq 0$. Then:*

- (i) *if V is tempered,* $\operatorname{Re} \lambda|\mathfrak{a}_Q = 0$*;*
- (ii) *if V is square-integrable, $\mathfrak{a}_Q = 0$ and λ is regular.*

Proof. (i) By 3.10.(ii), $w\lambda|\mathfrak{a}_{Q_o}$ is a restricted exponent for any $w \in W$ transversal to Q of maximal possible length. Choose w which satisfies the conditions of 11.4.(ii). Since ν is a linear combination of (restricted) fundamental weights corresponding to simple roots from $\Pi_o - \Psi$ with negative coefficients, we see that $w\lambda|\mathfrak{a}_{Q_o}$ is in the tempered cone only if $\nu = 0$. This in turn implies that $\operatorname{Re}\lambda|\mathfrak{a}_Q = 0$.

(ii) Since V is tempered, by (i) it follows that $\nu = 0$. Let $w \in W$ be the element transversal to Q of maximal possible length constructed in the proof of 11.4.(ii). The argument there can be sharpened as follows. Since $u \in W_\Theta$,

$$w(\mathfrak{a}_Q) = uv(\mathfrak{a}_Q) = u(\mathfrak{a}_{Q_v}) = \mathfrak{a}_{Q_v},$$

Φ_w consists of Q_v-imaginary roots by 11.3.(ii). Since Q_v is a Langlands orbit, Q_v-imaginary roots are generated by the set of simple Q_v-imaginary roots by 5.10. Their non-zero restrictions to $\mathfrak{a}_{Q_o}$ form a subset Θ_o of the set Π_o of all simple restricted roots such that their span contains Φ_w. Hence, $\Theta_o \subset \Psi$ and as in the proof of 11.4.(ii)

$$w\lambda|\mathfrak{a}_{Q_o} = \sum_{\beta\in\Theta_o} c_\beta \beta,$$

where $c_\beta \geq 0$. If $w\lambda|\mathfrak{a}_{Q_o}$ is in the interior of tempered cone, Θ_o must be equal to Π_o. Since the roots in Θ_o vanish on $\mathfrak{a}_{Q_v}$, this is possible only if $\mathfrak{a}_{Q_v} = 0$. This in turn implies that $\mathfrak{a}_Q = 0$.

It remains to show that λ is regular. Since $\mathfrak{a}_Q = 0$, the orbit Q is closed by 6.16 and all roots are Q-imaginary. Therefore, $\mathcal{V} = \mathcal{L}(Q,\tau) = \mathcal{I}(Q,\tau)$ for some irreducible K-homogeneous connection τ on Q. Assume that $\alpha\check{}(\lambda) = 0$ for $\alpha \in \Pi$. If α is compact, $LI_{s_\alpha}(D(\mathcal{V})) = D(\mathcal{V})[1]$ by 7.5. This in turn implies, by 2.17, that $\Gamma(X,\mathcal{V}) = 0$ contradicting our assumption. If α is noncompact, s_α is transversal to Q by 6.7. Also, $Q' = Q_{s_\alpha}$ is a K-orbit such that $\mathfrak{a}_{Q'}$ is spanned by $\alpha\check{}$ by 6.8. Hence, $\lambda|\mathfrak{a}_{Q'} = 0$. The argument from the preceding paragraph implies that there exists $w \in W$ transversal to Q' of maximal possible length such that $w\lambda|\mathfrak{a}_{Q_o}$ is not in the interior of the tempered cone. By 6.6, ws_α is transversal to Q and $\ell(ws_\alpha) = \ell(w) + 1 = \operatorname{codim} Q' + 1 = \operatorname{codim} Q$, i.e., it has the maximal possible length. In addition, $ws_\alpha\lambda|\mathfrak{a}_{Q_o} = w\lambda|\mathfrak{a}_{Q_o}$ is not in the interior of the tempered cone, contradicting square-integrability of $\Gamma(X,\mathcal{V})$. Hence, λ must be regular. □

Finally, by combining 11.5 and 11.6, we get the following result which explains the meaning of the vanishing of the Langlands invariant κ_V.

Corollary 11.7. *Let V be an irreducible Harish-Chandra module. Then the following conditions are equivalent:*

(i) *V is tempered;*
(ii) $\kappa_V = 0$.

12. Tempered Harish-Chandra modules

In this section we reprove some "classical" results about tempered Harish-Chandra modules. These results are certainly well-known, but our arguments are completely new and we think much simpler and conceptual than the traditional ones.

Let V be an irreducible Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta, K)$. Let $\lambda \in \theta$ be strongly antidominant. As we discussed in the last section, there exists a unique pair (Q,τ) consisting of a K-orbit Q and an irreducible K-homogeneous connection

τ on K compatible with $\lambda+\rho$, such that $V \cong \Gamma(X, \mathcal{L}(Q,\tau))$. The orbit Q determines an involution σ_Q on $\mathfrak{h}^*$. As before, we put

$$\Sigma_1 = \{\alpha \in \Sigma \mid \operatorname{Re}\alpha\check{}(\lambda) = \operatorname{Re}(\sigma_Q\alpha)\check{}(\lambda) = 0\}.$$

The following sufficient condition is useful in determining if a root is in Σ_1.

Observation 12.1. *Let λ be strongly antidominant. Let α be a root such that $\operatorname{Re}\alpha\check{}(\lambda_Q) = 0$. Assume that α is either in $D_-(Q)$ or Q-real. Then α is in Σ_1.*

Proof. First, $\alpha\check{}(\lambda) - (\sigma_Q\alpha)\check{}(\lambda)$ is imaginary, i.e.,

$$\operatorname{Re}\alpha\check{}(\lambda) = \operatorname{Re}(\sigma_Q\alpha)\check{}(\lambda).$$

This immediately implies the statement if α is Q-real. In the other case, since $-\sigma_Q\alpha \in \Sigma^+$ and λ is strongly antidominant, it follows that $\operatorname{Re}(\sigma_Q\alpha)\check{}(\lambda) \geq 0$, and $\operatorname{Re}\alpha\check{}(\lambda) = 0$. ☐

In particular, if $\operatorname{Re}\lambda_Q = 0$, all roots $\alpha \in D_-(Q)$ and all Q-real roots are in Σ_1. Hence, 12.1 has the following consequence which was first proved by Ivan Mirković [17].

Theorem 12.2. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant. Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda+\rho$. Assume that $\operatorname{Re}\lambda_Q = 0$. Then $\Gamma(X, \mathcal{L}(Q,\tau)) \neq 0$ implies that $\mathcal{I}(Q,\tau)$ is irreducible, i.e., $\mathcal{L}(Q,\tau) = \mathcal{I}(Q,\tau)$.*

Proof. As we already remarked, all roots $\alpha \in D_-(Q)$ and all Q-real roots are in Σ_1. Therefore, by 12.1, for all Q-complex positive roots $\alpha \in D_-(Q)$ we have $\alpha\check{}(\lambda) \neq 0$. In addition, for all Q-real roots the SL_2-parity condition is satisfied. Hence, $\mathcal{I}(Q,\tau)$ is irreducible by 8.7. ☐

Theorem 12.2, in conjuction with 11.7, provides also a classification of the tempered irreducible Harish-Chandra modules. Specifically, by 11.7, the condition $\operatorname{Re}\lambda_Q = 0$ is equivalent to the temperedness of the Harish-Chandra module $\Gamma(X, \mathcal{L}(Q,\tau))$. Thus 12.2 explains the simplicity of the classification of tempered irreducible Harish-Chandra modules: every tempered irreducible Harish-Chandra module is the space of global sections of an irreducible standard Harish-Chandra sheaf.

In combination with 10.2, we get the following result.

Corollary 12.3. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant. Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q compatible with $\lambda+\rho$. Assume that $\operatorname{Re}\lambda_Q = 0$. Then $\Gamma(X, \mathcal{I}(Q,\tau))$ is a direct sum of tempered irreducible Harish-Chandra modules.*

If $\Gamma(X, \mathcal{I}(Q,\tau))$ is reducible, the SL_2-parity condition for τ fails for some Q-real root α.

The situation becomes especially simple in the case of square-integrable irreducible Harish-Chandra modules. We reprove Harish-Chandra's celebrated results [10]. First, we have his criterion for existence of square-integrable Harish-Chandra modules.

Theorem 12.4. *Assume that $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$ contains square-integrable Harish-Chandra modules. Then*

(i) $\operatorname{rank}\mathfrak{g} = \operatorname{rank} K$*;*

(ii) *the orbit θ is regular and real.*

Proof. Assume that V is an irreducible square-integrable Harish-Chandra module. Then, by the above discussion there exist a strongly antidominant $\lambda \in \theta$, a K-orbit Q in X and an irreducible K-homogeneous connection τ on Q compatible with $\lambda + \rho$, such that $V = \Gamma(X, \mathcal{L}(Q, \tau))$. By 11.6.(ii), this implies that λ is regular and that $\mathfrak{a}_Q = 0$. The latter condition this is equivalent with the equality of ranks and, by 7.1, it also implies that λ is real. □

Harish-Chandra's enumeration of the discrete series is thus equivalent to the following result. By 12.4, we assume that $\operatorname{rank} \mathfrak{g} = \operatorname{rank} K$. As we remarked in the proof of 12.4, if $\Gamma(X, \mathcal{L}(Q, \tau))$ is square-integrable, we have $\mathfrak{a}_Q = 0$ and $\sigma_Q = 1$. Hence, all Borel subalgebras in Q are σ-stable. By 6.16, the K-orbit Q is necessarily closed. The stabilizer of a point in Q in K is a Borel subgroup of K. Therefore, on Q there exists an irreducible K-homogeneous connection $\tau_{Q,\lambda}$ compatible with $\lambda + \rho$ if and only if $\lambda + \rho$ specializes to the differential of a character of this Borel subgroup. The connection $\tau_{Q,\lambda}$ is completely determined by $\lambda + \rho$. In this case, the standard module $\mathcal{I}(Q, \tau_{Q,\lambda})$ is irreducible.

Since θ is real, it contains a unique strongly antidominant λ. It determines a subset $\mathcal{O}_\theta$ of closed K-orbits Q in X which allow an irreducible K-homogeneous connection compatible with $\lambda + \rho$. For $Q \in \mathcal{O}_\theta$, the global sections of $\mathcal{I}(Q, \tau_{Q,\lambda})$ form an irreducible Harish-Chandra module by the equivalence of categories. By 11.5.(ii), $\Gamma(X, \mathcal{I}(Q, \tau_{Q,\lambda}))$ is square-integrable.

Theorem 12.5. *The map $\mathcal{O}_\theta \longmapsto \Gamma(X, \mathcal{I}(Q, \tau_{Q,\lambda}))$ is a bijection between closed K-orbits in X and equivalence classes of irreducible square-integrable Harish-Chandra modules in $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$.*

By definition, the *discrete series* is the set of equivalence classes of irreducible square-integrable Harish-Chandra modules.

Now we relax the regularity condition. Then we have to consider vanishing of global sections of irreducible Harish-Chandra sheaves. The next result is an obvious consequence of 9.1.

Theorem 12.6. *Suppose that* $\operatorname{rank} \mathfrak{g} = \operatorname{rank} K$. *Let λ be strongly antidominant, Q a closed K-orbit in X and τ an irreducible K-homogeneous connection compatible with $\lambda + \rho$. Then:*

(i) $\Gamma(X, \mathcal{I}(Q, \tau)) \neq 0$ *if and only if there exists no compact Q-imaginary root $\alpha \in \Pi$ such that $\alpha\check{}(\lambda) = 0$;*
(ii) *if* $\Gamma(X, \mathcal{I}(Q, \tau)) \neq 0$*, this is a tempered irreducible Harish-Chandra module.*

These Harish-Chandra modules constitute the *limits of discrete series* [18].

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