

ON WEAKLY SYMMETRIC PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT. Weakly symmetric space theory is a natural generalization of the theory of Riemannian symmetric spaces. It includes a theory of weakly symmetric Riemannian nilmanifolds. Much of the recent progress there has been based on the geodesic orbit property and that fact that the nilpotent groups in question are abelian or two-step nilpotent. Here we concentrate on the geodesic orbit property for pseudo-Riemannian manifolds, obtaining sharp results on the structure of geodesic orbit (in particular weakly symmetric) Lorentzian nilmanifolds. Suppose that the geodesic orbit nilmanifold is G/H with $G = N \rtimes H$ and N nilpotent. Then Theorem 4.2 shows that N either is at most 2-step nilpotent as in the Riemannian situation, or is 4-step nilpotent, but cannot be 3-step nilpotent. Examples show that these bounds on are the best possible. Surprisingly, Theorem 5.1 shows that N is at most 2-step nilpotent when the metric is degenerate on $[\mathfrak{n}, \mathfrak{n}]$. Both theorems give additional structural information.

Keywords: Weakly symmetric space; Pseudo-Riemannian manifold; Geodesic Orbit Space; Weakly symmetric Lorentzian nilmanifold

1. INTRODUCTION

Weakly symmetric Riemannian manifolds were introduced by A. Selberg [11]. They give a natural extension of the theory of Riemannian symmetric spaces. Selberg defined a connected Riemannian manifold (M, g) to be *weakly symmetric* if there exist a subgroup G of the isometry group $I(M, g)$ that is transitive on M and an isometry $\mu \in I(M, g)$ with these properties. First, $\mu^2 \in G$ and $\mu G \mu^{-1} = G$. Second, for any two points $p, q \in M$ there exists an isometry $\phi \in G$ with $\phi(p) = \mu(q)$ and $\phi(q) = \mu(p)$. Of course, any Riemannian symmetric space is weakly symmetric. This definition is somewhat complicated, and several people gave more transparent geometric characterizations. J. Berndt and L. Vanhecke [2] showed that a Riemannian homogeneous space M is weakly symmetric if and only if for any two points $p, q \in M$ there is an isometry of M that exchanges p and q . Z. I. Szabó [12] introduced the notion of symmetric ray space, where for every maximal geodesic γ and point $m \in \gamma$ there is an isometry that preserves γ with m as isolated fixed point; the authors of [2] showed that this condition is equivalent to the weakly symmetric condition. As noted by W. Ziller [16] now a Riemannian manifold (M, g) is weakly symmetric if and only if, given $x \in M$ and a nonzero tangent vector $\xi \in T_x(M)$ there exists $s_{x,\xi} \in I(M, g)$ such that $s_{x,\xi}(x) = x$ and $ds_{x,\xi}(\xi) = -\xi$. See the survey article [7] for a discussion of weakly symmetric spaces, D'Atri spaces and geodesic orbit spaces, and [16] for a number of new examples.

The point of Selberg's introduction of Riemannian weakly symmetric spaces $M = G/H$ is that the algebra $\mathcal{D}(G/H)$ of G -invariant differential operators on G/H is commutative, generalizing the well known fact for Riemannian symmetric spaces. This is a special case of the notion of commutative space: if G is a separable locally compact group and H is a compact subgroup, then G/H is called commutative (or (G, H) is a Gelfand pair) if the convolution algebra $L^1(H \backslash G/H)$ is commutative. The two notions are equivalent if G is a connected Lie group. See [13] for an exposition. Also (again see [13]) G/H is commutative if and only if the left regular representation of G on $L^2(G/H)$ is multiplicity free. So that multiplicity free condition applies in particular to weakly symmetric Riemannian manifolds $M = G/H$ independent of choice of G -invariant Riemannian metric.

Weakly symmetric pseudo-Riemannian manifolds were studied by Z. Chen and J. A. Wolf in [3, 4, 14]. They showed that many results from the Riemannian case can be generalized to weakly symmetric pseudo-Riemannian manifolds, but some need additional hypotheses. There are many open problems for the theory of weakly symmetric pseudo-Riemannian manifolds. In this paper, we mainly study several equivalent characterizations and obtain a complete structure theorem for weakly symmetric Lorentz nilmanifolds.

2. WEAKLY SYMMETRIC PSEUDO-RIEMANNIAN MANIFOLDS

As mentioned in the Introduction, there are a number of characterizations equivalent to the definition of weakly symmetric for a Riemannian manifold. The paper [3] of Z. Chen and J. A. Wolf concentrates on the condition most accessible by Lie algebra methods.

Definition 2.1. Let (M, g) be a pseudo-Riemannian manifold. Suppose that for every $x \in M$ and every nonzero tangent vector $\xi \in T_x M$, there is an isometry $\phi = \phi_{x, \xi}$ of M such that $\phi(x) = x$ and $d\phi(\xi) = -\xi$. Then (M, g) is a *weakly symmetric pseudo-Riemannian manifold*. In particular, if $\phi_{x, \xi}$ is independent of ξ , M is symmetric.

In other words, if γ is a maximal geodesic and $m \in \gamma$ there is an isometry of M which is a non-trivial involution on γ with m as fixed point.

A connected homogeneous pseudo-Riemannian manifold need not be geodesically convex, but any two points can be joined by a broken geodesic. Thus

Proposition 2.2. Let (M, g) be a connected pseudo-Riemannian manifold. If (M, g) is weakly symmetric, then for any $x, y \in M$ there is an isometry that interchanges x and y . In particular if (M, g) is weakly symmetric then it is homogeneous.

A. Selberg's original definition of weakly symmetric space holds also for pseudo-Riemannian manifolds:

Definition 2.3. Let (M, g) be a pseudo-Riemannian manifold. If there exists a subgroup G of the isometry group $I(M)$ of M acting transitively on M and an involutive isometry μ of (M, g) with $\mu G = G\mu$ such that whenever $x, y \in M$ there exists $\phi \in G$ with $\phi(x) = \mu(y)$ and $\phi(y) = \mu(x)$, then (M, g) is a *weakly symmetric pseudo-Riemannian manifold*.

Going segment by segment along broken geodesics, as in the Riemannian case we have

Proposition 2.4. A pseudo-Riemannian manifold (M, g) is weakly symmetric if and only if for any two points $x, y \in M$ there is an isometry of M mapping x to y and y to x .

Recall the De Rham-Wu decomposition theorem [15]. Let (M, g) be a complete simply connected pseudo-Riemannian manifold, $x \in M$, and $T_x(M) = T_{x,0} \oplus \cdots \oplus T_{x,r}$ a decomposition of the tangent space at x into holonomy invariant mutually orthogonal subspaces, where the holonomy group at x is trivial on $T_{x,0}$ and irreducible on the other $T_{x,i}$. Suppose that the pseudo-Riemannian metric g has nondegenerate restriction to $T_{x,i}$ for each index i . Then (M, g) is isometric to a pseudo-Riemannian direct product $(M_0, g_0) \times \cdots \times (M_r, g_r)$, where $x = (x_0, \dots, x_r)$ and for each index, (M_i, g_i) has tangent space T_{x_i} at x_i . As in the Riemannian case (M_i, g_i) is the maximal integral manifold through x of the distribution obtained by parallel translating $T_{x,i}$ M and equipped with the metric g_i induced by g . Thus

Proposition 2.5. Let (M, g) be a complete simply connected pseudo-Riemannian manifold, $x \in M$, and $T_x(M) = T_{x,0} \oplus \cdots \oplus T_{x,r}$ a decomposition of the tangent space at x into holonomy invariant mutually orthogonal subspaces, where the holonomy group at x is trivial on $T_{x,0}$ and irreducible on the other $T_{x,i}$. Suppose that the pseudo-Riemannian metric g has nondegenerate restriction to $T_{x,i}$ for each index i and let $(M, g) = (M_0, g_0) \times \cdots \times (M_r, g_r)$ be the De Rham-Wu decomposition. Then (M, g) is weakly symmetric if and only if each of the pseudo-Riemannian manifolds (M_i, g_i) is weakly symmetric.

Definition 2.6. Let G be a connected Lie group and H be a closed subgroup. Suppose that σ is an automorphism of G such that $\sigma(p) \in Hp^{-1}H \forall p \in G$. Then G/H is called a *weakly symmetric coset space*, (G, H) is called a *weakly symmetric pair*, and σ is called a *weak symmetry* of G/H .

It is easy to see that a weakly symmetric pseudo-Riemannian manifold is a weakly symmetric coset space (G, H) where G is the isometry group.

In the context of homogeneous spaces G/H where G is a connected Lie group and H is a compact subgroup, one often says that G/H is *commutative* when the algebra of all G -invariant differential operators is commutative. That is a special case (where G is a connected Lie group) of the correct definition: G is a separable locally compact group, H is a compact subgroup, and the convolution algebra $L^1(H \backslash G/H)$ is commutative. Selberg [11] proved that a Riemannian weakly symmetric space $M = G/H$ is a commutative space, but Lauret [9] found commutative spaces that are not weakly symmetric.

3. GEODESICS IN PSEUDO-RIEMANNIAN WEAKLY SYMMETRIC SPACES

In this section we discuss questions of the geodesic orbit property. We will need that in the proofs of our main results, Theorems 4.2 and 5.1. First we recall some background and some results of Z. Chen and J. A. Wolf from [3].

Definition 3.1. A pseudo-Riemannian manifold M is called a *geodesic orbit space* if every maximal geodesic in M is an orbit of a one-parameter group of isometries of M .

Note that pseudo-Riemannian geodesic orbit spaces are geodesically complete and homogeneous. The first principal result in [3, §4] is

Proposition 3.2. Weakly symmetric pseudo-Riemannian manifolds are geodesic orbit spaces.

This is phrased in [3] as saying that every maximal geodesic is homogeneous. The concept of a homogeneous geodesic is well known in the Riemannian case; see [8]. In the pseudo-Riemannian case the generalized version comes from [5]:

Definition 3.3. Let $M = G/H$ be a homogeneous pseudo-Riemannian manifold, $p = 1H \in G/H$ the base point, and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive ($[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$) decomposition. Let $s \mapsto \gamma(s)$ be a geodesic through p defined on an open interval J . Then γ is *homogeneous* if there exist

- 1) a diffeomorphism $t \mapsto \phi(t)$ from the real line onto J and
- 2) a vector $X \in \mathfrak{g}$ such that $\gamma(\phi(t)) = \exp(tX)(p)$ for all $t \in \mathbb{R}$.

The vector X is then called a *geodesic vector*.

The formula for geodesic vectors in the pseudo-Riemannian case appeared in [5, 6, 10]:

Lemma 3.4. (Geodesic Lemma). *Let $M = G/H$ be a pseudo-Riemannian homogeneous space. Suppose that there is a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. Let $p = 1H \in G/H$ and $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ is a geodesic curve with respect to some parameter s if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z \rangle$$

for all $Z \in \mathfrak{m}$, where k is a constant. If $k = 0$, then t is an affine parameter for γ . If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for γ , and γ is a null curve in M .

Definition 3.5. A homogeneous pseudo-Riemannian space $M = G/H$ is called a **g.o. space**, if every geodesic of M is homogeneous.

Let $M = G/H$, $G = I(M)$, be a pseudo-Riemannian homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. From the Geodesic Lemma, M is a geodesic orbit space if and only if for each $X \in \mathfrak{m}$, there exists $A \in \mathfrak{h}$ such that $\langle [X + A, Z]_{\mathfrak{m}}, X \rangle = k \langle X, Z \rangle$ for any $Z \in \mathfrak{m}$.

4. LORENTZ GEODESIC ORBIT AND WEAKLY SYMMETRIC NILMANIFOLDS, I

We start with a useful lemma.

Lemma 4.1. [1] *Let $B \in \mathfrak{so}(n-1, 1)$. In a suitable basis $\{e_i\}$ of \mathbb{R}^n , either*

- (1) *B is semisimple, $B = \begin{pmatrix} \mu & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & -\mu \end{pmatrix}$ with $C \in \mathfrak{so}(n-2)$ and $\mu \leq 0$, where $\langle e_1, e_n \rangle = 1$, $\langle e_i, e_j \rangle = \delta_{ij}$ ($i, j = 2, \dots, n-1$) and the other scalar products vanish, or*
- (2) *B is not semisimple, $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + C$ with $C \in \mathfrak{so}(n-3)$, where $\mathbf{0}$ fills in with zeroes, $-\langle e_1, e_3 \rangle = \langle e_2, e_2 \rangle = 1$, $\langle e_i, e_j \rangle = \delta_{ij}$ ($i, j = 4, \dots, n$) and the other scalar products vanish.*

There is a typographical error in [1]; in case (2) there, one should assume $r = e_1$ and $q = e_2$.

From Lemma 4.1, any matrix $H \in \mathfrak{so}(n-1, 1)$ has $n-2$ purely imaginary eigenvalues and two non-zero real eigenvalues $\pm\mu$, or has n purely imaginary eigenvalues, viewing 0 as purely imaginary. Now we can state the first of our two main results, the case where the metric is nondegenerate on $[\mathfrak{n}, \mathfrak{n}]$.

Theorem 4.2. *Let $(M = G/H, \langle \cdot, \cdot \rangle)$ be a connected Lorentz geodesic orbit nilmanifold, where $G = N \rtimes H$ with N nilpotent. (For example, by Proposition 3.2, $(M, \langle \cdot, \cdot \rangle)$ could be a connected weakly symmetric Lorentz nilmanifold with $G = I(N)^0$). Suppose that there is a reductive decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$. Then N is at most 4-step nilpotent.*

Identify \mathfrak{n} with the tangent space at $1H$ and let \mathfrak{v} denote the orthocomplement of $[\mathfrak{n}, \mathfrak{n}]$ in \mathfrak{n} . Suppose that $\langle \cdot, \cdot \rangle$ is nondegenerate on $[\mathfrak{n}, \mathfrak{n}]$. Then $\text{ad}(x) = 0$ for any $x \in \mathfrak{v}$, or there is a basis $\{x, \tilde{x}_1, \dots, \tilde{x}_s\}$ of \mathfrak{v} ($s \geq 1$) such that

$$\text{ad}(x)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ \mathbf{0} & & & & \\ \mathbf{0} & & & & \mathbf{0} \end{pmatrix}; \quad \text{ad}(\tilde{x}_1)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} \mathbf{0} & a_1 & a_2 & \dots & a_p \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \mathbf{0} \\ 0 & 0 & a_p & & \end{pmatrix};$$

$$\text{ad}(\tilde{x}_i)|_{[\mathfrak{n}, \mathfrak{n}]} = 0 \text{ for } 2 \leq s, 2 \leq i \leq s; \text{ and } \text{ad}([y, z])|_{[\mathfrak{n}, \mathfrak{n}]} = 0 \text{ for all } y, z \in \mathfrak{n}.$$

Proof. For every $X \in \mathfrak{n}$, there exists $A \in \mathfrak{h}$ such that $\langle [X + A, Z]_{\mathfrak{n}}, X \rangle = k \langle X, Z \rangle$ for any $Z \in \mathfrak{n}$. Since $\mathfrak{v} = [\mathfrak{n}, \mathfrak{n}]^\perp \cap \mathfrak{n}$ relative to $\langle \cdot, \cdot \rangle$, and \mathfrak{n} and $[\mathfrak{n}, \mathfrak{n}]$ are ideals in \mathfrak{g} , \mathfrak{v} is $\text{Ad}_G(H)$ -invariant.

For any $\eta \in [\mathfrak{n}, \mathfrak{n}]$, by the Geodesic Lemma, there exists A_η such that

$$\langle [\eta + A_\eta, \xi]_{\mathfrak{n}}, \eta \rangle = k \langle \eta, \xi \rangle = 0$$

for any $\xi \in \mathfrak{v}$. It follows that $\langle [\xi, \eta]_{\mathfrak{n}}, \eta \rangle = 0$. That is, for any $\xi \in \mathfrak{v}$ and $\eta, \zeta \in [\mathfrak{n}, \mathfrak{n}]$,

$$\langle [\xi, \eta], \zeta \rangle + \langle \eta, [\xi, \zeta] \rangle = 0 \tag{4.1}$$

We are assuming that $[\mathfrak{n}, \mathfrak{n}]$ is non-degenerate. Then $\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] \oplus \mathfrak{v}$. If $[\mathfrak{n}, \mathfrak{n}]$ is positive or negative definite, then ([3, Theorem 4.12]) \mathfrak{n} is commutative or 2-step nilpotent. Those cases aside, suppose that $[\mathfrak{n}, \mathfrak{n}]$ is indefinite. Write $\dim[\mathfrak{n}, \mathfrak{n}] = p + 3$. Fix $x \in \mathfrak{v}$. By Lemma 4.1, $[\mathfrak{n}, \mathfrak{n}]$ has a basis $\{e_1, e_2, \dots, e_{p+3}\}$ in which the inner product has matrix

$$\langle \cdot, \cdot \rangle_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} 0 & 0 & -1 & & \\ 0 & 1 & 0 & & \\ -1 & 0 & 0 & & \\ \mathbf{0} & & & & I_p \end{pmatrix} \tag{4.2}$$

and if $\text{ad}(x) \neq 0$ it has matrix

$$\text{ad}(x)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \mathbf{0} & & & \mathbf{0} \end{pmatrix} \tag{4.3}$$

If $\text{ad}(x) = 0$ for any $x \in \mathfrak{v}$, then \mathfrak{n} is 2-step nilpotent since \mathfrak{v} generates \mathfrak{n} . In the following, we assume that there exists $x \in \mathfrak{v}$ such that $\text{ad}(x) \neq 0$.

Let $y \in \mathfrak{v}$. The matrix of $\text{ad}(y)$ with respect to the given basis on $[\mathfrak{n}, \mathfrak{n}]$ has form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $D \in \mathbb{R}^{p \times p}$. By Lemma 4.1, in a possibly different basis, we have either $\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ or $\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]} = 0$.

If $\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]} \neq 0$, we have the rank $r(\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]}) = 2$. If $D \neq 0$ Take a real number $\lambda \gg 0$ and consider the matrix of $\text{ad}(\lambda x + y)|_{[\mathfrak{n}, \mathfrak{n}]}$. For $\lambda \gg 0$ the rank $r(\text{ad}(\lambda x + y)) \geq 3$, which is a contradiction. Thus $D = 0$.

Express $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. In the basis $\{e_1, e_2, \dots, e_{p+3}\}$, $\text{ad}(\lambda x + y)|_{[\mathfrak{n}, \mathfrak{n}]}$ has matrix $\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \mathbf{0} \end{pmatrix}$ where $\tilde{A} = \begin{pmatrix} a_{11} & a_{12} + \lambda & a_{13} \\ a_{21} & a_{22} & a_{23} + \lambda \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. If $a_{31} \neq 0$ and $\lambda \gg 0$ then $\det(\tilde{A}) \neq 0$. Then $r(\text{ad}(\lambda x + y)|_{[\mathfrak{n}, \mathfrak{n}]}) \geq 3$, which is a contradiction. Thus $a_{31} = 0$. Similarly the first column of C and the third row of B vanish. We will need these constraints on the matrix $\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} A & B \\ C & \mathbf{0} \end{pmatrix}$.

Since the metric on N has matrix $\begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & I_p \end{pmatrix}$ where $W = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, by equation (4.1), we have

$$\begin{pmatrix} A^t & C^t \\ B^t & \mathbf{0} \end{pmatrix} \begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & I_p \end{pmatrix} + \begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & I_p \end{pmatrix} \begin{pmatrix} A & B \\ C & \mathbf{0} \end{pmatrix} = 0.$$

In other words,

$$A^t W = -W A \text{ and } C^t = -W B. \quad (4.4)$$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ 0 & 0 & \dots & 0 \end{pmatrix}$ and $C^t = \begin{pmatrix} 0 & 0 & \dots & 0 \\ c_{21} & c_{22} & \dots & c_{2p} \\ c_{31} & c_{32} & \dots & c_{3p} \end{pmatrix}$. By (4.4), $A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{12} \\ 0 & a_{21} & -a_{11} \end{pmatrix}$, $c_{2i} = -b_{2i}$ and $c_{3i} = b_{1i}$, for $i = 1, \dots, p$. Thus

$$\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} a_{11} & a_{12} & 0 & | & b_{11} & b_{12} & \dots & b_{1p} \\ a_{21} & 0 & a_{12} & | & b_{21} & b_{22} & \dots & b_{2p} \\ 0 & a_{21} & -a_{11} & | & 0 & 0 & \dots & 0 \\ \hline 0 & -b_{21} & b_{11} & | & & & & \\ \vdots & \vdots & \vdots & | & & & & \\ 0 & -b_{2p} & b_{1p} & | & & & & \end{pmatrix}. \quad (4.5)$$

Since $\text{ad}(\mathfrak{n})$ preserves both $[\mathfrak{n}, \mathfrak{n}]$ and $\mathfrak{v} = [\mathfrak{n}, \mathfrak{n}]^\perp$, we have $\text{ad}([x, y])|_{[\mathfrak{n}, \mathfrak{n}]} = [\text{ad}(x)|_{[\mathfrak{n}, \mathfrak{n}]}, \text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]}]$. Combining (4.3) and (4.5), now,

$$\text{ad}([x, y])|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} a_{21} & -a_{11} & 0 & | & b_{21} & b_{22} & \dots & b_{2p} \\ 0 & 0 & -a_{11} & | & 0 & 0 & \dots & 0 \\ 0 & 0 & a_{21} & | & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & b_{21} & | & & & & \\ \vdots & \vdots & \vdots & | & & & & \\ 0 & 0 & b_{2p} & | & & & & \end{pmatrix}.$$

As $\text{ad}([x, y])|_{[\mathfrak{n}, \mathfrak{n}]}$ is nilpotent, its eigenvalues all are zero. Thus $a_{21} = 0$. Similarly, from (4.5), $a_{11} = 0$. Moreover, the matrix of $(\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]})^2$ is

$$(\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]})^2 = \begin{pmatrix} 0 & -\sum_{i=1}^p b_{1i}b_{2i} & a_{12}^2 + \sum_{i=1}^p b_{1i}^2 & | & b_{21} & b_{22} & \dots & b_{2p} \\ 0 & -\sum_{i=1}^p b_{2i}^2 & \sum_{i=1}^p b_{1i}b_{2i} & | & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & | & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & -a_{12}b_{21} & | & -b_{21}^2 & -b_{21}b_{22} & \dots & -b_{21}b_{2p} \\ \vdots & \vdots & \vdots & | & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -a_{12}b_{2p} & | & -b_{2p}b_{21} & -b_{2p}b_{22} & \dots & -b_{2p}^2 \end{pmatrix}. \quad (4.6)$$

From that we compute the trace $\text{Tr}((\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]})^2) = -2 \sum_1^p b_{2i}^2$. As $(\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]})^2$ is nilpotent, it has trace 0, so $b_{21} = \dots = b_{2p} = 0$.

From these calculations we have

$$\text{ad}([x, y])|_{[\mathfrak{n}, \mathfrak{n}]} = 0 \text{ for all } y \in \mathfrak{v}. \quad (4.7)$$

Writing $a(y)$ for a_{12} and $b_j(y)$ for b_{1j} we also have

$$\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]} = \left(\begin{array}{ccc|cccc} 0 & a(y) & 0 & b_1(y) & b_2(y) & \cdots & b_p(y) \\ 0 & 0 & a(y) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & b_1(y) & & & & \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & b_p(y) & & & & \end{array} \right), \quad \text{for all } y \in \mathfrak{v}. \quad (4.8)$$

Initially (4.8) requires y to be linearly independent of x , but it holds for all $y \in \mathfrak{v}$ with $a(y) = 1$ and $b_j(y) = 0$.

We continue to simplify the structure of $\text{ad}(y)|_{[\mathfrak{n}, \mathfrak{n}]}$. For the moment assume $\dim \mathfrak{v} = s+1 \geq 2$. Extend $\{x\}$ to a basis $\{x, x_1, \dots, x_s\}$ of \mathfrak{v} . Using (4.8)

$$\text{ad}(x_i)|_{[\mathfrak{n}, \mathfrak{n}]} = \left(\begin{array}{ccc|cccc} 0 & a(x_i) & 0 & b_1(x_i) & b_2(x_i) & \cdots & b_p(x_i) \\ 0 & 0 & a(x_i) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & b_1(x_i) & & & & \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & b_p(x_i) & & & & \end{array} \right) \text{ for } 1 \leq i \leq s.$$

From this point on, in the proof of Theorem 4.2, we will make successive modifications of the basis $\{x, x_1, \dots, x_s - a(x_s)x\}$, along the lines of Gauss Elimination. To avoid complicated notation we use $\{x, \tilde{x}_1, \dots, \tilde{x}_s\}$ for each of the successive modifications.

In the basis $\{x, \tilde{x}_1, \dots, \tilde{x}_s\} := \{x, x_1 - a(x_1)x, \dots, x_s - a(x_s)x\}$ we now have

$$\text{ad}(\tilde{x}_i)|_{[\mathfrak{n}, \mathfrak{n}]} = \left(\begin{array}{ccc|cccc} 0 & 0 & 0 & b_1(\tilde{x}_i) & b_2(\tilde{x}_i) & \cdots & b_p(\tilde{x}_i) \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & b_1(\tilde{x}_i) & & & & \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & b_p(\tilde{x}_i) & & & & \end{array} \right) \text{ for } 1 \leq i \leq s. \quad (4.9)$$

From (4.9) we compute $[\text{ad}(\tilde{x}_i), \text{ad}(\tilde{x}_j)]|_{[\mathfrak{n}, \mathfrak{n}]} = [\text{ad}(\tilde{x}_i)|_{[\mathfrak{n}, \mathfrak{n}]}, \text{ad}(\tilde{x}_j)|_{[\mathfrak{n}, \mathfrak{n}]}] = 0$ for $1 \leq i, j \leq s$. Also, from (4.3) together with (4.9), $[\text{ad}(x), \text{ad}(\tilde{x}_j)]|_{[\mathfrak{n}, \mathfrak{n}]} = [\text{ad}(x)|_{[\mathfrak{n}, \mathfrak{n}]}, \text{ad}(\tilde{x}_j)|_{[\mathfrak{n}, \mathfrak{n}]}] = 0$. Thus

$$[[\mathfrak{v}, \mathfrak{v}], [\mathfrak{n}, \mathfrak{n}]] = 0. \quad (4.10)$$

Assume $s \geq 2$. Write $[x, \tilde{x}_i] = \sum_{k=1}^{p+3} a_i^k e_k$ and $[\tilde{x}_i, \tilde{x}_j] = \sum_{k=1}^{p+3} a_{i,j}^k e_k$. Then

$$\begin{aligned} [\tilde{x}_i, [x, \tilde{x}_j]] &= \sum_{k=1}^{p+3} a_j^k [\tilde{x}_i, e_k] = a_j^3 \sum_{\ell=1}^p b_\ell(\tilde{x}_i) e_{\ell+3} + \left(\sum_{k=1}^p a_j^{k+3} b_k(\tilde{x}_i) \right) e_1, \\ [x, [\tilde{x}_j, \tilde{x}_i]] &= \sum_{k=1}^{p+3} a_{j,i}^k [x, e_k] = a_{j,i}^2 e_1 + a_{j,i}^3 e_2, \text{ and} \\ [\tilde{x}_j, [\tilde{x}_i, x]] &= -\sum_{k=1}^{p+3} a_i^k [\tilde{x}_j, e_k] = -a_i^3 \sum_{\ell=1}^p b_\ell(\tilde{x}_j) e_{\ell+3} + \left(\sum_{k=1}^p a_i^{k+3} b_k(\tilde{x}_j) \right) e_1 \end{aligned} \quad (4.11)$$

The first and third terms here have no e_2 component. From the Jacobi Identity $[x, [\tilde{x}_j, \tilde{x}_i]]$ has no e_2 component, i.e. $a_{j,i}^3 = 0$, so $[x, [\tilde{x}_j, \tilde{x}_i]] = a_{j,i}^2 e_1$ and $[\tilde{x}_j, \tilde{x}_i]$ has no e_3 component.

Suppose that $[x, \tilde{x}_j]$ has nonzero e_3 component. At least one of those e_3 components is nonzero because \mathfrak{v} generates \mathfrak{n} . We next modify the basis $\{x, \tilde{x}_1, \dots, \tilde{x}_s\}$ of \mathfrak{v} by (1) permuting the $\{\tilde{x}_j\}$ if necessary so that $[x, \tilde{x}_1]$ has nonzero e_3 component, and (2) if $j > 1$ and $[x, \tilde{x}_j]$ has nonzero e_3 component then subtract a multiple of \tilde{x}_1 from \tilde{x}_j so that $[x, \tilde{x}_j]$ has e_3 component zero. Then (4.9), and thus (4.11), still hold for the modified \tilde{x}_i .

We have arranged $a_{j,i}^3 = 0$ for $1 \leq i, j \leq s$, $a_1^3 \neq 0$, and $a_k^3 = 0$ for $k > 1$. Thus, in (4.11), $[\tilde{x}_i, [x, \tilde{x}_j]]$ is a multiple of e_1 when $j > 1$, $[x, [\tilde{x}_j, \tilde{x}_i]]$ is a multiple of e_1 in general, and $[\tilde{x}_j, [\tilde{x}_i, x]]$ is a multiple of e_1 when $i > 1$. From the Jacobi Identity, if $i > 1$ then $[\tilde{x}_i, [x, \tilde{x}_1]]$ is a multiple of e_1 . Again from (4.11) $a_1^3 \sum_{\ell=1}^p b_\ell(\tilde{x}_i) e_{\ell+3} = 0$, and since $a_1^3 \neq 0$ this says that each $b_\ell(\tilde{x}_i) = 0$. Going back to (4.9),

$$\text{if } i > 1 \text{ then } \text{ad}(\tilde{x}_i)|_{[\mathfrak{n}, \mathfrak{n}]} = 0.$$

In summary we see that \mathfrak{n} has a very simple structure. Associated with an appropriate basis $\{x, \tilde{x}_1, \dots, \tilde{x}_s\}$ of \mathfrak{v} ($s \geq 2$), we have

$$\text{ad}(x)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ \mathbf{0} & & & \mathbf{0} & \\ & & & & \mathbf{0} \end{pmatrix}; \quad \text{ad}(\tilde{x}_1)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} \mathbf{0} & a_1 & a_2 & \dots & a_p \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & a_1 & & \\ \vdots & \vdots & & \mathbf{0} & \\ 0 & 0 & a_p & & \end{pmatrix}; \quad (4.12)$$

$$\text{ad}(\tilde{x}_i)|_{[\mathfrak{n}, \mathfrak{n}]} = 0 \text{ for } 2 \leq i \leq s; \text{ and } \text{ad}([y, z])|_{[\mathfrak{n}, \mathfrak{n}]} = 0 \text{ for all } y, z \in \mathfrak{n}.$$

In particular $[\mathfrak{n}, \mathfrak{n}]$ is abelian. Thus \mathfrak{n} is at most 4-step nilpotent. That completes the proof of Theorem 4.2. \square

Example 4.3. Consider the connected Lorentz nilmanifold $(M = G/H, \langle \cdot, \cdot \rangle)$ with $G = N \rtimes H$, N nilpotent and $[\mathfrak{n}, \mathfrak{n}]$ non-degenerate, where $\{x_1, x_2, e_1, e_2, e_3\}$ is a basis of \mathfrak{n} whose non-zero Lie brackets are $[x_1, x_2] = e_3, [x_1, e_2] = e_1, [x_1, e_3] = e_2$. A calculation shows that N is 4-step nilpotent.

Remark 4.4. By Theorem 4.2 and Example 4.3, \mathfrak{n} is at most 2-step nilpotent, or exactly 4-step nilpotent, but cannot be 3-step nilpotent.

5. LORENTZ GEODESIC ORBIT AND WEAKLY SYMMETRIC NILMANIFOLDS, II

The second of our two main results, the case where the metric is degenerate on $[\mathfrak{n}, \mathfrak{n}]$, is as follows. The result contrasts with Theorem 4.2, and essentially coincides with the situation for Riemannian manifolds.

Theorem 5.1. *Let $(M = G/H, \langle \cdot, \cdot \rangle)$ be a connected Lorentz geodesic orbit nilmanifold. (For example, by Proposition 3.2, $(M, \langle \cdot, \cdot \rangle)$ could be a connected weakly symmetric Lorentz nilmanifold with $G = I(N)^0$). Suppose that $G = N \rtimes H$ with N nilpotent. Suppose further that there is a reductive decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, where $[\mathfrak{n}, \mathfrak{n}]$ is degenerate and the action of $\text{Ad}(H)|_{\mathfrak{n}}$ is completely reducible on \mathfrak{n} . Then \mathfrak{n} is at most 2-step nilpotent.*

Furthermore, there is a basis $\{e_1, \dots, e_p, e_{p+1}\}$ of $[\mathfrak{n}, \mathfrak{n}]$ and a basis $\{v_0; v_1, \dots, v_s\}$ of a vector space complement \mathfrak{a} to $[\mathfrak{n}, \mathfrak{n}]$ in \mathfrak{n} with the following properties.

- (1) $\mathfrak{v}_1 := \text{Span}(e_1, \dots, e_p)$ and $\mathfrak{v}_2 := \text{Span}(v_1, \dots, v_s)$ are both positive definite or both negative definite and are $\text{Ad}(H)$ -invariant,
- (2) $[\mathfrak{n}, \mathfrak{n}] \cap [\mathfrak{n}, \mathfrak{n}]^\perp = e_{p+1}\mathbb{R}$ and $\mathfrak{a} \cap \mathfrak{a}^\perp = v_0\mathbb{R}$ are $\text{Ad}(H)$ -invariant,
- (3) $\mathfrak{w} := \text{Span}(e_{p+1}, v_0)$ is of signature $(1, 1)$,
- (4) $\mathfrak{n} = \mathfrak{v}_1 \oplus \mathfrak{w} \oplus \mathfrak{v}_2$ is an $\text{Ad}(H)$ -invariant orthogonal direct sum,
- (5) $\text{ad}(x)|_{[\mathfrak{n}, \mathfrak{n}]} = 0$ for any $x \in \mathfrak{a}$.

Proof. Let $\dim[\mathfrak{n}, \mathfrak{n}] = p + 1$. Since \mathfrak{n} is of Lorentz signature and $[\mathfrak{n}, \mathfrak{n}]$ is degenerate, $\dim([\mathfrak{n}, \mathfrak{n}] \cap [\mathfrak{n}, \mathfrak{n}]^\perp) = 1$. So we have $e_{p+1} \neq 0$ spanning $[\mathfrak{n}, \mathfrak{n}] \cap [\mathfrak{n}, \mathfrak{n}]^\perp$, and $e_{p+1}^\perp = \mathfrak{v} + e_{p+1}\mathbb{R}$ where \mathfrak{v} is

positive or negative definite. Now $\mathfrak{v} = \mathfrak{v}_1 + \mathfrak{v}_2$, $\text{Ad}(H)$ -invariant orthogonal direct sum, where $\mathfrak{v}_1 = \mathfrak{v} \cap [\mathfrak{n}, \mathfrak{n}]$. Thus $\mathfrak{w} := \mathfrak{v}^\perp$ is spanned by e_{p+1} and a null vector v_0 with $\langle e_{p+1}, v_0 \rangle = 1$ and $\text{Ad}(H)v_0 \in v_0\mathbb{R}$. Choose orthonormal bases $\{e_1, \dots, e_p\}$ of \mathfrak{v}_1 and $\{v_1, \dots, v_s\}$ of \mathfrak{v}_2 . With those, we have constructed a basis of \mathfrak{n} that satisfies conditions (1) through (4) above. Note that the inner product on \mathfrak{w} has matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So the metric on \mathfrak{n} has the matrix

$$\langle \cdot, \cdot \rangle_{\mathfrak{n}} = \begin{pmatrix} I_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 1 & \mathbf{0} \\ \mathbf{0} & 1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_s \end{pmatrix}$$

under the basis $\{e_1, e_2, \dots, e_p; e_{p+1}, v_0; v_1, \dots, v_s\}$. In particular, the metric on $[\mathfrak{n}, \mathfrak{n}]$ has the matrix $\langle \cdot, \cdot \rangle_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} I_p & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}$ which is degenerate.

Let $x \in \mathfrak{n}$. Then $\text{ad}(x)$ preserves $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{v}_1 + e_{p+1}\mathbb{R}$ and $\text{ad}(x)|_{[\mathfrak{n}, \mathfrak{n}]}$ has matrix, relative to $\{e_1, \dots, e_p; e_{p+1}\}$, of the form $\begin{pmatrix} A & B \\ C & d \end{pmatrix} = \begin{pmatrix} A(x) & B(x) \\ C(x) & d(x) \end{pmatrix}$.

First we consider $\text{ad}(v_0)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} A(v_0) & B(v_0) \\ C(v_0) & d(v_0) \end{pmatrix}$. By the Geodesic Lemma, there exists $a_{v_0} \in \mathfrak{h}$ such that $\langle [v_0 + a_{v_0}, e_i], v_0 \rangle = k\langle v_0, e_i \rangle = 0$ for $1 \leq i \leq p$. Since H is completely reducible on \mathfrak{g} we have $[a_{v_0}, e_i] \in \mathfrak{v}_1$. Now $\langle [a_{v_0}, e_i], v_0 \rangle = 0$, so

$$\langle [v_0, e_i], v_0 \rangle = 0.$$

Now $[v_0, e_i] = C_i(v_0)e_{p+1} + \sum_{j=1}^p a_{ji}e_j$ for any $1 \leq i \leq p$. So

$$\langle [v_0, e_i], v_0 \rangle = \langle C_i(v_0)e_{p+1}, v_0 \rangle = C_i(v_0), \quad 1 \leq i \leq p.$$

It forces $C_i(v_0) = 0$, i.e. $C(v_0) = \mathbf{0}$. Now $\text{ad}(v_0)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} A(v_0) & B(v_0) \\ \mathbf{0} & d(v_0) \end{pmatrix}$. Furthermore, for any $e \in \mathfrak{v}_1$, by the Geodesic Lemma, there exists $a_e \in \mathfrak{h}$ such that $\langle [e + a_e, v_0], e \rangle = k\langle v_0, e \rangle$. Since v_0 is a one dimensional submodule, we know $\langle [a_e, v_0], e \rangle = 0$. Hence $\langle [e, v_0], e \rangle = 0$. It follows that

$$\langle [v_0, e_i], e_j \rangle + \langle e_i, [v_0, e_j] \rangle = 0, \quad 1 \leq i, j \leq p.$$

Then we have $a_{ij} + a_{ji} = 0$, that says $A(v_0)^t = -A(v_0)$. Since $\text{ad}(v_0)$ is nilpotent, we have $A(v_0) = \mathbf{0}$ and $d(v_0) = 0$. Thus,

$$\text{ad}(v_0)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} \mathbf{0} & B(v_0) \\ \mathbf{0} & 0 \end{pmatrix}.$$

Next consider $\text{ad}(v)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} A(v) & B(v) \\ C(v) & d(v) \end{pmatrix}$. For any $v \in \mathfrak{v}_2$, we write equation (4.1) as

$$\begin{pmatrix} A(v)^t & C(v)^t \\ B(v)^t & d \end{pmatrix} \begin{pmatrix} I_p & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} + \begin{pmatrix} I_p & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} \begin{pmatrix} A(v) & B(v) \\ C(v) & d \end{pmatrix} = 0.$$

It follows that $A(v)^t = -A(v)$ and $B(v) = \mathbf{0}$. Since $\text{ad}(v)|_{[\mathfrak{n}, \mathfrak{n}]}$ is nilpotent, now we have $A(v) = \mathbf{0}$ and $d = 0$, so $\text{ad}(v)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ C(v) & 0 \end{pmatrix}$. We now apply the Geodesic Lemma to $\text{ad}(v + v_0)|_{[\mathfrak{n}, \mathfrak{n}]}$. That gives us $a_{v+v_0} \in \mathfrak{h}$ such that $\langle [v + v_0 + a_{v+v_0}, e_i], v + v_0 \rangle = k\langle v + v_0, e_i \rangle = 0$ for $1 \leq i \leq p$. Since $[a_{v+v_0}, e_i] \in [\mathfrak{h}, \mathfrak{v}_1] \subset \mathfrak{v}_1$,

$$\langle [v + v_0, e_i], v + v_0 \rangle = 0.$$

On the other hand, since $[v_0, e_i] = 0$ for any $1 \leq i \leq p$, we have

$$\begin{aligned} \langle [v + v_0, e_i], v + v_0 \rangle &= \langle [v, e_i], v + v_0 \rangle = \langle C_i(v)e_{p+1}, v + v_0 \rangle \\ &= \langle C_i(v)e_{p+1}, v_0 \rangle \\ &= C_i(v). \end{aligned}$$

This forces $C_i(v) = 0$, i.e. $C(v) = \mathbf{0}$. Thus $\text{ad}(v)|_{[\mathfrak{n}, \mathfrak{n}]} = 0$.

Furthermore for any $v \in \mathfrak{v}_2$, first we get $\text{ad}(v)|_{\mathfrak{v}_1 \oplus \mathfrak{w}} = \begin{pmatrix} \mathbf{0} & B_1(v) \\ \mathbf{0} & 0 \end{pmatrix}$ in the basis $\{e_1, \dots, e_{p+1}, v_0\}$ since $\text{ad}(v)|_{[\mathfrak{n}, \mathfrak{n}]} = 0$. By the Geodesic Lemma to $\text{ad}(v)|_{\mathfrak{v}_1 \oplus \mathfrak{w}}$ and the fact that \mathfrak{v}_2 and $\mathfrak{v}_1 \oplus \mathfrak{w}$ are

$\text{Ad}(H)$ -invariant, we know $\text{ad}(v)|_{\mathfrak{v}_1 \oplus \mathfrak{w}} \in \mathfrak{so}(p+1, 1)$. By Lemma 4.1, we have $\text{ad}(v)|_{\mathfrak{v}_1 \oplus \mathfrak{w}} = 0$. That is, $[v, v_0] = 0$ for any $v \in \mathfrak{v}_2$, then for any $v \in \mathfrak{a}$.

Since \mathfrak{a} generates \mathfrak{n} , $\text{ad}(v)|_{[\mathfrak{n}, \mathfrak{n}]} = 0$ for any $v \in \mathfrak{v}_2$, and $\text{ad}(v_0)|_{[\mathfrak{n}, \mathfrak{n}]} = \begin{pmatrix} \mathbf{0} & B(v_0) \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, there exist $y, z \in \mathfrak{a}$ such that $[y, z] = \sum_{i=1}^{p+1} a_i e_i$ with $a_{p+1} \neq 0$. Then

$$[v_0, [y, z]] = a_{p+1} [v_0, e_{p+1}] = a_{p+1} \sum_{i=1}^p B_i(v_0) e_i.$$

From the Jacobi Identity and the fact $[v, v_0] = 0$ for any $v \in \mathfrak{a}$, we have

$$[v_0, [y, z]] = [[v_0, y], z] + [y, [v_0, z]] = 0,$$

it forces $B(v_0) = \mathbf{0}$. That is $\text{ad}(v_0)|_{[\mathfrak{n}, \mathfrak{n}]} = 0$.

Now we know $\text{ad}(x)|_{[\mathfrak{n}, \mathfrak{n}]} = 0$ for any $x \in \mathfrak{a}$, and thus also for any $x \in \mathfrak{n}$. Thus \mathfrak{n} is at most 2-step nilpotent since \mathfrak{a} generates \mathfrak{n} . \square

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