ON THE GEODESIC ORBIT PROPERTY FOR LORENTZ MANIFOLDS

ZHIQI CHEN, JOSEPH A. WOLF AND SHAOXIANG ZHANG

Abstract. The geodesic orbit property has been studied intensively for Riemannian manifolds. Geodesic orbit spaces are homogeneous and allow simplifications of many structural questions using the Lie algebra of the isometry group. Weakly symmetric Riemannian manifolds are geodesic orbit spaces. Here we define the property "naturally reductive" for pseudo–Riemannian manifolds and note that those manifolds are geodesic orbit spaces. A few years ago two of the authors proved that weakly symmetric pseudo–Riemannian manifolds are geodesic orbit spaces. In particular that result applies to pseudo–Riemannian Lorentz manifolds. Our main results are Theorems 4.2 and 5.1. In the Riemannian case the nilpotent isometry group for a geodesic orbit nilmanifold is abelian or 2–step nilpotent. Examples show that this fails dramatically in the pseudo–Riemannian case. Here we concentrate on the geodesic orbit property for Lorentz nilmanifolds G/H with $G = N \rtimes H$ and N nilpotent. When the metric is nondegenerate on $[\mathfrak{n}, \mathfrak{n}]$, Theorem 4.2 shows that N either is at most 2–step nilpotent as in the Riemannian situation, or is 4–step nilpotent, but cannot be 3–step nilpotent. Examples show that these bounds are the best possible. Surprisingly, Theorem 5.1 shows that N is at most 2–step nilpotent when the metric is degenerate on $[\mathfrak{n}, \mathfrak{n}]$. Both theorems give additional structural information and specialize to naturally reductive and to weakly symmetric Lorentz nilmanifolds.
Mathematics Subject Classification(2010): 53C25, 22E60.
Key Words: Geodesic Orbit Space; Lorentz nilmanifold; Weakly Symmetric Space; Naturally Reductive Space; Pseudo-Riemannian Manifold.

1. Introduction

A Riemannian manifold is called a *geodesic orbit space* if every geodesic is the orbit of a one parameter group of isometries. They are homogeneous, thus amenable to study by Lie algebra methods, and have been studied intensively by many mathematicians. Two particular cases are of special interest are *naturally reductive spaces* and *weakly symmetric spaces*. Here we study extension of results on these topics from Riemannian manifolds to pseudo–Riemannian manifolds, in particular to Lorentz manifolds. Some of the results, especially for geodesic orbit Lorentz manifolds, are quite surprising.

A connected Riemannian homogeneous space $M = G/H$ is a *nilmanifold* if the isometry group G has a connected nilpotent subgroup N that is transitive on M. In that case N is the nilradical of G, and G is the semidirect product $N \rtimes H$. If M is a geodesic orbit Riemannian manifold then N is commutative or 2–step nilpotent. This fails dramatically for connected pseudo–Riemannian geodesic orbit nilmanifolds. Our main results form a very strong extension of the 2–step nilpotent theorem from Riemannian geodesic orbit spaces to Lorentz geodesic orbit spaces. Of course these results hold for special cases such as naturally reductive spaces and, more importantly, weakly symmetric spaces.

Consider a geodesic orbit Lorentz nilmanifold $M = G/H$ with $G = N \rtimes H$ and N nilpotent. Then N is a closed normal subgroup of G and is simply transitive on M, so we can view M as the group N with a left invariant $\mathrm{Ad}(H)$–invariant Lorentz metric.

Our first main result, Theorem 4.2, applies to the case where the metric is nondegenerate on $[\mathfrak{n}, \mathfrak{n}]$. It says that N either is at most 2–step nilpotent as in the Riemannian situation, or is 4–step nilpotent, but cannot be 3–step nilpotent. Examples show that these bounds are the best possible. Our second main result, Theorem 5.1, shows that N is at most 2–step nilpotent when the metric is degenerate on $[\mathfrak{n}, \mathfrak{n}]$. As one expects, both theorems give additional structural information and specialize to naturally reductive and to weakly symmetric Lorentz nilmanifolds.

In Section 2 we indicate the background on geodesic orbit spaces. We describe the particular cases of naturally reductive spaces and weakly symmetric spaces. The emphasis is on extending definitions and results from the Riemannian case to the pseudo–Riemannian cases. Weakly symmetric spaces are the most important special case (see [16]), so in Section 3 we go into more detail on that. Sections 4 and 5 contain our main results and their proofs.

2. Geodesics in Pseudo-Riemannian Manifolds

In this section we discuss the geodesic orbit property for Riemannian manifolds and extensions to the pseudo–Riemannian setting. We will need that in the proofs of our main results, Theorems 4.2 and 5.1. First we recall some background and some results of the first two authors from [4].

Definition 2.1. A pseudo-Riemannian manifold M is called a *geodesic orbit space* if every maximal geodesic in M is an orbit of a one-parameter group of isometries of M. $\diamondsuit$

Pseudo-Riemannian geodesic orbit spaces are geodesically complete and homogeneous [4]. Let G be a transitive Lie group of isometries of M, say $M = G/H$ with base point $x_0 = 1H$. We say that a nonzero element ξ in The Lie algebra $\mathfrak{g}$ is a *geodesic vector* if the image of $t \mapsto \gamma(\varphi(t)) = \exp(t\xi)x_0$ is a maximal geodesic, where φ is a diffeomorphism of $\mathbb{R}$ onto an open interval in $\mathbb{R}$. Then γ is a *homogeneous geodesic*. In the Riemannian case $\varphi(t) = t$ and t is the affine parameter.

Suppose that we have a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (vector space direct sum) with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, which of course is automatic in the Riemannian case where H is compact. As usual we write $\pi_{\mathfrak{h}}$ and $\pi_{\mathfrak{m}}$ for projections to the summands, and $\xi_{\mathfrak{h}}$ and $\xi_{\mathfrak{m}}$ for the components of an element $\xi \in \mathfrak{g}$. Then M is a geodesic orbit space (*relative to* $\mathfrak{g}$ *and* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$) if and only if, for every $\xi \in \mathfrak{m}$, there is an $\eta \in \mathfrak{h}$ such that $\xi + \eta$ is a geodesic vector. In other words, a homogeneous pseudo-Riemannian manifold M is a geodesic orbit space if every geodesic of M is homogeneous.

Proposition 2.2. (**Geodesic Lemma**). *Let $M = G/H$ be a pseudo-Riemannian homogeneous space. Suppose that there is a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. Let $x_0 = 1H \in G/H$ and $\xi \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(t\xi)(x_0)$ is a geodesic curve with respect to some parameter s if and only if*

$$\langle [\xi, \zeta]_{\mathfrak{m}}, \xi_{\mathfrak{m}} \rangle = k \langle \xi_{\mathfrak{m}}, \zeta \rangle \tag{2.1}$$

for all $\zeta \in \mathfrak{m}$, where k is a constant. If $k = 0$, then t is an affine parameter for γ. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for γ, and γ is a null curve in M.

Corollary 2.3. *Given a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, the geodesic orbit property M is equivalent to the following condition. If $\xi \in \mathfrak{m}$ there exist $\alpha \in \mathfrak{h}$ and a constant k such that, if $\zeta \in \mathfrak{m}$ then $\langle [\xi + \alpha, \zeta]_{\mathfrak{m}}, \xi \rangle = k \langle \zeta, \xi \rangle$.*

For the notion of homogeneous geodesic and the formula in the Geodesic Lemma see [11] for the Riemannian case, and then [6], [7] and [14] for the pseudo–Riemannian case.

——o——

A homogeneous Riemannian manifold $M = G/H$, with reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, is *naturally reductive* (with respect to G) if $\pi_{\mathfrak{m}} \cdot \mathrm{ad}(\xi)|_{\mathfrak{m}}$ is skew symmetric for every $\xi \in \mathfrak{m}$. In terms of the inner product on $\mathfrak{m}$ this condition is $\langle [\xi, \eta]_{\mathfrak{m}}, \zeta \rangle + \langle \eta, [\xi, \zeta]_{\mathfrak{m}} \rangle = 0$ for all $\xi, \eta, \zeta \in \mathfrak{m}$. The case $\xi = \eta$ is: $\xi, \zeta \in \mathfrak{m} \Rightarrow \langle \xi, [\xi, \zeta]_{\mathfrak{m}} \rangle = 0$. As noted in [8, Proposition 1.7(a)], following [11] this says that every $\xi \in \mathfrak{m}$ is a homogeneous vector. However we have a simpler treatment that avoids [8] and [11], and is valid for the pseudo–Riemannian case as well.

Definition 2.4. Let $M = G/H$ be a pseudo-Riemannian homogeneous space. Suppose that there is a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. Let $x_0 = 1H \in G/H$ and $\xi \in \mathfrak{g}$ and let L be the group of linear transformations of $\mathfrak{m}$ that preserve the inner product. If $\mathrm{ad}(\xi)|_{\mathfrak{m}}$ belongs to the Lie algebra of L for every $\xi \in \mathfrak{m}$, then M is **naturally reductive** (with respect to G and the reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$). ♢

Remark 2.5. The naturally reductive property depends on the choice of transitive isometry group and the reductive decomposition, even in the Riemannian case. See [13]. We thank Yurii Nikonorov for that and other references. ♢

As in the Riemannian case, the defining condition for "naturally reductive" in terms of the inner product and the Lie algebra $\mathfrak{l}$ of L is $\mathrm{ad}(\eta)|_{\mathfrak{m}} \in \mathfrak{l}$ for all $\eta \in \mathfrak{m}$, i.e. $\langle [\xi,\zeta]_{\mathfrak{m}}, \eta\rangle + \langle \zeta, [\xi,\eta]_{\mathfrak{m}}\rangle = 0$ for all $\xi,\eta,\zeta \in \mathfrak{m}$. The case $\xi = \eta$ says $\langle [\xi,\zeta]_{\mathfrak{m}}, \xi\rangle = 0$ for all ζ, $\xi \in \mathfrak{m}$. In other words if $\xi,\zeta \in \mathfrak{m}$ then 2.1 holds with $k = 0$. We have proved

Proposition 2.6. [9, pp. 200–202] *Naturally reductive homogeneous pseudo–Riemannian manifolds are pseudo–Riemannian geodesic orbit spaces.*

——o——

Selberg introduced an important extension of the class of Riemannian symmetric spaces: Riemannian weakly symmetric spaces [15]. As mentioned in the Introduction, there are a number of characterizations equivalent to the definition of weak symmetry for a Riemannian manifold. The paper [4] of Chen and Wolf uses the characterization that is geometrically most accessible.

Definition 2.7. Let (M, g) be a pseudo-Riemannian manifold. Suppose that for every $x \in M$ and every nonzero tangent vector $\xi \in T_xM$, there is an isometry $\phi = \phi_{x,\xi}$ of M such that $\phi(x) = x$ and $d\phi(\xi) = -\xi$. Then (M, g) is a *weakly symmetric pseudo-Riemannian manifold.* In particular, if $\phi_{x,\xi}$ is independent of ξ then M is symmetric. ♢

Riemannian symmetric spaces are geodesic orbit spaces. A few years ago the first two authors extended this result to weak symmetry and indefinite metric:

Proposition 2.8. [4, §4] Weakly symmetric pseudo-Riemannian manifolds are geodesic orbit spaces.

In other words, if (M, g) is a weakly symmetric pseudo–Riemannian manifold, and if γ is a maximal geodesic and $x \in \gamma$, there is an isometry of M which is a non-trivial involution on γ with x as fixed point.

——o——

There are many other classes of Riemannian manifolds related to Riemannian geodesic orbit spaces, for example normal homogeneous spaces, D'Atri spaces and Damek-Ricci spaces. See the survey article [10] and the references there for a discussion of these classes and their relation to weakly symmetric spaces and geodesic orbit spaces, [19] and [16] for a number of examples, and the comprehensive book [2] on Riemannian geodesic orbit spaces. Rather than digress to consider these various classes in any depth, in Section 3 below we only sketch some important background material applicable to weakly symmetric pseudo–Riemannian manifolds.

3. Weakly Symmetric Pseudo-Riemannian Manifolds

In this section we sketch some of the main results on weakly symmetric pseudo–Riemannian manifolds. A. Selberg's original definition of weakly symmetric space holds also for pseudo-Riemannian manifolds, but it is rather complicated[1] so we use Definition 2.7 above.

[1]Selberg: Let (M, g) be a Riemannian manifold. If there exists a subgroup G of the isometry group $I(M, g)$ of M acting transitively on M and an involutive isometry μ of (M, g) with $\mu G = G\mu$ such that whenever $x, y \in M$ there exists $\phi \in G$ with $\phi(x) = \mu(y)$ and $\phi(y) = \mu(x)$, then (M, g) is a *weakly symmetric Riemannian manifold.*

A connected homogeneous pseudo-Riemannian manifold need not be geodesically convex, but any two points can be joined by a broken geodesic. Going segment by segment along broken geodesics, as in the Riemannian case we have

Proposition 3.1. A pseudo-Riemannian manifold (M, g) is weakly symmetric if and only if for any two points x, $y \in M$ there is an isometry of M mapping x to y and y to x.

Recall the De Rham-Wu decomposition theorem [18]. Let (M, g) be a complete simply connected pseudo-Riemannian manifold, $x \in M$, and $T_x(M) = T_{x,0} \oplus \cdots \oplus T_{x,r}$ a decomposition of the tangent space at x into holonomy invariant mutually orthogonal subspaces, where the holonomy group at x is trivial on $T_{x,0}$ and irreducible on the other $T_{x,i}$. Suppose that the pseudo-Riemannian metric g has nondegenerate restriction to $T_{x,i}$ for each index i . Then (M, g) is isometric to a pseudo-Riemannian direct product $(M_0, g_0) \times \cdots \times (M_r, g_r)$, where $x = (x_0, \ldots, x_r)$ and for each index, (M_i, g_i) has tangent space $T_{x,i}$ at x_i . As in the Riemannian case (M_i, g_i) is the maximal integral manifold through x of the distribution obtained by parallel translating $T_{x,i}\ M$ and equipped with the metric g_i induced by g. Thus

Proposition 3.2. Let (M, g) be a complete simply connected pseudo-Riemannian manifold, $x \in M$, and $T_x(M) = T_{x,0} \oplus \cdots \oplus T_{x,r}$ a decomposition of the tangent space at x into holonomy invariant mutually orthogonal subspaces, where the holonomy group at x is trivial on $T_{x,0}$ and irreducible on the other $T_{x,i}$. Suppose that the pseudo-Riemannian metric g has nondegenerate restriction to $T_{x,i}$ for each index i and let $(M, g) = (M_0, g_0) \times \cdots \times (M_r, g_r)$ be the De Rham-Wu decomposition. Then (M, g) is weakly symmetric if and only if each of the pseudo-Riemannian manifolds (M_i, g_i) is weakly symmetric.

Definition 3.3. Let G be a connected Lie group and H be a closed subgroup. Suppose that σ is an automorphism of G such that $\sigma(p) \in Hp^{-1}H$, $\forall p \in G$. Then G/H is called a *weakly symmetric coset space*, (G, H) is called a *weakly symmetric pair*, and σ is called a *weak symmetry* of G/H. $\diamondsuit$

It is easy to see that a weakly symmetric pseudo-Riemannian manifold is a weakly symmetric coset space G/H where G is the isometry group.

In the context of homogeneous spaces G/H where G is a connected Lie group and H is a compact subgroup, one often says that G/H is *commutative* when the algebra of all G-invariant differential operators is commutative. That is a special case (where G is a connected Lie group) of the definition in the setting of topological groups: G is a separable locally compact group, H is a compact subgroup, and the convolution algebra $L^1(H\backslash G/H)$ is commutative. Selberg [15] proved that a Riemannian weakly symmetric space $M = G/H$ is a commutative space, but Lauret [12] found commutative spaces that are not weakly symmetric.

4. Lorentz Geodesic Orbit and Weakly Symmetric Nilmanifolds, I

We start with a useful lemma.

Lemma 4.1. [1] *Let $B \in \mathfrak{so}(n-1, 1)$. In a suitable basis $\{e_i\}$ of $\mathbb{R}^n$, either*

(1) B *is semisimple* , $B = \begin{pmatrix} \mu & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & -\mu \end{pmatrix}$ *with $C \in \mathfrak{so}(n-2)$ and $\mu \le 0$, where*

$\langle e_1, e_n \rangle = 1$, $\langle e_i, e_j \rangle = \delta_{ij}$ $(i, j = 2, \cdots, n-1)$ *and the other scalar products vanish, or*

(2) B *is not semisimple* , $B = \begin{pmatrix} \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & C \end{pmatrix}$ *where $C \in \mathfrak{so}(n-3)$, $\mathbf{0}$ is the zero matrix of appropriate size;* $-\langle e_1, e_3 \rangle = \langle e_2, e_2 \rangle = 1, \langle e_i, e_j \rangle = \delta_{ij}(i, j = 4, \cdots, n)$, *and the other scalar products vanish.*

There is a typographical error in [1]; in case (2) there, one should assume $r = e_1$ and $q = e_2$.

From Lemma 4.1, any matrix $H \in \mathfrak{so}(n-1,1)$ has $n-2$ purely imaginary eigenvalues and two non-zero real eigenvalues $\pm\mu$, or has n purely imaginary eigenvalues, viewing 0 as purely imaginary. Now we can state the first of our two main results, the case where the metric is nondegenerate on $[\mathfrak{n},\mathfrak{n}]$.

Theorem 4.2. *Let $(M = G/H, \langle\, ,\, \rangle)$ be a connected Lorentz geodesic orbit nilmanifold, where $G = N \rtimes H$ with N nilpotent. Then N is abelian, or 2-step nilpotent, or 4-step nilpotent.*

Note that $G = N \rtimes H$ defines the reductive decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$. Identify $\mathfrak{n}$ with the tangent space at $1H$ and let $\mathfrak{v}$ denote the orthocomplement of $[\mathfrak{n},\mathfrak{n}]$ in $\mathfrak{n}$. Suppose that $\langle\cdot,\cdot\rangle$ is nondegenerate on $[\mathfrak{n},\mathfrak{n}]$. Then either $\mathrm{ad}(x) = 0$ for any $x \in \mathfrak{v}$, or there is a basis $\{x, \tilde{x}_1, \cdots, \tilde{x}_s\}$ of $\mathfrak{v}$ $(s \geq 1)$ such that

$$\mathrm{ad}(x)|_{[\mathfrak{n},\mathfrak{n}]} = \begin{pmatrix} \begin{smallmatrix} 0&1&0\\0&0&1\\0&0&0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}; \qquad \mathrm{ad}(\tilde{x}_1)|_{[\mathfrak{n},\mathfrak{n}]} = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} a_1 & a_2 & \cdots & a_p \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 0 & a_1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_p \end{smallmatrix} & \mathbf{0} \end{pmatrix};$$

$$\mathrm{ad}(\tilde{x}_i)|_{[\mathfrak{n},\mathfrak{n}]} = 0 \textit{ for } 2 \leq s,\ 2 \leq i \leq s; \textit{ and } \mathrm{ad}([y,z])|_{[\mathfrak{n},\mathfrak{n}]} = 0 \textit{ for all } y, z \in \mathfrak{n}.$$

Proof. We may assume that $[\mathfrak{n},\mathfrak{n}] \neq 0$. Note that $\mathfrak{v} = [\mathfrak{n},\mathfrak{n}]^\perp \cap \mathfrak{n}$ is $Ad_G(H)$-invariant, where of course orthogonality is relative to $\langle\cdot,\cdot\rangle$. For any $\eta \in [\mathfrak{n},\mathfrak{n}]$, the Geodesic Lemma provides $\alpha_\eta \in \mathfrak{h}$ such that $\langle[\eta + \alpha_\eta, \xi]_\mathfrak{n}, \eta\rangle = k\langle\eta,\xi\rangle = 0$ for every $\xi \in \mathfrak{v}$. It follows that $\langle[\xi,\eta]_\mathfrak{n},\eta\rangle = 0$. That is, for any $\xi \in \mathfrak{v}$ and $\eta, \zeta \in [\mathfrak{n},\mathfrak{n}]$,

$$\langle[\xi,\eta],\zeta]\rangle + \langle\eta,[\xi,\zeta]\rangle = 0 \tag{4.1}$$

We are assuming that $\langle\cdot,\cdot\rangle$ is nondegenerate on $[\mathfrak{n},\mathfrak{n}]$. Then $\mathfrak{n} = [\mathfrak{n},\mathfrak{n}] + \mathfrak{v}$. If $[\mathfrak{n},\mathfrak{n}]$ is positive or negative definite, then ([4, Theorem 4.12]) $\mathfrak{n}$ is commutative or 2-step nilpotent. Those cases aside, we suppose that $[\mathfrak{n},\mathfrak{n}]$ is indefinite.

We are going to prove that $[\mathfrak{n},\mathfrak{n}]$ is abelian. Since $[\mathfrak{n},\mathfrak{n}]$ is nilpotent, this is automatic if $\dim[\mathfrak{n},\mathfrak{n}] < 3$, so we may suppose $\dim[\mathfrak{n},\mathfrak{n}] \geq 3$. Write $\dim[\mathfrak{n},\mathfrak{n}] = p+3$ with $p \geq 0$. Fix $x \in \mathfrak{v}$. By Lemma 4.1, $[\mathfrak{n},\mathfrak{n}]$ has a basis $\{e_1, e_2, \cdots, e_{p+3}\}$ in which the inner product has matrix

$$\langle\cdot,\cdot\rangle_{[\mathfrak{n},\mathfrak{n}]} = \begin{pmatrix} \begin{smallmatrix} 0&0&-1\\0&1&0\\-1&0&0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & I_p \end{pmatrix} \tag{4.2}$$

and if $\mathrm{ad}(x) \neq 0$ it has matrix

$$\mathrm{ad}(x)|_{[\mathfrak{n},\mathfrak{n}]} = \begin{pmatrix} \begin{smallmatrix} 0&1&0\\0&0&1\\0&0&0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{4.3}$$

If $\mathrm{ad}(x) = 0$ for every $x \in \mathfrak{v}$, then $\mathfrak{n}$ is 2-step nilpotent, because $\mathfrak{v}$ generates $\mathfrak{n}$. In the following we assume that there exists $x \in \mathfrak{v}$ such that $\mathrm{ad}(x) \neq 0$.

Let $y \in \mathfrak{v}$. The matrix of $\mathrm{ad}(y)$ in the given basis on $[\mathfrak{n},\mathfrak{n}]$ has form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where $D \in \mathbb{R}^{p\times p}$. By Lemma 4.1, in a possibly modified basis, either $\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]} = \begin{pmatrix} \begin{smallmatrix} 0&1&0\\0&0&1\\0&0&0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ or $\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]} = \mathbf{0}$.

If $\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]} \neq 0$ the rank $r(\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]}) = 2$. If $D \neq 0$ let $\lambda \gg 0$ be a large real number and consider the matrix of $\mathrm{ad}(\lambda x + y)|_{[\mathfrak{n},\mathfrak{n}]}$. For λ it has rank ≥ 3. This contradiction shows $D = \mathbf{0}$.

Express $A = \left(\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{smallmatrix}\right)$. In the basis $\{e_1, e_2, \cdots, e_{p+3}\}$, $\mathrm{ad}(\lambda x + y)|_{[\mathfrak{n},\mathfrak{n}]}$ has matrix $\left(\begin{smallmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \mathbf{0} \end{smallmatrix}\right)$ where $\tilde{A} = \left(\begin{smallmatrix} a_{11} & a_{12}+\lambda & a_{13} \\ a_{21} & a_{22} & a_{23}+\lambda \\ a_{31} & a_{32} & a_{33} \end{smallmatrix}\right)$. If $a_{31} \neq 0$ and $\lambda \gg 0$ then $\det(\tilde{A}) \neq 0$, so $r(\mathrm{ad}(\lambda x + y)|_{[\mathfrak{n},\mathfrak{n}]}) \geq 3$. This contradiction shows $a_{31} = 0$. Similarly the first column of C and the third row of B vanish. We will need these constraints on the matrix $\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]} = \left(\begin{smallmatrix} A & B \\ C & \mathbf{0} \end{smallmatrix}\right)$.

The metric on N has matrix $\begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & I_p \end{pmatrix}$ where $W = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, by equation (4.1). Thus

$$\begin{pmatrix} A^t & C^t \\ B^t & \mathbf{0} \end{pmatrix}\begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & I_p \end{pmatrix} + \begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & I_p \end{pmatrix}\begin{pmatrix} A & B \\ C & \mathbf{0} \end{pmatrix} = \mathbf{0}.$$

In other words,

$$A^t W = -WA \text{ and } C^t = -WB. \tag{4.4}$$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ and $C^t = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & c_{2p} \\ c_{31} & c_{32} & \cdots & c_{3p} \end{pmatrix}$. By (4.4), $A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{12} \\ 0 & a_{21} & -a_{11} \end{pmatrix}$. Further. $c_{2i} = -b_{2i}$ and $c_{3i} = b_{1i}$ for $i = 1, \cdots, p$. It follows that

$$\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]} = \left(\begin{array}{ccc|cccc} a_{11} & a_{12} & 0 & b_{11} & b_{12} & \cdots & b_{1p} \\ a_{21} & 0 & a_{12} & b_{21} & b_{22} & \cdots & b_{2p} \\ 0 & a_{21} & -a_{11} & 0 & 0 & \cdots & 0 \\ \hline 0 & -b_{21} & b_{11} & & & & \\ \vdots & \vdots & \vdots & & \mathbf{0} & & \\ 0 & -b_{2p} & b_{1p} & & & & \end{array}\right). \tag{4.5}$$

Since $\mathrm{ad}(\mathfrak{n})$ preserves both $[\mathfrak{n},\mathfrak{n}]$ and $\mathfrak{v} = [\mathfrak{n},\mathfrak{n}]^\perp$, we have $\mathrm{ad}([x,y])|_{[\mathfrak{n},\mathfrak{n}]} = [\mathrm{ad}(x)|_{[\mathfrak{n},\mathfrak{n}]}, \mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]}]$. Combining (4.3) and (4.5), now,

$$\mathrm{ad}([x,y])|_{[\mathfrak{n},\mathfrak{n}]} = \left(\begin{array}{ccc|cccc} a_{21} & -a_{11} & 0 & b_{21} & b_{22} & \cdots & b_{2p} \\ 0 & 0 & -a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{21} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & b_{21} & & & & \\ \vdots & \vdots & \vdots & & \mathbf{0} & & \\ 0 & 0 & b_{2p} & & & & \end{array}\right).$$

As $\mathrm{ad}([x,y])|_{[\mathfrak{n},\mathfrak{n}]}$ is nilpotent, its eigenvalues all are zero. Thus $a_{21} = 0$. Similarly, from (4.5), $a_{11} = 0$. Moreover, the matrix of $(\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]})^2$ is

$$(\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]})^2 = \left(\begin{array}{ccc|cccc} 0 & -\sum_{i=1}^p b_{1i}b_{2i} & a_{12}^2 + \sum_{i=1}^p b_{1i}^2 & b_{21} & b_{22} & \cdots & b_{2p} \\ 0 & -\sum_{i=1}^p b_{2i}^2 & \sum_{i=1}^p b_{1i}b_{2i} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & -a_{12}b_{21} & -b_{21}^2 & -b_{21}b_{22} & \cdots & -b_{21}b_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -a_{12}b_{2p} & -b_{2p}b_{21} & -b_{2p}b_{22} & \cdots & -b_{2p}^2 \end{array}\right). \tag{4.6}$$

From that we compute the trace $\mathrm{Tr}((\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]})^2) = -2\sum_1^p b_{2i}^2$. As $(\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]})^2$ is nilpotent, it has trace 0, so $b_{21} = \cdots = b_{2p} = 0$.

From these calculations we have

$$\mathrm{ad}([x,y])|_{[\mathfrak{n},\mathfrak{n}]} = 0 \text{ for all } y \in \mathfrak{v}. \tag{4.7}$$

Writing $a(y)$ for a_{12} and $b_j(y)$ for b_{1j} we also have

$$\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]} = \left(\begin{array}{ccc|cccc} 0 & a(y) & 0 & b_1(y) & b_2(y) & \cdots & b_p(y) \\ 0 & 0 & a(y) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & b_1(y) & & & & \\ \vdots & \vdots & \vdots & & \mathbf{0} & & \\ 0 & 0 & b_p(y) & & & & \end{array}\right), \quad \text{for all } y \in \mathfrak{v}. \tag{4.8}$$

Initially (4.8) requires y to be linearly independent of x, but it holds for all $y \in \mathfrak{v}$ with $a(y) = 1$ and $b_j(y) = 0$.

We continue to simplify the structure of $\mathrm{ad}(y)|_{[\mathfrak{n},\mathfrak{n}]}$. For the moment assume $\dim \mathfrak{v} = s+1 \geq 2$. Extend $\{x\}$ to a basis $\{x, x_1, \cdots, x_s\}$ of $\mathfrak{v}$. Using (4.8)

$$\mathrm{ad}(x_i)|_{[\mathfrak{n},\mathfrak{n}]} = \left(\begin{array}{ccc|cccc} 0 & a(x_i) & 0 & b_1(x_i) & b_2(x_i) & \cdots & b_p(x_i) \\ 0 & 0 & a(x_i) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & b_1(x_i) & & & & \\ \vdots & \vdots & \vdots & & \mathbf{0} & & \\ 0 & 0 & b_p(x_i) & & & & \end{array}\right) \text{ for } 1 \leq i \leq s.$$

From this point on, in the proof of Theorem 4.2, we will make successive modifications of the basis $\{x, x_1, \cdots, x_s - a(x_s)x\}$, along the lines of Gauss Elimination. To avoid complicated notation we use $\{x, \tilde{x}_1, \cdots, \tilde{x}_s\}$ for each of the successive modifications.

In the basis $\{x, \tilde{x}_1, \cdots, \tilde{x}_s\} := \{x, x_1 - a(x_1)x, \cdots, x_s - a(x_s)x\}$ we now have

$$\mathrm{ad}(\tilde{x}_i)|_{[\mathfrak{n},\mathfrak{n}]} = \left(\begin{array}{ccc|cccc} 0 & 0 & 0 & b_1(\tilde{x}_i) & b_2(\tilde{x}_i) & \cdots & b_p(\tilde{x}_i) \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & b_1(\tilde{x}_i) & & & & \\ \vdots & \vdots & \vdots & & \mathbf{0} & & \\ 0 & 0 & b_p(\tilde{x}_i) & & & & \end{array}\right) \text{ for } 1 \leq i \leq s. \tag{4.9}$$

From (4.9) we compute $[\mathrm{ad}(\tilde{x}_i), \mathrm{ad}(\tilde{x}_j)]|_{[\mathfrak{n},\mathfrak{n}]} = [\mathrm{ad}(\tilde{x}_i)|_{[\mathfrak{n},\mathfrak{n}]}, \mathrm{ad}(\tilde{x}_j)|_{[\mathfrak{n},\mathfrak{n}]}] = 0$ for $1 \leq i, j \leq s$. Also, from (4.3) together with (4.9), $[\mathrm{ad}(x), \mathrm{ad}(\tilde{x}_j)]|_{[\mathfrak{n},\mathfrak{n}]} = [\mathrm{ad}(x)|_{[\mathfrak{n},\mathfrak{n}]}, \mathrm{ad}(\tilde{x}_j)|_{[\mathfrak{n},\mathfrak{n}]}] = 0$. Thus

$$[[\mathfrak{v},\mathfrak{v}],[\mathfrak{n},\mathfrak{n}]] = 0. \tag{4.10}$$

Assume $s \geq 2$. Write $[x, \tilde{x}_i] = \sum_{k=1}^{p+3} a_i^k e_k$ and $[\tilde{x}_i, \tilde{x}_j] = \sum_{k=1}^{p+3} a_{i,j}^k e_k$. Then

$$\begin{aligned} [\tilde{x}_i, [x, \tilde{x}_j]] &= \sum\nolimits_{k=1}^{p+3} a_j^k [\tilde{x}_i, e_k] = a_j^3 \sum\nolimits_{\ell=1}^{p} b_\ell(\tilde{x}_i) e_{\ell+3} + \left(\sum\nolimits_{k=1}^{p} a_j^{k+3} b_k(\tilde{x}_i)\right) e_1\,, \\ [x, [\tilde{x}_j, \tilde{x}_i]] &= \sum\nolimits_{k=1}^{p+3} a_{j,i}^k [x, e_k] = a_{j,i}^2\, e_1 + a_{j,i}^3\, e_2\,, \text{ and} \\ [\tilde{x}_j, [\tilde{x}_i, x]] &= -\sum\nolimits_{k=1}^{p+3} a_i^k [\tilde{x}_j, e_k] = -a_i^3 \sum\nolimits_{\ell=1}^{p} b_\ell(\tilde{x}_j) e_{\ell+3} + \left(\sum\nolimits_{k=1}^{p} a_i^{k+3} b_k(\tilde{x}_j)\right) e_1 \end{aligned} \tag{4.11}$$

The first and third terms here have no e_2 component. From the Jacobi Identity $[x, [\tilde{x}_j, \tilde{x}_i]]$ has no e_2 component, i.e. $a_{j,i}^3 = 0$, so $[x, [\tilde{x}_j, \tilde{x}_i]] = a_{j,i}^2\, e_1$ and $[\tilde{x}_j, \tilde{x}_i]$ has no e_3 component.

Suppose that $[x, \tilde{x}_j]$ has nonzero e_3 component. At least one of those e_3 components is nonzero because $\mathfrak{v}$ generates $\mathfrak{n}$. We next modify the basis $\{x, \tilde{x}_1, \ldots, \tilde{x}_s\}$ of $\mathfrak{v}$ by (1) permuting the $\{\tilde{x}_j\}$ if necessary so that $[x, \tilde{x}_1]$ has nonzero e_3 component, and (2) if $j > 1$ and $[x, \tilde{x}_j]$ has nonzero e_3 component then subtract a multiple of $\tilde{x}_1$ from $\tilde{x}_j$ so that $[x, \tilde{x}_j]$ has e_3 component zero. Then (4.9), and thus (4.11), still hold for the modified $\tilde{x}_i$.

We have arranged $a_{j,i}^3 = 0$ for $1 \leq i, j \leq s$, $a_1^3 \neq 0$, and $a_k^3 = 0$ for $k > 1$. Thus, in (4.11), $[\tilde{x}_i, [x, \tilde{x}_j]]$ is a multiple of e_1 when $j > 1$, $[x, [\tilde{x}_j, \tilde{x}_i]]$ is a multiple of e_1 in general, and $[\tilde{x}_j, [\tilde{x}_i, x]]$ is a multiple of e_1 when $i > 1$. From the Jacobi Identity, if $i > 1$ then $[\tilde{x}_i, [x, \tilde{x}_1]]$ is a multiple of e_1. Again from (4.11) $a_1^3 \sum_{\ell=1}^{p} b_\ell(\tilde{x}_i) e_{\ell+3} = 0$, and since $a_1^3 \neq 0$ this says that each $b_\ell(\tilde{x}_i) = 0$. Going back to (4.9),

$$\text{if } i > 1 \text{ then } \mathrm{ad}(\tilde{x}_i)|_{[\mathfrak{n},\mathfrak{n}]} = 0.$$

In summary we see that $\mathfrak{n}$ has a very simple structure. Associated with an appropriate basis $\{x, \tilde{x}_1, \cdots, \tilde{x}_s\}$ of $\mathfrak{v}$ $(s \geq 2)$,

$$\mathrm{ad}(x)|_{[\mathfrak{n},\mathfrak{n}]} = \begin{pmatrix} \begin{smallmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}; \qquad \mathrm{ad}(\tilde{x}_1)|_{[\mathfrak{n},\mathfrak{n}]} = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} a_1 & a_2 & \cdots & a_p \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0&0&a_1 \\ \vdots&\vdots&\vdots \\ 0&0&a_p \end{smallmatrix} & \mathbf{0} \end{pmatrix}; \tag{4.12}$$

$$\mathrm{ad}(\tilde{x}_i)|_{[\mathfrak{n},\mathfrak{n}]} = 0 \text{ for } 2 \leq i \leq s; \text{ and } \mathrm{ad}([y,z])|_{[\mathfrak{n},\mathfrak{n}]} = 0 \text{ for all } y, z \in \mathfrak{n}.$$

In particular $[\mathfrak{n},\mathfrak{n}]$ is abelian. Thus $\mathfrak{n}$ is at most 4-step nilpotent.

In order to see that $\mathfrak{n}$ cannot be 3 step nilpotent we use the structure just described. Suppose $[\mathfrak{n},\mathfrak{n}] \neq 0 \neq [\mathfrak{n},[\mathfrak{n},\mathfrak{n}]]$. Then, by construction, $[\mathfrak{n},\mathfrak{n}] = Span\{e_1, ..., e_{p+3}\}$. By (4.12), if $p = 0$, $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] = Span\{e_1, e_2\}$; if $p \neq 0$, $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] = Span\{e_1, e_2, \sum_{i=1}^{p} a_i e_{3+i}\}$. Note that e_3 is not contained in this span. Continuing with (4.12) we see $[\mathfrak{n},[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]]] = Span\{e_1\}$. This eliminates the possibility of 3–step nilpotence and completes the proof of Theorem 4.2. $\square$

Remark 4.3. In connection with Theorem 4.2, there are many examples where $\mathfrak{n}$ is abelian or 2–step nilpotent, but we have not been able to construct an example where it is 4–step nilpotent. So, at the moment, 4–step nilpotence is only a necessary condition.

5. Lorentz Geodesic Orbit and Weakly Symmetric Nilmanifolds, II

The second of our two main results, the case where the metric is degenerate on $[\mathfrak{n},\mathfrak{n}]$, is as follows. The result contrasts with Theorem 4.2, and essentially coincides with the situation for Riemannian manifolds.

Theorem 5.1. *Let $(M = G/H, \langle,\rangle)$ be a connected Lorentz geodesic orbit nilmanifold. Suppose that $G = N \rtimes H$ with N nilpotent. Suppose further that there is a reductive decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$, where $[\mathfrak{n},\mathfrak{n}]$ is degenerate and the action of $\mathrm{Ad}(H)|_{\mathfrak{n}}$ is completely reducible on $\mathfrak{n}$. Then $\mathfrak{n}$ is at most 2-step nilpotent.*

Furthermore, there is a basis $\{e_1, \cdots, e_p; e_{p+1}\}$ of $[\mathfrak{n},\mathfrak{n}]$ and a basis $\{v_0; v_1, \cdots, v_s\}$ of a vector space complement $\mathfrak{a}$ to $[\mathfrak{n},\mathfrak{n}]$ in $\mathfrak{n}$ with the following properties.

(1) $\mathfrak{v}_1 := \mathrm{Span}(e_1, \cdots, e_p)$ and $\mathfrak{v}_2 := \mathrm{Span}(v_1, \cdots, v_s)$ are both positive definite or both negative definite and are $\mathrm{Ad}(H)$–invariant,

(2) $[\mathfrak{n},\mathfrak{n}] \cap [\mathfrak{n},\mathfrak{n}]^\perp = e_{p+1}\mathbb{R}$ and $\mathfrak{a} \cap \mathfrak{a}^\perp = v_0\mathbb{R}$ are $\mathrm{Ad}(H)$–invariant,

(3) $\mathfrak{w} := \mathrm{Span}(e_{p+1}, v_0)$ is of signature $(1,1)$,

(4) $\mathfrak{n} = \mathfrak{v}_1 + \mathfrak{w} + \mathfrak{v}_2$ is an $\mathrm{Ad}(H)$–invariant orthogonal direct sum,

(5) $\mathrm{ad}(x)|_{[\mathfrak{n},\mathfrak{n}]} = 0$ for any $x \in \mathfrak{a}$.

Proof. Let $\dim[\mathfrak{n},\mathfrak{n}] = p+1$. Since $\mathfrak{n}$ is of Lorentz signature and $[\mathfrak{n},\mathfrak{n}]$ is degenerate, $\dim([\mathfrak{n},\mathfrak{n}] \cap [\mathfrak{n},\mathfrak{n}]^\perp) = 1$. So we have $e_{p+1} \neq 0$ spanning $[\mathfrak{n},\mathfrak{n}] \cap [\mathfrak{n},\mathfrak{n}]^\perp$, and $e_{p+1}^\perp = \mathfrak{v} + e_{p+1}\mathbb{R}$ where $\mathfrak{v}$ is positive or negative definite. Now $\mathfrak{v} = \mathfrak{v}_1 + \mathfrak{v}_2$, $\mathrm{Ad}(H)$–invariant orthogonal direct sum, where $\mathfrak{v}_1 = \mathfrak{v} \cap [\mathfrak{n},\mathfrak{n}]$. Thus $\mathfrak{w} := \mathfrak{v}^\perp$ is spanned by e_{p+1} and a null vector v_0 with $\langle e_{p+1}, v_0\rangle = 1$ and $\mathrm{Ad}(H)v_0 \in v_0\mathbb{R}$. Choose orthonormal bases $\{e_1, \cdots, e_p\}$ of $\mathfrak{v}_1$ and $\{v_1, \cdots, v_s\}$ of $\mathfrak{v}_2$. With those, we have constructed a basis of $\mathfrak{n}$ that satisfies conditions (1) through (4) above. Note that the inner product on $\mathfrak{w}$ has matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. So the metric on $\mathfrak{n}$ has the matrix

$$\langle\cdot,\cdot\rangle_{\mathfrak{n}} = \begin{pmatrix} I_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 1 & \mathbf{0} \\ \mathbf{0} & 1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_s \end{pmatrix}$$

under the basis $\{e_1, e_2, \cdots, e_p; e_{p+1}, v_0; v_1, \cdots, v_s\}$. In particular, the metric on $[\mathfrak{n},\mathfrak{n}]$ has the matrix $\langle\cdot,\cdot\rangle_{[\mathfrak{n},\mathfrak{n}]} = \left(\begin{smallmatrix} I_p & \mathbf{0} \\ \mathbf{0} & 0 \end{smallmatrix}\right)$ which is degenerate.

Let $x \in \mathfrak{n}$. Then $\mathrm{ad}(x)$ preserves $[\mathfrak{n},\mathfrak{n}] = \mathfrak{v}_1 + e_{p+1}\mathbb{R}$ and $\mathrm{ad}(x)|_{[\mathfrak{n},\mathfrak{n}]}$ has matrix, relative to $\{e_1, \cdots, e_p; e_{p+1}\}$, of the form $\left(\begin{smallmatrix} A & B \\ C & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} A(x) & B(x) \\ C(x) & d(x) \end{smallmatrix}\right)$.

First we consider $\mathrm{ad}(v_0)|_{[n,n]} = \left(\begin{smallmatrix} A(v_0) & B(v_0) \\ C(v_0) & d(v_0) \end{smallmatrix}\right)$. By the Geodesic Lemma, there exists $a_{v_0} \in \mathfrak{h}$ such that $\langle [v_0 + a_{v_0}, e_i], v_0\rangle = k\langle v_0, e_i\rangle = 0$ for $1 \le i \le p$. Since H is completely reducible on $\mathfrak{g}$ we have $[a_{v_0}, e_i] \in \mathfrak{v}_1$. Now $\langle [a_{v_0}, e_i], v_0\rangle = 0$, so

$$\langle [v_0, e_i], v_0\rangle = 0.$$

Now $[v_0, e_i] = C_i(v_0)e_{p+1} + \sum_{j=1}^{p} a_{ji}e_j$ for any $1 \le i \le p$. So

$$\langle [v_0, e_i], v_0\rangle = \langle C_i(v_0)e_{p+1}, v_0\rangle = C_i(v_0), \quad 1 \le i \le p.$$

It forces $C_i(v_0) = 0$, i.e. $C(v_0) = \mathbf{0}$. Now $\mathrm{ad}(v_0)|_{[n,n]} = \left(\begin{smallmatrix} A(v_0) & B(v_0) \\ \mathbf{0} & d(v_0) \end{smallmatrix}\right)$. Furthermore, for any $e \in \mathfrak{v}_1$, by the Geodesic Lemma, there exists $a_e \in \mathfrak{h}$ such that $\langle [e + a_e, v_0], e\rangle = k\langle v_0, e\rangle$. Since v_0 is a one dimensional submodule, we know $\langle [a_e, v_0], e\rangle = 0$. Hence $\langle [e, v_0], e\rangle = 0$. It follows that

$$\langle [v_0, e_i], e_j\rangle + \langle e_i, [v_0, e_j]\rangle = 0, \quad 1 \le i, j \le p.$$

Then we have $a_{ij} + a_{ji} = 0$, that says $A(v_0)^t = -A(v_0)$. Since $\mathrm{ad}(v_0)$ is nilpotent, we have $A(v_0) = \mathbf{0}$ and $d(v_0) = 0$. Thus,

$$\mathrm{ad}(v_0)|_{[\mathfrak{n},\mathfrak{n}]} = \begin{pmatrix} \mathbf{0} & B(v_0) \\ \mathbf{0} & 0 \end{pmatrix}.$$

Next consider $\mathrm{ad}(v)|_{[n,n]} = \left(\begin{smallmatrix} A(v) & B(v) \\ C(v) & d(v) \end{smallmatrix}\right)$ for any $v \in \mathfrak{v}_2$. We write equation (4.1) as

$$\begin{pmatrix} A(v)^t & C(v)^t \\ B(v)^t & d \end{pmatrix}\begin{pmatrix} I_p & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} + \begin{pmatrix} I_p & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}\begin{pmatrix} A(v) & B(v) \\ C(v) & d \end{pmatrix} = 0.$$

It follows that $A(v)^t = -A(v)$ and $B(v) = \mathbf{0}$. Since $\mathrm{ad}(v)|_{[\mathfrak{n},\mathfrak{n}]}$ is nilpotent, now we have $A(v) = \mathbf{0}$ and $d = 0$, so $\mathrm{ad}(v)|_{[\mathfrak{n},\mathfrak{n}]} = \left(\begin{smallmatrix} \mathbf{0} & \mathbf{0} \\ C(v) & 0 \end{smallmatrix}\right)$. We now apply the Geodesic Lemma to $\mathrm{ad}(v + v_0)|_{[\mathfrak{n},\mathfrak{n}]}$. That gives us $a_{v+v_0} \in \mathfrak{h}$ such that $\langle [v + v_0 + a_{v+v_0}, e_i], v + v_0\rangle = k\langle v + v_0, e_i\rangle = 0$ for $1 \le i \le p$. Since $[a_{v+v_0}, e_i] \in [\mathfrak{h}, \mathfrak{v}_1] \subset \mathfrak{v}_1$, it follows that

$$\langle [v + v_0, e_i], v + v_0\rangle = 0.$$

On the other hand, since $[v_0, e_i] = 0$ for any $1 \le i \le p$, we have

$$\begin{aligned} \langle [v + v_0, e_i], v + v_0\rangle &= \langle [v, e_i], v + v_0\rangle = \langle C_i(v)e_{p+1}, v + v_0\rangle \\ &= \langle C_i(v)e_{p+1}, v_0\rangle \\ &= C_i(v). \end{aligned}$$

This forces $C_i(v) = 0$, i.e. $C(v) = \mathbf{0}$. Thus $\mathrm{ad}(v)|_{[\mathfrak{n},\mathfrak{n}]} = 0$.

Furthermore for any $v \in \mathfrak{v}_2$, first we get $\mathrm{ad}(v)|_{\mathfrak{v}_1+\mathfrak{w}} = \left(\begin{smallmatrix} \mathbf{0} & B_1(v) \\ \mathbf{0} & 0 \end{smallmatrix}\right)$ in the basis $\{e_1, \cdots, e_{p+1}, v_0\}$ since $\mathrm{ad}(v)|_{[\mathfrak{n},\mathfrak{n}]} = 0$. By the Geodesic Lemma to $\mathrm{ad}(v)|_{\mathfrak{v}_1+\mathfrak{w}}$ and the fact that $\mathfrak{v}_2$ and $\mathfrak{v}_1 + \mathfrak{w}$ are $\mathrm{Ad}(H)$-invariant, we know $\mathrm{ad}(v)|_{\mathfrak{v}_1+\mathfrak{w}} \in \mathfrak{so}(p + 1, 1)$. By Lemma 4.1, we have $\mathrm{ad}(v)|_{\mathfrak{v}_1+\mathfrak{w}} = 0$. That is, $[v, v_0] = 0$ for any $v \in \mathfrak{v}_2$, then for any $v \in \mathfrak{a}$.

Since $\mathfrak{a}$ generates $\mathfrak{n}$, $\mathrm{ad}(v)|_{[\mathfrak{n},\mathfrak{n}]} = 0$ for any $v \in \mathfrak{v}_2$, and $\mathrm{ad}(v_0)|_{[\mathfrak{n},\mathfrak{n}]} = \left(\begin{smallmatrix} \mathbf{0} & B(v_0) \\ \mathbf{0} & 0 \end{smallmatrix}\right)$, there exist $y, z \in \mathfrak{a}$ such that $[y, z] = \sum_{i=1}^{p+1} a_ie_i$ with $a_{p+1} \ne 0$. Then

$$[v_0, [y, z]] = a_{p+1}[v_0, e_{p+1}] = a_{p+1}\sum_{i=1}^{p} B_i(v_0)e_i.$$

From the Jacobi Identity and the fact $[v, v_0] = 0$ for any $v \in \mathfrak{a}$, we have

$$[v_0, [y, z]] = [[v_0, y], z] + [y, [v_0, z]] = 0,$$

it forces $B(v_0) = \mathbf{0}$. That is $\mathrm{ad}(v_0)|_{[\mathfrak{n},\mathfrak{n}]} = 0$.

Now we know $\mathrm{ad}(x)|_{[\mathfrak{n},\mathfrak{n}]} = 0$ for any $x \in \mathfrak{a}$, and thus also for any $x \in \mathfrak{n}$. Thus $\mathfrak{n}$ is at most 2-step nilpotent since $\mathfrak{a}$ generates $\mathfrak{n}$. □

Remark 5.2. By Proposition 2.6, Theorem 4.2 holds in particular where $(M, \langle\, ,\, \rangle)$ is a naturally reductive Lorentz nilmanifold. By Proposition 2.8, Theorem 5.1 holds in particular where $(M, \langle\, ,\, \rangle)$ is a connected weakly symmetric Lorentz nilmanifold with $G = I(M)^0$). ♢

6. Acknowledgements

ZC was partially supported by National Natural Science Foundation of China (11931009 and 12131012) and Natural Science Foundation of Tianjin (19JCYBJC30600). JAW was partially supported by the Simons Foundation. SZ was partially supported by National Natural Science Foundation of China (12071228 and 51535008), and SZ thanks the China Scholarship Council for support at University of California, Berkeley and he also thanks U. C. Berkeley for hospitality.

References

[1] D. V. Alekseevskii, *Self-similar Lorentzian manifolds*, Ann. Global Anal. Geom. 3 (1985), 59–84.
[2] V. N. Berestovskii & Yu. G. Nikonorov, "Riemannian Manifolds and Homogeneous Geodesics" , Springer Monographs in Mathematics, Springer, Cham, 2020
[3] J. Berndt & L. Vanhecke, *Geometry of weakly symmetric spaces*, J. Math. Soc. Japan 48 (1996), 745–760.
[4] Z. Chen & J. A. Wolf, *Pseudo-Riemannian weakly symmetric manifolds*, Ann. Global Anal. Geom. 41 (2012), 381–390.
[5] Z. Chen & J. A. Wolf, *Semisimple weakly symmetric pseudo-Riemannian manifolds*, Abh. Math. Semin. Univ. Hambg. 88 (2018), no. 2, 331–369.
[6] Z. Dusek & O. Kowalski, *Light-like homogeneous geodesics and the geodesic lemma for any signature*, Publ. Math. Debrecen 71 1–2 (2007), 245–252.
[7] J. Figueroa-O'Farrill, P. Meessen & S. Philip, *Homogeneity and plane-wave limits*, J. High Energy Physics 05 (2005), 050.
[8] C. S. Gordon, *Homogeneous Riemannian manifolds whose geodesics are orbits,* in "Topics in Geometry, in memory of Joseph D'Atri", Progress in Nonlinear Differential Equations 20, Birkhäuser, Basel, (1996), 155–174.
[9] S. Kobayashi & K. Nomizu, "Foundations of Differential Geometry, Volume II", Interscience Publishers (Wiley), 1969.
[10] O. Kowalski, F. Prüfer & L. VanHecke, *D'Atri spaces*, in "Topics in Geometry, in memory of Joseph D'Atri", Progress in Nonlinear Differential Equations 20, Birkhäuser, Basel, (1996), 241–284.
[11] O. Kowalski & L. Vanhecke, *Riemannian manifolds with homogeneous geodesics*, Boll. Un. Math. Ital. B (7) 5 (1991), 189–246.
[12] J. Lauret, *Commutative spaces which are not weakly symmetric*, Bull. London Math. Soc. 30 (1998), 29–36.
[13] Y. Nikolayevsky & Yu. G. Nikonorov, *On invariant Riemannian metrics on Ledger—Obata spaces*, Manuscripta Mathematica, 158 (2019), 353–370.
[14] S. Philip, *Penrose limits of homogeneous spaces*, J. Geom. Phys. 56 (2006), 1516–1533.
[15] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.) 20 (1956), 47–87.
[16] J. A. Wolf, "Harmonic analysis on commutative spaces", Mathematical Surveys and Mongraphs, 142. American Mathematical Society, 2007.
[17] J. A. Wolf & Z. Chen, *Weakly symmetric pseudo-Riemannian nilmanifolds*, J. Differential Geom., to appear. {arXiv: 1806.07847 (math DG)}
[18] H. Wu, *On the de Rham decomposition theorem*, Illinois J. Math. 8 (1964), 291–311.
[19] W. Ziller, *Weakly symmetric spaces*, in "Topics in Geometry, in memory of Joseph D'Atri", Progress in Nonlinear Differential Equations 20, Birkhäuser, Basel, (1996), 355–368.

(Zhiqi Chen) School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou, 510520, P.R. China. *email:* `chenzhiqi@nankai.edu.cn`

(Joseph A. Wolf) Department of Mathematics, University of California, Berkeley, CA 94720-3840, U. S. A. *email:* `jawolf@math.berkeley.edu`

(Shaoxiang Zhang) College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao 266590, P.R. China. Corresponding author. *email:* `zhangshaoxiang@mail.nankai.edu.cn`