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Partial Dirac cohomology and tempered representations

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Abstract

The tempered representations of a real reductive Lie group G are naturally partitioned into series associated with conjugacy classes of Cartan subgroups H of G. We define partial Dirac cohomology, apply it for geometric construction of various models of these H–series representations, and show how this construction fits into the framework of geometric quantization and symplectic reduction.

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1. Introduction

Let G be a real reductive Lie group. Then every conjugacy class of Cartan subgroups $H \subset G$ defines a series of unitary representations that enters into the Plancherel formula in an essential manner. For simplicity of exposition we assume that G is of Harish–Chandra class, and later we will relax that condition to include groups such as the universal covering group of $SU(p,q)$. Then the Cartan subgroup H defines a cuspidal parabolic subgroup $MAN \subset G$, meaning that $H = TA$ where T is a compact Cartan subgroup of M and A is split over $\mathbb{R}$, so M has discrete series representations. The H–series of unitary G-representations consists of the unitarily induced $\mathrm{Ind}_{MAN}^G(\eta \otimes e^{i\sigma} \otimes 1)$ where $\eta \in \widehat{M}_{disc}$ (unitary equivalence classes of discrete series representations of M), and $\sigma \in \mathfrak{a}^*$ (so that $e^{i\sigma}$ is a unitary character on A). Dirac cohomology was studied in [15], settling Vogan's Conjecture. Dirac cohomology reveals the infinitesimal characters of Harish-Chandra modules attached to the H–series for H as compact as possible, i.e. for the situation where T is a Cartan subgroup of a maximal compact subgroup $K \subset G$. That is usually called the *fundamental series* of G, and it is the *discrete series* just when $H = T$.

Here we study the partial Dirac cohomology with respect to H in general, and express it as a direct integral of H-series representations. We say "partial" because we are using Dirac cohomology of representations of the subgroup M for the purpose of constructing representations of G. As an application, we incorporate partial Dirac cohomology into geometric quantization [19] to construct unitary G-representations, and we show that the occurrences of H-series representations are controlled by the image of the moment map.

There are two approaches. One is to work directly with the class of general real reductive Lie groups introduced and studied in [26] and updated in [29]. The other is to first work with groups of Harish–Chandra class, and then show how the results extend *mutatis mutandis* to general real reductive Lie groups. For clarity of exposition we do the second of these. Thus, in Sections 3 through 7, G will be of Harish–Chandra class. In Sections 8 through 10, G will be a

general real reductive Lie group.

Our main results are stated in Section 2 as Theorem 2.9, Theorem 2.10, Theorem 2.12 and as their extensions to the class of general real reductive Lie groups.

The sections of this article are arranged as follows.

In Section 2 we develop the background material, give a detailed introduction, and formulate our main results.

In Section 3 we study the Plancherel decomposition of $L^2(G/N)$ and use it to prove Proposition 3.8. In Section 4 we use Proposition 3.8 to prove Theorem 2.9.

In Section 5 we study several L^2-function spaces and prove Proposition 5.6. In Section 6 we use Proposition 5.6 to prove Theorem 2.10.

In Section 7 we perform symplectic reduction and prove Theorem 2.12.

In Section 8 we formulate the extension of the theorems from groups of Harish-Chandra class to general real reductive Lie groups.

In Section 9 we develop a method of reducing questions of general real reductive Lie groups G to the case where $G^\dagger$ has compact center. In Section 10 we use the tools of Section 9 to reduce the proofs of the theorems of Section 8, to the arguments used for groups of Harish–Chandra class.

2. Statement of results

As usual we use the lower case Gothic letters for the real Lie algebras, and add the subscript $\mathbb{C}$ for their complexifications. So for example $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{g}_{\mathbb{C}}$ is its complexification.

If L is a Lie group then $\widehat{L}$ denotes its unitary dual. If $\pi \in \widehat{L}$ then $\mathscr{H}_\pi$ denotes its representation space. If Z is a central subgroup of L and $\zeta \in \widehat{Z}$ is a unitary character then $\widehat{L}_\zeta = \{\pi \in \widehat{L} | \pi|_Z \text{ is a multiple of } \zeta\}$.

The *Harish–Chandra class* of Lie groups consists of all real reductive Lie groups G such that

the identity component G^0 has finite index in G, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (2.1)
the derived group $G' = [G^0, G^0]$ of G^0 has finite center, and
if $g \in G$ then $\mathrm{Ad}(g)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$.

The Lie algebra of G is $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the semisimple part and $\mathfrak{z}$ is the center. The second condition in (2.1) is that the semisimple part of G^0 (the connected subgroup of G^0

with Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$) has finite center.

Let H be a Cartan subgroup of G and θ a Cartan involution that stabilizes H. See [26] (or [29]) for existence of Cartan involutions of groups of Harish–Chandra class. Then the fixed point set $K = G^\theta$ is a maximal compact subgroup of G. The Lie algebra $\mathfrak{g}$ of G is the sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of ± 1 eigenspaces of θ. We have decompositions $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ of the Lie algebra, $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$, and $H = T \times A$ on the group level where $T = H \cap K$ and $A = \exp(\mathfrak{a})$. The centralizer $Z_G(A) = M \times A$ where T is a compact Cartan subgroup of M. Further, M is of Harish–Chandra class. Choose a positive $\mathfrak{a}$–root system $\Sigma^+_{\mathfrak{g},\mathfrak{a}}$ on $\mathfrak{g}$. Let $\mathfrak{n}$ denote the sum of the positive $\mathfrak{a}$–root spaces and let $N = \exp(\mathfrak{n})$. Then the parabolic subgroup $P = MAN$ of G is *a cuspidal parabolic subgroup associated to H.* Cuspidal parabolics are characterized by the fact that M has a compact Cartan subgroup, in other words that M has discrete series representations.

When G has discrete series representations, in other words when G has a compact Cartan subgroup T, we avoid dealing with projective representations as follows. Replace G by a double covering if necessary so that $e^{\rho_\mathfrak{g}}$ is a well defined unitary character on T^0, where $\rho_\mathfrak{g} := \frac{1}{2} \sum_{\alpha \in \Sigma^+_{\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}}} \alpha$. Then we say that G is *acceptable.* Acceptability is independent of the choice of compact Cartan subgroup T because any two are $\mathrm{Ad}(G^0)$–conjugate, and independent of the choice of positive $\mathfrak{t}$–root system because any two such $\rho_\mathfrak{g}$ differ by a linear combination of $\mathfrak{t}$–roots. See [26] for the construction of acceptable double covers.

We now assume that all our groups M (including G when it has a compact Cartan subgroup) are acceptable. Thus for each M, $e^{\rho_\mathfrak{m}}$ is a well defined unitary character on T^0 where $\rho_\mathfrak{m}$ is half the sum over a positive $\mathfrak{t}_\mathbb{C}$–root system of $\mathfrak{m}_\mathbb{C}$.

Since M has the compact Cartan subgroup T it has discrete series representations $\eta_{\chi,\lambda}$ parameterized as follows. $\lambda \in i\mathfrak{t}^*$ belongs to the weight lattice for finite dimensional representations of M. In other words it satisfies the integrality condition that e^λ is a well defined character on T^0. Further, λ is regular in the sense that $\langle \lambda, \alpha \rangle \neq 0$ for every root $\alpha \in \Sigma_{\mathfrak{m}_\mathbb{C}, \mathfrak{t}_\mathbb{C}}$. Let Δ_{M^0} denote the Weyl denominator, $\Delta_{M^0} = \prod_{\alpha \in \Sigma^+_{\mathfrak{m}_\mathbb{C}, \mathfrak{t}_\mathbb{C}}} (e^{\alpha/2} - e^{-\alpha/2})$. The corresponding discrete series representation for M^0 and its distribution character are

$$\eta^0_\lambda \in \widehat{M^0}_{disc} \text{ with } \Theta_{\eta^0_\lambda} = \pm \frac{1}{\Delta_{M^0}} \sum_{w \in W_{M^0}} \det(w) e^{w(\lambda)} \text{ on } (M^0)'' \cap T^0 \qquad (2.2)$$

where $(M^0)''$ is the M–regular set in M^0. The representation η^0_λ has infinitesimal character with Harish–Chandra parameter λ and central character $e^{\lambda - \rho_\mathfrak{m}}|_{Z_{M^0}}$. This last is why we require M to be acceptable. See Harish–Chandra [9, Theorems 13 and 16], or [26, Theorem 3.4.7] ([29, Theorem 3.4.4]) for an extension to general real reductive Lie groups.

Define $\varpi(\lambda) := \prod_{\alpha \in \Sigma^+_{\mathfrak{m}_\mathbb{C}, \mathfrak{t}_\mathbb{C}}} \langle \alpha, \lambda \rangle$. Then η^0_λ has formal degree $\deg(\eta^0_\lambda) = |\varpi(\lambda)|$. This replaces the degree that appears in the Peter–Weyl Theorem for compact groups.

Suppose that $\chi \in \widehat{Z_M(M^0)}$ agrees with η_λ^0 on the center Z_{M^0}, so $\chi \in \widehat{Z_M(M^0)}_\zeta$ and $\eta_\lambda^0 \in \widehat{M^0}_\zeta$ for the same $\zeta = e^{\lambda - \rho_{\mathfrak{m}}}|_{Z_{M^0}} \in \widehat{Z_{M^0}}$. Define $M^\dagger := Z_M(M^0)M^0$. Then

$$\eta_{\chi,\lambda}^\dagger := \chi \otimes \eta_\lambda^0 \in \widehat{M^\dagger}_{disc} \text{ with } \Theta_{\eta_{\chi,\lambda}^\dagger}(zm) = \text{trace } \chi(z)\Theta_{\eta_\lambda^0}(m) \qquad (2.3)$$

for $z \in Z_M(M^0)$ and $m \in M^0$. It has infinitesimal character of Harish–Chandra parameter λ, central character $e^{\lambda - \rho_{\mathfrak{m}}}|_{Z_{M^0}}$, and formal degree $\deg(\eta_{\chi,\lambda}^\dagger) = \deg(\chi)|\varpi(\lambda)|$.

The discrete series representations of M are the $\eta_{\chi,\lambda} := \text{Ind}_{M^\dagger}^M(\eta_{\chi,\lambda}^\dagger) \in \widehat{M}_{disc}$. Their distribution characters are supported in $M^\dagger$, where

$$\eta_{\chi,\lambda} := \text{Ind}_{M^\dagger}^M(\eta_{\chi,\lambda}^\dagger) \text{ with } \Theta_{\eta_{\chi,\lambda}}(zm) = \sum_{xM^\dagger \in M/M^\dagger} \text{trace } \chi(x^{-1}zx)\Theta_{\eta_\lambda^0}(x^{-1}mx) \qquad (2.4)$$

for $z \in Z_M(M^0)$ and $m \in M^0$. The representation $\eta_{\chi,\lambda}$ has infinitesimal character of Harish–Chandra parameter λ and formal degree $\deg(\eta_{\chi,\lambda}) = |M/M^\dagger| \deg(\chi)|\varpi(\lambda)|$. Every discrete series representation of M is one of the $\eta_{\chi,\lambda}$. Further $\eta_{\chi,\lambda} \simeq \eta_{\chi',\lambda'}$ just when they have equivalent restrictions to $M^\dagger$.

Let $\sigma \in \mathfrak{a}^*$. Then $e^{i\sigma}$ is a unitary character on A, and

$$\pi_{\chi,\lambda,\sigma} := \text{Ind}_{MAN}^G(\eta_{\chi,\lambda} \otimes e^{i\sigma} \otimes 1) = \text{Ind}_{M^\dagger AN}^G(\eta_{\chi,\lambda}^\dagger \otimes e^{i\sigma} \otimes 1) \qquad (2.5)$$

is a unitary representation of G. These representations form the *H–series*. The H–series depends only on the G–conjugacy class of H. The discrete series is the case where H is compact (i.e. $H = T$), and the principal series is the case where H is as noncompact as possible. Let $Car(G)$ denote the set of G–conjugacy classes of Cartan subgroups. It is finite. Each $\{H\} \in Car(G)$ contributes a term in the Plancherel and Fourier Inversion formulae of G, and those formulae are the sums of those terms. See Harish–Chandra ([10], [11], [12]).

The various H–series representations $\pi_{\chi,\lambda,\sigma}$ are called the *standard tempered representations* of G. Plancherel-almost-all of them are irreducible. For example if σ is regular relative to the $\mathfrak{a}$–roots of $\mathfrak{g}$ then $\pi_{\chi,\lambda,\sigma}$ is irreducible. In any case, every standard tempered representation of G is a finite sum of irreducibles, and every tempered representation of G is a summand of a standard tempered representation.

Since $\text{Ad}_G(N)$ is a unipotent group of linear transformations of $\mathfrak{g}$, it preserves Haar measure dg and thus defines a G–invariant measure $d(gN)$ on G/N. Consider

$$L^2(G/N) = \left\{ f : G/N \longrightarrow \mathbb{C} \,\middle|\, \int_{G/N} |f(g)|^2 \, d(gN) < \infty \right\}.$$

Since M normalizes N, it acts on G/N from the right, and it preserves $d(gN)$. So the action of $G \times M$ on G/N leads to a unitary representation of $G \times M$ on $L^2(G/N)$. We now show how the

Plancherel decomposition of $L^2(M)$ as an $M \times M$ module induces a Plancherel decomposition of $L^2(G/N)$ as $G \times M$ module.

Since $\mathfrak{g}$ is real reductive, it has a non-degenerate symmetric $\mathrm{Ad}(G)$–invariant bilinear form b. We choose b to be the Killing form on $[\mathfrak{g}, \mathfrak{g}]$, negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

Let $\mathfrak{m}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{s}_\mathbb{C}$, orthogonal direct sum. Let S be the spin module for the Clifford algebra $C(\mathfrak{s}_\mathbb{C})$. Let

$$D_M = D_{(\mathfrak{m}_\mathbb{C}, \mathfrak{m}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C})} + iD_{diag;(\mathfrak{m}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} \tag{2.6}$$

be the modified Dirac operator as defined in [5, (1.1)]. It induces a densely defined symmetric operator, whose closure is a symmetric operator

$$\mathbb{D}_M : L^2(G/N) \otimes S \longrightarrow L^2(G/N) \otimes S. \tag{2.7}$$

Since M is of Harish–Chandra class, $T = Z_M(M^0)T^0$, so every irreducible unitary representation of $H = T \times A$ has form

$$\chi_{\lambda,\sigma} := \chi \otimes e^\lambda \otimes e^{i\sigma} \ , \lambda \in i\mathfrak{t}^*, \sigma \in \mathfrak{a}^*, \chi \in \widehat{Z_M(M^0)} \text{ agrees with } e^\lambda \text{ on } Z_{M^0}. \tag{2.8}$$

If G is a linear group then H is commutative and $\widehat{H}$ is the set of all unitary characters on H. In any case these representations are finite dimensional.

The intersection $K \cap M$ is a maximal compact subgroup of M. Denote its Weyl group by $W_{K \cap M}$. Then $W_{K \cap M} = W(M, T) = N_M(T)/Z_M(T) \cong N_{K \cap M}(T)/Z_{K \cap M}(T)$. In general if ψ is a unitary representation we write $\mathcal{H}_\psi$ for its representation space. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces we write $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ for their projective tensor product. If one or both of the $\mathcal{H}_i$ is finite dimensional then it is the ordinary tensor product. Also note that if $\chi \in \widehat{Z_M(M^0)}$ then $\bar{\chi}$ is the same as the dual χ^*. Further the action of $W_{K \cap M}$ on $\widehat{Z_M(M^0)}$ is $w : \chi \mapsto \chi \cdot w^{-1}$.

Theorem 2.9

Let G be a real reductive Lie group of Harish–Chandra class. Recall the notation (2.4) and (2.5) for the representations $\eta_{\chi,\lambda}$ of M and $\pi_{\chi,\lambda,\sigma}$ of G. Then $\mathbb{D}_M$ has kernel

$$\sum_{\eta_{\chi,\lambda} \in \widehat{M}_{disc}} \left(\int_{\sigma \in \mathfrak{a}^*} \mathcal{H}_{\pi_{\chi,\lambda,\sigma}} \, d\sigma \right) \otimes \left(\sum_{w \in W_{K \cap M}} \mathcal{H}_{(\bar{\chi} \cdot w^{-1} \otimes e^{-w\lambda})} \right),$$

and the natural action of $G \times T$ on that kernel is the direct integral

$$\sum_{\eta_{\chi,\lambda} \in \widehat{M}_{disc}} \left(\int_{\sigma \in \mathfrak{a}^*} \pi_{\chi,\lambda,\sigma} \, d\sigma \right) \otimes \left(\sum_{w \in W_{K \cap M}} (\bar{\chi} \cdot w^{-1} \otimes e^{-w\lambda}) \right).$$

We shall incorporate partial Dirac cohomology into the setting of symplectic geometry. In particular, we consider geometric quantization on

$$X = G \times \mathfrak{h},$$

where a symplectic manifold (X, ω) with symmetry leads to a unitary representation $\pi_{(X,\omega)}$ [19]. For our case, compare [27].

There is a natural action of $G \times G$ on X given by the left and right actions of G. In Theorem 6.2, we recall a systematic construction of some $(G \times H)$–invariant symplectic forms ω on X. There is a complex line bundle $\mathbb{L}$ over X whose Chern class is the cohomology class of ω, equipped with a connection whose curvature is ω, and also equipped with an invariant Hermitian structure. One may use the Hermitian structure and the invariant measure on X to construct a unitary representation of $G \times T$ on the Hilbert space of L^2-sections of $\mathbb{L}$. That space of L^2-sections is too big to yield an irreducible representation, so extra conditions are imposed, a process called polarization that cuts the number of variables in half. For example holomorphic sections of $\mathbb{L}$ would be considered if X is complex. Here X is not complex, but it has a partial complex structure in the sense that the fibers of

$$\pi : X \longrightarrow G/H$$

are complex (see (5.1)). A section of $\mathbb{L}$ is called π–*holomorphic* or *partially holomorphic* if it is holomorphic on each fiber of π. In [4], using the partial Dolbeault cohomology introduced by one of us ([25], [26], or see [29]), this idea is applied to construct some of the unitary representations $\pi_{\chi,\lambda,\sigma} \otimes (\bar{\chi} \cdot w^{-1} \otimes e^{-w\lambda})$ on higher degree cohomology. We shall apply the Dirac operator as in [28] to simplify the polarization process, constructing $\mathcal{H}_{(X,\omega)}$ without involving higher degree Dolbeault cohomology.

Consider the inclusion

$$\imath : G \longrightarrow X \,,\ \imath(g) = (g, 0).$$

Then $\imath^* \mathbb{L}$ is a line bundle on G. Let $L^2(\mathbb{L})^N$ (resp. $L^2(\imath^* \mathbb{L})^N$) denote the right N-invariant sections f of $\mathbb{L}$ (resp. $\imath^* \mathbb{L}$) such that (f, f) is integrable over $\int_{G/N \times \mathfrak{h}} (\cdot)\, d(gN)\, dx$ (resp. $\int_{G/N} (\cdot)\, d(gN)$). As in (2.7), we have a Dirac operator $\mathbb{D}_{\mathbb{L}}$ on $L^2(\imath^* \mathbb{L})^N \otimes S$. Define

$$\mathcal{H}_{(X,\omega)} = \{f \in L^2(\mathbb{L})^N \otimes S\, |\, \imath^* f \in \text{Ker } \mathbb{D}_{\mathbb{L}} \text{ and } f \text{ is } \pi\text{–holomorphic}\},$$
$$\pi_{(X,\omega)} :\ \text{natural representation of } G \text{ on } \mathcal{H}_{(X,\omega)}\,.$$

The action of $G \times H$ on the symplectic manifold (X, ω) is Hamiltonian. In particular the right H action has a canonical moment map [8]

$$\Phi : X \longrightarrow \mathfrak{h}^*.$$

We modify it by i so that the image set satisfies $\mathrm{Im}(i\Phi) \subset i\mathfrak{h}^*$.

By Theorem 6.2 and (6.3), each $\lambda + i\sigma \in \mathrm{Im}(i\Phi)$ is MA–regular, so it determines the family of all representations $\eta_{\chi,\lambda} \otimes e^{i\sigma} \in \widehat{M}_{disc} \otimes \widehat{A}$ as described in (2.4). Thus it determines the family of all H–series representations $\pi_{\chi,\lambda,\sigma}$ of G, as in (2.5). Recall $\rho_\mathfrak{a}$, half the sum of positive $\mathfrak{a}$-roots relative to N. Let $d\lambda$ be the counting measure.

Theorem 2.10

Let G be a real reductive Lie group of Harish–Chandra class. As a unitary representation space for $G \times T$,

$$\mathscr{H}_{(X,\omega)} \cong \int\limits_{\substack{\lambda+i\sigma \in \mathrm{Im}(i\Phi) \\ \exp(\lambda+i\sigma) \in \widehat{H^0}}} \sum\nolimits_{\chi \in \widehat{Z_M(M^0)}_\lambda} \left(\mathrm{Ind}_P^G(\mathscr{H}_{\bar{\chi}\otimes\eta_{-\lambda}^0} \otimes e^{-i\sigma+\rho_\mathfrak{a}} \otimes 1) \otimes \mathscr{H}_{(\chi\otimes e^\lambda)} \right) \, d\lambda \, d\sigma$$

where $\widehat{Z_M(M^0)}_\lambda$ denotes the elements of $\widehat{Z_M(M^0)}$ that agree with e^λ on Z_{M^0}.

In Corollary 6.11, we use the case $A = 1$ of Theorem 2.10 to correct an error in [5, Theorem B].

According to Gelfand, a *model* of a compact Lie group is a multiplicity-free unitary representation which contains every equivalence class of irreducible representation [6]. This notion has been extended to models of various series of unitary G-representations, including the H-series [4, §4]. In Remark 6.12, we recall this idea and briefly sketch how Theorem 2.10 leads to models of H-series.

We perform symplectic reduction [20]. Let $\mu \in \mathrm{Im}(i\Phi) \subset i\mathfrak{h}^*$. There exists a unique $v \in \mathfrak{h}$ such that $\Phi^{-1}(\mu) = G \times \{v\}$. Let

$$\imath : \Phi^{-1}(\mu) \hookrightarrow X \text{ and } \jmath : \Phi^{-1}(\mu) \longrightarrow G/H^0 \tag{2.11}$$

respectively be the natural inclusion and fibration by H^0. Then there is a unique G-invariant symplectic form ω_μ on G/H^0 such that $\jmath^*\omega_\mu = \imath^*\omega$. Write $X_\mu = G/H^0$. The process

$$(X,\omega) \rightsquigarrow (X_\mu, \omega_\mu)$$

is called symplectic reduction with respect to μ, and (X_μ, ω_μ) is called the symplectic quotient.

Suppose that $e^\mu \in \widehat{H^0}$. We use the spinor bundle and Dirac cohomology to construct a unitary representation $\pi_{(X_\mu,\omega_\mu)}$ of G in (7.4). Let $\mathscr{H}_{(X,\omega)}$ be the representation space and $\mathscr{H}_{(X,\omega),\mu}$ its μ-component in the direct integral decomposition of $\mathscr{H}_{(X,\omega)}$. The next theorem says that our construction satisfies the principle *quantization commutes with reduction* proposed by Guillemin and Sternberg [7].

Theorem 2.12

Let G be a real reductive Lie group of Harish–Chandra class. Suppose that $\mu \in \mathrm{Im}(i\varPhi)$ with $e^\mu \in \widehat{H^0}$. As unitary representations of G,

$$\pi_{(X_\mu, \omega_\mu)} \cong \pi_{(X,\omega),\mu} \text{ and } \mathcal{H}_{(X_\mu, \omega_\mu)} \cong \mathcal{H}_{(X,\omega),\mu}.$$

Finally, we show how our results extend from the Harish-Chandra class of Lie groups G to the class of general real reductive groups introduced in [26].

(a) the Lie algebra $\mathfrak{g}$ of G is reductive,   (2.13)
 (b) if $g \in G$ then $\mathrm{Ad}(g)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$, and
(c) G has a closed normal abelian subgroup Z such that
    (i) Z centralizes G^0 , i.e. $Z \subset Z_G(G^0)$
    (ii) $|G/ZG^0| < \infty$ and
    (iii) $Z \cap G^0$ is co-compact in Z_{G^0} .

These conditions are inherited by reductive components of cuspidal parabolic subgroups, and the class of groups that satisfy them includes both Harish–Chandra's class and all connected real semisimple Lie groups. See [26, §0.3] for details.

3. A Plancherel decomposition for $L^2(G/N)$

In this section, we obtain a Plancherel decomposition of $L^2(G/N)$ as $G \times M$ module. More precisely, we show that parabolic induction from M to G on the left side Plancherel decomposition of $L^2(M)$ (as a $M \times M$ module) gives a $G \times M$ module Plancherel decomposition of $L^2(G/N)$.

We first recall the general setting of induced representations. Let X be a Lie group. Let dx be the left invariant measure on X, and it is unique up to positive scalar multiplication. For each $x \in X$, we let R_x be the right action by x. So $R_x dx$ is again a left invariant measure on X, and there exists a positive number $\delta_X(x)$ such that $R_x dx = \delta_X(x) dx$. The resulting multiplicative group homomorphism

$$\delta_X : X \longrightarrow \mathbb{R}^\times$$

is the *modular function of X*. If X is abelian, discrete, compact, nilpotent or reductive, then it is *unimodular*, namely $\delta_X \equiv 1$.

Let Y be a closed subgroup of X. Define

$$\delta_Y^X : Y \longrightarrow \mathbb{R}^\times \quad \text{by} \quad \delta_Y^X(y) = \delta_Y(y)^{1/2} \delta_X(y)^{-1/2}.$$

If $\eta \in \widehat{Y}$ then the *(unitarily) induced representation* $\pi = \mathrm{Ind}_Y^X(\eta)$ is the natural left translation action of X on the space $\mathscr{H}_\pi$ given by

$$\{f : X \to \mathscr{H}_\eta \mid f(xy) = \delta_Y^X(y)\pi_y(f(x)) \text{ for all } x \in X, y \in Y \text{ and } \|f\| \in L^2(X/Y, \mathscr{H})\}.$$

Note that δ_Y^X compensates any failure of left Haar measure on X to define an X–invariant Radon measure on X/Y. Of course

$$\text{if } \delta_Y^X \equiv 1, \text{ i.e. if } \delta_X|_Y = \delta_Y, \text{ then } \mathscr{H}_\pi \cong L^2(X/Y) \,\widehat{\otimes}\, \mathscr{H}_\eta \,. \tag{3.1}$$

We now consider our setting, namely G, K, M, A, N, T, P and H as given in the Introduction. Consider the direct product $G \times M$, with subgroups

$$M_{diag} \subset NM_{diag} \subset G \times M.$$

Here M_{diag} is the diagonal subgroup isomorphic to M, so NM_{diag} consists of (nm, m) where $n \in N$ and $m \in M$. Note that NM_{diag} is a well defined subgroup of $G \times M$ because M normalizes N. We write the elements of the quotient $(G \times M)/NM_{diag}$ as $[g, m]$. There is an action of $G \times M$ on $(G \times M)/NM_{diag}$, where the right action of M is given by $R_{m_1}[g, m] = [g, m_1 m]$ for all $m_1 \in M$ and $[g, m] \in (G \times M)/NM_{diag}$.

Consider the exponential map $\exp : \mathfrak{a} \to A$. As usual $\rho_\mathfrak{a} \in \mathfrak{a}^*$ denotes half the sum of positive $\mathfrak{a}$–roots determined by N, so $2\rho_\mathfrak{a}(v) = \mathrm{trace}\,(\mathrm{ad}(v) : \mathfrak{n} \to \mathfrak{n})$ for $v \in \mathfrak{a}$. It defines the quasi–character

$$e^{\rho_\mathfrak{a}} : A \to \mathbb{R}^\times \text{ by } e^{\rho_\mathfrak{a}}(\exp(v)) = e^{\rho_\mathfrak{a}(v)} \text{ for all } v \in \mathfrak{a}.$$

Proposition 3.2

We have a $(G \times M)$–equivariant diffeomorphism

$$\phi : (G \times M)/NM_{diag} \longrightarrow G/N \text{ defined by } \phi([g, m]) = gm^{-1}N,$$

and it equips $(G \times M)/NM_{diag}$ with a $(G \times M)$–invariant measure.

Proof

Since M normalizes N, we have a transitive action τ of $G \times M$ on G/N, given by $\tau_{(g,m)}xN = gxm^{-1}N$. We have $\tau_{(g,m)}eN = eN$ if and only if $(g, m) \in NM_{diag}$, so the stabilizer of the identity coset $eN \in G/N$ is NM_{diag}. This leads to the $(G \times M)$-equivariant diffeomorphism ϕ of this proposition.

We have $G = KMAN$. Let dg, dk, dm, da and dn be their Haar measures. Then $dg = e^{2\rho_\mathfrak{a}}\,dk\,dm\,da\,dn$ [18, Proposition 8.44]. The G-invariant measure on G/N is $e^{2\rho_\mathfrak{a}}\,dk\,dm\,da$,

and it is invariant under the right action of M. Therefore, ϕ induces a $(G \times M)$–invariant measure on $(G \times M)/NM_{diag}$. $\square$

Since G and M are unimodular, their direct product $G \times M$ is also unimodular. The left invariant measure on $P = MAN$ is

$$e^{2\rho_a}\, dm\, da\, dn. \tag{3.3}$$

The subgroup NM_{diag} of $G \times M$ is isomorphic to the subgroup NM of G. By (3.3), NM has Haar measure $dn\, dm$, so it is unimodular. Hence $\delta^{G \times M}_{NM_{diag}} \equiv 1$, and by (3.1),

$$\mathrm{Ind}^{G \times M}_{NM_{diag}}(1) = L^2((G \times M)/NM_{diag}). \tag{3.4}$$

The modular function of P is $\delta_P(man) = e^{2\rho_a}(a)$ [18, VIII-4], so its restriction to MN is trivial. Therefore,

$$\delta^{P \times M}_{MN \times M} \equiv 1 \text{ and } \delta^{MN \times M}_{NM_{diag}} \equiv 1. \tag{3.5}$$

We apply induction in stages and get (see for example [16, Thm.2.47])

$$
\begin{aligned}
\mathrm{Ind}^{G \times M}_{NM_{diag}}(1) &= \mathrm{Ind}^{G \times M}_{P \times M}\mathrm{Ind}^{P \times M}_{MN \times M}\mathrm{Ind}^{MN \times M}_{NM_{diag}}(1) &&\text{by [16, Thm.2.47]} \\
&= \mathrm{Ind}^{G \times M}_{P \times M}\mathrm{Ind}^{P \times M}_{MN \times M}(L^2(M) \otimes 1) &&\text{by (3.1) and (3.5)} \\
&= \mathrm{Ind}^{G \times M}_{P \times M}(L^2(A, L^2(M)) \otimes 1). &&\text{by (3.1) and (3.5)}
\end{aligned}
\tag{3.6}
$$

By the Plancherel theorem (see for example [22, Thm.7.9]),

$$L^2(A : L^2(M)) = \int_{\sigma \in \mathfrak{a}^*} L^2(M) \otimes e^{i\sigma}\, d\sigma,$$

where $d\sigma$ is Lebesgue measure on $\mathfrak{a}^*$. Therefore, (3.6) becomes

$$\mathrm{Ind}^{G \times M}_{NM_{diag}}(1) = \int_{\sigma \in \mathfrak{a}^*} \mathrm{Ind}^{G \times M}_{P \times M}(L^2(M) \otimes e^{i\sigma} \otimes 1)d\sigma. \tag{3.7}$$

Proposition 3.8

As the Hilbert spaces for unitary representations of $G \times M$,

$$L^2(G/N) = \int_{\sigma \in \mathfrak{a}^*} \mathrm{Ind}^{G \times M}_{P \times M}(L^2(M) \otimes e^{i\sigma} \otimes 1)\, d\sigma.$$

Proof

This follows from Proposition 3.2, (3.4) and (3.7). $\square$

4. Partial Dirac cohomology of $L^2(G/N)$

In this section we calculate the partial Dirac cohomology for the representation of $G \times M$ on $L^2(G/N)$ with respect to $D_M = D_{(\mathfrak{m}_\mathbb{C}, \mathfrak{m}_\mathbb{C} \cap \mathfrak{t}_\mathbb{C})} + iD_{diag;(\mathfrak{m}_\mathbb{C} \cap \mathfrak{t}_\mathbb{C}, \mathfrak{t}_\mathbb{C})}$, the modified Dirac operator of (2.6). We then apply it to prove Theorem 2.9.

If $N = 1$ and $M = G$, it reduces to the calculation of Dirac cohomology of $L^2(G)$ with respect to the Dirac operator $\widetilde{D}_{(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})}$. That was done in [5]. Therefore, we use the phrase "partial Dirac cohomology" for the case when $N \neq 1$. The essential part is the calculation of the Dirac cohomology of discrete series representations of M.

Now we recall the relevant results of Dirac cohomology of discrete series representations. Let M be an acceptable real reductive Lie group of Harish–Chandra class, for example the group M in the Iwasawa decomposition $P = MAN$ of a cuspidal parabolic subgroup of our group G. We described the discrete series $\widehat{M}_{disc}$ of M in the discussion leading up to (2.4). It consists of the representations $\eta_{\chi,\lambda} = \mathrm{Ind}_{M^\dagger}^M(\eta_{\chi,\lambda}^\dagger)$ specified there. The parameterization is that T^0 is a compact Cartan subgroup of M^0, λ belongs to the M^0–regular subset of the lattice $\Lambda = \{\nu \in \mathfrak{t}^* \,|\, e^{i\nu} \in \widehat{T^0}\}$, and $\chi \in Z_M(M^0)$ agrees with $e^{\lambda + \rho_\mathfrak{m}}$ on Z_{M^0}. Harish–Chandra's construction and characterization of the discrete series (for acceptable groups of Harish–Chandra class) is

Theorem 4.1

The discrete series $\widehat{M}_{disc}$ consists of the equivalence classes of representations $\eta_{\chi,\lambda}$ where λ runs over the set of M–regular elements in the lattice $\Lambda \subset \mathfrak{t}^*$ and $\chi \in \widehat{Z_M(M^0)}$ agrees with $e^{\lambda + \rho_\mathfrak{m}}$ on the center of M^0. Discrete series representations $\eta_{\chi,\lambda} \simeq \eta_{\chi',\lambda'}$ if and only if (χ, λ) and (χ', λ') are in the same orbit of the Weyl group W_K.

We denote by $V_{\chi,\lambda}$ the Harish–Chandra module of the $(K \cap M)$–finite vectors in $\mathcal{H}_{\eta_{\chi,\lambda}}$. The elements of $V_{\chi,\lambda}$ are C^∞ vectors (in fact real analytic vectors), and $V_{\chi,\lambda}$ is dense in $\mathcal{H}_{\eta_{\chi,\lambda}}$. Thus the modified Dirac operator $D = \widetilde{D}_{(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})}$ is a densely defined symmetric operator

$$D : \mathcal{H}_{\eta_{\chi,\lambda}} \otimes S \to \mathcal{H}_{\eta_{\chi,\lambda}} \otimes S.$$

We recall a standard fact in functional analysis that a densely defined symmetric operator is closable and its closure is also symmetric. If A^* denotes the adjoint of a densely defined symmetric operator A, then the closure $c\ell(A) = (A^*)^*$ and $c\ell(A)$ is also symmetric [21, Lemma 20.1]. Thus, D is closable and its closure $c\ell(D)$ is also symmetric. It follows that $\mathbf{Ker}\ c\ell(D)$ is a closed subspace of the Hilbert space $\mathcal{H}_{\eta_{\chi,\lambda}} \otimes S$. We define the Dirac cohomology $H_D(\mathcal{H}_{\eta_{\chi,\lambda}})$ of an irreducible unitary representation $\eta_{\chi,\lambda}$ to be $\mathbf{Ker}\ c\ell(D)$. The following proposition shows that the Dirac cohomology of an irreducible representation is equal to the Dirac cohomology of its Harish-Chandra module. It was proved in [5, Prop. 3.2] for the case of connected semisimple Lie groups of finite center.

Proposition 4.2

The kernel of $c\ell(D) : \mathcal{H}_{\eta_{\chi,\lambda}} \otimes S \to \mathcal{H}_{\eta_{\chi,\lambda}} \otimes S$ coincides with the kernel of $D : V_{\chi,\lambda} \otimes S \to V_{\chi,\lambda} \otimes S$. Thus

$$\text{Ker } c\ell(D) = H_D(V_{\chi,\lambda}) = \sum_{w \in W_{K \cap M}} \mathcal{H}_{\chi \cdot w^{-1}} \otimes \mathbb{C}_{w\lambda}.$$

Proof

If M is connected and semisimple the assertion is [5, Prop. 3.2]. But the argument of [5, Prop. 3.2] goes through without change, and without requiring semisimplicity because η_λ^0 restricts to the center Z_{M^0} as a multiple of some fixed unitary character ζ. Thus the assertion holds for connected M.

Consider the case $M = Z_M(M^0)M^0$. The argument of [5, Prop. 3.2] still shows that $\text{Ker } c\ell(D) = \text{Ker } D$, and conjugation by elements of $Z_M(M^0)$ makes no change in D, so $H_D(V_{\chi,\lambda}) = \sum_{w \in W_{K \cap M^0}} \mathcal{H}_{\chi \cdot w^{-1}} \otimes \mathbb{C}_{w\lambda}$. That gives us the assertion for $M = M^\dagger$.

Finally consider the general case. There $\eta_{\chi,\lambda}|_{M^\dagger} = \sum_{xM^\dagger \in M/M^\dagger} \eta_{\chi \cdot \text{Ad}(x), \lambda \cdot \text{Ad}(x)}^\dagger$ because $\eta_{\chi,\lambda}$ is induced from the normal subgroup $M^\dagger$. We use the result for $M^\dagger$ to write this as $\eta_{\chi,\lambda}|_{M^\dagger} = \sum_{w \in W_{K \cap M}} \mathcal{H}_{\chi \cdot w^{-1}} \otimes \mathbb{C}_{w\lambda}$. The assertion follows. $\square$

Recall that the Dirac operator D is in $U(\mathfrak{m}_{\mathbb{C}}) \otimes C(\mathfrak{s}_{\mathbb{C}})$. We first consider

$$D : C^\infty(M) \otimes S \to C^\infty(M) \otimes S.$$

Then D induces a densely defined symmetric operator on the Hilbert space $L^2(M) \otimes S$, and the closure of D defines a closed symmetric operator

$$\mathbb{D} : L^2(M) \otimes S \to L^2(M) \otimes S.$$

Then $\text{Ker } \mathbb{D}$ is a closed subspace in $L^2(M) \otimes S$. We define the Dirac cohomology $H_D(L^2(M))$ of $L^2(M)$ to be $\text{Ker } \mathbb{D}$. It follows from the fact that D is T-invariant that $\text{Ker } \mathbb{D}$ is a $(M \times T)$–module. The following theorem was proved for the case of connected M as [5, Theorem 3.3]. Since M is of Harish–Chandra class, the argument there goes through to prove

$\text{Ker } \mathbb{D} = \sum_{\eta_{\chi,\lambda} \in \widehat{M}_{disc}} \mathcal{H}_{\eta_{\chi,\lambda}} \otimes H_D(V_{\chi,\lambda}^*)$. Using Proposition 4.2, now,

Theorem 4.3

We have the following orthogonal sum decomposition as representation space of $(M \times T)$:

$$\text{Ker } \mathbb{D} = \sum_{\eta_{\chi,\lambda} \in \widehat{M}_{disc}} \mathcal{H}_{\eta_{\chi,\lambda}} \otimes \sum_{w \in W_{K \cap M}} (\mathcal{H}_{\chi^* \cdot w^{-1}} \otimes \mathbb{C}_{-w \cdot \lambda}).$$

Now we calculate the partial Dirac cohomology of $L^2(G/N)$. We are working with a Cartan involution θ of G, $K = G^\theta$ is a maximal compact subgroup, $H = T \times A$ is a θ–stable Cartan

subgroup, $P = MAN$ is an associated cuspidal parabolic subgroup so M has compact Cartan subgroup T and $MA = Z_G(A)$, and $G = KMAN$.

Let $\mathfrak{m}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{s}_\mathbb{C}$ where $\mathfrak{s}_\mathbb{C}$ is the sum of the $\mathfrak{t}_\mathbb{C}$–root spaces in $\mathfrak{m}_\mathbb{C}$. It is the orthogonal decomposition of $\mathfrak{m}_\mathbb{C}$ with respect to an invariant form of $\mathfrak{g}$. Let S be the spin module for the Clifford algebra $C(\mathfrak{s}_\mathbb{C})$. Then

$$D_M = D_{\mathfrak{m}_\mathbb{C}, \mathfrak{t}_\mathbb{C} \cap \mathfrak{m}_\mathbb{C}} + i D_{\delta; \mathfrak{t}_\mathbb{C} \cap \mathfrak{m}_\mathbb{C}, \mathfrak{t}_\mathbb{C}}$$

is the *modified Dirac operator* as defined in [5, (1.1)]. It induces a densely defined symmetric operator,

$$\mathbb{D}_M : L^2(G/N) \otimes S \longrightarrow L^2(G/N) \otimes S.$$

Proof of Theorem 2.9

By Proposition 3.8, the representation space of the regular representation of $G \times M$ on $L^2(G/N)$ has direct integral decomposition

$$L^2(G/N) = \int_{\sigma \in \mathfrak{a}^*} \mathrm{Ind}_{P \times M}^{G \times M}(L^2(M) \otimes e^{i\sigma} \otimes 1) \, d\sigma.$$

This decomposition splits into a discrete spectrum and a continuous spectrum. The discrete part corresponds to the discrete spectrum of the Plancherel decomposition of $L^2(M)$ with summation over all discrete series of M,

$$L^2(G/N)_{disc} = \sum_{\eta_{\chi,\lambda} \in \widehat{M}_{disc}} \left(\int_{\sigma \in \mathfrak{a}^*} \mathrm{Ind}_P^G\big((\mathscr{H}_{\eta_{\chi,\lambda}} \otimes \mathscr{H}_{\eta_{\chi,\lambda}}^*) \otimes e^{i\sigma} \otimes 1\big) \, d\sigma \right) \deg(\eta_{\chi,\lambda}).$$

Here $\deg(\eta_{\chi,\lambda})$ is the formal degree of $\eta_{\chi,\lambda}$. The continuous spectrum corresponds to a direct integral of other tempered (i.e. H–series) representations of M.

Lemma 4.4

Only the discrete spectrum in $L^2(M)$ contributes to $\mathrm{Ker}\ \mathbb{D}_M$.

Proof

It follows from the Plancherel decomposition that

$$\mathrm{Ker}\ \mathbb{D}_M = \int_{\eta \in \widehat{M}} \big(\int_{\sigma \in \mathfrak{a}^*} \mathrm{Ind}_P^G(\mathscr{H}_\tau \otimes e^{i\sigma} \otimes 1) \, d\sigma \big) \otimes \mathrm{Ker}\ \{\overline{D_M} : \mathscr{H}_\eta^* \otimes S \to \mathscr{H}_\eta^* \otimes S\} d\mu(\eta)$$

where μ is Plancherel measure on $\widehat{M}$. Therefore, the representation $\eta \otimes \eta^*$ of $M \times M$ contributes to $\mathrm{Ker}\ \mathbb{D}_M$ if and only if the Dirac operator $\overline{D_M}$ acts on $\mathscr{H}_\eta^* \otimes S$ with nonzero

kernel. This condition is equivalent to the Harish-Chandra module of $\mathcal{H}_\eta^*$ having nonzero Dirac cohomology, as we showed in Proposition 4.2. It follows that $\mathcal{H}_\eta^*$ must have a real infinitesimal character ξ, in the sense that if ξ is in the dominant chamber then $\theta(\xi) = \xi$. Therefore, only the discrete series can contribute to $\mathrm{Ker}\ \mathbb{D}_M$, since other tempered representations with real infinitesimal characters have Plancherel measure 0 by Harish-Chandra's Plancherel Theorem. □

From Lemma 4.4 we obtain the orthogonal sum decomposition

$$\mathrm{Ker}\ \mathbb{D}_M = \sum_{\eta_{\chi,\lambda} \in \widehat{M}_{disc}} \left(\int_{\sigma \in \mathfrak{a}^*} \mathrm{Ind}_P^G(\mathcal{H}_{\eta_{\chi,\lambda}} \otimes e^{i\sigma} \otimes 1)\, d\sigma \right) \deg(\eta_{\chi,\lambda}) \otimes H_D(\mathcal{H}_{\eta_{\chi,\lambda}}^*).$$

Then by substituting the Dirac cohomology $H_D(\mathcal{H}_\lambda^*)$ of the discrete series representation $\mathcal{H}_{\eta_{\chi,\lambda}}^*$ of M, we obtain

$$\mathrm{Ker}\ \mathbb{D}_M = \sum_{\eta_{\chi,\lambda} \in \widehat{M}_{disc}} \left(\int_{\sigma \in \mathfrak{a}^*} \mathrm{Ind}_P^G(\mathcal{H}_{\eta_{\chi,\lambda}} \otimes e^{i\sigma} \otimes 1)\, d\sigma \right) \deg(\eta_{\chi,\lambda}) \tag{4.5}$$

$$\otimes \left(\sum_{w \in W_{K \cap M}} \mathcal{H}_{\chi^* \cdot w^{-1}} \otimes \mathbb{C}_{-w\lambda} \right).$$

This completes the proof of Theorem 2.9. □

We note that $\mathbb{D}_M$ is $G \times T$-invariant, so (4.5) is an orthogonal decomposition of $(G \times T)$–modules.

5. L^2-functions

In this section, we study certain L^2-function spaces. The key result is Proposition 5.6, which will be used later.

Recall some notation. We fix a Cartan involution θ of G and the (± 1)–eigenspace decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$; $\mathfrak{k}$ is the Lie algebra of the fixed point set $K = G^\theta$. We fix a θ–stable Cartan subgroup $H = T \times A$ where $T = H \cap K$ and $A = \exp(\mathfrak{a})$, $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$. Then $H^0 = T^0 \times \exp(\mathfrak{a}) \cong T^0 \times \mathfrak{a}$, and $it \cong \exp(it)$ (group isomorphisms), and $\mathfrak{h} = \mathfrak{t} + \mathfrak{a} \cong i\mathfrak{h} = it \times i\mathfrak{a}$ (real vector space isomorphism), leading to identification of $T_\mathbb{C}^0 = T^0 \times \exp(it)$ with $(\mathbb{C}/\mathbb{Z})^n$, $n = \dim_\mathbb{R} T$, and of $\mathfrak{a}_\mathbb{C}$ with $\mathbb{C}^m$, $m = \dim_\mathbb{R} A$.

From that, there is a unique complex structure on $H^0 \times \mathfrak{h}$ such that the map

$$(t \exp v, x + y) \mapsto (t \exp ix, v + iy) \text{ for all } t \in T^0, x \in \mathfrak{t} \text{ and } v, y \in \mathfrak{a} \tag{5.1}$$

is a holomorphic diffeomorphism of $H^0 \times \mathfrak{h}$ onto $T_\mathbb{C}^0 \times \mathfrak{a}_\mathbb{C} \cong (\mathbb{C}/\mathbb{Z})^n \times \mathbb{C}^m$. This uses

$t \exp v \in H^0$, $x + y \in \mathfrak{h}$, $t \exp ix \in T_\mathbb{C}^0$ and $v + iy \in \mathfrak{a}_\mathbb{C}$.

Let $\mathscr{H}_{(H^0 \times \mathfrak{h})}$ denote the resulting space of holomorphic functions on $H^0 \times \mathfrak{h}$. The space $\widehat{H^0}$ of unitary characters on H^0 consists of the e^μ, $\mu \in i\mathfrak{h}^*$, that are well defined on H^0. Any such e^μ extends uniquely to a holomorphic homomorphism

$$e_\mathbb{C}^\mu : H^0 \times \mathfrak{h} \to \mathbb{C}^\times,$$

and $e_\mathbb{C}^\mu \in \mathscr{H}_{(H^0 \times \mathfrak{h})}$. We write $\mathbb{C}_\mu$ for the 1–dimensional space spanned by $e_\mathbb{C}^\mu$.

Fix a strictly convex function

$$F : \mathfrak{h} \longrightarrow \mathbb{R}.$$

Namely F is a smooth function such that under any linear coordinates (x_i) on $\mathfrak{h}$, the Hessian matrix $(\frac{\partial^2 F}{\partial x_i \partial x_j})$ is positive definite. We also identify it with an H^0-invariant function on $H^0 \times \mathfrak{h}$, and a G-invariant function on $G/N \times \mathfrak{h}$. For functions on $\mathfrak{h}$, $H^0 \times \mathfrak{h}$ and $G/N \times \mathfrak{h}$, the L^2-norm $\|\cdot\|^2$ refers to square-integration against e^{-F} times invariant measure. For instance we define the weighted Bergman space

$$\mathscr{H}_{(H^0 \times \mathfrak{h}, e^{-F})}^2 = \left\{ f \in \mathscr{H}_{(H^0 \times \mathfrak{h})} \,\middle|\, \int_{H^0 \times \mathfrak{h}} |f(h, x)|^2 e^{-F(x)} \, dh \, dx < \infty \right\},$$

where dx is the Lebesgue measure on $\mathfrak{h}$. The holomorphic functions form a closed subspace in the L^2-space, so $\mathscr{H}_{(H^0 \times \mathfrak{h}, e^{-F})}^2$ is a Hilbert space.

Let $F' : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ be the gradient mapping of F. The image $\mathrm{Im}(\frac{i}{2}F') \subset i\mathfrak{h}^*$. (The factor i is added so that the image lies in $i\mathfrak{h}^*$.) Since F is strictly convex, $\mathrm{Im}(\frac{i}{2}F')$ is a convex open set. Let $d\mu$ be the product of counting measure on $\{\lambda \in i\mathfrak{t}^* | e^\lambda \in \widehat{T^0}\}$ and Lebesgue measure on $i\mathfrak{a}^*$. It is a normalization of Haar measure on $\widehat{H^0}$.

Theorem 5.2

[2, Thm.1.2] *We have an isomorphism of unitary H^0-representations*

$$\mathscr{H}_{(H^0 \times \mathfrak{h}, e^{-F})}^2 = \int_{\substack{\mu \in \mathrm{Im}(\frac{i}{2}F') \\ e^\mu \in \widehat{H^0}}} \mathbb{C}_\mu \, d\mu.$$

We next study functions on $Y = G/N \times \mathfrak{h}$. There is a natural embedding and a fibration

$$\imath : G/N \hookrightarrow Y \ , \ \pi : Y \longrightarrow G/H^0 N. \tag{5.3}$$

Here $\imath(g) = (g, 0)$ for all $g \in G/N$ and π is the natural quotient. Given $f : Y \longrightarrow S$, we let

$\imath^* f : G/N \longrightarrow S$ be its pullback to G/N. Each fiber of π is diffeomorphic to $H^0 \times \mathfrak{h}$, so by (5.1), it has a complex structure. We say that f is π–holomorphic or *partially holomorphic* if it is holomorphic on each fiber of π. It is the same to consider holomorphic properties on the fibers of $Y \longrightarrow G/H^0 N$ and $Y \longrightarrow G/HN$, because their fibers have the same connected components. Let

$$\mathscr{H}_{(Y)} \otimes S = \{f : Y \longrightarrow S | \imath^* f \in \operatorname{Ker} \mathbb{D}_M \text{ and } f \text{ is } \pi\text{-holomorphic}\}.$$

Recall that F is a strictly convex function on $\mathfrak{h}$. Let

$$\mathscr{H}_{(Y,e^{-F})}^2 \otimes S = \left\{ f \in \mathscr{H}_{(Y)} \otimes S \middle| \int_Y |f(g,x)|^2 e^{-F(x)} d(gN) \, dx < \infty \right\}. \tag{5.4}$$

The right H^0-action on G/N leads to the direct integral decomposition

$$\operatorname{Ker} \mathbb{D}_M = \int_{e^\mu \in \widehat{H^0}} (\operatorname{Ker} \mathbb{D}_M)_\mu \, d\mu. \tag{5.5}$$

No single integrand $(\operatorname{Ker} \mathbb{D}_M)_\mu$ is contained in $\operatorname{Ker} \mathbb{D}_M$ (it is only weakly contained) because it has $d\mu$–measure 0. Thus $f_\mu \in (\operatorname{Ker} \mathbb{D}_M)_\mu$ is generally not square-integrable over G/N. It transforms by e^μ under the right H^0-action, namely $R_h^* f_\mu = e^\mu(h) f_\mu$.

Proposition 5.6

The map $\imath^* : \mathscr{H}_{(Y,e^{-F})}^2 \otimes S \longrightarrow \operatorname{Ker} \mathbb{D}_M$ *is injective. It defines a* $G \times T^0$*–equivariant isomorphism*

$$\mathscr{H}_{(Y,e^{-F})}^2 \otimes S \cong \int_{\mu \in \operatorname{Im}(\frac{i}{2} F'), e^\mu \in \widehat{H^0}} (\operatorname{Ker} \mathbb{D}_M)_\mu \, d\mu.$$

Proof

We first show that $\imath^*$ is injective. Suppose that $f_1, f_2 \in \mathscr{H}_{(Y,e^{-F})}^2 \otimes S$ satisfy $\imath^* f_1 = \imath^* f_2$, namely f_1 and f_2 agree on G/N. Fix $g \in G/N$, and we have

$$f_1(gh, 0) = f_2(gh, 0) \text{ for all } h \in H^0. \tag{5.7}$$

Being π–holomorphic, f_1, f_2 are holomorphic on the fiber $(gH^0, \mathfrak{h})$ of π. So together with (5.7), we have

$$f_1(gh, x) = f_2(gh, x) \text{ for all } (h, x) \in H^0 \times \mathfrak{h}. \tag{5.8}$$

Since (5.8) holds for all $g \in G/N$, it follows that $f_1 = f_2$. So $\imath^*$ is injective. It is clear that $\imath^*$ intertwines with the action of $G \times T^0$. It remains to prove that

$$\imath^*(\mathcal{H}^2_{(Y,e^{-F})} \otimes S) = \int\limits_{\mu \in \mathrm{Im}(\frac{i}{2}F'),\, e^\mu \in \widehat{H^0}} (\mathrm{Ker}\ \mathbb{D}_M)_\mu\, d\mu. \tag{5.9}$$

We first check the $\subset$ part of (5.9). Let $f \in \mathcal{H}^2_{(Y,e^{-F})} \otimes S$. Then $\imath^* f \in \mathrm{Ker}\ \mathbb{D}_M$, so by (5.5), we write

$$f(g,0) = \int\limits_{e^\mu \in \widehat{H^0}} f_\mu(g)\, d\mu,$$

where $f_\mu \in (\mathrm{Ker}\ \mathbb{D}_M)_\mu$ transforms by e^μ under the right action of H^0. We claim that

$$f(g,x) = \int\limits_{e^\mu \in \widehat{H^0}} f_\mu(g) e_{\mathbb{C}}^{\mu(x)}\, d\mu. \tag{5.10}$$

The function $f_\mu(g) e_{\mathbb{C}}^{\mu(x)}$ transforms by $e_{\mathbb{C}}^\mu$ under the right action of $H^0 \times \mathfrak{h}$, so it is holomorphic on the fiber $(gH^0, \mathfrak{h})$ of π. Hence for each $g \in G/N$, both sides of (5.10) agree on $(gH^0, 0)$ and are holomorphic on $(gH^0, \mathfrak{h})$, so they agree on $(gH^0, \mathfrak{h})$. This holds for each g, which proves (5.10) as claimed.

The restriction of f to $H^0 \times \mathfrak{h}$ belongs to $\mathcal{H}^2_{(H^0 \times \mathfrak{h}, e^{-F})} \otimes S$. By Theorem 5.2, it is a direct integral over $\{\mu \in \mathrm{Im}(\frac{i}{2}F')\}_{e^\mu \in \widehat{H^0}}$. So in (5.10), $f_\mu = 0$ for $\mu \notin \mathrm{Im}(\frac{i}{2}F')$. This proves the $\subset$ part of (5.9).

Next we prove the $\supset$ part of (5.9). Pick

$$f^0 = \int\limits_J f_\mu\, d\mu \in \int\limits_J (\mathrm{Ker}\ \mathbb{D}_M)_\mu\, d\mu \subset \mathrm{Ker}\ \mathbb{D}_M \subset L^2(G/N) \otimes S, \tag{5.11}$$

where $J \subset \mathrm{Im}(\frac{i}{2}F')$ is Borel–measurable and $e^J \subset \widehat{H^0}$. We may assume that J is bounded, as the members of $\mathrm{Ker}\ \mathbb{D}_M$ are Hilbert space sums of such elements. Let

$$f : Y \longrightarrow S \text{ defined by } f(g,x) = \int\limits_J f_\mu(g) e_{\mathbb{C}}^{\mu x}\, d\mu.$$

Here $(\imath^* f)(g) = f(g,0) = f^0(g)$, so $\imath^* f \in \mathrm{Ker}\ \mathbb{D}_M$. Also, $(g,x) \mapsto f_\mu(g) e_{\mathbb{C}}^{\mu(x)}$ is a π-holomorphic function for each μ, so f is π-holomorphic. Hence $f \in \mathcal{H}_{(Y)} \otimes S$. We want to show that $f^0 \in \imath^*(\mathcal{H}^2_{(Y,e^{-F})} \otimes S)$ in (5.9), so it remains to check that

$$\|f\|^2 = \int\limits_Y |f(g,x)|^2 e^{-F(x)}\, d(gN)\, dx < \infty. \tag{5.12}$$

The condition $\imath^* f \in \mathrm{Ker}\ \mathbb{D}_M$ implies that, in particular,

$$f(\cdot, 0) \in L^2(G/N) \otimes S. \tag{5.13}$$

The holomorphic homomorphism $e_{\mathbb{C}}^{\mu(x)} : H^0 \times \mathfrak{h} \to \mathbb{C}^\times$ maps the H^0 and $\mathfrak{h}$ components to S^1 and $\mathbb{R}^\times$ respectively. Fix $x \in \mathfrak{h}$. Since J is bounded, the set $\{e_{\mathbb{C}}^{2\mu(x)}\}_{\mu \in J}$ is bounded above by some $m = m_x$. We have

$$\|f(\cdot,x)\|^2 \quad = \int_J \left\|f_\mu e_{\mathbb{C}}^{\mu(x)}\right\|^2 d\mu = \int_J \int_{G/N} |f_\mu(g)|^2 e_{\mathbb{C}}^{2\mu(x)} d(gN)\, d\mu \tag{5.14}$$
$$\leq m \int_J \int_{G/N} |f_\mu(g)|^2 d(gN)\, d\mu = m \int_J \|f_\mu\|^2 d\mu = m\|f(\cdot,0)\|^2.$$

By (5.13) and (5.14), for all $x \in \mathfrak{h}$,

$$f(\cdot,x) \in L^2(G/N) \otimes S. \tag{5.15}$$

Let $\mathscr{H}_{(\mathfrak{h})}$ denote the analytic functions on $\mathfrak{h}$. By (5.15), we can define

$$\mathscr{H}_{(Y)} \otimes S \longrightarrow \mathscr{H}_{(\mathfrak{h})} \otimes L^2(G/N) \otimes S \text{ by } f \mapsto \tilde{f}, \tag{5.16}$$

where $\tilde{f}(x) = f(\cdot,x)$ for all $x \in \mathfrak{h}$. Let

$$\mathscr{H}^2_{(\mathfrak{h},e^{-F})} \otimes L^2(G/N) \otimes S = \left\{ k \in \mathscr{H}_{(\mathfrak{h})} \otimes L^2(G/N) \otimes S \, \middle| \, \int_{\mathfrak{h}} |k(x)|^2 e^{-F(x)} dx < \infty \right\}.$$

For $f \in \mathscr{H}_{(Y)} \otimes S$, we have

$$\|f\|^2 \quad = \int_Y |f(g,x)|^2 e^{-F(x)} d(gN)\, dx = \int_{\mathfrak{h}} \left(\int_{G/N} |f(g,x)|^2 d(gN) \right) e^{-F(x)} dx \tag{5.17}$$
$$= \int_{\mathfrak{h}} \|\tilde{f}(x)\|^2 e^{-F(x)} dx = \|\tilde{f}\|^2.$$

Hence (5.16) leads to a norm preserving map

$$\mathscr{H}^2_{(Y,e^{-F})} \otimes S \longrightarrow \mathscr{H}^2_{(\mathfrak{h},e^{-F})} \otimes L^2(G/N) \otimes S.$$

Write $\tilde{f} = k \otimes v$, where $k \in \mathscr{H}_{(\mathfrak{h})}$ and $v \in L^2(G/N) \otimes S$. Since (5.16) intertwines with the right $\mathfrak{h}$-action, we have $k = \int_J k_\mu \, d\mu$, where $k_\mu \in \mathbb{C}_\mu$. By Theorem 5.2, the function $(h,x) \mapsto \int_J k_\mu(x) e^\mu(h)\, d\mu$ is square integrable over $\int_{H^0 \times \mathfrak{h}} (\cdot) e^{-F(x)} dh\, dx$ because $J \subset \mathrm{Im}(\frac{i}{2}F')$. So k is square integrable over $\int_{\mathfrak{h}} (\cdot) e^{-F(x)} dx$. It implies that $\|\tilde{f}\| < \infty$, and hence $\|f\| < \infty$ by (5.17). We have proved (5.12), and therefore $f^0 = \imath^* f \in \imath^*(\mathscr{H}^2_{(Y,e^{-F})} \otimes S)$ in (5.11). This proves the $\supset$ part of (5.9). The proposition follows. $\square$

6. Geometric quantization

In this section, we incorporate partial Dirac cohomology into symplectic geometry, and prove Theorem 2.10. The intended symplectic manifold is

$$X = G \times \mathfrak{h}.$$

We first recall some results from [4, §3] on the symplectic geometry of X.

Let $\Omega^\bullet$ denote the de Rham complex of differential forms. Superscript denotes group invariance. So for example $\Omega^1(H^0 \times \mathfrak{h})^{H^0}$ consists of the H^0-invariant 1-forms on $H^0 \times \mathfrak{h}$.

Let $F : \mathfrak{h} \longrightarrow \mathbb{R}$ be a smooth function, and let F' be its gradient map. We may also regard F as a function on $H^0 \times \mathfrak{h}$ or $G \times \mathfrak{h}$ by invariance on the first component. As in (5.1) and (5.3), each fiber of

$$\pi : X \longrightarrow G/H^0 \tag{6.1}$$

inherits a complex structure from $H^0_{\mathbb{C}} = H^0 \times \mathfrak{h}$. As usual δ and $\bar{\partial}$ denote its Dolbeault operators. Let

$$\beta = -\tfrac{i}{2}(\partial - \bar{\partial})F \in \Omega^1(H^0 \times \mathfrak{h})^{H^0}.$$

Let $\mathfrak{h}^* \hookrightarrow \mathfrak{g}^*$ be the inclusion whose image consists of all linear functionals on $\mathfrak{g}$ that annihilate all the root spaces of $\mathfrak{h}$. It leads to the inclusion

$$\jmath : \Omega^1(H^0 \times \mathfrak{h})^{H^0} \hookrightarrow \Omega^1(X)^G.$$

Theorem 6.2

[4, Thm.3.1] *The 2-form* $\omega = d(\jmath\beta) \in \Omega^2(X)^{G \times H^0}$ *is symplectic if and only if* F' *is a local diffeomorphism and its image* $\mathrm{Im}(F') \subset \mathfrak{h}^*_{\mathrm{reg}}$.

Here $\mathfrak{h}^*_{\mathrm{reg}}$ consists of the elements of $\mathfrak{h}^*$ that are not perpendicular to any root. We shall fix a strictly convex function F whose gradient has image $\mathrm{Im}(F') \subset \mathfrak{h}^*_{\mathrm{reg}}$. The strictly convex condition implies that F' is a local diffeomorphism, so the 2–form ω constructed above is symplectic.

The $G \times H^0$-action on X preserves ω and is Hamiltonian, and the right H^0-action has a canonical moment map [8, §11]

$$\Phi : X \longrightarrow \mathfrak{h}^* \text{ given by } \Phi(g, x) = \tfrac{1}{2}F'(x) \tag{6.3}$$

for all $(g, x) \in G \times \mathfrak{h} = X$ [4, Prop.3.2]. The conventions in here and [4] differ by a factor 2, namely [4] uses $\beta = -i(\partial - \bar{\partial})F$ and $\Phi(g, x) = F'(x)$.

We next perform geometric quantization [19] on the symplectic manifold (X, ω). There is a complex line bundle $\mathbb{L} \to X$ whose Chern class of $\mathbb{L}$ is the cohomology class $[\omega]$. The construction in Theorem 6.2 shows that ω is exact, so $[\omega] = 0$ and $\mathbb{L}$ is topologically trivial. However, $\mathbb{L}$ has interesting geometry, as it has a connection ∇ whose curvature is ω, as well as an invariant Hermitian structure. If $W \subset X$ is a submanifold with a complex structure, we say that a section f of $\mathbb{L}$ is holomorphic on W if $\nabla_\xi f$ vanishes on W whenever ξ is an anti-holomorphic vector field on W. Each fiber of π of (6.1) is complex, and we say that f is π–*holomorphic* or *partially holomorphic* if it is holomorphic on each fiber of π.

There is a natural action of $G \times G$ on X, given by the left and right actions on the G-component. It lifts to a representation of $G \times G$ on the sections of $\mathbb{L} \to X$.

Proposition 6.4

[4, Cor.3.4] *There exists a $(G \times G)$–invariant non-vanishing section f_0 of $\mathbb{L}$ which is π-holomorphic, and $(f_0, f_0)(g, x) = e^{-F(x)}$ for all $(g, x) \in G \times \mathfrak{h} = X$.*

Recall $Y = G/N \times \mathfrak{h}$. If a section f of $\mathbb{L}$ is invariant under the right action of N, then so is (f, f), and we identify (f, f) with a function on Y. Let

$$L^2(\mathbb{L})^N = \left\{ N\text{-invariant sections } f \text{ of } \mathbb{L} \,\Big|\, \int\limits_{G/N \times \mathfrak{h}} (f, f)(g, x)\, d(gN)\, dx < \infty \right\}. \tag{6.5}$$

Then the action of $G \times T^0$ on $L^2(\mathbb{L})^N$ is a unitary representation.

Let f_0 denote the section in Proposition 6.4. For all $f \in C^\infty(G \times \mathfrak{h})^N$, we have

$$\int\limits_Y (ff_0, ff_0)(g, x)\, d(gN)\, dx = \int\limits_Y |f(g, x)|^2 e^{-F(x)}\, d(gN)\, dx,$$

so the trivialization $ff_0 \mapsto f$ defines a $(G \times T^0)$–equivariant isometry

$$L^2(\mathbb{L})^N \cong L^2(Y, e^{-F}). \tag{6.6}$$

Let ι denote both embeddings $G \hookrightarrow X$ and $G/N \hookrightarrow Y$, where $\iota(g) = (g, 0)$. So $\iota^* \mathbb{L}$ is a line bundle on G. Since f_0 is G-invariant, we can normalize it so that $(f_0, f_0)(g, 0) = 1$ for all $g \in G$. Then for all $f \in C^\infty(G)^N$,

$$\int\limits_{G/N} (f(\iota^* f_0), f(\iota^* f_0))(g)\, d(gN) = \int\limits_{G/N} |f(g)|^2\, d(gN),$$

so the trivialization $ff_0 \mapsto f$ leads to an isometry

$$L^2(\iota^* \mathbb{L})^N \otimes S \cong L^2(G/N) \otimes S. \tag{6.7}$$

Since (6.7) is $(G \times M)$–equivariant, it induces an operator

$$\mathbb{D}_{\mathbb{L}} : L^2(\iota^* \mathbb{L})^N \otimes S \longrightarrow L^2(\iota^* \mathbb{L})^N \otimes S$$

such that (6.7) intertwines $\mathbb{D}_{\mathbb{L}}$ and $\mathbb{D}_M$. Let

$$\mathscr{H}^2(\mathbb{L})^N \otimes S = \{ f \in L^2(\mathbb{L})^N \otimes S \mid \iota^* f \in \text{Ker } \mathbb{D}_{\mathbb{L}} \text{ and } f \text{ is } \pi\text{–holomorphic} \}.$$

Since (6.7) also preserves the π–holomorphic property, together with (6.6), they imply

$$\mathscr{H}^2(\mathbb{L})^N \otimes S \cong \mathscr{H}^2_{(Y,e^{-F})} \otimes S. \tag{6.8}$$

Proof of Theorem 2.10

By Proposition 5.6, (6.3) and (6.8),

$$\mathscr{H}_{(X,\omega)} \cong \mathscr{H}^2(\mathbb{L})^N \otimes S \cong \mathscr{H}^2_{(Y,e^{-F})} \otimes S \cong \int_{\substack{\mu \in \mathrm{Im}(i\Phi) \\ e^\mu \in \widehat{H^0}}} (\mathrm{Ker}\ \mathbb{D}_M)_\mu\ d\mu. \tag{6.9}$$

Write $\mu = \lambda + i\sigma \in \mathrm{Im}(i\Phi)$, where $e^\lambda \in \widehat{T^0}$. By Theorem 6.2 and (6.3), μ is MA–regular. Let $\rho_{\mathfrak{a}}$ be half the sum of positive $\mathfrak{a}$–roots relative to N. Set $w = 1$ and replace λ by $-\lambda$ in Theorem 2.9, then pick out the integrand which contains $e^{-i\sigma+\rho_{\mathfrak{a}}}$ and e^λ to get

$$(\mathrm{Ker}\ \mathbb{D}_M)_{\lambda+i\sigma} \cong \sum_{\chi \in \widehat{Z_M(M^0)}_\lambda} \mathrm{Ind}^G_{M^\dagger AN}(\mathscr{H}_{\eta_{\chi,-\lambda}} \otimes e^{-i\sigma+\rho_{\mathfrak{a}}} \otimes 1) \otimes \mathscr{H}_{(\bar\chi \otimes e^\lambda)}. \tag{6.10}$$

The $\rho_{\mathfrak{a}}$-shift in (6.10) is due to the definition of unitarily induced representation; see for example [17, VII §1]. The theorem follows from (6.9) and (6.10). □

We take this opportunity to revise an error in [5]. For G connected and with compact Cartan subgroup, [5, Thm.B] has a false expression

$$\mathscr{H}_{(X,\omega)} = \left(\sum_{\lambda \in \mathrm{Im}(\Phi), \pi_\lambda \in \widehat{G}_{disc}} \mathscr{H}_{\eta_\lambda}\right) \otimes \left(\sum_{w \in W_K} \mathbb{C}_{-w\lambda}\right),$$

as the summation over W_K should not appear.

Corollary 6.11

(Erratum to [5, Thm.B]) *If G is connected and has a compact Cartan subgroup, then*

$$\mathscr{H}_{(X,\omega)} \cong \sum_{\lambda \in \mathrm{Im}(i\Phi), e^\lambda \in \widehat{T^0}} \mathscr{H}_{\eta_{-\lambda}} \otimes \mathbb{C}_\lambda.$$

Proof

Suppose that G has a compact Cartan subgroup. Then $A = 1$, so the $\lambda + i\sigma$ of Theorem 2.10 becomes λ, and χ does not occur because G is connected, so $\mathrm{Ind}^G_P(\mathscr{H}_{\eta_{\chi,-\lambda}} \otimes e^{-i\sigma+\rho_{\mathfrak{a}}} \otimes 1) \otimes \mathscr{H}_{(\bar\chi \otimes e^\lambda)}$ becomes $\mathscr{H}_{\eta_{-\lambda}} \otimes \mathbb{C}_\lambda$. □

Remark 6.12

Models of tempered representations. According to Gelfand, a *model* of a compact Lie group is a unitary representation which contains every equivalence class of irreducible representation once [6]. For non-compact reductive Lie groups, this notion extends to models of discrete series [1] and principal series [3]. We briefly sketch the construction of models of standard

tempered representations in [4, §4], taking advantage of the fact that we make similar construction in Theorem 2.10.

There exist $(G \times H^0)$–invariant symplectic forms $\omega_1, \ldots, \omega_m$ on X with moment maps $\Phi_1, \ldots, \Phi_m : X \longrightarrow i\mathfrak{h}^*$, such that the image of

$$\bigcup_{j=1}^{m} \{\lambda + i\sigma \in \mathrm{Im}(\Phi_j) | e^{\lambda + i\sigma} \in \widehat{H^0}\}/W_G \longrightarrow \{\text{sum of tempered representations}\}$$

given by $\lambda + i\sigma \mapsto \sum_{\chi \in \widehat{Z_M(M^0)}_\lambda} \mathrm{Ind}_P^G(\mathscr{H}_{\eta_{\chi,-\lambda}} \otimes e^{-i\sigma + \rho_a} \otimes 1)$

is multiplicity-free and contains every equivalence class of standard tempered representations (thus almost every tempered representation – the missing tempered representations have Plancherel measure zero). By Theorem 2.10, $\sum_1^m \mathscr{H}_{(X, \omega_j)}$ is a model of tempered representations in the sense that every standard tempered representation occurs once.

7. Symplectic reduction

Let ω be a $(G \times H)$–invariant symplectic form on $X = G \times \mathfrak{h}$. Let $\mu \in \mathrm{Im}(i\Phi) \subset i\mathfrak{h}^*$ where $e^\mu \in \widehat{H^0}$. In this section, we carry out symplectic reduction [20] for the right action of H^0 to obtain the symplectic quotient (X_μ, ω_μ). Then we apply geometric quantization to (X_μ, ω_μ) and prove Theorem 2.12.

Recall symplectic reduction from [4, §5]. The moment map (6.3) of the right action of H^0 is $\Phi : X \longrightarrow \mathfrak{h}^*$. There is a unique $v \in \mathfrak{h}$ such that $(i\Phi)^{-1}(\mu) = G \times \{v\} \subset X$. Let ι and $\jmath$ be the maps in (2.11). Then there is a unique G–invariant symplectic form ω_μ on G/H^0 such that $\jmath^* \omega_\mu = \iota^* \omega$. We have

$$\omega_\mu = d\mu \in \Omega^2(G/H^0)^G. \tag{7.1}$$

As $\mu \in \mathfrak{h}^*$, $d\mu \in \wedge^2 \mathfrak{h}^* \subset \wedge^2 \mathfrak{g}^* \cong \Omega^2(G)^G$. Furthermore $d\mu$ lies in the image of the natural injection $\Omega^2(G/H^0)^G \hookrightarrow \Omega^2(G)^G$, which explains (7.1). The notation $d\mu$ does not imply that ω_μ is exact (for example if G/H^0 is compact, it cannot have an exact symplectic form) because μ does not lie in the image of $\Omega^1(G/H^0)^G \hookrightarrow \Omega^1(G)^G$. We obtain the symplectic quotient

$$(X_\mu, \omega_\mu) = (G/H^0, d\mu).$$

We shall incorporate Dirac cohomology into the geometric quantization of (X_μ, ω_μ), so we modify the line bundle for the spinor bundle over $G/H^0 N$,

$$B_\mu = G \times_\mu S \text{ defined by } [ghn, s] = [g, \chi_\mu^{-1}(h)s] \in B_\mu$$

for all $g \in G$, $hn \in H^0 N$ and $s \in S$. Here S is the same spinor as (6.7). A section f of B_μ can be

identified with a function $\psi : G/N \longrightarrow S$ such that $\psi(gh) = e^{\mu}(h)\psi(g)$ for all $h \in H$, given by $f(g) = [g, \psi(g)]$. This gives a Hermitian structure on the sections by $(f, f) = (\psi, \psi)$.

Recall that $G = KMAN$. Here $G/H^0 N$ has no G–invariant measure because $H^0 N$ is not unimodular, nevertheless $G/H^0 N$ has a measure $d(gH^0 N)$ which is K and M-invariant ([18, Prop.8.44], [4, p.2748]). Let

$$L^2(B_\mu) = \left\{ \text{sections } f \text{ of } B_\mu \middle| \int\limits_{G/H^0 N} (f, f)\, d(gH^0 N) < \infty \right\}.$$

The above correspondence $f \mapsto \psi$ leads to a G-equivariant map

$$L^2(B_\mu) \cong (L^2(G/N) \otimes S)_\mu. \tag{7.2}$$

In (7.2), $(L^2(G/N) \otimes S)_\mu$ is the μ-component of the direct integral decomposition of $L^2(G/N) \otimes S$. It is not a subspace, but is only weakly contained there. In (2.7), $\mathbb{D}_M$ stabilizes each $(L^2(G/N) \otimes S)_\mu$, and we let $\mathbb{D}_{M,\mu}$ denote the resulting operator. It induces an operator D_μ on $L^2(B_\mu)$, such that (7.2) intertwines D_μ with $\mathbb{D}_{M,\mu}$. Hence

$$\text{Ker } D_\mu \cong \text{Ker } \mathbb{D}_{M,\mu}. \tag{7.3}$$

We define the quantization on the symplectic quotient as

$$\mathcal{H}_{(X_\mu, \omega_\mu)} = \text{Ker } D_\mu. \tag{7.4}$$

Proof of Theorem 2.12

Let $\mu = \lambda + i\sigma$ belong to the image $\text{Im}(i\Phi)$, where $e^\mu \in \widehat{H^0}$. Then

$$
\begin{aligned}
\mathcal{H}_{(X,\omega),\mu} &\cong \sum\nolimits_{\chi \in \widehat{Z_M(M^0)}_\lambda} \text{Ind}_P^G(\mathcal{H}_{\eta_{\chi,-\lambda}} \otimes e^{-i\sigma+\rho_a} \otimes 1) && \text{by Theorem 2.10} \\
&\cong \text{Ker } \mathbb{D}_{M,\mu} && \text{by Theorem 2.9} \\
&\cong \text{Ker } D_\mu && \text{by (7.3)} \\
&= \mathcal{H}_{(X_\mu, \omega_\mu)}. && \text{by (7.4)}
\end{aligned}
$$

This proves the theorem. $\square$

8. Background for general real reductive groups

In this section we extend Theorem 2.9, Theorem 2.10, Theorem 2.12 from groups of Harish-Chandra class to general real reductive Lie groups. Recall that the latter class, introduced in [26], is given by

(a) the Lie algebra $\mathfrak{g}$ of G is reductive,   (8.1)

 (b) if $g \in G$ then $\mathrm{Ad}(g)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$, and

(c) G has a closed normal abelian subgroup Z such that

     (i) Z centralizes G^0 , i.e. $Z \subset Z_G(G^0)$

    (ii) $|G/ZG^0| < \infty$ and

    (iii) $Z \cap G^0$ is co-compact in Z_{G^0} .

Without loss of generality we always assume $Z_{G^0} \subset Z$, so (iii) becomes $Z \cap G^0 = Z_{G^0}$.

As Z centralizes G^0 it centralizes $\mathfrak{g}$. Thus Z centralizes every Cartan subalgebra of $\mathfrak{g}$ and so it is contained in every Cartan subgroup of G. The point is that, by definition, "Cartan subgroup" means the centralizer of a Cartan subalgebra. We have to be careful here because it can happen that two G–conjugate Cartan subgroups of G may fail to be G^0–conjugate.

Given a unitary character $\zeta \in \widehat{Z}$, define

$$\widehat{G}_\zeta = \{\pi \in \widehat{G} | \pi(gz) = \zeta(z)\pi(g) \text{ for } g \in G \text{ and } z \in Z\} \tag{8.2}$$

and

$$L^2(G/Z; \zeta) = \left\{ f : G \to \mathbb{C} \left| \begin{array}{l} f \text{ is measurable} \\ f(gz) = \zeta(z)^{-1} f(g) \ a.\,e. \ z \in Z, g \in G \\ \int_{G/Z} |f(gZ)|^2 d(gZ) < \infty \end{array} \right. \right\}. \tag{8.3}$$

Induction by stages gives $L^2(G/Z; \zeta) = \mathrm{Ind}_Z^G \mathbb{C}_\zeta$ and $L^2(G) = \int_{\zeta \in \widehat{Z}} L^2(G/Z; \zeta) d\zeta$. But this is a bit redundant for $\widehat{G}$. If $g \in G$ and $\zeta' = \mathrm{Ad}^*(g)\zeta$ then $\mathrm{Ad}^*(g) : \widehat{G}_\zeta \to \widehat{G}_{\zeta'}$ is a bijection (and homeomorphism for the hull–kernel topology) from $\widehat{G}_\zeta$ onto $\widehat{G}_{\zeta'}$, sending $\pi' \in \widehat{G}_{\zeta'}$ to an equivalent representation $\pi \in \widehat{G}_\zeta$. This depends only on $gZG^0 \in G/ZG^0$. Thus

$$\mathrm{Ad}^*(g) : L^2(G/Z; \zeta') \cong L^2(G/Z; \zeta) \text{ and } \widehat{G} = \bigcup_{\zeta \in (\widehat{Z}/\mathrm{Ad}^*(G/ZG^0))} \widehat{G}_\zeta.$$

Here note that $\widehat{Z}/\mathrm{Ad}^*(G/ZG^0)$ is finite by (8.1)(ii).

If Z is noncompact the coefficients of a representation $\pi \in \widehat{G}_\zeta$ cannot be square integrable over G. So we consider square integrability over G/Z. This is well defined because G/ZG^0 is finite. More precisely, consider a coefficient $f_{\pi,u,v}(g) = \langle u, \pi(g)v \rangle$, $u, v \in \mathcal{H}_\pi$. Then $f_{\pi,u,v}(gz) = \zeta(z)^{-1} f_{\pi,u,v}(g)$, so $|f_{\pi,u,v}(gz)| = |f_{\pi,u,v}(g)|$ for $g \in G$ and $z \in Z$, and $|f_{\pi,u,v}|$ is defined on G/Z. We say that $f_{\pi,u,v}$ is *square integrable* or *square integrable modulo Z* if $f_{\pi,u,v} \in L^2(G/Z)$. The following are equivalent for $\pi \in \widehat{G}_\zeta$.

(a) there exist nonzero $u, v \in \mathcal{H}_\pi$ with $f_{\pi,u,v} \in L^2(G/Z)$
(b) $f_{\pi,u,v} \in L^2(G/Z)$ for every $u, v \in L^2(G/Z)$
(c) π is a (discrete) summand of the left regular representation of G on $L^2(G/Z; \zeta)$

Then we say that π is a *relative discrete series representation* of G. The relative discrete series representations in $\widehat{G}_\zeta$ form the subset denoted $\widehat{G}_{\zeta,disc}$.

From Harish–Chandra's famous result, G has relative discrete series representations if and only if G/Z has a compact Cartan subgroup. When G satisfies (8.1), Levi components of cuspidal parabolic subgroups also satisfy (8.1). Thus we can construct the various tempered series more or less in the same way as when G is of Harish–Chandra class. Further, the Plancherel formula depends only on these tempered series. See [26], or [29] for an update, or [13] and [14] for a short direct proof.

Let H be a Cartan subgroup of G. As for Harish–Chandra class it is stable under a Cartan involution θ, leading to a decomposition $H = T \times A$ and cuspidal parabolic subgroup MAN. Here T is a Cartan subgroup of M, $Z \subset T$, and T/Z is compact. Thus, given $\zeta \in \widehat{Z}$ we have the part $\widehat{M}_{\zeta,disc}$ of the relative discrete series of M that transforms by ζ, as follows. Let $\lambda \in it^*$ such that (i) e^λ is well defined on T^0, (ii) λ is regular for $\Sigma_{\mathfrak{m}_\mathbb{C}, \mathfrak{t}_\mathbb{C}}$, and (iii) $\zeta|_{Z \cap T^0} = e^\lambda_{Z \cap T^0}$. Then as in (2.2) we have

$$\eta^0_\lambda \in \widehat{M^0}_{\zeta,disc} \text{ with } \Theta_{\eta^0_\lambda} = \pm \frac{1}{\Delta_{M^0}} \sum_{w \in W_{M^0}} \det(w) e^{w(\lambda)} \text{ on } (M^0)'' \cap T^0 , \tag{8.4}$$

as in (2.3), where we avoid clutter by writing ζ instead of $\zeta|_{(Z \cap M^0)}$. Now we have

$$\eta^\dagger_{\chi,\lambda} := \chi \otimes \eta^0_\lambda \in \widehat{M^\dagger}_{\zeta,disc} \text{ with } \Theta_{\eta^\dagger_{\chi,\lambda}}(zm) = \text{trace } \chi(z)\Theta_{\eta^0_\lambda}(m), \tag{8.5}$$

and as in (2.4) we have $\eta_{\chi,\lambda} \in \widehat{M}_{\zeta,disc}$ given by

$$\eta_{\chi,\lambda} := \text{Ind}^M_{M^\dagger}(\eta^\dagger_{\chi,\lambda}) \text{ with } \Theta_{\eta_{\chi,\lambda}}(zm) = \sum_{xM^\dagger \in M/M^\dagger} \text{trace } \chi(x^{-1}zx)\Theta_{\eta^0_\lambda}(x^{-1}mx). \tag{8.6}$$

As before, $\eta_{\chi,\lambda}$ has infinitesimal character of Harish–Chandra parameter λ and formal degree $\deg(\eta_{\chi,\lambda}) = |M/M^\dagger| \deg(\chi)|\varpi(\lambda)|$, and $\eta_{\chi,\lambda} \cong \eta_{\chi',\lambda'}$ just when their $M^\dagger$–restrictions are equivalent. Every representation in $\widehat{M}_{\zeta,disc}$ is one of the $\eta_{\chi,\lambda}$ just described, and the relative discrete series $\widehat{M}_{disc} = \bigcup_{\zeta \in \widehat{Z}} \widehat{M}_{\zeta,disc}$.

Similarly, if $\sigma \in \mathfrak{a}^*$, so $e^{i\sigma} \in \widehat{A}$,

$$\pi_{\chi,\lambda,\sigma} := \mathrm{Ind}_{MAN}^G(\eta_{\chi,\lambda} \otimes e^{i\sigma} \otimes 1) = \mathrm{Ind}_{M^\dagger AN}^G(\eta_{\chi,\lambda}^\dagger \otimes e^{i\sigma} \otimes 1) \tag{8.7}$$

is a unitary representation of G, and these representations (ζ fixed, χ and λ variable) form the H–series part of $\widehat{G}_\zeta$. That H–series part depends only on the G–conjugacy class of (H, ζ), and as these vary we sweep out all but a set of measure zero in $\widehat{G}$.

As usual we fix a Cartan involution θ on G, a splitting $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ into (± 1)–eigenspaces of $d\theta$, and a nondegenerate $\mathrm{Ad}(G)$–invariant symmetric bilinear form b on $\mathfrak{g}$ that is negative definite on $\mathfrak{k}$, positive definite on $\mathfrak{p}$ and satisfies $b(\mathfrak{k}, \mathfrak{p}) = 0$. Our θ–stable Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ and we have the corresponding cuspidal parabolic subalgebra $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ of $\mathfrak{g}$, with an orthogonal direct sum decomposition $\mathfrak{m} = \mathfrak{t} + \mathfrak{s}$. Using the spin module S for the Clifford algebra $C(\mathfrak{s}_\mathbb{C})$ the modified Dirac operator $D_M = D_{(\mathfrak{m}_\mathbb{C}, \mathfrak{k}_\mathbb{C} \cap \mathfrak{m}_\mathbb{C})} + iD_{diag;(\mathfrak{k}_\mathbb{C} \cap \mathfrak{m}_\mathbb{C}, \mathfrak{k}_\mathbb{C})}$ of (2.6) is defined as in [5, (1.1)] and its closure $\mathbb{D}_M$ as in (2.7). This is possible because D_M is defined in terms of the Lie algebra, so its construction is the same as the construction for groups of Harish–Chandra class.

To be precise, note that Z centralizes N because $N \subset G^0$ and Z centralizes G^0. Further Z centralizes $\mathfrak{s}$ and thus also the spin module S. Now

$$L^2(G/N) = \int_{\zeta \in \widehat{Z}} L^2(G/NZ; \zeta) d\zeta \text{ and} \tag{8.8}$$
$$L^2(G/N) \otimes S = \int_{\zeta \in \widehat{Z}} \left\{ L^2(G/NZ; \zeta) \otimes S \right\} d\zeta$$

as unitary G–module, and

$$\mathbb{D}_M = \int_{\zeta \in \widehat{Z}} \mathbb{D}_{\zeta,M} d\zeta \text{ where } \mathbb{D}_{\zeta,M} : L^2(G/NZ; \zeta) \otimes S \to L^2(G/NZ; \zeta) \otimes S. \tag{8.9}$$

Using $T = Z_M(M^0)$ every irreducible unitary representation of H has form $\chi_{\lambda,\sigma} := \chi \otimes e^\lambda \otimes e^{i\sigma}$ as in (2.8). In particular each such $\chi_{\lambda,\sigma}$ is finite dimensional.

The extension of Theorem 2.9 to general real reductive Lie groups has both a relative formulation and an absolute formulation. The relative formulation is

Theorem 8.10

Let G be a general real reductive Lie group as in (2.13). Let $\zeta \in \widehat{Z}$. In the notation of (8.8) and (8.9),

$$\mathrm{Ker}(\mathbb{D}_{\zeta,M}) = \sum_{\eta_{\chi,\lambda} \in \widehat{M}_{\zeta,disc}} \left(\int_{\sigma \in \mathfrak{a}^*} \mathscr{H}_{\pi_{\chi,\lambda,\sigma}} d\sigma \right) \otimes \left(\sum_{w \in W_{K \cap M}} \mathscr{H}_{(\bar{\chi} \cdot w^{-1} \otimes e^{-w\lambda})} \right)$$

and the natural action of $G \times T$ on $\mathrm{Ker}(\mathbb{D}_{\zeta,M})$ is

$$\sum\nolimits_{\eta_{\chi,\lambda}\in\widehat{M}_{\zeta,disc}}\left(\int_{\sigma\in\mathfrak{a}^*}\pi_{\chi,\lambda,\sigma}\,d\sigma\right)\otimes\left(\sum\nolimits_{w\in W_{K\cap M}}(\bar{\chi}\cdot w^{-1}\otimes e^{-w\lambda})\right).$$

The absolute formulation of our extension of Theorem 2.9 is

Corollary 8.11

In the notation of (8.8) *and* (8.9),

$$\mathrm{Ker}(\mathbb{D}_M)=\int_{\zeta\in\widehat{Z}}\left\{\sum\nolimits_{\eta_{\chi,\lambda}\in\widehat{M}_{\zeta,disc}}\left(\int_{\sigma\in\mathfrak{a}^*}\mathscr{H}_{\pi_{\chi,\lambda,\sigma}}\,d\sigma\right)\right.$$
$$\left.\otimes\left(\sum\nolimits_{w\in W_{K\cap M}}\mathscr{H}_{(\bar{\chi}\cdot w^{-1}\otimes e^{-w\lambda})}\right)\right\}d\zeta$$

and the natural action of $G\times T$ *on* $\mathrm{Ker}(\mathbb{D}_M)$ *is*

$$\int_{\zeta\in\widehat{Z}}\left\{\sum\nolimits_{\eta_{\chi,\lambda}\in\widehat{M}_{\zeta,disc}}\left(\int_{\sigma\in\mathfrak{a}^*}\pi_{\chi,\lambda,\sigma}\,d\sigma\right)\otimes\left(\sum\nolimits_{w\in W_{K\cap M}}(\bar{\chi}\cdot w^{-1}\otimes e^{-w\lambda})\right)\right\}d\zeta.$$

Corollary 8.11 is an immediate consequence of Theorem 8.10; one just integrates over $\widehat{Z}$ modulo the conjugation action of G/ZG^0. We will prove Theorem 8.10 in Section 10. This will use an extension, in Section 9, of a reduction method developed in [26].

The discussion leading to the statement of Theorem 2.10, and also Theorem 6.2, is valid for general real reductive Lie groups with essentially no modification. We have $(G\times H)$–invariant symplectic forms ω on $X=G\times\mathfrak{h}$. For each such ω we have a complex line bundle $\mathbb{L}\to X$ with $c_1(\mathbb{L})=\omega$. The inclusion $\imath:G\to X$, $\imath(g)=(g,0)$, pulls $\mathbb{L}$ back to a line bundle $\imath^*\mathbb{L}\to G$. Fix $\zeta\in\widehat{Z}$. Then we have

$L^2(\mathbb{L})_{\zeta}^N$: measurable sections f of $\mathbb{L}\to X$ such that
$$f(gnz,x)=\zeta(z)^{-1}f(g,x)\text{ and }\int_{G/NZ\times\mathfrak{h}}|f(g,x)|^2 d(gNZ)dx<\infty,\text{ and}$$
$L^2(\imath^*\mathbb{L})_{\zeta}^N$: measurable sections f of $\imath^*\mathbb{L}\to G$ such that
$$f(gnz)=\zeta(z)^{-1}f(g)\text{ and }\int_{G/NZ}|f(g)|^2 d(gNZ)<\infty.$$

Let $\mathbb{D}_{\zeta,\mathbb{L}}$ denote the Dirac operator on $L^2(\imath^*\mathbb{L})_{\zeta}^N\otimes S$. Then we have a unitary representation $\pi_{\zeta,(X,\omega)}$ with representation space

$$\mathscr{H}_{\zeta,(X,\omega)}=\{f\in L^2(\mathbb{L})_{\zeta}^N\,|\,\imath^*f\in\mathrm{Ker}\,\mathbb{D}_{\zeta,\mathbb{L}}\text{ and }f\text{ is }\pi\text{–holomorphic}\}.$$

Now recall that the action of $G\times H$ on (X,ω) is Hamiltonian, so the right action of H has moment map $\Phi:X\to\mathfrak{h}^*$ as in the case where G is of Harish–Chandra class. The relative version of the extension of Theorem 2.10 is

Theorem 8.12

Let G be a general real reductive Lie group as in (2.13). *Fix $\zeta \in \widehat{Z}$. The unitary representation space $\mathcal{H}_{\zeta,(X,\omega)}$ for $G \times T$ is*

$$\int_{\substack{\lambda+i\sigma \in \mathrm{Im}(i\Phi) \\ \exp(\lambda+i\sigma) \in \widehat{H^0_\zeta}}} \sum_{\chi \in \widehat{Z_M(M^0)}_\lambda} \left(\mathrm{Ind}_P^G(\mathcal{H}_{\bar\chi \otimes \eta^0_{-\lambda}} \otimes e^{-i\sigma+\rho_\mathfrak{a}} \otimes 1) \otimes \mathcal{H}_{(\chi \otimes e^\lambda)} \right) \, d\lambda \, d\sigma$$

where $\widehat{Z_M(M^0)}_\lambda$ denotes the elements of $Z_M(M^0)$ that agree with e^λ on Z_{M^0}, and thus agree with ζ on Z.

Now we sum over $\widehat{Z}$. Let $\mathbb{D}_\mathbb{L}$ denote the Dirac operator on $L^2(\imath^*\mathbb{L})^N \otimes S$ and $\pi_{(X,\omega)}$ the corresponding representation on

$$\mathcal{H}_{(X,\omega)} = \{f \in L^2(\mathbb{L})^N \,|\, \imath^* f \in \mathrm{Ker}\,\mathbb{D}_\mathbb{L} \text{ and } f \text{ is } \pi\text{--holomorphic}\}.$$

The absolute version of the extension of Theorem 2.10 is

Corollary 8.13

The unitary representation space $\mathcal{H}_{(X,\omega)}$ for $G \times T$ is

$$\int_{\substack{\lambda+i\sigma \in \mathrm{Im}(i\Phi) \\ \exp(\lambda+i\sigma) \in \widehat{H^0}}} \sum_{\chi \in \widehat{Z_M(M^0)}_\lambda} \left(\mathrm{Ind}_P^G(\mathcal{H}_{\bar\chi \otimes \eta^0_{-\lambda}} \otimes e^{-i\sigma+\rho_\mathfrak{a}} \otimes 1) \otimes \mathcal{H}_{(\chi \otimes e^\lambda)} \right) \, d\lambda \, d\sigma$$

where $\widehat{Z_M(M^0)}_\lambda$ denotes the elements of $Z_M(M^0)$ that agree with e^λ on Z_{M^0}.

Again, we will prove Theorem 8.12 in Section 10, using an extension, from Section 9, of a reduction method developed in [26].

Theorem 2.12, the principle that quantization commutes with reduction, is valid as stated for general real reductive Lie groups. We will go over the argument toward the end of Section 10.

9. Reduction to the case of compact center

In this section we state and prove Theorem 9.5, which will reduce the proofs of Theorem 8.10, Theorem 8.12 to the case where $G^\dagger = Z_G(G^0)G^0$ has compact center, so that we can identify discrete series representations by lowest K–type.

Since Z centralizes G^0 we have $Z_{G^0} \subset Z \subset Z_G(G^0)$. Let $\zeta \in \widehat{Z}$, $\chi \in \widehat{Z_G(G^0)}_\zeta$, $\mathcal{H}_\chi$ its representation space, and $U = U(\mathcal{H}_\chi)$ the unitary group of $\mathcal{H}_\chi$. Recall that $\dim \mathcal{H}_\chi < \infty$, so U is compact. Of course we have the defining representation $1_U \in \widehat{U}$. It is the usual representation of $U(\mathcal{H}_\chi)$ on $\mathcal{H}_\chi$, given by $1_U(z) = z$. If L is any closed subgroup of G of the form $Z_G(G^0)L^0$

then we denote

$$L[\chi] = (U \times L)/\{(\chi(z)^{-1}, z)|z \in Z_G(G^0)\}. \tag{9.1}$$

In particular we have the quotient groups

$$G^\dagger[\chi] = (U \times G^\dagger)/\{(\chi(z)^{-1}, z)|z \in Z_G(G^0)\}, \tag{9.2}$$
$$G[\chi] = (U \times G)/\{(\chi(z)^{-1}, z)|z \in Z_G(G^0)\}.$$

Note that $G^\dagger[\chi]$ is the identity component of $G[\chi]$. They are general real reductive Lie groups as in (2.13). We write p for the restriction to G of the projection $(U \times G) \to G[\chi]$ and also for the restriction to $G^\dagger$ of $(U \times G^\dagger) \to G^\dagger[\chi]$. Then p induces an isomorphism $G/G^\dagger \cong G[\chi]/G^\dagger[\chi]$.

Lemma 9.3

$G^\dagger[\chi]$ *is a connected reductive Lie group with Lie algebra* $\mathfrak{u} \oplus (\mathfrak{g}^\dagger/\mathfrak{z})$. *It has compact center U. For appropriate normalizations of Haar measures,* $f \mapsto f \cdot p$ *defines an equivariant isometry of* $L^2(G^\dagger[\chi]/U, 1_U)$ *onto* $L^2(G^\dagger/Z_G(G^0), \chi)$.

Proof

We follow the proof of [26, Lemma 3.3.2], which is the case $G^\dagger = ZG^0$. There $\chi = \zeta$ and U is the circle group $U(1)$. If $f \in L^2(G^\dagger[\chi]/U, 1_U)$, $z \in Z_G(G^0)$ and $g \in G^\dagger$ then

$$(f \cdot p)(gz) = f(1, gz) = f(\chi(z), g) = \chi(z)^{-1}f(1, g) = \chi(z)^{-1}(f \cdot p)(g)$$

and

$$\int\limits_{G^\dagger/Z_G(G^0)} |(f \cdot p)(g)|^2 d(gZ_G(G^0)) = \int\limits_{(U \times G^\dagger)/(U \times Z_G(G^0))} |f(u, g)|^2 d(uU \times gZ_G(G^0))$$
$$= \int\limits_{G^\dagger[\chi]/U} |f(\bar{g})|^2 d(\bar{g}U).$$

Thus $f \mapsto f \cdot p$ is an isometric injection of $L^2(G^\dagger[\chi]/U, 1_U)$ into $L^2(G^\dagger/Z_G(G^0), \chi)$. It is surjective because any $f' \in L^2(G^\dagger/Z_G(G^0), \chi)$ has inverse image $f(z, g) = \chi(z)^{-1}f'(g)$. $\square$

Let $P = MAN$ be a cuspidal parabolic subgroup of G associated to a Cartan subgroup $H = T \times A$. Recall some properties of $M^\dagger = Z_M(M^0)M^0$. First, $\widehat{M^\dagger}$ consists of the $\varphi \otimes \eta^0$ where $\varphi \in \widehat{Z_M(M^0)}$ agrees with $\eta^0 \in \widehat{M^0}$ on Z_{M^0}. Here we write φ instead of χ to avoid the possibility of confusion in Section 10, but to avoid cluttered notation we continue to use U for $U(\mathcal{H}_\varphi)$. $\widehat{M^\dagger}_\varphi$ denotes the subset of $\widehat{M^\dagger}$ corresponding to a fixed φ. The relative discrete series of $M^\dagger$ consists of the $\varphi \otimes \eta^0$ where $\eta^0 \in \widehat{M^0}_{disc}$, i.e. the representations $\eta^\dagger_{\varphi,\lambda} = \varphi \otimes \eta^0_\lambda$ of $M^\dagger$ as in (8.5), and $\widehat{M^\dagger}_{\varphi,disc} = \widehat{M^\dagger}_\varphi \cap \widehat{M^\dagger}_{disc}$. As in (9.2) we have

$$M^\dagger[\varphi] = (U \times M^\dagger)/\{(\varphi(z)^{-1}, z) | z \in Z_M(M^0)\}, \qquad (9.4)$$
$$M[\varphi] = (U \times M)/\{(\varphi(z)^{-1}, z) | z \in Z_M(M^0)\}.$$

The key observation of this section is this extension of [26, Theorem 3.3.3].

Theorem 9.5

The map $\varepsilon_\chi^\dagger : \widehat{G^\dagger[\chi]}_{1_U} \to \widehat{G^\dagger}_\chi$, given by $\varepsilon_\chi^\dagger(\psi) = \psi \cdot p$, is a well defined bijection and maps $\widehat{G^\dagger[\chi]}_{1_U, disc}$ onto $\widehat{G^\dagger}_{\chi, disc}$. It carries Plancherel measure of $\widehat{G^\dagger[\chi]}_{1_U}$ to Plancherel measure of $\widehat{G^\dagger}_\chi$. Distribution characters satisfy $\Theta_{\varepsilon_\chi^\dagger(\psi)} = \Theta_\psi \cdot p$.

Similarly, $\varepsilon_\varphi^\dagger : \widehat{M^\dagger[\varphi]}_{1_U} \to \widehat{M^\dagger}_\varphi$, given by $\varepsilon_\varphi^\dagger(\psi) = \psi \cdot p$, is a well defined bijection and maps $\widehat{M^\dagger[\varphi]}_{1_U, disc}$ onto $\widehat{M^\dagger}_{\varphi, disc}$. It carries Plancherel measure of $\widehat{M^\dagger[\varphi]}_{1_U}$ to Plancherel measure of $\widehat{M^\dagger}_\varphi$. Distribution characters satisfy $\Theta_{\varepsilon_\varphi^\dagger(\psi)} = \Theta_\psi \cdot p$.

Our argument for Lemma 9.3 was a perturbation of the proof of the special case [26, Lemma 3.3.2]. Similarly, the proof of [26, Theorem 3.3.3], which occupies most of [26, Section 3.3], goes through with no serious change, yielding the argument for Theorem 9.5. One needs only be careful about noncommutativity of U when $\dim \mathscr{H}_\chi > 1$ or $\dim \mathscr{H}_\varphi > 1$.

10. Proofs for general real reductive groups

Recall that K denotes the fixed point set of the Cartan involution θ of G, so $Z_{G^0} \subset Z \subset Z_G(G^0) \subset K$. K/Z is a maximal compact subgroup of G/Z and $K/Z_G(G^0)$ is a maximal compact subgroup of $G/Z_G(G^0)$. Since K is the normalizer of its Lie algebra $\mathfrak{k}$ it meets every component of G. Also, $K \cap G^0 = K^0$ and $K \cap G^\dagger = Z_G(G^0)K^0$. We will write $K^\dagger$ for this group $Z_G(G^0)K^0$. The construction (9.1) gives us

$$K^\dagger[\chi] = (U \times K^\dagger)/\{(\chi(z)^{-1}, z) | z \in Z_G(G^0)\}, \qquad (10.1)$$
$$K[\chi] = (U \times K)/\{(\chi(z)^{-1}, z) | z \in Z_G(G^0)\}.$$

Lemma 10.2

$K^\dagger[\chi]$ is a maximal compact subgroup of $G^\dagger[\chi]$, $K[\chi]$ is a maximal compact subgroup of $G[\chi]$, and $\mathfrak{p}$ induces isomorphisms $G/G^\dagger \cong G[\chi]/G^\dagger[\chi] \cong K[\chi]/K^\dagger[\chi]$.

Denote $K_M = K \cap M$ and $K_M^\dagger = K \cap M^\dagger$, so $K_M^\dagger = Z_M(M^0)K_M^0$. Now

$$K_M^\dagger[\varphi] = (U \times K_M^\dagger)/\{(\varphi(z)^{-1}, z) | z \in Z_M(M^0)\}, \qquad (10.3)$$
$$K_M[\varphi] = (U \times K_M)/\{(\varphi(z)^{-1}, z) | z \in Z_M(M^0)\}.$$

Applying the character formulas of (8.4) and (8.5), and using the map $\varepsilon_\varphi^\dagger$ of Theorem 9.5, we have the following.

Lemma 10.4

Let $\varphi \in \widehat{Z_M(M^0)}_\lambda$ such that $\eta_{\varphi,\lambda}^\dagger \in \widehat{M^\dagger}_{\varphi,disc}$. Let $\psi_{\varphi,\lambda}^\dagger \in \widehat{M^\dagger[\varphi]}_{1_U,disc}$ such that $\varepsilon_\varphi^\dagger(\psi_{\varphi,\lambda}^\dagger) = \eta_{\varphi,\lambda}^\dagger$ Then the restriction of the character of $\psi_{\varphi,\lambda}^\dagger$ from $M^\dagger[\varphi]$ to $K_M^\dagger[\varphi]$ is equal to the character of the restriction of $\psi_{\varphi,\lambda}^\dagger$ from $M^\dagger[\varphi]$ to $K_M^\dagger[\varphi]$. In other words the relative discrete series characters satisfy

$$\Theta_{\psi_{\varphi,\lambda}^\dagger}|_{K_M^\dagger[\varphi]} = \Theta_{\psi_{\varphi,\lambda}^\dagger|_{K_M^\dagger[\varphi]}}$$

and thus

$$\Theta_{\eta_{\varphi,\lambda}^\dagger}|_{K_M^\dagger} = \Theta_{\eta_{\varphi,\lambda}^\dagger|_{K_M^\dagger}}.$$

We use Theorem 9.5 and the character formula in (8.6) to carry the result of Lemma 10.4 from $M[\varphi]$ to M.

Proposition 10.5

Let $\varphi \in \widehat{Z_M(M^0)}_\lambda$ such that $\eta_{\varphi,\lambda} \in \widehat{M}_{\varphi,disc}$. Let $\psi_{\varphi,\lambda} \in \widehat{M[\varphi]}_{1_U,disc}$ such that $\varepsilon_\varphi\psi_{\varphi,\lambda} = \eta_{\varphi,\lambda}$. Then

$$\Theta_{\psi_{\varphi,\lambda}}|_{K_M[\varphi]} = \Theta_{\psi_{\varphi,\lambda}|_{K_M[\varphi]}}$$

and thus

$$\Theta_{\eta_{\varphi,\lambda}}|_{K_M} = \Theta_{\eta_{\varphi,\lambda}|_{K_M}}.$$

By K_M–type of $\eta_{\varphi,\lambda}$ we mean, as usual, an irreducible summand τ of $\eta_{\varphi,\lambda}|_{K_M}$. It has form $\tau = \mathrm{Ind}_{K_M^\dagger}^{K_M} \tau^\dagger$ where $\tau^\dagger = \varphi \otimes \tau^0$ with $\tau^0 \in \widehat{K_M^0}$. With respect to a positive root system τ^0 has some highest weight ν, and ν also is the highest weight of $\tau^\dagger = \varphi \otimes \tau^0$. The restriction of τ to $K_M^\dagger$ is the sum over $M/M^\dagger$ of conjugates $\tau^\dagger \cdot \mathrm{Ad}\,(m)^{-1}$ of $\tau^\dagger$. For brevity we will say that ν is the highest weight of τ.

We use the character formulas of (8.4), (8.5) and (8.6), or we can rely on [23] or [24], for the following corollary. It extends a case of Theorem [15, Theorem 5.3].

Corollary 10.6

Let $\varphi \in \widehat{Z_M(M^0)}_\lambda$ such that $\eta_{\varphi,\lambda} \in \widehat{M}_{\varphi,disc}$, where we choose the positive root system so that $\langle \alpha, \lambda|_t \rangle \geqq 0$ for all $\mathfrak{h}$–roots of $\mathfrak{n}$. Let ρ_{nonc} denote half the sum of the noncompact roots of $\mathfrak{n}$. Then $\eta_{\varphi,\lambda}$ has lowest K_M –type of highest weight $\lambda + 2\rho_{nonc}$.

Proof of Theorems 8.10 and 8.12

The delicate point in the proofs of Theorem 2.9, Theorem 2.10 is their dependence on [15,

Proposition 5.4]. The argument of [15, Proposition 5.4] relies on [23] (or see [24]) for the existence of a K–type of a certain highest weight $\lambda + \rho_{nonc}$ in Dirac cohomology modules of groups of Harish–Chandra class. Corollary 10.6 provides the corresponding existence result for the groups M. Now our arguments for Theorem 2.9, Theorem 2.10 go through with only minor changes for G and the representations in the $\widehat{G}_\zeta$. Theorem 8.10, Theorem 8.12 follow. □

Proof of Theorem 2.12 for general real reductive Lie groups

The discussion in Section 7 goes through with only trivial changes for general real reductive Lie groups. The point is that $Z \subset Z_G(G^0) \subset H$ because they centralize $\mathfrak{h}$, so we can replace H^0 by ZH^0 in the integration that defines $L^2(B_\mu)$. Then we proceed relative to $\zeta \in \widehat{Z}$ as usual with $e^\mu = \zeta$ on $Z \cap H^0$ and $\lambda = \mu + i\sigma$. That gives us relative versions of (7.3) and (7.4), and now the proof goes as in Section 7:

$$
\begin{aligned}
\mathscr{H}_{(X,\omega),\mu} &\cong \sum\nolimits_{\chi \in \widehat{Z_M(M^0)}_\lambda} \mathrm{Ind}_P^G(\mathscr{H}_{\eta_{\chi,-\lambda}} \otimes e^{-i\sigma+\rho_{\mathfrak{a}}} \otimes 1) &&\text{by Theorem 8.12} \\
&\cong \mathrm{Ker}\ \mathbb{D}_{M,\mu} &&\text{by Theorem 8.10} \\
&\cong \mathrm{Ker}\ D_\mu &&\text{by the extension of (7.3)} \\
&= \mathscr{H}_{(X_\mu,\omega_\mu)}. &&\text{by the extension of (7.4)}
\end{aligned}
$$

As before, this proves the theorem. □

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Data availability

No data was used for the research described in the article.

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