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Local and Global Homogeneity for Manifolds of Positive Curvature

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Abstract

We study globally homogeneous Riemannian quotients $\Gamma\backslash(M, ds^2)$ of homogeneous Riemannian manifolds (M, ds^2). The homogeneity conjecture is that $\Gamma\backslash(M, ds^2)$ is (globally) homogeneous if and only if (M, ds^2) is homogeneous and every $\gamma \in \Gamma$ is of constant displacement on (M, ds^2). We provide further evidence for that conjecture by (i) verifying it for normal homogeneous Riemannian manifolds of positive curvature and (ii) showing that in most cases the normality condition can be dropped.

17.1 Introduction

In this chapter we show that a certain conjecture, concerning global homogeneity for locally homogeneous Riemannian manifolds, holds for positively curved manifolds whose universal Riemannian cover is a normal homogeneous space of strictly positive curvature. That is Theorem 17.4.5. It depends on certain classifications (see Table 17.1 on p. 376) and on earlier results [32] concerning isotropy-split fibrations. We then explore the possibility of dropping the normality requirement. Theorem 17.5.4 eliminates the normality condition on the Riemannian metric for most of the positively curved Riemannian manifolds. That requires a modification (17.5.1) of the isotropy-splitting condition of [32].

We start by describing the homogeneity conjecture. Let (M, ds^2) be a connected, simply connected Riemannian homogeneous space. Let $\pi : M \to M'$ be a Riemannian covering. In other words, $\pi : M \to M'$ is a topological covering space that is a local isometry. Then the base of the covering must have form $M' = \Gamma\backslash M$, where Γ is a discontinuous group of isometries of M such that only the identity element has a fixed point. Clearly M', with the induced Riemannian metric ds'^2 from $\pi : M \to M'$, is locally homogeneous. We ask when (M', ds'^2) is globally homogeneous.

If $M' = \Gamma\backslash M$ is homogeneous then [27] every element $\gamma \in \Gamma$ is of constant displacement $\delta_\gamma(x) = dist(x, \gamma x)$ on M. For the isometry group $G' = \mathbf{I}(\Gamma\backslash M, ds'^2) = N_G(\Gamma)/\Gamma$, where $N_G(\Gamma)$ is the normalizer of Γ in the isometry group $G = \mathbf{I}(M, ds^2)$. Since Γ is discrete, the identity component $N_G(\Gamma)^0$ centralizes Γ in G. This centralizer $Z_G(\Gamma)$ is transitive on M. If $x, y \in M$ and $\gamma \in \Gamma$ we write $y = g(x)$ with $g \in Z_G(\Gamma)$, and we see that $\delta_\gamma(y) = dist(y, \gamma y) = dist(gx, \gamma gx) = dist(gx, g\gamma x) = dist(x, \gamma x) = \delta_\gamma(x)$. That is the easy half of the

Homogeneity conjecture *Let M be a connected, simply connected Riemannian homogeneous manifold and $M \to \Gamma\backslash M$ a Riemannian covering. Then $\Gamma\backslash M$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement on M.*

The first case is implicit in the thesis of Georges Vincent [25, Sect. 10.5]; he noted that the linear transformations $\mathrm{diag}\{R(\theta), \ldots, R(\theta)\}$, $R(\theta) = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$, are of constant displacement on the sphere S^{2n-1}. If Γ is the cyclic group of order k, generated by $\mathrm{diag}\{R(2\pi/k), \ldots, R(2\pi/k)\}$, then it has centralizer $U(n)$ in $SO(2n)$ for $k > 2$, all of $SO(2n)$ for $k \leqq 2$, so its centralizer is transitive on S^{2n-1}, and $\Gamma\backslash S^{2n-1}$ is homogeneous. Vincent did not consider homogeneity, but he referred to such linear transformations as *Clifford translations* ("translation au sens de Clifford") and examined space forms $\Gamma\backslash S^{2n-1}$ where Γ is a cyclic group $\langle \mathrm{diag}\{R(2\pi/k), \ldots, R(2\pi/k)\}\rangle$.

This was extended to a proof of the homogeneity conjecture, first for spherical space forms [28] and then for locally symmetric Riemannian manifolds [29] by the author. The proof in [29] used classification and case by case checking. This was partially improved by Freudenthal [19] and Ozols ([20], [21], [22]), who gave direct proofs for the case where Γ is contained in the identity component of $\mathbf{I}(M, ds^2)$.

Since then a number of special cases of the homogeneity conjecture have been verified. The ones known by the author are the case [30] where M is of non-positive sectional curvature, extended by Druetta [18] to the case where M has no focal points; Cámpoli's work ([7], [8]) on the case where M is a Stieffel manifold and ds^2 is the normal Riemannian metric;

the case [17] where M admits a transitive semisimple group of isometries that has no compact factor; the case [31] where M admits a transitive solvable group of isometries; the case [32] where M has a fibration such as that of Stieffel manifolds over Grassmann manifolds; and the case [34] where every element of Γ is close to the identity and M belongs to a certain class of Riemannian normal homogeneous spaces. Incidently, the homogeneity conjecture is valid for locally symmetric Finsler manifolds as well [9].

There has also been a lot of work on the infinitesimal version of constant displacement isometries. These are the Killing vector fields of constant length. For example, see papers ([3], [4], [5]) of Berestovskii and Nikonorov, and, in the Finsler manifold setting, of Deng and Xu ([10], [11], [12], [13], [14], [15], [16]).

In this chapter we verify the homogeneity conjecture for (i) the case where (M, ds^2) is a normal homogeneous Riemannian manifold of strictly positive curvature and (ii) most cases where (M, ds^2) is a homogeneous Riemannian manifold, not necessarily normal, of strictly positive curvature. The three cases from which we have not yet eliminated the normality requirement are the odd dimensional spheres $M = G/H = SU(m+1)/SU(m), Sp(m+1)/Sp(m), SU(2)/\{1\}$ with possibly non-standard metrics. In effect, this is a progress report.

Added in proof: the author now has eliminated the normality requirement for these three cases, see [33].

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17.2 The Classification for Positive Curvature

The connected, simply connected homogeneous Riemannian manifolds of positive sectional curvature were classified by Marcel Berger [6], Nolan Wallach [26], Simon Aloff and Nolan Wallach [1], and Lionel Bérard-Bergery [2]. Their isometry groups were worked out by Krishnan Shankar [23]. The spaces and the isometry groups are listed in the first two columns of Table 17.1. When there is a fibration that will be relevant to our verification of the homogeneity conjecture, it will also be listed in the first column.

Table 17.1Isometry groups of csc homogeneous spaces of positive curvature and fibrations over symmetric spaces

	$M = G/H$	$\mathbf{I}(M, ds^2)$
(1)	$S^n = SO(n+1)/SO(n)$	$O(n+1)$
(2)	$P^m(\mathbb{C}) = SU(m+1)/U(m)$	$PSU(m+1) \rtimes \mathbb{Z}_2$
(3)	$P^k(\mathbb{H}) = Sp(k+1)/(Sp(k) \times Sp(1))$	$Sp(k+1)/\mathbb{Z}_2)$
(4)	$P^2(\mathbb{O}) = F_4/Spin(9)$	F_4
(5)	$S^6 = G_2/SU(3)$	$O(7)$
(6)	$P^{2m+1}(\mathbb{C}) = Sp(m+1)/(Sp(m) \times U(1))$ $P^{2m+1}(\mathbb{C}) \to P^m(\mathbb{H})$	$(Sp(m+1)/\mathbb{Z}_2) \times \mathbb{Z}_2$
(7)	$F^6 = SU(3)/T^2$ $F^6 \to P^2(\mathbb{C})$	$(PSU(3) \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$
(8)	$F^{12} = Sp(3)/(Sp(1) \times Sp(1) \times Sp(1))$ $F^{12} \to P^2(\mathbb{H})$	$(Sp(3)/\mathbb{Z}_2) \times \mathbb{Z}_2$
(9)	$F^{24} = F_4/Spin(8)$ $F^{24} \to P^2(\mathbb{O})$	F_4
(10)	$M^7 = SO(5)/SO(3)$	$SO(5)$
(11)	$M^{13} = SU(5)/(Sp(2) \times_{\mathbb{Z}_2} U(1)$ $M^{13} \to P^4(\mathbb{C})$	$PSU(5) \rtimes \mathbb{Z}_2$
(12)	$N_{1,1} = (SU(3) \times SO(3))/U^*(2)$	$(PSU(3) \rtimes \mathbb{Z}_2) \times SO(3)$
(13)	$N_{k,\ell} = SU(3)/U(1)_{k,\ell}$ $(k,\ell) \neq (1,1), 3 \mid (k^2 + \ell^2 + k\ell)$ $N_{k,\ell} \to P^2(\mathbb{C})$	$(PSU(3) \rtimes \mathbb{Z}_2) \times (U(1) \rtimes \mathbb{Z}_2)$
(14)	$N_{k,\ell} = SU(3)/U(1)_{k,\ell}$ $(k,\ell) \neq (1,1), 3 \nmid (k^2 + \ell^2 + k\ell)$ $N_{k,\ell} \to P^2(\mathbb{C})$	$U(3) \rtimes \mathbb{Z}_2$
(15)	$S^{2m+1} = SU(m+1)/SU(m)$ $S^{2m+1} \to P^m(\mathbb{C})$	$U(m+1) \rtimes \mathbb{Z}_2$
(16)	$S^{4m+3} = Sp(m+1)/Sp(m)$ $S^{4m+3} \to P^m(\mathbb{H})$	$Sp(m+1) \times_{\mathbb{Z}_2} Sp(1)$
(17)	$S^3 = SU(2)$ $S^3 \to P^1(\mathbb{C}) = S^2$	$O(4)$
(18)	$S^7 = Spin(7)/G_2$	$O(8)$
(19)	$S^{15} = Spin(9)/Spin(7)$ $S^{15} \to S^8$	$Spin(9)$

Most of the embeddings $H \hookrightarrow G$ in Table 17.1 are obvious, but a few might need explanation. For (9), $Spin(8) \hookrightarrow Spin(9) \hookrightarrow F_4$. For (10), $SO(3) \hookrightarrow SO(5)$ is the irreducible representation of highest weight 4λ, where λ is the fundamental highest weight; the tangent space representation is the irreducible representation of highest weight 6λ. For (11), $Sp(2) \hookrightarrow SU(4)$ so $Sp(2) \times U(1)$ maps to $U(5)$ with kernel $\{\pm(I,1)\}$ and image in $SU(5)$. The $\mathbb{Z}_2$ for the non-identity component of $\mathbf{I}(M, ds^2)$ here corresponds to complex conjugation on $SU(5)$. For (12), $U^*(2)$ is the image of $U(2) \hookrightarrow (SU(3) \times SO(3))$, given by $h \mapsto (\alpha(h), \beta(h))$, where $\alpha(h) = \begin{pmatrix} h & 0 \\ 0 & 1/\det(h) \end{pmatrix}$ and β is the projection $U(2) \to U(2)/(center) \cong SO(3)$. For (13) and (14),

$$U(1) = H \hookrightarrow G = SU(3) \text{ is } e^{i\theta} \mapsto \operatorname{diag}\{e^{ik\theta}, e^{i\ell\theta}, e^{-i(k+\ell)\theta}\}.$$

For (19), $Spin(7) \hookrightarrow Spin(8) \hookrightarrow Spin(9)$.

Note that the first four spaces $M = G/H$ of Table 17.1 are Riemannian symmetric spaces with $G = \mathbf{I}(M)^0$. The fifth space is $S^6 = G_2/SU(3)$, where the isotropy group $SU(3)$ is irreducible on the tangent space, so the only invariant metric is the one of constant positive curvature; thus it is isometric to a Riemannian symmetric space. In view of [29] we have

Proposition 17.2.1 *The homogeneity conjecture is valid for the entries* (1) *through* (5) *of* Table 17.1.

17.3 Positive Curvature and Isotropy Splitting

Some of the entries of Table 17.1 are of positive Euler characteristic, i.e. have $\operatorname{rank} H = \operatorname{rank} G$. In those cases every element of $G = \mathbf{I}(M, ds^2)^0$ is conjugate to an element of H, hence has a fixed point on M. Those are the table entries (6), (7), (8) and (9). Each is isotropy-split with fibration over a projective (thus Riemannian symmetric) space, as defined in [32, (1.1)]. The homogeneity conjecture follows for these (M, ds^2) where ds^2 is the normal Riemannian metric [32, Cor. 5.7]:

Proposition 17.3.1 *The homogeneity conjecture is valid for the entries* (6) *through* (9) *of* Table 17.1*, where ds^2 is the normal Riemannian metric on M.*

The argument of Proposition 17.3.1 applies with only obvious changes to a number of other table entries, using [32, Thm. 6.1] instead of [32, Cor. 5.7]. That gives us

Proposition 17.3.2 *The homogeneity conjecture is valid for the entries* (11), (13), (14), (15), (16), (17) and (19) *of* Table 17.1*, where* ds^2 *is the normal Riemannian metric on* M*.*

17.4 The Three Remaining Positive Curvature Cases

In positive curvature it remains only to verify the homogeneity conjecture for table entries $M = G/H$ given by (10) $M^7 = SO(5)/SO(3)$, (12) $N_{1,1} = (SU(3) \times SO(3))/U^*(2)$, and (18) $S^7 = Spin(7)/G_2$. For (10) and (18), H is irreducible on the tangent space of M. In particular for (18) ds^2 must be the constant positive curvature metric on S^7, where the homogeneity conjecture is known:

Lemma 17.4.1 *The homogeneity conjecture is valid for entry* (18) *of* Table 17.1*, where* ds^2 *is the invariant Riemannian metric on* M*.*

Now consider the case $M^7 = G/H = SO(5)/SO(3)$. There H acts irreducibly on the tangent space, so the only invariant metric is a normal one. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ where $\mathfrak{h} \perp \mathfrak{m}$ and $\mathfrak{m}$ represents the tangent space at $x_0 = 1H \in G/H$. If $\eta \in \mathfrak{m}$ then $t \mapsto \exp(t\eta)$ is a geodesic based at x_0.

Suppose that we have an isometry γ of some constant displacement $d > 0$. As $\mathbf{I}(M, ds^2) = SO(5)$ is connected, we have $\xi \in \mathfrak{m}$ such that $\sigma(t) = \exp(t\xi)x_0\,, 0 \leqq t \leqq 1$, is a minimizing geodesic in M from x_0 to $\gamma(x_0)$. Let X denote the Killing vector field on M corresponding to ξ. Note that $||X_{x_0}|| = ||\xi|| = d$. Let $g \in G$ and $y = gx_0 \in M$. Then $t \mapsto g\sigma(t) = g\exp(t\xi)x_0$ is a minimizing geodesic in M from y to $g\gamma(x_0) = \mathrm{Ad}(g)(\gamma)(y)$. Since $\mathrm{Ad}(g)(\gamma)$ has the same constant displacement d as γ we have $||(g_*X)_{x_0}|| = d$. But $||(g_*X)_{x_0}|| = ||X_y||$, in other words $||X_y|| = ||X_{x_0}||$. Thus X is a Killing vector field of constant length on $SO(5)/SO(3)$. There is no such nonzero vector field [34], so γ does not exist. We have proved

Lemma 17.4.2 *There is no isometry* $\neq 1$ *of constant displacement on the manifold* M^7 *with normal Riemannian metric. In particular the homogeneity conjecture is valid for entry* (10) *of* Table 17.1*, where* ds^2 *is the normal Riemannian metric on* M^7*.*

Now consider the case (12) of $N_{1,1} = G/H = (SU(3)\times SO(3))/U^*(2)$. Let γ be an isometry of constant displacement $d > 0$ on $N_{1,1}$. Suppose that γ^2 is also an isometry of constant displacement. The argument of Lemma 17.4.1 shows that γ cannot belong to the identity component $G = SU(3) \times SO(3)$ of $\mathbf{I}(N_{1,1})$. Further, γ^2 belongs to that identity component, so the argument of Lemma 17.4.1 shows that $\gamma^2 = 1$.

Now $\gamma = (g_1, g_2)\nu$, where $g_1 \in SU(3)$, $g_2 \in SO(3)$, $\nu^2 = 1$, $\mathrm{Ad}(\nu)$ is complex conjugation on $SU(3)$, and $\mathrm{Ad}(\nu)$ is the identity on $SO(3)$. The centralizer of ν is $K := ((SO(3) \times SO(3)) \cup (SO(3) \times SO(3))\nu$. Let T_1 (resp. T_2) be a maximal torus of the first (resp. second) $SO(3)$. Following de Siebenthal [24] we may assume $g_i \in T_i$, where we replace γ by a conjugate. Compute $\gamma^2 = (g_1, g_2)(\overline{g_1}, g_2) = (g_1^2, g_2^2)$. We have reduced our considerations to the cases where g_1 is either the identity matrix I_3 or the matrix $I_3' := \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{smallmatrix}\right)$, and also g_2 is either I_3 or I_3'.

Recall that $H = U^*(2)$ is the image of $U(2) \hookrightarrow (SU(3)\times SO(3))$, given by $h \mapsto (\alpha(h), \beta(h))$, where $\alpha(h) = \left(\begin{smallmatrix} h & 0 \\ 0 & 1/\det(h) \end{smallmatrix}\right)$ and β is the projection $U(2) \to U(2)/(center) \cong SO(3)$. Further, $\mathbf{I}(L_{1,1}) = G \cup G\nu$ and its isotropy subgroup is $H \cup H\nu$. Observe that

$$\begin{aligned}&\text{if } (g_1, g_2) = (I_3\,, I_3) \text{ then } \gamma = (\alpha(I_2), \beta(I_2))\nu \in H\nu, \text{ and}\\ &\text{if } (g_1, g_2) = (I_3'\,, I_3) \text{ then } \gamma = (\alpha(-I_2), \beta(-I_2))\nu \in H\nu.\end{aligned}$$

We can replace I_3' by its $SO(3)$ conjugate $I_3'' = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. Set $I_2'' = \left(\begin{smallmatrix} -1 & 0 \\ 0 & +1 \end{smallmatrix}\right)$. Note that

$$\text{if } (g_1, g_2) = (I_3''\,, I_3'') \text{ then } \gamma = (\alpha(I_2''), \beta(I_2''))\nu \in H\nu.$$

When $\gamma \in H\nu$ it cannot be of nonzero constant displacement. We have reduced our considerations to the case $\gamma = (I_3\,, I_3')\nu$, or equivalently to one of its conjugates. Compute

$$\begin{aligned}&\left(\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}, I_3\right) \cdot (I_3\,, I_3')\nu \cdot \left(\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}, I_3\right)^{-1}\\ &= \left(\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}, I_3\right) \cdot (I_3\,, I_3') \cdot \left(\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}, I_3\right)\nu\\ &= (I_3'\,, I_3')\nu \in H\nu,\end{aligned}$$

so again γ cannot be of nonzero constant displacement. We have proved

Lemma 17.4.3 *There is no isometry $\gamma \neq 1$ of constant displacement on the manifold $N_{1,1}$ with normal Riemannian metric, for which γ^2 is also of constant displacement. In particular the homogeneity conjecture is valid for entry* (12) *of* Table 17.1 *with normal Riemannian metric on* $N_{1,1}$.

Combining Lemmas 17.4.1, 17.4.2 and 17.4.3 we have

Proposition 17.4.4 *The homogeneity conjecture is valid for entries* (10), (12) *and* (18) *of* Table 17.1*, where ds^2 is the normal Riemannian metric on M.*

Finally, combine Propositions 17.2.1, 17.3.1, 17.3.2 and 17.4.4 to obtain our main result:

Theorem 17.4.5 *Let (M, ds^2) be a connected, simply connected normal homogeneous Riemannian manifold of strictly positive sectional curvature. Then the homogeneity conjecture is valid for (M, ds^2).*

17.5 Dropping Normality in Positive Curvature

In several cases one can eliminate the normality requirement of Theorem 17.4.5. Of course this is automatic when the adjoint action of H on the tangent space $\mathfrak{g}/\mathfrak{h}$ is irreducible; there, every invariant Riemannian metric on $M = G/H$ is normal, so Theorem 17.4.5 applies. Those are the spaces given by entries (1), (2), (3), (4), (5), (10), (11), (12) and (18) of Table 17.1. Some other cases require an extension of certain results from [32] concerning normal Riemannian homogeneous spaces.

The homogeneity conjecture was verified by Wolf [32] for a class of normal Riemannian homogeneous spaces $M = G/H$ that fiber over homogeneous spaces $M' = G/K$, where H is a local direct factor of K. We are going to weaken the normality conditions in such a way that the results still apply to some of the homogeneous Riemannian manifolds of positive sectional curvature. In [32] the metrics on M and M' were required to be the normal Riemannian metrics defined by the Killing form of G. Instead, we look at Riemannian surjections $\pi : M \to M'$, with fiber F, as follows.

Condition 17.5.1 Let G is a compact, connected, simply connected Lie group, and $H \subset K$ be closed, connected subgroups of G such that

(i) $M = G/H$, $M' = G/K$, and $F = H\backslash K$,
(ii) $\pi : M \to M'$ by $\pi(gH) = gK$, right action of K,
(iii) M' and F are Riemannian symmetric spaces, and
(iv) the tangent spaces $\mathfrak{m}'$ for M', $\mathfrak{m}''$ for F and $(\mathfrak{m}'+\mathfrak{m}'')$ for M satisfy $\mathfrak{m}' \perp \mathfrak{m}''$.

We first modify [32, Lem. 5.2]:

Lemma 17.5.2 *Assume $\pi : M \to M'$ and F satisfy Condition 17.5.1. Then the fiber F of $M \to M'$ is totally geodesic in M In particular it is a geodesic orbit space, and any geodesic of M tangent to F at some point is of the form $t \mapsto \exp(t\xi)x$ with $x \in F$ and $\xi \in \mathfrak{m}''$.*

Proof Consider the restriction $\mathrm{Ad}_{\mathfrak{g}}|_{\mathfrak{h}}$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}' + \mathfrak{m}''$, where $\mathfrak{h}$ acts by its adjoint representation (on itself), by the restriction of the isotropy representation of $\mathfrak{k}$ on $\mathfrak{m}'$, and by its isotropy representation on $\mathfrak{m}''$. If $\xi \in \mathfrak{m}''(\subset \mathfrak{k})$ and $\eta \in \mathfrak{m}'(\subset \mathfrak{g})$ then $[\xi,\eta] \in \mathfrak{m}'$, so $\langle [\xi,\eta]_{\mathfrak{m}'+\mathfrak{m}''}, \xi\rangle = 0$ because $\langle \xi, \mathfrak{m}'\rangle = 0$. If $\xi, \eta \in \mathfrak{m}''$ then $\mathrm{ad}\,(\eta)$ is antisymmetric so $[\xi,\eta] \perp \xi$, i.e. $\langle [\xi,\eta]_{\mathfrak{m}'+\mathfrak{m}''}, \xi\rangle = 0$. We have just shown that if $\xi \in \mathfrak{m}''$ then $t \mapsto \exp(t\xi)H$ is a geodesic in M based at $1H$. Since it is a typical geodesic in the symmetric space $F = H\backslash K$, we conclude that F is totally geodesic in M. □

Now we set up the conditions for applying (17.5.1) in the possible absence of normality.

Lemma 17.5.3 *Suppose that (M, ds^2) is one of the entries of* Table 17.1 *for which $G = \mathbf{I}(M, ds^2)^0$, that (M, ds^2) satisfies Condition 17.5.1, and that $\pi : (M, ds^2) \to (M', ds'^2)$ is a Riemannian submersion. Then the homogeneity conjecture holds for (M, ds^2).*

Proof Running through the last column of Table 17.1 we see something surprising: in each case, $\mathrm{rank}\, G = \mathrm{rank}\, K$, in other words $\chi(M') > 0$. That noted, we use the starting point of the proof of [32, Prop. 5.4]. There one takes $\gamma \in G$ of constant displacement on M to be of the form $(g, r(k))$, with $g \in G$ acting on the left on $M = G/H$ and $r(k)$ given by the right action of the normalizer of H in G. The metric ds'^2 is the usual one in each case because it is G-invariant. The argument of [32, Prop. 5.4] adapts to show that $\Gamma \cap G$ is central in G. Further, the proof of [32, Lem. 5.5] carries through for the intersection of Γ with other components of $\mathbf{I}(M, ds^2)$. Thus, if Γ is a group of isometries of constant displacement on (M, ds^2), then Γ centralizes G. The homogeneity conjecture follows for (M, ds^2). $\square$

The argument of Lemma 17.5.3 does not apply directly to entry (6) of Table 17.1, so we give another argument specific to that case. There $\operatorname{rank} H = \operatorname{rank} G$, so every element of G has a fixed point on M. If $\Gamma \neq \{1\}$ is a subgroup of $\mathbf{I}(M, ds^2)$ acting freely on M then $\Gamma = \{1, g\nu\}$, where $g \in G$ and ν denotes complex conjugation. As $(g\nu)^2 = 1$ we have $g \cdot {}^t g^{-1} = g\overline{g} = \pm I$ in terms of matrices. Then ${}^t g = cg$ for some $c \in \mathbb{C}$ and $g = c^2 g$ so $c = \pm 1$. If $c = 1$ then $g = {}^t g$. In that case g is diagonalized by a real matrix and $g\nu$ has a fixed point. Thus $c = -1$ and we may assume $g = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Thus $g\nu$ centralizes G, and that proves the homogeneity conjecture for case (6) of Table 17.1.

A small modification of the argument of Lemma 17.5.3 applies to entry (19), where $G = \mathbf{I}(M, ds^2)$. We simply note that the proof of [32, Thm. 6.1] proves the homogeneity conjecture for case (19) of Table 17.1.

Now we run through the entries of Table 17.1. As noted above, in cases (1), (2), (3), (4), (5), (10), (11), (12) and (18) the group H is irreducible on the tangent space of M, so the metric is the normal Riemannian metric, and the homogeneity conjecture follows from the normal metric case, Theorem 17.4.5. And we just dealt with entries (6) and (19). Now we run through the other entries.

In table entries (7), (8) and (9), $G = \mathbf{I}(M, ds^2)^0$, and in entries (13) and (14) we may assume $G = U(3) = \mathbf{I}(M, ds^2)^0$. For those entries, $\mathfrak{m}' \perp \mathfrak{m}''$ because the representations of H on those spaces have no common summand, and $\pi : (M, ds^2) \to (M', ds'^2)$ is a Riemannian submersion because (M', ds'^2) is an irreducible Riemannian symmetric space. Thus Lemma 17.5.3 applies to (7), (8), (9), (13) and (14).

Consider the table entries for which M is an odd sphere. Cases (1), (18) and (19) have already been dealt with, leaving (15), (16) and (17). Combining all the results of this section we have

Theorem 17.5.4 *Let (M, ds^2) be a connected, simply connected homogeneous Riemannian manifold of strictly positive curvature, where ds^2 is not required to be the normal Riemannian metric. Suppose that (M, ds^2) is not one of the entries* (15), (16) *or* (17) *of* Table 17.1. *Then the homogeneity conjecture is valid for (M, ds^2).*

Added in proof: the author recently proved the homogeneity conjecture for the entries (15), (16) and (17) of Table 17.1. So those entries need not be excluded in Theorem 17.5.3, see [33]

References

[1] S. Aloff & N. R. Wallach, An infinite family of distinct 7-manifolds admitting positively curved Riemannian structure, Bull. Amer. Math. Soc. **81** (1975), 93–97.

[2] L. Bérard-Bergery, Les variétés riemanniennes homogènes simplement connexes de dimension impaire à courbure strictement positive, J. Math. Pures Appl. **55** (1976), 47–67.

[3] V. N. Berestovskii & Y. G. Nikonorov, Killing vector fields of constant length on locally symmetric Riemannian manifolds. Transform. Group. **13** (2008), 25–45.

[4] V. N. Berestovskii & Y. G. Nikonorov, On Clifford–Wolf homogeneous Riemannian manifolds, Doklady Math. **78** (2008), 807–810.

[5] V. N. Berestovskii & Y. G. Nikonorov, Clifford–Wolf homogeneous Riemannian manifolds. J. Differ. Geom. **82** (2009), 467–500.

[6] M. Berger, Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive, Ann. Scuola Norm. Sup. Pisa **15** (1961), 179–246.

[7] O. Cámpoli, Clifford isometries of compact homogeneous Riemannian manifolds, Revista Unión Math. Argentina **31** (1983), 44–49.

[8] O. Cámpoli, Clifford isometries of the real Stieffel manifolds, Proc. Amer. Math. Soc. **97** (1986), 307–310.

[9] S. Deng & J. A. Wolf, Locally symmetric homogeneous Finsler spaces, Intern. Mat. Res. Not. (IMRN) **2013** (2013), 4223–4242.

[10] S. Deng & M. Xu, Clifford–Wolf homogeneous Randers spaces, J. Lie Theory **23** (2013), 837–845.

[11] S. Deng & M. Xu, Clifford–Wolf homogeneous Randers spheres, Israel J. Math. **199** (2014), 507–525.

[12] S. Deng & M. Xu, Clifford–Wolf translations of left invariant Randers metrics on compact Lie groups, Quart. J. Math. **65** (2014), 133–148.

[13] S. Deng & M. Xu, Clifford–Wolf translations of Finsler spaces, Forum Math. **26** (2014), 1413–1428.

[14] S. Deng & M. Xu, Left invariant Clifford–Wolf homogeneous (α, β)-metrics on compact semisimple Lie groups, Transform. Group. **20** (2015), 395–416.

[15] S. Deng & M. Xu, Clifford–Wolf homogeneous Finsler metrics on spheres, Ann. Mat. Pura Appl. **194** (2015), 759–766.

[16] S. Deng & M. Xu, (α_1, α_2)–metrics and Clifford–Wolf homogeneity. J. Geom. Anal. **26** (2016), 2282–2321.

[17] I. Dotti Miatello, R. J. Miatello & J. A. Wolf, Bounded isometries and homogeneous Riemannian quotient manifolds, Geom. Dedicata **21** (1986), 21–28.

[18] M. J. Druetta, Clifford translations in manifolds without focal points, Geom. Dedicata **14** (1983), 95–103.

[19] H. Freudenthal, Clifford–Wolf-Isometrien symmetrischer Räume, Math. Ann. **150** (1963), 136–149.

[20] V. Ozols, Critical points of the displacement function of an isometry, J. Differ. Geom. **3** (1969), 411–432.

[21] V. Ozols, Critical sets of isometries, Proc. Sympos. Pure Math. **27** (1973), 375–378.

[22] V. Ozols, Clifford translations of symmetric spaces, Proc. Amer. Math. Soc. **44** (1974), 169–175.

[23] K. Shankar, Isometry groups of homogeneous spaces with positive sectional curvature, Differ. Geom. Appl. **14** (2001), 57–78.

[24] J. de Siebenthal, Sur les groupes de Lie compacts non connexes, Comment. Math. Helv. **31** (1956), 41–89.

[25] G. Vincent, Les groupes linéaires finis sans points fixes, Comment. Math. Helv. **20** (1947), 117–171.

[26] N. R. Wallach, Compact homogeneous Riemannian manifolds with strictly positive curvature, Ann. Math. **96** (1972), 277–295.

[27] J. A. Wolf, Sur la classification des variétés riemanniènnes homogènes à courbure constante, C. R. Acad. Sci. Paris, **250** (1960), 3443–3445.

[28] J. A. Wolf, Vincent's conjecture on Clifford translations of the sphere, Comment. Math. Helv. **36** (1961), 33–41.

[29] J. A. Wolf, Locally symmetric homogeneous spaces. Comment. Math. Helv. **37** (1962), 65–101.

[30] J. A. Wolf, Homogeneity and bounded isometries in manifolds of negative curvature, Ill. J. Math. **8** (1964), 14–18.

[31] J. A. Wolf, Bounded isometries and homogeneous quotients, J. Geom. Anal. **26** (2016), 1–9.

[32] J. A. Wolf, Homogeneity for a class of Riemannian quotient manifolds, Differ. Geom. Appl. **56** (2018), 355–372.

[33] J. A. Wolf, Local and Global Homogeneity for Three Obstinate Spheres, preprint, arXiv:2005.09702 [math.DG], 2020.

[34] M. Xu & J. A. Wolf, Killing vector fields of constant length on Riemannian normal homogeneous spaces, Transform. Group. **21** (2016), 871–902.