# UNITARY REPRESENTATIONS, $L^2$ DOLBEAULT COHOMOLOGY, AND WEAKLY SYMMETRIC PSEUDO–RIEMANNIAN NILMANIFOLDS

JOSEPH A. WOLF

*To the memory of Bert Kostant, a good friend and mathematical pioneer*

Abstract. We combine recent developments on weakly symmetric pseudo–riemannian nilmanifolds with with geometric methods for construction of unitary representations on square integrable Dolbeault cohomology spaces. This runs parallel to construction of discrete series representations on spaces of square integrable harmonic forms with values in holomorphic vector bundles over flag domains. Some special cases had been described by Satake in 1971 and the author in 1975. Here we develop a theory of pseudo–riemannian nilmanifolds of complex type. They can be viewed as the nilmanifold versions of flag domains. We construct the associated square integrable (modulo the center) representations on holomorphic cohomology spaces over those domains and note that there are enough such representations for the Plancherel and Fourier Inversion Formulae there. Finally, we note that the most interesting such spaces are weakly symmetric pseudo–riemannian nilmanifolds, so we discuss that theory and give classifications for three basic families of weakly symmetric pseudo–riemannian nilmanifolds of complex type.

## 1. Introduction

This paper records and expands on a surprising observation. It has long been known that the standard tempered representations of semisimple Lie groups — which are enough for the Plancherel and Fourier Inversion Formulae there — can be realized on partially holomorphic cohomology spaces over flag domains. See [14] and [19]. Here we develop a theory of pseudo–riemannian nilmanifolds of complex type, a sort of nilmanifold version of flag domains. We then construct the associated square integrable (modulo the center) representations on holomorphic cohomology spaces over those domains and note that there are enough such representations for the Plancherel and Fourier Inversion Formulae there. Finally, we note that many of the interesting such spaces are weakly symmetric pseudo–riemannian nilmanifolds, so we discuss that theory and give classifications for three basic families of weakly symmetric pseudo–riemannian nilmanifolds of complex type.

The Bott–Borel–Theorem of the 1950's [2] gave complex geometric realizations for representations of compact Lie groups. In the early 1960's Kirillov described the unitary dual for nilpotent Lie groups in terms of coadjoint orbits [6]. Kostant saw the correspondence between those two theories and developed a common generalization, *geometric quantization*. In the framework of geometric quantization, the Bott–Borel–Theorem uses totally complex polarizations and the Kirillov theory uses real polarizations. On the other hand, the infinite dimensional irreducible unitary representations of Heisenberg groups have complex realizations, on spaces of Hermite polynomials.

The extension of Bott–Borel–Weil to noncompact groups, perhaps inspired by Harish-Chandra's holomorphic discrete series, was made plausible when Andreotti and Vesentini [1] initiated the study of square integrable Dolbeault cohomology. The extension to noncompact real semisimple Lie groups, then called the

*Date*: file /texdata/submitted/dolbeault-nilpotent/nil-flag.tex submitted 25 October 2018, some expository clarifications and typos corrected later, final edit 15 September 2019.

2010 *Mathematics Subject Classification.* 22E45, 43A80, 32M15, 53B30, 53B35.

*Key words and phrases.* weakly symmetric space, pseudo–riemannian manifold, homogeneous manifold, Lorentz manifold, trans–Lorentz manifold.

Research partially supported by a Simons Foundation grant.

Langlands Conjecture, was the realization of square integrable (discrete series) representations of semisimple Lie groups on square integrable $\overline{\partial}$ cohomology (and certain variations) for holomorphic hermitian vector bundles $\mathbb{E} \to D$ over flag domains. It was carried out by a number of people; see Harish-Chandra [5], Narasimhan and Okamoto [8], Schmid [10, 11], Wolf [14, 15] and Wong [21, 22].

Here we address the corresponding problem for a class of connected unimodular Lie groups of the form $G = N \rtimes H$ where $N$ is a two-step nilpotent Lie group and $H$ is a closed reductive subgroup of $G$. In the language of geometric quantization, we are looking for representations of $N$ defined by totally complex polarizations and their extension to $G$. The first example is the case where $N$ is the Heisenberg group of dimension $2n+1$ and $H = U(n)$, or more generally $U(p,q)$ with $p+q=n$. There, the Fock representations of $N$ extend to $G$ without Mackey obstruction, for purely geometric reasons [16]. Also in [16], this leads to the Plancherel and Fourier Inversion Formulae for $G = N \rtimes H$ and for certain similar semidirect product groups.

These Heisenberg group examples belong to a much larger family, the weakly symmetric riemannian (and pseudo–riemannian) nilmanifolds, studied in [20]. Many members of that larger family enjoy special properties that combine complex geometry and real analysis. For example they inherit some curvature properties from [12]. To describe them we use the obvious decomposition

$$\mathfrak{n} = \mathfrak{z} + \mathfrak{v} \text{ where } \mathfrak{z} \text{ is the center and } \mathrm{Ad}(H)\mathfrak{v} = \mathfrak{v}. \tag{1.1}$$

Corresponding to (1.1), we will use the following notation on the dual spaces:

$$\mathfrak{z}^* \ni \zeta \rightsquigarrow \lambda_\zeta \in \mathfrak{n}^* \text{ by } \lambda_\zeta|_{\mathfrak{z}} = \zeta \text{ and } \lambda_\zeta|_{\mathfrak{v}} = 0, \quad \text{and} \quad \mathfrak{n}^* \ni \lambda \rightsquigarrow \zeta_\lambda = \lambda|_{\mathfrak{z}} \in \mathfrak{z}^*. \tag{1.2}$$

The basic conditions with which we'll deal are

$$\begin{aligned} &N \text{ has square integrable representations modulo its center,}\\ &\mathfrak{n} \text{ has an } \mathrm{Ad}(H)\text{–invariant symmetric bilinear form } b \text{ for which } \mathfrak{z} \perp \mathfrak{v},\\ &\text{the symmetric bilinear form } b \text{ has nondegenerate restriction } b|_{\mathfrak{v}} \text{ to } \mathfrak{v}, \text{ and}\\ &\mathfrak{v} \text{ has a complex vector space structure } J \text{ with } b(Ju,Jv) = b(u,v) \text{ for } u,v \in \mathfrak{z}. \end{aligned} \tag{1.3}$$

This will allow us to carry out the program

(1.4)

(a) define a pseudo–Kähler structure on $D = \exp(\mathfrak{v}) = N/Z$,

(b) construct holomorphic line bundles $\mathbb{E}_\zeta \to D = N/Z$ for almost every $\zeta \in \mathfrak{z}^*$,

(c) describe the corresponding representations $\pi_{\lambda_\zeta} \in \widehat{N}$ both on $L^2$ Dolbeault cohomology and on spaces of square integrable harmonic $\mathbb{E}_\zeta$–valued differential forms on $V$,

(d) use the underlying holomorphic structure to extend $\pi_{\lambda_\zeta}$ to a linear (not projective) representation of the $H$–stabilizer of $\zeta$, and

(e) use this explicit information for the Plancherel and Fourier Inversion formulae for $G$.

In Section 2 we review the algebraic and analytic structure of the nilpotent Lie groups $N$ that have irreducible square integrable unitary representations. Most weakly symmetric pseudo–riemannian manifolds can be viewed as group manifolds of that sort. These square integrable representations are the nilpotent group analogs of discrete series representations of semisimple Lie groups. We discuss their structure along the lines of [7] and [17]. Those representations are basic to our geometric considerations.

In Section 3 we look at the domains $D = N/Z$ that satisfy (1.3). We'll view them as nilpotent group analogs of flag domains for semisimple ([13], [4]) Lie groups. We consider the circumstances under which we have invariant almost complex structures and pseudo–Kähler structures on $N/Z$. Those almost complex structures have constant coefficients in the coordinates of $\mathfrak{n}/\mathfrak{z}$, so obviously they are integrable, and the pseudo–kähler structure comes out of geometric quantization theory. The main point here is the construction of square integrable Dolbeault cohomology spaces for homogeneous holomorphic vector bundles over the domains $D$. We follow the flag domain idea for $N/Z$ and associate a $N$–homogeneous hermitian holomorphic line bundle $\mathbb{E}_\zeta$ to each "nonsingular" $\zeta \in \mathfrak{z}^*$. We realize the associated representation $\pi_{\lambda_\zeta}$ as the natural action of $N$ on a square integrable Dolbeault cohomology space $H_2^{0,\ell}(D;\mathbb{E}_\zeta)$ where $\ell$ is the number of negative eigenvalues of a certain hermitian form defined by $\zeta$.

In Section 4 we extend the constructions of Section 3 to semidirect product groups $G = N \rtimes H$. The model (which we discuss later) is the case where $G/H$ is a weakly symmetric pseudo–riemannian manifold of complex type. We are especially interested in case of the the group $H$ of all automorphisms of $N$ that preserve our pseudo–Kähler structure on the domain $D = N/Z = G/HZ$, where the $\mathbb{E}_\zeta \to D$ are $G$–homogeneous. Those cases occur quite often in the setting of weakly symmetric pseudo–riemannian manifolds. All the ingredients, in the geometric construction of $\pi_{\lambda_\zeta}$ and $H_2^{0,\ell}(D;\mathbb{E}_\zeta)$, are invariant under the $H$–stabilizer $H_\zeta$ of $\zeta$, so $\pi_{\lambda_\zeta}$ extends naturally from $N$ to a representation $\pi'_{\lambda_\zeta}$ of $N \rtimes H_\zeta$. A key point here is that the geometry lets us bypass the problem of the Mackey obstruction. Then of course we have the induced representations $\pi_{\tau,\zeta} := \mathrm{Ind}\,^G_{NH_\zeta}(\tau\widehat{\otimes}\pi'_{\lambda_\zeta})$, $\tau \in \widehat{H_\zeta}$. Since the $\pi_{\lambda_\zeta}$ support the Plancherel measure of $N$, the Mackey little–group method shows that the $\pi_{\tau,\zeta}$ support the Plancherel measure of $G$.

In Section 4 we also indicate the realization of the $\pi_{\tau,\zeta}$ both on square integrable partially holomorphic cohomology spaces and on spaces of square integrable partially harmonic bundle–valued spinors.

The geometric construction (4.9) of the $\pi_{\tau,\zeta}$ is parallel to that of the standard tempered representations of real reductive Lie groups ([14], or see [19]). The domain $D$ corresponds to a flag domain, $\pi_{\lambda_\zeta}$ corresponds to a relative discrete series representation of the Levi component of a parabolic subgroup, and the construction $\pi_{\tau,\zeta} = \mathrm{Ind}\,^G_{G_\zeta}(\tau\widehat{\otimes}\pi'_{\lambda_\zeta})$ corresponds to $L^2$ parabolic induction. In both settings one can use partially harmonic square integrable bundle–valued forms as in [15], instead of square integrable Dolbeault cohomology.

Finally, in Sections 5 and 6, we extract examples from the theory of weakly symmetric pseudo–riemannian nilmanifolds, listing the holomorphic cases from [20] and recording the signatures of invariant pseudo–Kählerian metrics. There, as in the semisimple setting, the Dolbeault cohomology degree is the number of negative eigenvalues of the invariant pseudo–Kählerian metric.

In Section 5 we review the notion of real form family $\{\{G_r/H_r\}\}$ of pseudo–riemannian weakly symmetric nilmanifolds associated to a riemannian weakly symmetric nilmanifold $G_r/H_r$. That is considerably more delicate than the semisimple case [3]. We introduce the notion of "complex type" for pseudo–riemannian weakly symmetric nilmanifolds. The pseudo–riemannian weakly symmetric nilmanifolds of complex type satisfy the Satake conditions (3.2) and (3.4), so the program (1.4) goes through for them. Table 5.3 lists the pseudo–riemannian weakly symmetric nilmanifolds of complex type for which $N$ is a Heisenberg group. It also shows that we need a maximality condition in order to have a usable listing for more general $N$, and Table 5.4 lists the maximal pseudo–riemannian weakly symmetric nilmanifolds of complex type for which $H$ acts irreducibly on $\mathfrak{n}/\mathfrak{z}$.

In Section 6 we consider the complete classification of maximal pseudo–riemannian weakly symmetric nilmanifolds of complex type. That is necessarily combinatorial and based on a further listing of indecomposable maximal pseudo–riemannian weakly symmetric spaces for which $H$ acts reducibly on $\mathfrak{n}/\mathfrak{z}$ and satisfies some technical conditions. That is carried out in Table 6.2.

In order to reduce clutter in the notation we will denote

$$\pi_\zeta := \pi_{\lambda_\zeta} \text{ for all } \zeta \in \mathfrak{z}^*. \tag{1.5}$$

This notation will be justified by Theorems 2.4 and 2.5 below.

## 2. Square Integrable Representations

In this Section we collect some information on square integrable representations for nilpotent Lie groups. The references are [7] and [17], with a summary in [18, Section 2]. These are the representations and nilpotent groups to which our results apply. The groups considered in this note are all of Type I with a countable basis for open sets, so there are no measure–theoretic complications.

First, if $B$ is a unimodular Lie group with center $Z$ and $\pi \in \widehat{B}$ we have the **central character** $\chi_\pi \in \widehat{Z}$ defined by $\pi(z) = \chi_\pi(x) \cdot 1$ for $z \in Z$. Given $u$ and $v$ in the representation space $\mathcal{H}_\pi$ we have the **matrix**

**coefficient** or **coefficient function** $f_{u,v} : x \mapsto \langle u, \pi(x)v\rangle$, and $|f_{u,v}|$ is a well defined function on $B/Z$. Fix Haar measures $\mu_B$ on $B$, $\mu_Z$ on $Z$ and $\mu_{B/Z}$ on $B/Z$ such that $d\mu_B = d\mu_Z\, d\mu_{B/Z}$ . Then these conditions are equivalent:

$$\begin{aligned} &(1) \text{ There exist nonzero } u, v \in \mathcal{H}_\pi \text{ with } |f_{u,v}| \in L^2(B/Z). \\ &(2) \ |f_{u,v}| \in L^2(B/Z) \text{ for all } u, v \in \mathcal{H}_\pi. \\ &(3) \ \pi \text{ is a discrete summand of the representation } \mathrm{Ind}\,_Z^B(\chi_\pi). \end{aligned} \tag{2.1}$$

When those conditions are satisfied for $\pi \in \widehat{B}$ we say that $\pi$ is **square integrable** (modulo $Z$). Then there is a number $\deg \pi > 0$, called the **formal degree** of $\pi$, such that

$$\int_{G/Z} f_{u,v}(x)\overline{f_{u',v'}(x)}d\mu_{G/Z}(xZ) = \tfrac{1}{\deg \pi}\langle u, u'\rangle\overline{\langle v, v'\rangle} \tag{2.2}$$

for all $u, u', v, v' \in \mathcal{H}_\pi$ . If $\pi_1, \pi_2 \in \widehat{G}$ are inequivalent and satisfy (2.1), and if $\chi_{\pi_1} = \chi_{\pi_2}$, then

$$\int_{G/Z} \langle u, \pi_1(x)v\rangle\overline{\langle u', \pi_2(x)v'\rangle}d\mu_{G/Z}(xZ) = 0 \tag{2.3}$$

for all $u, v \in \mathcal{H}_{\pi_1}$ and all $u', v' \in \mathcal{H}_{\pi_2}$ .

The main results of [7] shows exactly how this works for nilpotent Lie groups:

**Theorem 2.4.** *Let $N$ be a connected simply connected nilpotent Lie group with center $Z$, $\mathfrak{n}$ and $\mathfrak{z}$ their Lie algebras, and $\mathfrak{n}^*$ the linear dual space of $\mathfrak{n}$. Let $\lambda \in \mathfrak{n}^*$ and let $\pi_\lambda$ denote the irreducible unitary representation attached to the coadjoint orbit $\mathrm{Ad}^*(N)\lambda$ by the Kirillov theory* [6]*. Then the following conditions are equivalent.*

(1) *$\pi_\lambda$ satisfies the conditions of* (2.1).

(2) *The coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \{\nu \in \mathfrak{n}^* \mid \nu|_\mathfrak{z} = \lambda|_\mathfrak{z}\}$.*

(3) *The bilinear form $b_\lambda(x, y) = \lambda([x, y])$ on $\mathfrak{n}/\mathfrak{z}$ is nondegenerate.*

*The Pfaffian $\mathrm{Pf}(b_\lambda)$ is a polynomial function $P(\lambda|_\mathfrak{z})$ on $\mathfrak{z}^*$. The set of equivalence classes, of representations $\pi_\lambda$ for which these conditions hold, is parameterized by the set $\{\zeta \in \mathfrak{z}^* \mid P(\zeta) \neq 0\}$ (which is empty or Zariski open in $\mathfrak{z}^*$).*

We will say that the connected simply connected nilpotent Lie group $N$ is **square integrable** if it has a square integrable irreducible unitary representation, in other words if there exists $\lambda \in \mathfrak{n}^*$ such that $P(\lambda|_\mathfrak{z}) \neq 0$. We remark that the group $N$ is square integrable if and only if the universal enveloping algebra $\mathcal{U}(\mathfrak{z})$ is the center of $\mathcal{U}(\mathfrak{n})$ [7].

Recall that if $\zeta \in \mathfrak{z}^*$ then (1.5) $\pi_\zeta$ denotes the $\pi_\lambda$ for which $\lambda|_\mathfrak{v} = 0$ and $\lambda|_\mathfrak{z} = \zeta$.

**Theorem 2.5.** *Let $N$ be a square integrable connected simply connected nilpotent Lie group with center $Z$. Then Plancherel measure on $\widehat{N}$ is concentrated on $\{\pi_\zeta \mid P(\zeta) \neq 0\}$, where it is a positive multiple of the absolutely continuous measure $|P(\zeta)|d\zeta$. The formal degree $\deg \pi_\zeta = |P(\zeta)|$.*

Given $\zeta \in \mathfrak{z}^*$ with $P(\zeta) \neq 0$ and a Schwartz class function $f \in \mathcal{S}(N)$, $\mathcal{O}(\zeta)$ denotes the co-adjoint orbit $\mathrm{Ad}^*(N)(\lambda_\zeta) = \zeta + \mathfrak{z}^\perp$ , $f_\zeta = (f \cdot \exp)|_{\mathcal{O}(\zeta)}$ and $\widehat{f_\zeta}$ is the Fourier transform of $f_\zeta$ on $\mathcal{O}(\zeta)$.

**Theorem 2.6.** *Let $N$ be a square integrable connected simply connected nilpotent Lie group with center $Z$. Let $f \in \mathcal{S}(N)$. If $\zeta \in \mathfrak{z}^*$ with $P(\zeta) \neq 0$ then the distribution character of $\pi_\zeta$ is given by*

$$\Theta_{\pi_\zeta}(f) = \mathrm{trace} \int_N f(x)\pi_\zeta(x)d\mu_G(x) = c^{-1}|P(\zeta)|^{-1}\int_{\nu\in\mathcal{O}(\zeta)} \widehat{f_\zeta}\, d\nu \tag{2.7}$$

*where $c = d!2^d$ and $d = \dim(\mathfrak{n}/\mathfrak{z})/2$ , and $d\nu$ is ordinary Lebesgue measure on the affine space $\mathcal{O}(\zeta)$ . The Fourier Inversion formula for $N$ is*

$$f(x) = c \int_{\mathfrak{z}^*} \Theta_{\pi_\zeta}(r_x f)|P(\zeta)|\, d\zeta \text{ where } (r_x f)(y) = f(yx) \text{ (right translate).} \tag{2.8}$$

## 3. Holomorphic Line Bundles over Domains $N/Z$

Let $N$ be a connected, simply connected, nilpotent Lie group. Let $Z$ denote the center of $N$. Their Lie algebras satisfy $\mathfrak{n} = \mathfrak{z} + \mathfrak{v}$ where $\mathfrak{v}$ is a vector space complement to $\mathfrak{z}$ in $\mathfrak{n}$. Let $H$ be a reductive group of automorphisms on $N$. Consider the semidirect product

$$G = N \rtimes H, \text{ so } \mathfrak{g} = \mathfrak{z} + \mathfrak{v} + \mathfrak{h}. \tag{3.1}$$

Then automatically $\mathrm{Ad}(H)\mathfrak{z} = \mathfrak{z}$. We make our choice of $\mathfrak{v}$ so that $\mathrm{Ad}(H)\mathfrak{v} = \mathfrak{v}$. Later we will impose further conditions on these Lie algebras. For the moment we only require Satake's conditions.

First, we assume that $\mathfrak{v}$ has an $\mathrm{Ad}(H)$–invariant complex structure $J$ whose $(\pm i)$ eigenspaces $\mathfrak{v}_\pm$ satisfy

$$[\mathfrak{v}_+, \mathfrak{v}_+] \subset \mathfrak{v}_+ + \mathfrak{z}_\mathbb{C} \text{ and } [\mathfrak{v}_-, \mathfrak{v}_-] \subset \mathfrak{v}_- + \mathfrak{z}_\mathbb{C} \tag{3.2}$$

Then $[\mathfrak{h}_\mathbb{C}, \mathfrak{v}_\pm] \subset \mathfrak{v}_\pm$ , so each $\mathfrak{h}_\mathbb{C} + \mathfrak{v}_\pm + \mathfrak{z}_\mathbb{C}$ is a subalgebra of $\mathfrak{g}_\mathbb{C}$, very much like a parabolic, with nilradical $\mathfrak{v}_\pm + \mathfrak{z}_\mathbb{C}$ and Levi component $\mathfrak{h}_\mathbb{C}$ .

On the group level we suppose that $G$ is contained in its complexification $G_\mathbb{C}$, so $N_\mathbb{C} = Z_\mathbb{C} V_\mathbb{C}$ where $V = \exp(\mathfrak{v})$, and $G_\mathbb{C} = N_\mathbb{C} \rtimes H_\mathbb{C}$ . Let $V_\pm = \exp(\mathfrak{v}_\pm)$, so each $Z_\mathbb{C} V_\pm$ is a closed complex analytic subgroup of $N_\mathbb{C}$ . For convenience we denote

$$N_\pm = Z_\mathbb{C} V_\pm \text{ and } \mathfrak{n}_\pm = \mathfrak{z}_\mathbb{C} + \mathfrak{v}_\pm\, . \tag{3.3}$$

**Lemma 3.4.** $D = G/HZ = G_\mathbb{C}/H_\mathbb{C} N_- \cong N_\mathbb{C}/N_-$ .

*Proof.* From (3.3) we have $G \cap H_\mathbb{C} N_\pm = HZ$ and $H_\mathbb{C} \cap N_\pm = \{1\}$. By dimension, $GH_\mathbb{C} N_\pm$ is open in $G_\mathbb{C}$ and $G/HZ \simeq GH_\mathbb{C} N_\pm / H_\mathbb{C} N_\pm$ is open in $G_\mathbb{C}/H_\mathbb{C} N_\pm$ . Thus $D = G/HZ$ is an open $G$–orbit in the complex homogeneous space $G_\mathbb{C}/H_\mathbb{C} N_- \cong N_\mathbb{C}/N_-$ . But $G/HZ$ is an $N$–orbit, and orbits of unipotent groups on affine manifolds are closed. Thus $D = G/HZ$ is a closed $G$–orbit $G_\mathbb{C}/H_\mathbb{C} N_- \cong N_\mathbb{C}/N_-$ . The Lemma follows. □

We will work with $D$ in a way suggested by realization of discrete series of semisimple Lie groups representations over flag domains ([8], [10, 11], [14, 15]).

**Lemma 3.5.** *Let $\zeta \in \mathfrak{z}^*$. Consider the symmetric bilinear form $\beta_\zeta$ on $\mathfrak{v}$ given by $\beta_\zeta(u,v) = \lambda_\zeta([u, Jv])$. Then $\beta_\zeta$ is nondegenerate if and only if $\lambda_\zeta$ satisfies the conditions of* Theorem 2.4*.*

*Proof.* In the notation of Theorem 2.4, $\beta_\zeta$ is nondegenerate if and only if $b_{\lambda_\zeta}$ is nondegenerate. □

Now let $\zeta \in \mathfrak{z}^*$ with $\beta_\zeta$ nondegenerate. Then

$$\chi_\zeta := \exp(i\lambda_\zeta|_{N_-}) \text{ is a holomorphic character on } N_-\, . \tag{3.6}$$

Using Lemma 3.4, $\chi_\zeta$ defines

$$\begin{aligned} &\mathbb{E}_\zeta \to D = G/HZ = G_\mathbb{C}/H_\mathbb{C} N_- \cong N_\mathbb{C}/N_- \ : \\ &\quad G_\mathbb{C}\text{–homogeneous holomorphic line bundle associated to } \chi_\zeta\, . \end{aligned} \tag{3.7}$$

Then we have

$$C_c^{p,q}(D; \mathbb{E}_\zeta) : \text{ compactly supported } \mathbb{E}_\zeta\text{–valued } (p,q) \text{ forms on } D \tag{3.8}$$

Choose an $N$–invariant positive definite hermitian inner product $\gamma$ on (the fibers of ) $\mathbb{E}_\zeta$ . Then we have the usual Hodge–Kodaira orthocomplementation operator $\sharp$ sending $\mathbb{E}_\zeta$–valued $(p,q)$ forms to $\mathbb{E}_\zeta^*$–valued

$(n-p,n-q)$–forms, $n=\dim_{\mathbb{C}} D$, and the formal adjoint $\overline{\partial}^* = -\sharp\partial\sharp$ of $\overline{\partial} : C_c^{p,q}(D;\mathbb{E}_\zeta) \to C_c^{p,q+1}(D;\mathbb{E}_\zeta)$. That gives us the Hodge–Kodaira–Laplace operator

$$\square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}. \tag{3.9}$$

The closure and the adjoint of $\square$ are equal, giving a self–adjoint extension (which is also denoted $\square$) to

$$L_2^{p,q}(D;\mathbb{E}_\zeta) : \text{ square integrable } \mathbb{E}_\zeta\text{–valued } (p,q) \text{ forms on } D. \tag{3.10}$$

That defines the space of square integrable harmonic $\mathbb{E}_\zeta$–valued $(p,q)$ forms on $D$:

$$H_2^{p,q}(D;\mathbb{E}_\zeta) = \{\omega \in L_2^{p,q}(D;\mathbb{E}_\zeta) \mid \square(\omega) = 0\}. \tag{3.11}$$

By elliptic regularity of $\square$, $H_2^{p,q}(D;\mathbb{E}_\zeta)$ consists of $C^\infty$ forms, in fact $C^\infty$ Schwartz class forms, There the domain of $\square$ is all of $\mathcal{S}^{p,q}(D;\mathbb{E}_\zeta)$, and $H_2^{p,q}(D;\mathbb{E}_\zeta) \subset \mathcal{S}^{p,q}(D;\mathbb{E}_\zeta)$. As defined, $H_2^{p,q}(D;\mathbb{E}_\zeta)$ depends on the choice of positive definite hermitian inner product $\gamma$, but we can avoid that issue using the orthogonal decomposition of Andreotti and Vesentini [1]:

$$L_2^{p,q}(D;\mathbb{E}_\zeta) = c\ell\ \overline{\partial}^* L_2^{p,q+1}(D;\mathbb{E}_\zeta) + c\ell\ \overline{\partial} L_2^{p,q-1}(D;\mathbb{E}_\zeta) + H_2^{p,q}(D;\mathbb{E}_\zeta) \tag{3.12}$$

where $c\ell$ denotes $L^2$ closure. Thus we may (and do) identify $H_2^{p,q}(D;\mathbb{E}_\zeta)$ as a Hilbert space completion of square integrable Dolbeault cohomology based on smooth Schwartz class forms,

$$H_2^{p,q}(D;\mathbb{E}_\zeta) \cong \text{Kernel}\left(\overline{\partial} : \mathcal{S}^{p,q}(D;\mathbb{E}_\zeta) \to \mathcal{S}^{p,q+1}(D;\mathbb{E}_\zeta)\right)/\text{Image}\left(\overline{\partial} : \mathcal{S}^{p,q-1}(D;\mathbb{E}_\zeta) \to \mathcal{S}^{p,q}(D;\mathbb{E}_\zeta)\right).$$

The theorem of Satake [9, Proposition 1]and Okamoto (unpublished), in this setting, can be reformulated as follows. Consider the hermitian form

$$\gamma_\zeta(u,v) = \lambda_\zeta([u,Jv]) + i\,\lambda_\zeta([u,v]) \text{ where } u,v \in \mathfrak{v}. \tag{3.13}$$

Here $\lambda_\zeta([u,Jv]) = \beta_\zeta(u,v)$ is real symmetric on $\mathfrak{v}$ of some signature $(2k,2\ell)$ and $\lambda_\zeta([u,v])$ is antisymmetric, so $\gamma_\zeta(u,v)$ is (complex) hermitian on $(\mathfrak{v},J)$ of corresponding signature $(k,\ell)$. Thus $k$ is the dimension of any maximal positive definite subspace of $(\mathfrak{v},J)$ and $\ell$ is the dimension of any maximal negative definite subspace. Note that $\beta_\zeta$ is nondegenerate if and only if $P(\zeta) \neq 0$.

**Proposition 3.14.** *Let $\zeta \in \mathfrak{z}^*$ such that the hermitian form $\gamma_\zeta(u,v)$ on $\mathfrak{v}$ is nondegenerate. Let $n = \dim_{\mathbb{C}}(\mathfrak{v},J)$ and let $(k,\ell)$ for the signature of $\gamma_\zeta$ on $(\mathfrak{v},J)$. Then $H_2^{0,q}(D;\mathbb{E}_\zeta) = 0$ for $q \neq \ell$ and the natural action of $N$ on $H_2^{0,\ell}(D;\mathbb{E}_\zeta)$ is the irreducible unitary representation $\pi_\zeta$ with central character $\chi_\zeta$ .*

In view of Lemma 3.5 and the decomposition (3.12), Proposition 3.14 shows that the square integrable cohomology representation of $N$ corresponding to $\zeta$ is independent (up to unitary equivalence) of the choice of $J$. The cohomology degree, however, will depend on choice of $J$, as seen in [16]. Further, since $N$ satisfies the conditions of Theorem 2.4, the Plancherel measure and the Plancherel Formula and Fourier Inversion theorems for $N$ are given by Theorems 2.5 and 2.6.

## 4. Extension to the Semidirect Product Group

In this section we extend the results of Section 3 from the nilpotent group $N$ to the semidirect product group $G = N \rtimes H$.

**Lemma 4.1.** *Let $\zeta \in \mathfrak{z}^*$ with $P(\zeta) \neq 0$. Define $H_\zeta = \{h \in H \mid \mathrm{Ad}^*(h)\zeta = \zeta\}$ and $G_\zeta = NH_\zeta$ . Then $G_\zeta$ is the subgroup of $G$ for which $\pi_\zeta \cdot \mathrm{Ad}(g)$ is equivalent to $\pi_\zeta$. In other words, $G_\zeta$ is the Mackey little–group in $G$ for $\pi_\zeta$ .*

*Proof.* Let $h \in H$. If $\mathrm{Ad}^*(h)\zeta = \zeta$ then $\pi_{\lambda_\zeta} \cdot \mathrm{Ad}(h)$ is equivalent to $\pi_{\lambda_\zeta}$ by Kirillov theory, in other words $\pi_\zeta \cdot \mathrm{Ad}(h)$ is equivalent to $\pi_\zeta$. If $\pi_\zeta \cdot \mathrm{Ad}(h)$ is equivalent to $\pi_\zeta$ then $\mathrm{Ad}^*(N)(\lambda_\zeta \cdot \mathrm{Ad}(h)) = \mathrm{Ad}^*(N)(\lambda_\zeta)$. As $H$ is reductive $\mathrm{Ad}^*(h)$ preserves $\mathfrak{z}^*$ and its complement $\mathfrak{v}^*$, so it preserves $\mathrm{Ad}^*(N)(\lambda_\zeta) \cap \mathfrak{z}^* = \{\zeta\}$. Now $H_\zeta$ is the $H$–stabilizer of $\pi_\zeta$, so $G_\zeta = NH_\zeta$ is the $G$–stabilizer. $\square$

Since $H_\zeta$ preserves every ingredient in the construction of $\pi_\zeta$ given by Proposition 3.14 we now have

**Proposition 4.2.** *$\pi_\zeta$ extends to a unitary representation $\pi'_\zeta$ of $G_\zeta$ on the representation space $\mathcal{H}_\zeta$ of $\pi_\zeta$.*

In view of Theorem 2.5, the Mackey little–group method gives us

**Proposition 4.3.** *Plancherel measure on $\widehat{G}$ is concentrated on the representations*

$$\pi_{\tau,\zeta} = \mathrm{Ind}_{G_\zeta}^{G}(\tau\widehat{\otimes}\pi'_\zeta) \text{ where } \zeta \in \mathfrak{z}^* \text{ with } P(\zeta) \neq 0 \text{ and where } \tau \in \widehat{H_\zeta}\,.$$

Now we extend Proposition 3.14 from $N$ to $G$. Let $\zeta \in \mathfrak{z}^*$ with $P(\zeta) \neq 0$. Let $\tau \in \widehat{H_\zeta}$ with representation space $E_\tau$, and let $\mathbb{E}_\tau \to D$ denote the corresponding $G_\zeta$–homogeneous holomorphic vector bundle. Similarly let $E_\zeta$ denote the complex line that is the representation space of $\chi_\zeta$; it led to our $G_\zeta$–homogeneous holomorphic line bundle $\mathbb{E}_\zeta \to D$. Recall the notation of Proposition 3.14. Then $E_\tau\widehat{\otimes}H_2^{0,\ell}(D;\mathbb{E}_\zeta)$ is the representation space of $\tau\widehat{\otimes}\pi'_\zeta$. Denote

$$(\mathbb{H}_{\tau,\zeta} = \mathbb{H}_\tau \otimes \mathbb{E}_\zeta) \to (D = G_\zeta/H_\zeta Z): \text{ associated vector bundle with fiber } E_\tau \otimes E_\zeta\,. \tag{4.4}$$

Express $D = G_\zeta/H_\zeta Z$. The isotropy $H_\zeta Z$ preserves the infinitesimal right action of the antiholomorphic tangent space $\mathfrak{v}_-$ of $D$, so $\mathfrak{v}_-$ acts on the right on smooth local sections of $\mathbb{H}_{\tau,\zeta} \to D$. In other words we have a well defined $\overline{\partial}$–operator on smooth local sections of $\mathbb{H}_{\tau,\zeta} \to D$. Thus

**Lemma 4.5.** *$\mathbb{H}_{\tau,\zeta} \to D$ is a hermitian $G_\zeta$–homogeneous holomorphic vector bundle with $\overline{\partial}$–operator given by the right action of $\mathfrak{v}_-$.*

Now we have the Hodge–Kodaira–Laplace operator $\square$ as in (3.9). As in that case, where $\tau$ is the trivial representation, $\square$ acts on the dense subspace $C_c^{p,q}(D;\mathbb{H}_{\tau,\zeta})$ of $L_2^{p,q}(D;\mathbb{H}_{\tau,\zeta})$–valued smooth $(p,q)$–forms on $D$. Note that its action only affects the $\mathbb{E}_\zeta$ component of the values of local sections. Thus, as before, the closure and adjoint of $\square$ are equal, so $\square$ is essentially self adjoint, and we have its kernel

$$H_2^{p,q}(D;\mathbb{H}_{\tau,\zeta}) = \{\omega \in L_2^{p,q}(D;\mathbb{H}_{\tau,\zeta}) \mid \square(\omega) = 0\}, \tag{4.6}$$

the space of $\mathbb{H}_{\tau,\zeta}$–valued square integrable harmonic $(p,q)$–forms on $D$. Applying Proposition 3.14 we have

**Proposition 4.7.** *Let $\zeta \in \mathfrak{z}^*$ such that $\gamma_\zeta$* (from 3.13) *is nondegenerate with signature $(k,\ell)$ on $\mathfrak{v}$. Then $H_2^{0,q}(D;\mathbb{H}_{\tau,\zeta}) = 0$ for $q \neq \ell$, and the natural action of $G_\zeta = H_\zeta N$ on $H_2^{0,\ell}(D;\mathbb{H}_{\tau,\zeta})$ is the unitary representation $\tau\widehat{\otimes}\pi'_\zeta$.*

Again we can apply [1] to see

$$L_2^{p,q}(D;\mathbb{H}_{\tau,\zeta}) = c\ell\; \overline{\partial}^* L_2^{p,q+1}(D;\mathbb{H}_{\tau,\zeta}) + c\ell\; \overline{\partial} L_2^{p,q-1}(D;\mathbb{H}_{\tau,\zeta}) + H_2^{p,q}(D;\mathbb{H}_{\tau,\zeta}) \tag{4.8}$$

where $c\ell$ denotes $L^2$ closure. Making use of elliptic regularity of $\square$ we identify $H_2^{p,q}(D;\mathbb{H}_{\tau,\zeta})$ as a Hilbert space of square integrable Dolbeault cohomology based on Schwartz class forms,

$$H_2^{p,q}(D;\mathbb{H}_{\tau,\zeta}) \cong \mathrm{Kernel}\,\big(\overline{\partial} : \mathcal{S}^{p,q}(D;\mathbb{H}_{\tau,\zeta}) \to \mathcal{S}^{p,q+1}(D;\mathbb{H}_{\tau,\lambda})\big)/\mathrm{Image}\,\big(\overline{\partial} : \mathcal{S}^{p,q-1}(D;\mathbb{H}_{\tau,\zeta}) \to \mathcal{S}^{p,q}(D;\mathbb{H}_{\tau,\zeta})\big).$$

Thus $H_2^{p,q}(D;\mathbb{H}_{\tau,\zeta})$ is a complete locally convex topological vector space and is independent of choice of the hermitian inner product on $\mathbb{E}_\zeta$ used to define $\square$ on $\mathbb{H}_{\tau,\lambda}$.

Now let us return to the representations $\pi_{\tau,\zeta} = \mathrm{Ind}_{G_\zeta}^{G}(\tau\widehat{\otimes}\pi'_\zeta) \in \widehat{G}$ of Proposition 4.3. The representation space of $\pi_{\tau,\zeta}$ is

$$\begin{aligned}&\mathcal{H}_{\pi_{\tau,\zeta}} = \{f : G \to H_2^{0,\ell}(D;\mathbb{H}_{\tau,\zeta}) \mid f(gx) = (\tau\widehat{\otimes}\pi'_\zeta)(x)^{-1}f(g) \text{ for } x \in G_\zeta\\ &\text{with inner product given by } ||f||^2 = \int_{G/G_\zeta} ||f(gG_\zeta)||^2 d(gG_\zeta) < \infty.\end{aligned} \tag{4.9}$$

The extension of Theorem 2.5 to $G$ is

**Theorem 4.10.** *Let $N$ be a square integrable connected simply connected nilpotent Lie group with center $Z$. Let $H$ be a reductive group of automorphisms of $N$ that preserves a nondegenerate symmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$. Let $G = N \rtimes H$ and suppose that* (3.2) *and* (3.4) *hold. Then Plancherel measure on $\widehat{G}$ is concentrated on $\{\pi_{\tau,\zeta} \mid \zeta \in \mathfrak{z}^*, P(\zeta) \neq 0, \text{ and } \tau \in \widehat{H_\zeta}\}$.*

The construction (4.9) of the $\pi_{\tau,\zeta}$ is analogous to that of the standard tempered representations of real reductive Lie groups ([14], [19]). For that, the domain $D$ corresponds to a flag domain, $\pi_\zeta$ corresponds to a relative discrete series representation of the Levi component of a parabolic subgroup, and the construction $\pi_{\tau,\zeta} = \mathrm{Ind}_{G_\zeta}^G(\tau \widehat{\otimes} \pi'_\zeta)$ corresponds to $L^2$ parabolic induction. In both settings, the geometric realizations can occur both on spaces of partially harmonic square integrable bundle–valued spinors and on square integrable partially holomorphic cohomology spaces.

## 5. Weakly Symmetric Pseudo–Riemannian Nilmanifolds

The theory of weakly symmetric pseudo–riemannian nilmanifolds provides many interesting examples of the spaces $G/H$, $G = N \rtimes H$, studied in Sections 3 and 4 above. We list the more accessible examples in Sections 5 and 6. Here we start by sketching some elements of the theory. In the next section we also discuss the classification.

Recall that a riemannian manifold $(S, ds^2)$ is **symmetric** if, given $x \in S$, there is an isometry $s_x$ of $(S, ds^2)$ such that $s_x(x) = x$ and $ds_x(\xi) = -\xi$ for every tangent vector $\xi \in T_x(S)$. It is **weakly symmetric** if, given $x \in S$ and $\xi \in T_x(S)$, there is an isometry $s_{x,\xi}$ of $(S, ds^2)$ such that $s_{x,\xi}(x) = x$ and $ds_{x,\xi}(\xi) = -\xi$. The obvious difference is that $s_{x,\xi}$ depends on $\xi$ as well as $x$. Many properties of symmetric spaces hold in the weakly symmetric setting, for example homogeneity, the geodesic orbit property, commutativity of the $L^1$ convolution algebra, the theory of spherical functions, and Plancherel and Fourier inversion formulae; see [17] for an exposition. But the associated Lie group theory and the classification theory are quite different.

There are many weakly symmetric riemannian nilmanifolds, i.e. weakly symmetric riemannian manifolds that admit a transitive nilpotent group of isometries. By contrast the only symmetric riemannian nilmanifolds are the flat ones; they are the products of flat tori and euclidean spaces. Here is the best known example of this phenomenon. Let $\mathfrak{n}$ be the Heisenberg Lie algebra of real dimension $2n+1$, $\mathfrak{n} = \mathrm{Im}\,\mathbb{C} + \mathbb{C}^n$ with composition $[(z,v),(z',v')] = \mathrm{Im}\,\langle v, v'\rangle$ where $\langle \cdot, \cdot \rangle$ is the usual positive definite hermitean inner product on $\mathbb{C}^n$. The unitary group $U(n)$ acts by isomorphisms, $h : (z,v) \mapsto (z, hv)$. That gives us the semidirect product group $G = N \rtimes U(n)$. That results in weakly symmetric $G$–invariant riemannian metrics on $N = G/U(n)$. None of the corresponding weakly symmetric spaces are symmetric. See [17] for an exposition of Yakimova's classification ([23], [24], [25]) of weakly symmetric riemannian manifolds.

The theory of weakly symmetric pseudo–riemannian manifolds is much more delicate, and in fact there are several competing definitions. We will consider the most accessible one, that of real forms of weakly symmetric riemannian manifolds.

Let $(M_r\,, ds_r^s)$ be a connected weakly symmetric riemannian manifold. Suppose that $M_r = G_r/H_r$ is a riemannian nilmanifold, in other words that $G_r = N_r \rtimes H_r$ where $N_r$ is a connected nilpotent Lie group acting transitively on $M_r$. The associated **real form family** $\{\{G_r/H_r\}\}$ consists of all $G/H$ with the same complexification $(G_r)_\mathbb{C}/(H_r)_\mathbb{C}$. In other words $H$ is a real form of $(H_r)_\mathbb{C}$, and $G = N \rtimes H$ where $N$ is an $\mathrm{Ad}(H)$–invariant real form of $(N_r)_\mathbb{C}$. See [20] for the definition and a discussion of the $\mathrm{Ad}(H)$–invariance condition. These $M = G/H$, with invariant pseudo–riemannian metric $ds^2$, are our weakly symmetric pseudo–riemannian nilmanifolds. Every weakly symmetric riemannian manifold is a commutative space, and we work a little bit more generally, assuming that $M_r$ is a commutative nilmanifold.

**Definition 5.1.** A weakly symmetric pseudo–riemannian nilmanifold $M = G/H$ is of **complex type** if it satisfies the conditions (3.2) and (3.4). $\diamondsuit$

**Example 5.2.** Let $M = G/H$ be a weakly symmetric pseudo–riemannian nilmanifold, say $G = N \rtimes H$ and $\mathfrak{n} = \mathfrak{z} + \mathfrak{v}$ as in (3.1). Suppose that $\mathrm{Ad}_G(H)$ is irreducible on $\mathfrak{v}$ and that $H$ has a central subgroup $T \cong U(1)$. Let $\zeta \in \mathfrak{t}$ such that $J := \mathrm{Ad}(\exp(\zeta))|_\mathfrak{v}$ has square $-I$. Then $J$ is an $\mathrm{Ad}(H)$–invariant complex structure on $\mathfrak{v}$ with which $M = G/H$ is of complex type. $\diamondsuit$

There are many cases, as we see in Table 5.3 below, of weakly symmetric pseudo–riemannian nilmanifolds $M_i = G_i/H_i$ with $G_i = N \rtimes H_i$ and $H_2 \subsetneqq H_1$. There in both cases we have the same $\mathfrak{n} = \mathfrak{z} + \mathfrak{v}$. If $M_1 = G_1/H_1$ is of complex type as defined by a central circle subgroup of $H_1$ as in Example 5.2, the

$\mathrm{Ad}(H_1)$–invariant complex structure $J$ on $\mathfrak{v}$ is $\mathrm{Ad}(H_2)$–invariant as well, so $M_2 = G_2/H_2$ is of complex type, and $\mathfrak{v}$ has the same signature for both.

We now extract a number of examples of weakly symmetric pseudo–riemannian nilmanifolds of complex type from [20], to which the results of Section 4 apply. Those are manifolds $(M, ds^2)$, $M = G/H$ with $G = N \rtimes H$, $N$ nilpotent and $H$ reductive in $G$, such that the pseudo–riemannian metric $ds^2$ is the real part of an $H$–invariant pseudo–Kähler metric. The first examples are, of course, those for which $N$ is a Heisenberg group and $H$ acts $\mathbb{R}$–irreducibly on $\mathfrak{v}$. Table 5.3 just below extracts them from [20, Table 4.2]. For the convenience of the reader who wants to check this passage to real forms we retain the numbering of real form families as in [20, Table 4.2]. For the signature of $\mathfrak{v}$ we give the signature $(2k, 2\ell)$ of the real symmetric bilinear form $\beta_\zeta$ on $\mathfrak{v}$ for a choice of nonzero $\zeta$; if we used $-\zeta$ instead, then the signature on $\mathfrak{v}$ would be the reverse, $(2\ell, 2k)$. Then, of course, the signature of the hermitian form $\gamma_\zeta$ on $\mathfrak{v}$ is $(k, \ell)$, and of $\gamma_{-\zeta}$ is $(\ell, k)$. For brevity we only list one of $(k, \ell)$ and $(\ell, k)$.

**Table 5.3 Irreducible Commutative Heisenberg Nilmanifolds** $(N \rtimes H)/H$

| | Group $H$ | $\mathfrak{v}$ and signature$(\mathfrak{v})$ | $\mathfrak{z}$ |
|---|---|---|---|
| 1 | $SU(r,s)$ | $\mathbb{C}^{r,s}$, $(2r,2s)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 2 | $U(r,s)$ | $\mathbb{C}^{r,s}$, $(2r,2s)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 3 | $Sp(k,\ell)$ | $\mathbb{C}^{2k,2\ell}$, $(4k,4\ell)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 4 | $U(1)\cdot Sp(k,\ell)$ | $\mathbb{C}^{2k,2\ell}$, $(4k,4\ell)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1)\cdot Sp(m;\mathbb{R})$ | $\mathbb{C}^{m,m}$, $(2m,2m)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 5 | $SO(2)\cdot SO(r,s), r+s \geqq 2$ | $\mathbb{R}^{2\times(r,s)}$, $(2r,2s)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1)\cdot SO^*(n), n$ even | $\mathbb{C}^n \simeq \mathbb{R}^{n,n}$, $(n,n)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 6 | $U(k,\ell)$ | $S^2_{\mathbb{C}}(\mathbb{C}^{k,\ell})$, $(k^2+k+\ell^2+\ell, 2k\ell))$ | $\mathrm{Im}\,\mathbb{C}$ |
| 7 | $SU(k,\ell), k+\ell$ odd | $\Lambda^2_{\mathbb{C}}(\mathbb{C}^{k,\ell})$, $(k^2-k+\ell^2-\ell, 2k\ell)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 8 | $U(k,\ell)$ | $\Lambda^2_{\mathbb{C}}(\mathbb{C}^{k,\ell})$, $(k^2-k+\ell^2-\ell, 2k\ell))$ | $\mathrm{Im}\,\mathbb{C}$ |
| 9 | $SU(k,\ell)\cdot SU(r,s)$ | $\mathbb{C}^{(k,\ell)\times(r,s)}$, $(2kr+2\ell s, 2ks+2\ell r)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $SL(\frac{m}{2};\mathbb{H})\cdot SL(\frac{n}{2};\mathbb{H})$ | $\mathbb{C}^{m\times n}$, $(mn,mn)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 10 | $S(U(k,\ell)\cdot U(r,s))$ | $\mathbb{C}^{(k,\ell)\times(r,s)}$, $(2kr+2\ell s, 2ks+2\ell r)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $S(GL(\frac{m}{2};\mathbb{H})\cdot GL(\frac{n}{2};\mathbb{H}))$ | $\mathbb{C}^{m\times n}$, $(mn,mn)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 11 | $U(a,b)\cdot Sp(k,\ell), a+b=2$ | $\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell}$, $(4ak+4b\ell, 4a\ell+4bk)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(a,b)\cdot Sp(m;\mathbb{R}), a+b=2$ | $\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2m}$, $(4m,4m)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 12 | $SU(a,b)\cdot Sp(k,\ell)$ | $\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell}$, $(4ak+4b\ell, 4a\ell+4bk)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 13 | $U(a,b)\cdot Sp(k,\ell), a+b=3$ | $\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell}$, $(4ak+4b\ell, 4a\ell+4bk)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(a,b)\cdot Sp(m;\mathbb{R}), a+b=3$ | $\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2m}$, $(6m,6m)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 14 | $U(a,b)\cdot Sp(k,\ell)$, $\substack{a+b=4\\k+\ell=4}$ | $\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell}$, $(4ak+4b\ell, 4a\ell+4bk)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(a,b)\cdot Sp(4;\mathbb{R}), a+b=4$ | $\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^8$, $(32,32)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 15 | $SU(k,\ell)\cdot Sp(r,s), r+s=4$ | $\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{C}^{2r,2s}$, $(4kr+4\ell s, 4ks+4\ell k)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 16 | $U(k,\ell)\cdot Sp(r,s)$, $\substack{k+\ell\geqq 3\\r+s=4}$ | $\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{C}^{2r,2s}$, $(4kr+4\ell s, 4ks+4\ell r)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(k,\ell)\cdot Sp(4;\mathbb{R}), k+\ell \geqq 3$ | $\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{C}^8$, $(8m,8m)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 17 | $U(1)\cdot Spin(7)$ | $\mathbb{C}^8$, $(16,0)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1)\cdot Spin(6,1)$ | $\mathbb{C}^{6,2}$, $(12,4)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1)\cdot Spin(5,2)$ | $\mathbb{C}^{6,2}$, $(12,4)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1)\cdot Spin(4,3)$ | $\mathbb{C}^{4,4}$, $(8,8)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 18 | $U(1)\cdot Spin(9)$ | $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^{16}$, $(32,0)$ | $\mathrm{Im}\,\mathbb{C}$ |

... *Table 5.3 continued on next page*

*Table 5.3 continued from previous page ...*

| | Group $H$ | $\mathfrak{v}$ and signature($\mathfrak{v}$) | $\mathfrak{z}$ |
|---|---|---|---|
| | $U(1) \cdot Spin(r,s), r+s=9$ | $\mathbb{C}^{8,8}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 19 | $Spin(10)$ | $\mathbb{C}^{16}$, $(32,0)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $Spin(8,2)$ | $\mathbb{C}^{8,8}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $Spin(6,4)$ | $\mathbb{C}^{8,8}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 20 | $U(1) \cdot Spin(10)$ | $\mathbb{C}^{16}$, $(32,0)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1) \cdot Spin(8,2)$ | $\mathbb{C}^{8,8}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1) \cdot Spin(6,4)$ | $\mathbb{C}^{8,8}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1) \cdot Spin^*(10)$ | $\mathbb{H}^{4,4}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 21 | $U(1) \cdot G_2$ | $\mathbb{C}^7$, $(14,0)$ | $\mathrm{Re}\,\mathbb{O}$ |
| | $U(1) \cdot G_{2,A_1A_1}$ | $\mathbb{C}^{3,4}$, $(6,8)$ | $\mathrm{Re}\,\mathbb{O}_{sp}$ |
| 22 | $U(1) \cdot E_6$ | $\mathbb{C}^{27}$, $(54,0)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1) \cdot E_{6,A_5A_1}$ | $\mathbb{C}^{15,12}$, $(30,24)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1) \cdot E_{6,D_5T_1}$ | $\mathbb{C}^{16,11}$, $(32,22)$ | $\mathrm{Im}\,\mathbb{C}$ |

In Table 5.3, every entry is contained in a "maximal" entry for which $\dim_{\mathbb{C}} \mathfrak{v} = m$ and $H$ is a subgroup of $U(m)$. There are so many of those, that it is best to restrict attention to the cases where the action of $H$ on $\mathfrak{v}$ is irreducible. The next examples are those for which the action of $H$ on $\mathfrak{v}$ is irreducible. We extract them from [20, Table 5.2]. Again we retain the numbering corresponding to real form families from that table. Also we omit the cases where $N$ is commutative, i.e. where $G/H = \mathbb{C}^n$.

**Table 5.4 Maximal Irreducible Weakly Symmetric Nilmanifolds**
**$(N \rtimes H, H)$ of Complex Type**

| | Group $H$ | $\mathfrak{v}$ and signature($\mathfrak{v}$) | $\mathfrak{z}$ |
|---|---|---|---|
| 4 | $U(1) \cdot SO(r,s), r+s \neq 4$ | $\mathbb{C}^{r,s}$, $(2r,2s)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1) \cdot SO^*(n), n=2m \neq 4$ | $\mathbb{C}^{m,m}$, $(2m,2m)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 5 | $SU(r,s)$, $r+s$ even | $\mathbb{C}^{r,s}$, $(2r,2s)$ | $\Lambda^2_{\mathbb{R}}(\mathbb{C}^{r,s}) \oplus \mathrm{Im}\,\mathbb{C}$ |
| | $U(r,s)$ | $\mathbb{C}^{r,s}$, $(2r,2s)$ | $\Lambda^2_{\mathbb{R}}(\mathbb{C}^{r,s}) \oplus \mathrm{Im}\,\mathbb{C}$ |
| 6 | $SU(r,s), r+s$ odd | $\mathbb{C}^{r,s}$, $(2r,2s)$ | $\Lambda^2_{\mathbb{R}}(\mathbb{C}^{r,s})$ |
| 7 | $SU(r,s), r+s$ odd | $\mathbb{C}^{r,s}$, $(2r,2s)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 8 | $U(r,s)$ | $\mathbb{C}^{r,s}$, $(2r,2s)$ | $\mathfrak{u}(r,s)$ |
| 9 | $(\{1\}$ or $U(1)) \cdot Sp(r,s)$ | $\mathbb{H}^{r,s}$, $(4r,4s)$ | $\mathrm{Re}\,\mathbb{H}_0^{(r,s)\times(r,s)} \oplus \mathrm{Im}\,\mathbb{H}$ |
| | $U(1) \cdot Sp(n;\mathbb{R})$ | $\mathbb{R}^{2n,2n}$, $(2n,2n)$ | $\mathrm{Re}\,\mathbb{H}_{sp,0}^{n\times n} \oplus \mathrm{Im}\,\mathbb{H}_{sp}$ |
| 10 | $U(r,s)$ | $S^2_{\mathbb{C}}(\mathbb{C}^{r,s})$, $(r(r+1)+s(s+1), 2rs)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 11 | $SU(r,s), r+s \geqq 3, r+s$ odd | $\Lambda^2_{\mathbb{C}}(\mathbb{C}^{r,s})$, $(r^2-r+s^2-s, 2rs)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(r,s), r+s \geqq 3$ | $\Lambda^2_{\mathbb{C}}(\mathbb{C}^{r,s})$, $(r^2-r+s^2-s, 2rs)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 12 | $U(1) \cdot Spin(7)$ | $\mathbb{O}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^8$, $(16,0)$ | $\mathbb{R}^7 \oplus \mathbb{R}$ |
| | $U(1) \cdot Spin(6,1)$ | $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{6,2}$, $(12,4)$ | $\mathbb{R}^{6,1} \oplus \mathbb{R}$ |
| | $U(1) \cdot Spin(5,2)$ | $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{6,2}$, $(12,4)$ | $\mathbb{R}^{5,2} \oplus \mathbb{R}$ |
| | $U(1) \cdot Spin(4,3)$ | $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{4,4}$, $(8,8)$ | $\mathbb{R}^{4,3} \oplus \mathbb{R}$ |
| 13 | $U(1) \cdot Spin(9)$ | $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{16}$, $(32,0)$ | $\mathbb{R}$ |
| | $U(1) \cdot Spin(8,1)$ | $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{8,8}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1) \cdot Spin(7,2)$ | $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}^{4,4}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1) \cdot Spin(6,3)$ | $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}^{4,4}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1) \cdot Spin(5,4)$ | $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}^{2,2}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 14 | $(\{1\}$ or $U(1)) \cdot Spin(10)$ | $\mathbb{C}^{16}$, $(32,0)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $Spin(9,1)$ | $\mathbb{R}^{16,16}$, $(16,16)$ | $\mathbb{R}$ |
| | $(\{1\}$ or $U(1)) \cdot Spin(8,2)$ | $\mathbb{C}^{8,8}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |

*... Table 5.4 continued on next page*

*Table 5.4 continued from previous page ...*

| | Group $H$ | $\mathfrak{v}$ and signature($\mathfrak{v}$) | $\mathfrak{z}$ |
|---|---|---|---|
| | $Spin(7,3)$ | $\mathbb{H}^{4,4}$, (16,16) | $\mathbb{R}$ |
| | $(\{1\}$ or $U(1))\cdot Spin(6,4)$ | $\mathbb{C}^{8,8}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $Spin(5,5)$ | $\mathbb{R}^{16,16}$, $(16,16)$ | $\mathbb{R}$ |
| | $U(1)\cdot Spin^*(10)$ | $\mathbb{H}^{4,4}$, $(16,16)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 15 | $U(1)\cdot G_2$ | $\mathbb{C}^7=\mathrm{Im}\,\mathbb{O}_{\mathbb{C}}$, $(14,0)$ | $\mathbb{R}=\mathrm{Re}\,\mathbb{O}$ |
| | $U(1)\cdot G_{2,A_1A_1}$ | $\mathbb{C}^{3,4}$, $(6,8)$ | $\mathrm{Re}\,\mathbb{O}_{sp}$ |
| 16 | $U(1)\cdot E_6$ | $\mathbb{C}^{27}$, $(54,0)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1)\cdot E_{6,A_5A_1}$ | $\mathbb{C}^{15,12}$, $(30,24)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $U(1)\cdot E_{6,D_5T_1}$ | $\mathbb{C}^{16,11}$, $(32,22)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 19 | $(\{1\}$ or $U(1))\cdot(SU(k,\ell)\cdot SU(r,s))$, $k+\ell,r+s\geqq 3, U(1)$ if $k+\ell=r+s$ | $\mathbb{C}^{(k,\ell)\times(r,s)}$, $(2kr+2\ell s, 2ks+2\ell r)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $(\{1\}$ or $U(1))\cdot SL(m;\mathbb{C})$ | $\mathfrak{gl}(m;\mathbb{C})$, $(m^2,m^2)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 20 | $(\{1\}$ or $U(1))\cdot(SU(2)\cdot SU(r,s))$, $r+s\geqq 2, U(1)$ if $r+s=2$ | $\mathbb{C}^{2\times(r,s)}$, $(4r,4s)$ | $\mathrm{Im}\,\mathbb{C}^{2\times 2}=\mathfrak{u}(2)$ |
| | $(\{1\}$ or $U(1))\cdot(SU(1,1)\cdot SU(r,s))$ | $\mathbb{C}^{(1,1)\times(r,s)}$, $(2r+2s,2r+2s)$ | $\mathfrak{u}(1,1)$ |
| 21 | $(\{1\}$ or $U(1))\cdot(Sp(2)\cdot SU(r,s))$, $r+s\geqq 3, U(1)$ if $r+s\leqq 4$ | $\mathbb{H}^2\otimes_{\mathbb{R}}\mathbb{C}^{r,s}$, $(16r,16s)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $(\{1\}$ or $U(1))\cdot(Sp(1,1)\cdot SU(r,s))$ | $\mathbb{H}^{1,1}\otimes_{\mathbb{R}}\mathbb{C}^{r,s}$, $(8r+8s,8r+8s)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $Sp(2;\mathbb{R})\cdot U(r,s))$ | $\mathbb{R}^{4,4}\otimes_{\mathbb{R}}\mathbb{C}^{r,s}$, $(8r+8s,8r+8s)$ | $\mathrm{Im}\,\mathbb{C}$ |
| 22 | $H=U(k,\ell)\cdot Sp(r,s), k+\ell=2$ | $\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{H}^{r,s}$, $(4kr+4\ell s, 4ks+4\ell r)$ | $\mathfrak{u}(k,\ell)$ |
| | $H=U(k,\ell)\cdot Sp(n;\mathbb{R}), k+\ell=2$ | $\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\otimes_{\mathbb{C}}(\mathbb{R}^{2n}\oplus\mathbb{R}^{2n})$, $(4n,4n)$ | $\mathfrak{u}(k,\ell)$ |
| 23 | $H=U(k,\ell)\cdot Sp(r,s)$, $k+\ell=3$ | $\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{H}^{r,s}$, $(4kr+4\ell s, 4ks+4\ell r)$ | $\mathrm{Im}\,\mathbb{C}$ |
| | $H=U(k,\ell)\cdot Sp(n;\mathbb{R})$, $k+\ell=3$ | $\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{C}^{2n}$, $(6n,6n)$ | $\mathrm{Im}\,\mathbb{C}$ |

## 6. Toward the Classification for Weakly Symmetric Pseudo–Riemannian Nilmanifolds of Complex Type

The classification of irreducible to indecomposable commutative spaces is due to Yakimova. It is combinatorial, based on her classification ([24], [25]; or see [17]) of indecomposable commutative spaces — subject to a few technical conditions. In this section we broaden the scope of Table 5.4 from irreducible to indecomposable commutative spaces, subject to those technical conditions. The technical conditions, which we explain just below, are that $(N\rtimes H, H)$ be indecomposable, principal, maximal and $Sp(1)$–saturated.

We work out the classification of weakly symmetric pseudo–riemannian nilmanifolds of complex type for the real form families corresponding to those indecomposable commutative spaces. This is the main non–combinatorial step in classifying all the weakly symmetric pseudo–riemannian nilmanifolds of complex type.

Since $G=N\rtimes H$ acts almost–effectively on $M=G/H$, the centralizer of $N$ in $H$ is discrete, in other words the representation of $H$ on $\mathfrak{n}$ has finite kernel. (In the notation of [25, Section 1.4] this says $H=L=L^0$ and $P=\{1\}$.) That simplifies the general definitions [25, Definition 6] of **principal** and [25, Definition 8] of $Sp(1)$–**saturated**, as follows. Decompose $\mathfrak{v}$ as a sum $\mathfrak{w}_1\oplus\cdots\oplus\mathfrak{w}_t$ of irreducible $\mathrm{Ad}(H)$–invariant subspaces. Then $(G,H)$ is **principal** if $Z_H^0=Z_1\times\cdots\times Z_m$ where $Z_i\subset GL(\mathfrak{w}_i)$, in other words $Z_i$ acts trivially on $\mathfrak{w}_j$ for $j\neq i$. Decompose $H=Z_H^0\times H_1\times\cdots\times H_m$ where the $H_i$ are simple. Suppose that whenever some $H_i$ acts nontrivially on some $\mathfrak{w}_j$ and $Z_H^0\times\prod_{\ell\neq i}H_\ell$ is irreducible on $\mathfrak{w}_j$, it follows that $H_i$ is trivial on $\mathfrak{w}_k$ for all $k\neq j$. Then $H_i\cong Sp(1)$ and we say that $(G,H)$ is $Sp(1)$–**saturated**. The group $Sp(1)$ will be more visible in the definition when we extend the definition to the cases where $H\neq L$.

In the following table, $\mathfrak{h}_{n;\mathbb{F}}$ is the Heisenberg algebra $\mathrm{Im}\,\mathbb{F}+\mathbb{F}^n$ of real dimension $(\dim_{\mathbb{R}}\mathbb{F}-1)+n\dim_{\mathbb{R}}\mathbb{F}$. Here $\mathbb{F}$ is the real, complex, quaternion or octonion algebra over $\mathbb{R}$, $\mathrm{Im}\,\mathbb{F}$ is its imaginary component, and

$$\mathfrak{h}_{n;\mathbb{F}}=\mathrm{Im}\,\mathbb{F}+\mathbb{F}^n \text{ with product } [(z_1,v_1),(z_2,v_2)]=(\mathrm{Im}\,(v_1\cdot v_2^*),0)$$

where the $v_i$ are row vectors and $v_2^*$ denotes the conjugate ($\mathbb{F}$ over $\mathbb{R}$) transpose of $v_2$. It is the Lie algebra of the (slightly generalized) Heisenberg group $H_{n;\mathbb{F}}$. Also in the table, in the listing for $\mathfrak{n}$ the summands in double parenthesis $((..))$ are the subalgebras $[\mathfrak{w},\mathfrak{w}]+\mathfrak{w}$ where $\mathfrak{w}$ is an $H$–irreducible subspace of $\mathfrak{v}$ with $[\mathfrak{w},\mathfrak{w}]\neq 0$, and the summands not in double parentheses are $H$–invariant subspaces $\mathfrak{w}\subset\mathfrak{z}$ with $[\mathfrak{w},\mathfrak{w}]=0$. Thus

$$\mathfrak{n}=\mathfrak{z}+\mathfrak{v} \text{ vector space direct sum, and } \mathfrak{z}=[\mathfrak{n},\mathfrak{n}]\oplus\mathfrak{u} \tag{6.1}$$

where the center $\mathfrak{u}$ is the sum of the summands listed for $\mathfrak{n}$ that are *not* enclosed in double parenthesis $((..))$.

As before, when we write $m/2$ it is assumed that $m$ is even, and similarly $n/2$ requires that $n$ be even. Further $k+\ell=m$ and $r+s=n$ where applicable.

**Table 6.2 Maximal Indecomposable Principal Commutative Nilmanifolds $(N\rtimes H,H)$**
**$N$ Nonabelian, Where the Action of $H$ on $\mathfrak{n}/[\mathfrak{z},\mathfrak{z}]$ is Reducible**

| | Group $H$ and<br>Algebra $\mathfrak{n}$ | $H$–module $\mathfrak{v}$ and<br>Signature($\mathfrak{v}$) | $[\mathfrak{n},\mathfrak{n}]$<br>$\mathfrak{u}$ |
|---|---|---|---|
| 1 | $U(r,s)$<br>$((\mathfrak{h}_{r+s;\mathbb{C}}))+\mathfrak{su}(r,s)$ | $\mathbb{C}^{r,s}$<br>$(2r,2s)$ | $\mathrm{Im}\,\mathbb{C}$<br>$\mathfrak{su}(r,s)$ |
| 2 | $U(r,s),(r,s)=(4,0)$ or $(2,2)$<br>$((\mathrm{Im}\,\mathbb{C}+\Lambda^2(\mathbb{C}^{r,s})+\mathbb{C}^{r,s}))+\Lambda^2(\mathbb{R}^{r,s})$ | $\mathbb{C}^{r,s}$<br>$(2r,2s)$ | $\Lambda^2(\mathbb{C}^{r,s})+\mathrm{Im}\,\mathbb{C}$<br>$\Lambda^2(\mathbb{R}^{r,s})$ |
| 3 | $U(1)\cdot SU(r,s)\cdot U(1)$<br>$((\mathfrak{h}_{n;\mathbb{C}}))+((\mathfrak{h}_{n(n-1)/2;\mathbb{C}}))$ | $\mathbb{C}^{r,s}\oplus\Lambda^2_{\mathbb{C}}(\mathbb{C}^{r,s})$<br>$(2r,2s)$<br>$\oplus(r^2-r+s^2-s,2rs)$ | $\mathrm{Im}\,\mathbb{C}\oplus\mathrm{Im}\,\mathbb{C}$<br>$\{0\}$ |
| 4 | $SU(r,s)$, $(r,s)=(4,0)$ or $(2,2)$<br>$((\mathrm{Im}\,\mathbb{C}+\mathrm{Re}\,\mathbb{H}^{2\times 2}+\mathbb{C}^{r,s}))+\Lambda^2(\mathbb{R}^{r,s})$ | $\mathbb{C}^{r,s}$<br>$(2r,2s)$ | $\mathrm{Im}\,\mathbb{C}\oplus\mathrm{Re}\,\mathbb{H}^{2\times 2}$<br>$\Lambda^2(\mathbb{R}^4)$ |
| 5 | $U(k,\ell)\times U(r,s)$<br>$k+\ell=2,(r,s)=(4,0)$ or $(2,2)$<br>$((\mathrm{Im}\,\mathbb{C}^{(k,\ell)\times(k,\ell)}+\mathbb{C}^{(k,\ell)\times(r,s)}))+\Lambda^2(\mathbb{R}^{r,s})$ | $\mathbb{C}^{(k,\ell)\times(r,s)}$<br>$(2kr+2\ell s,2ks+2\ell r)$ | $\mathrm{Im}\,\mathbb{C}^{(k,\ell)\times(k,\ell)}$<br>$\Lambda^2(\mathbb{R}^{r,s})$ |
| 6 | $S(U(k,\ell)\times U(r,s)),(k,\ell)=(4,0)$ or $(2,2)$<br>$((\mathfrak{h}_{4(r+s);\mathbb{C}}))+\Lambda^2(\mathbb{R}^{k,\ell})$ | $\mathbb{C}^{(k,\ell)\times(r,s)}$<br>$(2kr+2\ell s,2ks+2\ell r)$ | $\mathrm{Im}\,\mathbb{C}$<br>$\Lambda^2(\mathbb{R}^{k,\ell})$ |
| 7 | $U(k,\ell)\cdot U(r,s)$<br>$((\mathfrak{h}_{(k+\ell,r+s);\mathbb{C}}))+((\mathfrak{h}_{k+\ell;\mathbb{C}}))$ | $\mathbb{C}^{(k,\ell)\times(r,s)}\oplus\mathbb{C}^{k,\ell}$<br>$(2kr+2\ell s,2ks+2\ell r)$<br>$\oplus(2k,2\ell)$ | $\mathrm{Im}\,\mathbb{C}\oplus\mathrm{Im}\,\mathbb{C}$<br>$\{0\}$ |
| 8 | $U(1)\cdot Sp(r,s)\cdot U(1)$<br>$((\mathfrak{h}_{2(r+s);\mathbb{C}}))+((\mathfrak{h}_{2(r+s);\mathbb{C}}))$ | $\mathbb{C}^{2r,2s}\oplus\mathbb{C}^{2r,2s}$<br>$(4r,4s)\oplus(4r,4s)$ | $\mathrm{Im}\,\mathbb{C}\oplus\mathrm{Im}\,\mathbb{C}$<br>$\{0\}$ |
| | $U(1)\cdot Sp(n;\mathbb{R})\cdot U(1)$<br>$((\mathfrak{h}_{2n;\mathbb{C}}))\oplus((\mathfrak{h}_{2n;\mathbb{C}}))$ | $\mathbb{C}^{n,n}\oplus\mathbb{C}^{n,n}$<br>$(2n,2n)\oplus(2n,2n)$ | $\mathrm{Im}\,\mathbb{C}\oplus\mathrm{Im}\,\mathbb{C}$<br>$\{0\}$ |
| 11 | $Sp(k,\ell)\cdot(U(1)$ or $\{1\})\cdot Sp(r,s)$<br>$((\mathfrak{h}_{k+\ell;\mathbb{H}}))+\mathbb{H}^{(k,\ell)\times(r,s)}$ | $\mathbb{H}^{k,\ell}$<br>$(4k,4\ell)$ | $\mathrm{Im}\,\mathbb{H}=\mathfrak{sp}(1)$<br>$\mathbb{H}^{(k,\ell)\times(r,s)}$ |
| | $Sp(m;\mathbb{R})\cdot(U(1)$ or $\{1\})\cdot Sp(n;\mathbb{R})$<br>$((\mathfrak{h}_{m;\mathbb{H}}))+\mathbb{C}^{(m,m)\times(n,n)}$ | $\mathbb{C}^{m,m}$<br>$(2m,2m)$ | $\mathfrak{sp}(1;\mathbb{R})$<br>$\mathbb{C}^{(m,m)\times(n,n)}$ |
| 12 | $Sp(k,\ell)\cdot(U(1)$ or $\{1\})$<br>$((\mathfrak{h}_{k+\ell;\mathbb{H}}))+\mathrm{Re}\,\mathbb{H}_0^{(k,\ell)\times(k,\ell)}$ | $\mathbb{H}^{k,\ell}$<br>$(4k,4\ell)$ | $\mathrm{Im}\,\mathbb{H}=\mathfrak{sp}(1)$<br>$\mathrm{Re}\,\mathbb{H}_0^{(k,\ell)\times(k,\ell)}$ |
| | $Sp(m;\mathbb{R})\cdot(U(1)$ or $\{1\})$<br>$((\mathfrak{h}_{m;\mathbb{H}}))+\mathrm{Re}\,\mathbb{H}^{m\times m}_{sp,0}$ | $\mathbb{C}^{m,m}$<br>$(2m,2m)$ | $\mathrm{Im}\,\mathbb{H}$<br>$\mathrm{Re}\,\mathbb{H}^{m\times m}_{sp,0}$ |
| 13 | $Spin(k,7-k)\cdot(SO(2)$ or $\{1\})$<br>$((\mathfrak{h}_{1;\mathbb{O}}))+\mathbb{R}^{(k,7-k)\times 2}$, $4\leqq k\leqq 7$ | $\mathbb{R}^{q,8-q}$, $q=2[\frac{k+1}{2}]$<br>$(q,8-q)$ | $\mathbb{R}^{k,7-k}$<br>$\mathbb{R}^{(k,7-k)\times 2}$ |
| 14 | $U(1)\cdot Spin(k,7-k)$, $4\leqq k\leqq 7$<br>$((\mathfrak{h}_{7;\mathbb{C}}))+\mathbb{R}^{q,8-q}$, $q=2[\frac{k+1}{2}]$ | $\mathbb{C}^{2k,14-2k}$<br>$(2k,14-2k)$ | $\mathrm{Im}\,\mathbb{C}$<br>$\mathbb{R}^{q,8-q}$ |
| 15 | $U(1)\cdot Spin(k,7-k)$, $4\leqq k\leqq 7$<br>$((\mathfrak{h}_{8;\mathbb{C}}))+\mathbb{R}^7$ | $\mathbb{C}^{q,8-q}$, $q=2[\frac{k+1}{2}]$<br>$(2q,16-2q)$ | $\mathrm{Im}\,\mathbb{C}$<br>$\mathbb{R}^{k,7-k}$ |
| 16 | $U(1)\cdot Spin(k,8-k)\cdot U(1)$<br>$((\mathfrak{h}_{8;\mathbb{C}}))+((\mathfrak{h}_{8;\mathbb{C}}))$ | $\mathbb{C}^{k,\ell}_{+}\oplus\mathbb{C}^{k,\ell}_{-}$<br>$(2k,2\ell)\oplus(2k,2\ell)$ | $\mathrm{Im}\,\mathbb{C}\oplus\mathrm{Im}\,\mathbb{C}$<br>$\{0\}$ |

*. . . Table 6.2 continued on next page*

*Table 6.2 continued from previous page ...*

| | Group $H$ and Algebra $\mathfrak{n}$ | $H$–module $\mathfrak{v}$ and Signature($\mathfrak{v}$) | $[\mathfrak{n},\mathfrak{n}]$ $\mathfrak{u}$ |
|---|---|---|---|
| | $U(1)\cdot Spin^*(8)\cdot U(1)$<br>$((\mathfrak{h}_{8;\mathbb{C}}))+((\mathfrak{h}_{8;\mathbb{C}}))$ | $\mathbb{C}^{4,4}\oplus\mathbb{C}^{4,4}$<br>$(8,8)\oplus(8,8)$ | $\operatorname{Im}\mathbb{C}\oplus\operatorname{Im}\mathbb{C}$<br>$\{0\}$ |
| 17 | $U(1)\cdot Spin(2k,2\ell)$, $\substack{k=3,4,5\\ \ell=5-k}$<br>$((\mathfrak{h}_{16;\mathbb{C}}))+\mathbb{R}^{2k,2\ell}$ | $\mathbb{C}^{q,16-q}$, $q=2^{[\frac{k+1}{2}]+2}$<br>$(2q,32-2q)$ | $\operatorname{Im}\mathbb{C}$<br>$\mathbb{R}^{2k,2\ell}$ |
| | $U(1)\cdot Spin^*(10)$<br>$((\mathfrak{h}_{16;\mathbb{C}}))+\mathbb{R}^{10}$ | $\mathbb{C}^{8,8}$<br>$(16,16)$ | $\operatorname{Im}\mathbb{C}$<br>$\mathbb{R}^{10}$ |
| 18 | $(SU(k,\ell)$ or $U(k,\ell)$ or $U(1)Sp(\frac{m}{2}))\cdot SU(r,s)$<br>$k+\ell=m, r+s=2$<br>$((\mathfrak{h}_{2m;\mathbb{C}}))+\mathfrak{su}(r,s)$ | $\mathbb{C}^{(k,\ell)\times(r,s)}$<br>$(2kr+2\ell s,2ks+2\ell r)$ | $\operatorname{Im}\mathbb{C}$<br>$\mathfrak{su}(r,s)$ |
| | $Sp(m/2;\mathbb{R})\cdot U(r,s)$<br>$((\mathfrak{h}_{2m;\mathbb{C}}))+\mathfrak{su}(r,s)$ | $\mathbb{C}^{(m/2,m/2)\times(r,s)}$<br>$(2m,2m)$ | $\operatorname{Im}\mathbb{C}$<br>$\mathfrak{su}(r,s)$ |
| 19 | $(SU(k,\ell)$ or $U(k,\ell)$ or $U(1)Sp(\frac{m}{2}))\cdot U(r,s)$<br>$k+\ell=m, r+s=2$<br>$((\mathfrak{h}_{2m;\mathbb{C}}))+((\mathfrak{h}_{2;\mathbb{C}}))$ | $\mathbb{C}^{(k,\ell)\times(r,s)}\oplus\mathbb{C}^{r,s}$<br>$(2kr+2\ell s,2ks+2\ell r)$<br>$\oplus(2r,2s)$ | $\operatorname{Im}\mathbb{C}\oplus\operatorname{Im}\mathbb{C}$<br>$\{0\}$ |
| | $U(1)Sp(\frac{m}{2};\mathbb{R})\cdot U(r,s)$<br>$((\mathfrak{h}_{2m;\mathbb{C}}))+((\mathfrak{h}_{2;\mathbb{C}}))$ | $\mathbb{C}^{(\frac{m}{2},\frac{m}{2})\times(r,s)}\oplus\mathbb{C}^{r,s}$<br>$(2m,2m)\oplus(2r,2s)$ | $\operatorname{Im}\mathbb{C}\oplus\operatorname{Im}\mathbb{C}$<br>$\{0\}$ |
| 20 | $(SU(k,\ell),U(k,\ell),U(1)Sp(\frac{k}{2},\frac{\ell}{2}))\cdot SU(a,b)$<br>$\cdot(SU(r,s),U(r,s),U(1)Sp(\frac{r}{2},\frac{s}{2}))$<br>$k+\ell=m, a+b=2, r+s=n$<br>$((\mathfrak{h}_{2m;\mathbb{C}}))+((\mathfrak{h}_{2n;\mathbb{C}}))$ | $\mathbb{C}^{(k,\ell)\times(a,b)}\oplus\mathbb{C}^{(a,b)\times(r,s)}$<br>$(2(ak+b\ell),2(a\ell+bk))$<br>$\oplus(2(ar+bs),2(as+br))$ | $\operatorname{Im}\mathbb{C}\oplus\operatorname{Im}\mathbb{C}$<br>$\{0\}$ |
| | $(SU(k,\ell),U(k,\ell),U(1)Sp(\frac{k}{2},\frac{\ell}{2}))$<br>$\cdot SU(a,b)\cdot U(1)Sp(\frac{n}{2};\mathbb{R})$<br>$k+\ell=m, a+b=2, r+s=n$<br>$((\mathfrak{h}_{2m;\mathbb{C}}))+((\mathfrak{h}_{2n;\mathbb{C}}))$ | $\mathbb{C}^{(k,\ell)\times(a,b)}\oplus\mathbb{C}^{(a,b)\times(\frac{n}{2},\frac{n}{2})}$<br>$(2(ak+b\ell),2(a\ell+bk))\oplus(\frac{n}{2},\frac{n}{2})$ | $\operatorname{Im}\mathbb{C}\oplus\operatorname{Im}\mathbb{C}$<br>$\{0\}$ |
| | $U(1)Sp(\frac{m}{2};\mathbb{R})\cdot SU(a,b)\cdot U(1)Sp(\frac{n}{2};\mathbb{R})$<br>$k+\ell=m, a+b=2, r+s=n$<br>$((\mathfrak{h}_{2m;\mathbb{C}}))+((\mathfrak{h}_{2n;\mathbb{C}}))$ | $\mathbb{C}^{(m/2,m/2)\times(a,b)}$<br>$\oplus\mathbb{C}^{(a,b)\times(n/2,n/2)}$<br>$(2m,2m)\oplus(2n,2n)$ | $\operatorname{Im}\mathbb{C}\oplus\operatorname{Im}\mathbb{C}$<br>$\{0\}$ |
| 21 | $(SU(k,\ell),U(k,\ell),U(1)Sp(\frac{k}{2},\frac{\ell}{2}))$<br>$\cdot SU(a,b)\cdot U(r,s)$, $r,s$ even<br>$k+\ell=m, a+b=2, r+s=4$<br>$((\mathfrak{h}_{2m;\mathbb{C}}))+((\mathfrak{h}_{8;\mathbb{C}}))+\Lambda^2(\mathbb{R}^{r,s})$ | $\mathbb{C}^{(k,\ell)\times(a,b)}\oplus\mathbb{C}^{(a,b)\times(r,s)}$<br>$(2(ak+b\ell),2(a\ell+bk))$<br>$\oplus(2(ar+bs),2(as+br))$ | $\operatorname{Im}\mathbb{C}\oplus\operatorname{Im}\mathbb{C}$<br>$\Lambda^2(\mathbb{R}^{r,s})$ |
| | $U(1)Sp(m;\mathbb{R})\cdot SU(a,b))\cdot U(r,s))$<br>$a+b=2$, $(r,s)=(4,0)$ or $(2,2)$<br>$((\mathfrak{h}_{2m;\mathbb{C}}))+((\mathfrak{h}_{8;\mathbb{C}}))+\Lambda^2(\mathbb{R}^{r,s})$ | $\mathbb{C}^{(m,m)\times(a,b)}\oplus\mathbb{C}^{(a,b)\times(r,s)}$<br>$(4m,4m)\oplus(2(ar+bs),2(as+br))$ | $\operatorname{Im}\mathbb{C}\oplus\operatorname{Im}\mathbb{C}$<br>$\Lambda^2(\mathbb{R}^{r,s})$ |
| 22 | $U(a,b)\cdot U(r,s)$<br>$a+b=2$, $(r,s)=(4,0)$ or $(2,2)$<br>$((\mathfrak{h}_{8;\mathbb{C}}))+\Lambda^2(\mathbb{R}^{r,s})+\mathfrak{su}(2)$ | $\mathbb{C}^{(a,b)\times(r,s)}$<br>$(ar+bs,as+br)$ | $\operatorname{Im}\mathbb{C}$<br>$\Lambda^2(\mathbb{R}^{r,s})+\mathfrak{su}(2)$ |
| 23 | $U(k,\ell)\cdot U(a,b)\cdot U(r,s)$<br>$(k,\ell)=(4,0)$ or $(2,2)$, $a+b=2$,<br>$(r,s)=(4,0)$ or $(2,2)$<br>$\Lambda^2(\mathbb{R}^{k,\ell})+((\mathfrak{h}_{8;\mathbb{C}}))$<br>$+((\mathfrak{h}_{8;\mathbb{C}}))+\Lambda^2(\mathbb{R}^{r,s})$ | $\mathbb{C}^{(k,\ell)\times(a,b)}\oplus\mathbb{C}^{(a,b)\times(r,s)}$<br>$(ak+b\ell,a\ell+bk)$<br>$\oplus(ar+bs,as+br)$ | $\operatorname{Im}\mathbb{C}\oplus\operatorname{Im}\mathbb{C}$<br>$\Lambda^2(\mathbb{R}^{k,\ell})$<br>$+\Lambda^2(\mathbb{R}^{r,s})$ |
| 24 | $U(1)\cdot SU(k,\ell)\cdot U(1)$<br>$(k,\ell)=(4,0)$ or $(2,2)$<br>$((\mathfrak{h}_{4;\mathbb{C}}))+((\mathfrak{h}_{4;\mathbb{C}}))+\Lambda^2(\mathbb{R}^{k,\ell})$ | $\mathbb{C}^{k,\ell}\oplus\mathbb{C}^{k,\ell}$<br>$(2k,2\ell)\oplus(2k,2\ell)$ | $\operatorname{Im}\mathbb{C}\oplus\operatorname{Im}\mathbb{C}$<br>$\Lambda^2(\mathbb{R}^{k,\ell})$ |
| 25 | $(U(1),\{1\}))\cdot SU(k,\ell))\cdot(\{1\},U(1))$,<br>$k+\ell=4$<br>$((\mathfrak{h}_{4;\mathbb{C}}))+\Lambda^2(\mathbb{C}^{k,\ell})$ | $\mathbb{C}^{k,\ell}$<br>$(2k,2\ell)$ | $\operatorname{Im}\mathbb{C}$<br>$\Lambda^2(\mathbb{C}^{k,\ell})$ |

## References

[1] A. Andreotti & E. Vesentini, Carleman estimates for the Laplace–Beltrami operator on complex manifolds, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 81–130.

[2] R. Bott, Homogeneous vector bundles, Annals of Math. **66** (1957), 203–248.

[3] Z. Chen & J. A. Wolf, Semisimple weakly symmetric pseudo-riemannian manifolds, Abh. Math. Semin. Univ. Hamburg **88** (2018) pp. 331–369. the Abh. Math. Seminar Hamburg. {arXiv: 1707.01181}.

[4] G. Fels, A. T. Huckleberry & J. A. Wolf, "Cycle Spaces of Flag Domains: A Complex Geometric Viewpoint". Progress in Mathematics, vol. 245, Birkhäuser/Springer Boston, 2005.

[5] Harish–Chandra, Discrete series for semisimple Lie groups, I, Acta Math. **113** (1965), 241–317.

[6] A. A. Kirillov, Unitary representations of nilpotent Lie groups, Uspekhi Math. Nauk **17** (1962), 57–110 (English: Russian Math. Surveys **17** (1962), 53–104).

[7] C. C. Moore & J. A. Wolf, Square integrable representations of nilpotent groups, Trans. Amer. Math. Soc. **185** (1973), 445–462.

[8] M. S. Narasimhan & K. Okamoto, An analogue of the Borel–Weil–Bott theorem for hermitian symmetric pairs of noncompact type, Ann. of Math. **91** (1970), 486–511.

[9] I. Satake, Unitary representations of a semi–direct product of Lie groups on $\overline{\partial}$–cohomology spaces, Math. Ann. **190** (1971), 177–202.

[10] W. Schmid, On a conjecture of Langlands, Ann. Math. **93** (1971), 1–42.

[11] W. Schmid, $L^2$ cohomology and the discrete series, Ann. Math. **103** (1976), 375–394.

[12] J. A. Wolf, Curvature in nilpotent Lie groups, Proc. Amer. Math. Soc. **15** (1964), 271–274.

[13] J. A. Wolf, The action of a real semisimple group on a complex flag manifold, I: Orbit structure and holomorphic arc components. Bull. Amer. Math. Soc. **75** (1969), 1121–1237.

[14] J. A. Wolf, The action of a real semisimple group on a complex flag manifold, II: Unitary representations on partially holomorphic cohomology spaces. Memoirs of the American Mathematical Society, Number 138, 1974.

[15] J. A. Wolf, Partially harmonic spinors and representations of reductive Lie groups, J. Functional Analysis **15** (1974), 117–154.

[16] J. A. Wolf, Representations of certain semidirect product groups. J. Functional Analysis, **19** (1975), 339–372.

[17] J. A. Wolf, "Harmonic Analysis on Commutative Spaces", Mathematical Surveys and Monographs, **142**. American Mathematical Society, 2007.

[18] J. A. Wolf, Stepwise square integrable representations: the concept and some consequences, in "Lie Theory and Its Applications in Physics", Springer Proceedings in Mathematics and Statistics, Ed. V. Dobrev, vol. 191 (2016), 181–202.

[19] J. A. Wolf, Representations on Partially Holomorphic Cohomology Spaces, Revisited, Contemporary Mathematics, **714** (2018), 253–303.

[20] J. A. Wolf & Z. Chen, Weakly Symmetric Pseudo–Riemannian Nilmanifolds, to appear. {arXiv: 1806.07847 (math DG)}

[21] H.-W. Wong, Dolbeault cohomological realization of Zuckerman modules associated with finite rank representations, J. Functional Analysis **129** (1995), 428–454.

[22] H.-W. Wong, Cohomological induction in various categories and the maximal globalization conjecture, Duke J. Math. **96** (1999), 1–27.

[23] O. S. Yakimova, Weakly symmetric riemannian manifolds with reductive isometry group, Math. USSR Sbornik **195** (2004), 599–614.

[24] O. S. Yakimova, "Gelfand Pairs," Bonner Math. Schriften (Universität Bonn) **374**, 2005.

[25] O. S. Yakimova, Principal Gelfand pairs, Transformation Groups **11** (2006), 305–335.



Department of Mathematics, University of California, Berkeley, CA 94720–3840, U.S.A.

*Email address*: `jawolf@math.berkeley.edu`