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WEAKLY SYMMETRIC PSEUDO–RIEMANNIAN NILMANIFOLDS

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Abstract

In an earlier paper we developed the classification of weakly symmetric pseudo–Riemannian manifolds G/H, where G is a semisimple Lie group and H is a reductive subgroup. We derived the classification from the cases where G is compact. As a consequence we obtained the classification of semisimple weakly symmetric manifolds of Lorentz signature $(n-1,1)$ and trans–Lorentzian signature $(n-2,2)$. Here we work out the classification of weakly symmetric pseudo–Riemannian nilmanifolds G/H from the classification for the case $G = N \rtimes H$ with H compact and N nilpotent. It turns out that there is a plethora of new examples that merit further study. Starting with that Riemannian case, we see just when a given involutive automorphism of H extends to an involutive automorphism of G, and we show that any two such extensions result in isometric pseudo–Riemannian nilmanifolds. The results are tabulated in the last two sections of the paper.

1. Introduction

There have been a number of important extensions of the theory of Riemannian symmetric spaces. Weakly symmetric spaces, introduced by A. Selberg [15], play important roles in number theory, Riemannian geometry and harmonic analysis; see [22]. Pseudo–Riemannian symmetric spaces also appear in number theory, differential geometry, relativity, Lie group representation theory and harmonic analysis. We study the classification of weakly symmetric pseudo–Riemannian nilmanifolds. This is essentially different from the Riemannian symmetric case; the only simply connected Riemannian symmetric nilmanifolds are the Euclidean spaces. The work of Vinberg, Yakimova and others shows that there are many simply connected weakly symmetric Riemannian nilmanifolds,

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and here we will see that each of them leads to a large number of simply connected weakly symmetric pseudo–Riemannian nilmanifolds.

Recall that a Riemannian manifold (M, ds^2) is weakly symmetric if, for every $x \in M$ and $\xi \in M_x$, there is an isometry $s_{x,\xi}$ such that $s_{x,\xi}(x) = x$ and $ds_{x,\xi}(\xi) = -\xi$. In that case (M, ds^2) is homogeneous. Let G be the identity component of the isometry group, so M has expression G/H. Then the following are equivalent, and each follows from weak symmetry. (i) M is (G,H)–commutative, i.e. the convolution algebra $L^1(H\backslash G/H)$ is commutative, i.e. (G,H) is a Gelfand pair, (ii) the left regular representation of G on $L^2(G/H)$ is multiplicity free, and (iii) the algebra of G–invariant differential operators on M is commutative. If G is reductive then conversely commutativity implies weak symmetry, but that can fail for example when $G = N \rtimes H$ with N nilpotent. See [22] for a systematic treatment with proper references.

The situation is more complicated for pseudo–Riemannian weakly symmetric spaces. If (M, ds^2) is pseudo–Riemannian then the definition of weakly symmetric can require only the existence of isometries $s_{x,\xi}$ as above, either for all $\xi \in M_x$ (as in our paper [5]) or only for $\xi \in M_x$ such that $ds^2(\xi, \xi) \neq 0$. The definition, (i) above, of commutativity, is problematic for noncompact H because the convolution product involves integration over H. The multiplicity free condition, (ii) above, becomes $Hom_H(\pi_\infty, id) \leqq 1$ for every $\pi \in \widehat{G}$, where π_∞ is the space of C^∞ vectors for π; see the work [17], [7], [8], [9] and [10] of Thomas and van Dijk on generalized Gelfand pairs. Our class of weakly symmetric pseudo–Riemannian manifolds consists of the complexifications, and real forms of the complexifications, of weakly symmetric Riemannian manifolds. There is still quite a lot of work to be done to clarify the relations between these various extensions of "weakly symmetric Riemannian manifold" and the corresponding extensions of "commutative space".

The weakly symmetric spaces we study have the form G/H where H is reductive in G and G is a semidirect product $N \rtimes H$ with N nilpotent. We find a large number of interesting new examples of these spaces, in particular many new homogeneous Lorentz and trans–Lorentz (e.g. conformally Lorentz) manifolds.

Our analysis of weakly symmetric pseudo–Riemannian nilmanifolds is based on the work of (in chronological order) Wolf ([20], [21]), Carcano [4], Benson-Jenkins-Ratcliff ([1], [2], [3]), Gordon [12], Vinberg ([18], [19]) and Yakimova ([23], [24], [25]), as described in [22, Chapter 13]. Starting there we adapt the "real form family" method of Chen–Wolf [6] to the setting of nilmanifolds.

We first consider weakly symmetric pseudo–Riemannian nilmanifolds (M, ds^2), $M = G/H$ with $G = N \rtimes H$, N nilpotent and H reductive in G. We show that every space of that sort belongs to a family of such spaces associated to a weakly symmetric Riemannian manifold $M_r = G_r/H_r$ (r for Riemannian). There H_r is a compact real form of the complex Lie group $H_{\mathbb{C}}$, $G_r = N_r \rtimes H_r$ is a real form of $G_{\mathbb{C}}$ and where N_r is a real form of $N_{\mathbb{C}}$, and M_r is a weakly symmetric Riemannian nilmanifold. In fact, every weakly symmetric Riemannian manifold is a commutative space, and we work a bit more generally, assuming that M and M_r are commutative nilmanifolds.

Definition 1.1. The **real form family** of G_r/H_r consists of the complexification $(G_r)_{\mathbb{C}}/(H_r)_{\mathbb{C}}$ and all G'/H' with the same complexification $(G_r)_{\mathbb{C}}/(H_r)_{\mathbb{C}}$. The space $(G_r)_{\mathbb{C}}/(H_r)_{\mathbb{C}}$ is considered to be a real manifold. We denote that real form family by $\{\{G_r/H_r\}\}$. In this paper, a space $G/H \in \{\{G_r/H_r\}\}$ is called **weakly symmetric** just when G_r/H_r is a weakly symmetric Riemannian manifold.

Our classifications will consist of examinations of the various possible real form families. Proposition 1.2 reduces this to an examination of involutions of the groups G_r.

Proposition 1.2. *Let $G_r = N_r \rtimes H_r$ where N_r is nilpotent, H_r is compact, and $M_r = G_r/H_r$ is a commutative, connected, simply connected Riemannian nilmanifold. Let σ be an involutive automorphism of G_r that preserves H_r. Then there is a unique G'/H' in the real form family $\{\{G_r/H_r\}\}$ such that G' is connected and simply connected, H' is connected, and $\mathfrak{g}' = \mathfrak{g}_r^+ + \sqrt{-1}\mathfrak{g}_r^-$ with $\mathfrak{h}' = \mathfrak{h}_r^+ + \sqrt{-1}\mathfrak{h}_r^-$ in terms of the (± 1)–eigenspaces of σ on $\mathfrak{g}_r$ and $\mathfrak{h}_r$. Up to covering, every space $G'/H' \in \{\{G_r/H_r\}\}$ either is obtained in this way or is the real manifold underlying the complex structure of $(G_r)_{\mathbb{C}}/(H_r)_{\mathbb{C}}$.*

Proof. First let σ be an involutive automorphism of G_r that preserves H_r. In terms of the (± 1)–eigenspaces of σ on $\mathfrak{g}_r$, $\mathfrak{g}' = \mathfrak{g}_r^+ + \sqrt{-1}\mathfrak{g}_r^-$ is a well defined Lie algebra with nilradical $\mathfrak{n}' = \mathfrak{n}_r^+ + \sqrt{-1}\mathfrak{n}_r^-$ and Levi component $\mathfrak{h}' = \mathfrak{h}_r^+ + \sqrt{-1}\mathfrak{h}_r^-$. Let G' be the connected simply connected group with Lie algebra $\mathfrak{g}'$, and H' and N' the analytic subgroups for $\mathfrak{h}'$ and $\mathfrak{n}'$. Then H' is reductive in G', N' is the nilradical of G' and is simply transitive on G'/H', and $G'/H' \in \{\{G_r/H_r\}\}$.

Conversely let $G'/H' \in \{\{G_r/H_r\}\}$; we want to construct the involution σ as above. Without loss of generality we may assume that G' is connected and simply connected, that H' is the analytic subgroup for $\mathfrak{h}'$, and that $\mathfrak{g}'$ and $\mathfrak{h}'$ are stable under the complex conjugation ν of $(\mathfrak{g}_r)_{\mathbb{C}}$ over $\mathfrak{g}_r$. Then $\mathfrak{g}' = (\mathfrak{g}')^+ + (\mathfrak{g}')^-$ and $\mathfrak{h}' = (\mathfrak{h}')^+ + (\mathfrak{h}')^-$, eigenspaces under ν. Now $\mathfrak{g}_r = (\mathfrak{g}')^+ + \sqrt{-1}(\mathfrak{g}')^-$ and $\mathfrak{h}_r = (\mathfrak{h}')^+ + \sqrt{-1}(\mathfrak{h}')^-$, and $\nu_{\mathfrak{g}_r}$ is the desired involution. q.e.d.

In Section 2 we work out the relations between involutive automorphisms of H_r (which of course are known) and involutive automorphisms of G_r. This is a matter of understanding how to extend an involutive automorphism of H_r to an automorphism of G_r, finding the condition for that extension to be involutive, and seeing that involutive extensions (when they exist) are essentially unique.

We need some technical preparation on linear groups and bilinear forms in order to carry out our classifications. That is carried out in Section 3. Some of it is delicate.

In Section 4 we examine the case where N_r is a Heisenberg group. There are two distinct reasons for examining this Heisenberg case separately. First, it indicates our general method and illustrates the need for the maximality condition when we look at a larger class of groups N_r. Second, and more important, the study of harmonic analysis on $H_r \rtimes U(k,\ell)/U(k,\ell)$ is a developing topic, and it is important to have a number of closely related examples.

Section 5 contains our first main results. We interpret Vinberg's classification of maximal irreducible commutative nilmanifolds, Table 5.1, as the classification of all real form families for all maximal irreducible commutative nilmanifolds. Then we extend Vinberg's classification to a complete analysis (including signatures of invariant pseudo–Riemannian metrics) of the real form families of maximal irreducible commutative nilmanifolds. Many of these cases are delicate and rely on both the results of Section 2 and specific technical information worked out in Section 3. Table 5.2 is the classification. Then we extract some non-Riemannian cases of special interest from Table 5.2. Those are the cases of Lorentz signature $(n-1,1)$, important in physical applications, and trans–Lorentz signature $(n-2,2)$, the parabolic geometry extension of conformal geometry.

Finally in Section 6 we extend the results of Sections 5 to a larger collection of real form families, those where G_r/H_r is indecomposable and satisfies certain technical conditions. Those results rely on the methods developed for the maximal irreducible case. They are collected in Table 6.1. As corollaries we extract the cases of Lorentz signature $(n-1,1)$ and trans–Lorentz signature $(n-2,2)$.

2. Reduction of the real form question

Proposition 1.2 reduces the classification of spaces in a real form family $\{\{G_r/H_r\}\}$ to a classification of involutive automorphisms of G_r that preserve H_r. If this section we reduce it further to a classification of involutive automorphisms of H_r. For that we first review some facts about nilpotent groups that occur in commutative Riemannian nilmanifolds.

Let $M_r = G_r/H_r$ be a commutative nilmanifold, $G_r = N_r \rtimes H_r$ with N_r connected simply connected and nilpotent, and with H_r compact and connected. Then N_r is the nilradical of G_r and it is the only nilpotent subgroup of G_r that is transitive on M_r [**20**, Theorem 4.2]. Further, N_r is commutative or 2–step nilpotent ([**1**, Theorem 2.4], [**12**, Theorem

2.2]). Thus we can decompose $\mathfrak{n}_r = \mathfrak{z}_r + \mathfrak{v}_r$ where $\mathfrak{z}_r$ is the center and $\mathfrak{v}_r$ is an $\mathrm{Ad}(H_r)$–invariant complement. Following [22, Chapter 13] we say that

$$\begin{aligned} &G_r/H_r \text{ is } \textit{irreducible} \\ &\qquad \text{if } [\mathfrak{n}_r, \mathfrak{n}_r] = \mathfrak{z}_r \text{ and } H_r \text{ is irreducible on } \mathfrak{n}_r/[\mathfrak{n}_r, \mathfrak{n}_r] \\ &(G_r/Z)/(H_r \cap Z) \text{ is a } \textit{central reduction} \text{ of } G_r/H_r \\ &\qquad \text{if } Z \text{ is a closed central subgroup of } G_r \\ &G_r/H_r \text{ is } \textit{maximal} \text{ if it is not a nontrivial central reduction.} \end{aligned} \tag{2.1}$$

Now we employ a decomposition from [18] and [19]. Split $\mathfrak{z}_r = \mathfrak{z}_{r,0} \oplus [\mathfrak{n}_r, \mathfrak{n}_r]$ with $\mathrm{Ad}(H_r)\mathfrak{z}_{r,0} = \mathfrak{z}_{r,0}$. Also decompose $\mathfrak{n}_r = \mathfrak{z}_{r,0} \oplus ([\mathfrak{n}_r, \mathfrak{n}_r] + \mathfrak{v}_r)$ with $\mathfrak{v}_r = \mathfrak{v}_{r,1} + \cdots + \mathfrak{v}_{r,m}$ vector space direct sum where $\mathrm{Ad}(H_r)$ preserves and acts irreducibly on each $\mathfrak{v}_{r,i}$. The representations of $\mathrm{Ad}(H_r)$ on the $\mathfrak{v}_{r,i}$ are mutually inequivalent. Consider the subalgebras $\mathfrak{n}_{r,i} = [\mathfrak{v}_{r,i}, \mathfrak{v}_{r,i}] + \mathfrak{v}_{r,i}$ of $\mathfrak{n}_r$ generated by $\mathfrak{v}_{r,i}$. Then $[\mathfrak{v}_{r,i}, \mathfrak{v}_{r,j}] = 0$ for $i \neq j$ and $(\xi_0, \xi_1, \ldots, \xi_m) \mapsto (\xi_0 + \xi_1 + \cdots + \xi_m)$ defines an $\mathrm{Ad}(H_r)$–equivariant homomorphism of $\mathfrak{z}_r \oplus \mathfrak{n}_{r,1} \oplus \cdots \oplus \mathfrak{n}_{r,m}$ onto $\mathfrak{n}_r$ with central kernel.

Let $N_{r,i}$ be the analytic subgroup of N_r for $\mathfrak{n}_{r,i}$. Let $H_{r,i} = H_r/J_{r,i}$ where $J_{r,i}$ is the kernel of the adjoint action of H_r on $\mathfrak{n}_{r,i}$. Similarly let $J_{r,0}$ be the kernel of the adjoint action of H_r on $\mathfrak{z}_{r,0}$ and $H_{r,0} = H_r/J_{r,0}$. Let $G_{r,i} = N_{r,i} \rtimes H_{r,i}$ for $i \geqq 0$. As in [22, Section 13.4C], we summarize.

Proposition 2.2. *The representation of H_r on $\mathfrak{v}_{r,i}$ is irreducible. If $i \neq j$ then $[\mathfrak{n}_{r,i}, \mathfrak{n}_{r,j}] = 0$ and the representations of H_r on $\mathfrak{v}_{r,i}$ and $\mathfrak{v}_{r,j}$ are mutually inequivalent. $M_{r,0} = G_{r,0}/H_{r,0}$ is an Euclidean space, the other $M_{r,i} = G_{r,i}/H_{r,i}$ are irreducible commutative Riemannian nilmanifolds.*

The $\mathrm{Ad}(H_r)$–equivariant homomorphism of $\mathfrak{z}_{r,0} \oplus \mathfrak{n}_{r,1} \oplus \cdots \oplus \mathfrak{n}_{r,m}$ onto $\mathfrak{n}_r$ defines a Riemannian fibration

$$\gamma : \widetilde{M_r} = M_{r,0} \times M_{r,1} \times \cdots \times M_{r,m} \to M_r \tag{2.3}$$

with flat totally geodesic fibers that are intersections of the $\exp([\mathfrak{v}_{r,i}, \mathfrak{v}_{r,i}])$.

Theorem 2.4. *Let $M_r = G_r/H_r$ be a commutative nilmanifold, $G_r = N_r \rtimes H_r$ with N_r connected simply connected and nilpotent, and with H_r compact and connected. Let G'/H', $G''/H'' \in \{\{G_r/H_r\}\}$. If $H' \cong H''$, then $G' \cong G''$.*

Proof. Let σ', σ'' be the involutive automorphisms of G_r that define G' and G''. As $H' \cong H''$, their restrictions to $\mathfrak{h}_r$ belong to the same component of the automorphism group. Define $L = \mathrm{Ad}(H_r) \cup \sigma' \mathrm{Ad}(H_r)$. It is a compact group of linear transformations of $\mathfrak{g}_r$ that has one or two components, and $\sigma'' \in \sigma' \mathrm{Ad}(H_r)$. Let T be a maximal torus of the centralizer of σ' in $\mathrm{Ad}(H_r)$. A theorem of de Siebenthal [16] on compact disconnected Lie groups says that every element of $\sigma' \mathrm{Ad}(H_r)$

is $\mathrm{Ad}(H_r)$–conjugate to an element of $\sigma' T$. Thus we may replace σ'' by an $\mathrm{Ad}(H_r)$–conjugate and assume $\sigma'' \in \sigma' T$. That done, σ' and σ'' commute. Thus we may assume that $\nu := (\sigma')^{-1} \cdot \sigma''$ satisfies $\nu^2 = 1$.

We first consider the case where G_r/H_r is irreducible. In other words, following (2.1), $[\mathfrak{n}_r, \mathfrak{n}_r] = \mathfrak{z}_r$ and $\mathrm{Ad}(H_r)|_{\mathfrak{v}_r}$ is irreducible. Thus the commuting algebra of $\mathrm{Ad}(H_r)|_{\mathfrak{v}_r}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, so the only elements of square 1 in that commuting algebra are ± 1. As $\nu^2 = 1$, now $\nu|_{\mathfrak{v}_r} = \pm 1$. If $\nu = 1$ on $\mathfrak{v}_r$ then $\nu = 1$ on $\mathfrak{n}_r$, in other words $\sigma' = \sigma''$ on $\mathfrak{n}_r$. Then $\mathfrak{n}' = \mathfrak{n}''$. As σ' and σ'' commute, and as we have an isomorphism $f : H' \cong H''$, we extend f to an isomorphism $G' \to G''$ by the identity on $N' = N''$.

The other possibility is that $\nu = -1$ on $\mathfrak{v}_r$. As linear transformations of $\mathfrak{v}_r$, $\sigma' = c'$ and $\sigma'' = c''$ where $c'^2 = 1 = c''^2$ and $c'c'' = c''c'$. Again using irreducibility, $c' = \pm 1$ and $c'' = \pm 1$. But $c'c'' = \nu = -1$. So we may suppose $c' = 1$ and $c'' = -1$. Then $\mathfrak{n}' = \mathfrak{z}_r + \mathfrak{v}_r$ and $\mathfrak{n}'' = \mathfrak{z}_r + \sqrt{-1}\mathfrak{v}_r$. Now define $\varphi : \mathfrak{n}' \to \mathfrak{n}''$ by $\varphi(z, v) = (-z, \sqrt{-1}v)$. Compute $[\varphi(z_1, v_1), \varphi(z_2, v_2)] = [(-z_1, \sqrt{-1}v_1), (-z_2, \sqrt{-1}v_2)] = (-[v_1, v_2], 0) = \varphi([v_1, v_2], 0) = \varphi[(z_1, v_1), (z_2, v_2)]$. Thus $\varphi : \mathfrak{n}' \to \mathfrak{n}''$ is an isomorphism. It commutes with $(\mathrm{Ad}(H_r)_{\mathbb{C}})$, so it combines with $f : H' \cong H''$ to define an isomorphism $G' \to G''$.

That completes the proof of Theorem 2.4 in the case where G_r/H_r is irreducible. We now reduce the general case to the irreducible case, using Proposition 2.2, i.e., material from [**22**, Section 13.4C]. As the representations α_i of H_r on the $\mathfrak{v}_{r,i}$ are inequivalent, $\nu|_{\mathfrak{v}_r}$ permutes the $\mathfrak{v}_{r,i}$. If $\nu(\mathfrak{v}_{r,i}) = \mathfrak{v}_{r,j}$ with $i \neq j$ then ν defines an equivalence of α_i and α_j, contradicting inequivalence. Thus $\nu(\mathfrak{v}_{r,i}) = \mathfrak{v}_{r,i}$ for every i. As $\nu^2 = 1$ now $\nu|_{\mathfrak{v}_{r,i}} = \varepsilon_i = \pm 1$. As in the irreducible case $f : H' \cong H''$ together with the $\nu|_{\mathfrak{v}_{r,i}}$ defines an isomorphism $G' \cong G''$. q.e.d.

Theorem 2.5. *Let $M_r = G_r/H_r$ be a commutative, connected, simply connected Riemannian nilmanifold, say with $G_r = N_r \rtimes H_r$ where H_r is compact and connected and N_r is nilpotent. Let θ and H denote an involutive automorphism of H_r and the corresponding real form of $(H_r)_{\mathbb{C}}$. Consider the fibration $\gamma : \widetilde{G_r}/\widetilde{H_r} = \widetilde{M_r} \to M_r = G_r/H_r$ of (2.3). Then θ lifts to an involutive automorphism $\widetilde{\theta}$ of $\widetilde{H_r}$, and $\widetilde{\theta}$ extends to an automorphism σ of G_r such that $d\sigma(\mathfrak{v}_r) = \mathfrak{v}_r$.*

If $\sigma^2 = 1$ then the corresponding $(G, H) \in \{\{G_r/H_r\}\}$ is a homogeneous pseudo–Riemannian manifold. If we cannot choose σ so that $\sigma^2 = 1$ then θ and H do not correspond to an element of $\{\{G_r/H_r\}\}$.

Proof. In the notation leading to Proposition 2.2, $\mathfrak{v}_r = \mathfrak{v}_{r,1} + \cdots + \mathfrak{v}_{r,m}$ where $\mathrm{Ad}(H_r)$ acts on $\mathfrak{v}_{r,i}$ by an irreducible representation α_i, and the α_i are mutually inequivalent. As $\theta(H_r) = H_r$ the corresponding representations $\alpha'_i = \alpha_i \cdot \theta$ just form a permutation of the α_i, up to equivalence. If θ is inner then $\alpha'_i = \alpha_i$.

If $i \neq j$ with α_i equivalent to α'_j, let τ denote the intertwiner. So $\alpha'_i(h)\tau = \tau\alpha_j(h)$ and the intertwiner τ interchanges $\mathfrak{v}_{r,i}$ and $\mathfrak{v}_{r,j}$. On the other hand if α_i is equivalent to α'_i, the intertwiner τ satisfies $\alpha'_i(h)\tau = \tau\alpha_i(h)$ and $\tau\mathfrak{v}_{r,i} = \mathfrak{v}_{r,i}$. Thus $\alpha(\theta(h))\tau = \tau\alpha(h)$ for $h \in H_r$.

Define $\widetilde{H_r} = H_r \cup tH_r$ where $tht^{-1} = \theta(h)$ and t^2 belongs to the center of H_r. Define $\sigma(h) = \alpha(h)$ and $\sigma(th) = \tau\alpha(h)$ for $h \in H_r$ (in particular $\sigma(t) = \tau$). We check that σ is a representation of $\widetilde{H_r}$ on $\mathfrak{v}$:

(i) $\sigma(th_1)\sigma(th_2) = \tau\alpha(h_1)\tau\alpha(h_2) = \alpha(\theta h_1)\alpha(h_2) = \alpha(th_1th_2) = \sigma(th_1th_2)$,
(ii) $\sigma(th_1)\sigma(h_2) = \tau\alpha(h_1)\alpha(h_2) = \tau\alpha(h_1h_2) = \sigma(th_1h_2)$, and
(iii) $\sigma(h_1)\sigma(th_2) = \alpha(h_1)\tau\alpha(h_2) = \tau\alpha(\theta h_1)\alpha(h_2) = \tau\alpha(\theta(h_1)h_2) = \sigma(t\theta(h_1)h_2) = \sigma(h_1th_2)$.

Now we check that $\sigma(\widetilde{H_r})$ consists of automorphisms of $\mathfrak{n}_r$. Set $\sigma(t)$ equal to the identity on $\mathfrak{z}_{r,0}$. Since $[\mathfrak{v}_{r,i}, \mathfrak{v}_{r,j}] = 0$ for $i \neq j$ we extend $\sigma(t)$ to the subalgebras $\mathfrak{z}_{r,i} := [\mathfrak{v}_{r,i}, \mathfrak{v}_{r,i}]$ by $\Lambda^2(\alpha_i)$. In order that this be well defined on $[\mathfrak{n}_r, \mathfrak{n}_r]$ it suffices to know that $[\mathfrak{n}_r, \mathfrak{n}_r] = [\mathfrak{v}_r, \mathfrak{v}_r]$ is the direct sum of the $\mathfrak{z}_{r,i} = [\mathfrak{v}_{r,i}, \mathfrak{v}_{r,i}]$. That is clear if there is only one $[\mathfrak{v}_{r,i}, \mathfrak{v}_{r,i}]$, in other words if $\mathrm{Ad}(H_r)$ is irreducible on $\mathfrak{v}_r$. In general θ permutes the irreducible factors of the representation of H_r on $\mathfrak{v}_r$, so it permutes the $\mathfrak{v}_{r,i}$. Thus θ lifts to $\widetilde{H_r}$, and we apply the irreducible case result to the factors $M_{r,i}$. Thus θ extends to the automorphism σ of G_r.
q.e.d.

Remark 2.6. If θ extends to $\sigma \in \mathrm{Aut}(G_r)$ then evidently that extension is well defined on $\mathfrak{v}_r \rtimes H_r$. But the converse holds as well (and this will be important for us): If θ extends to $\alpha \in \mathrm{Aut}(\mathfrak{v}_r \rtimes H_r)$ then θ extends to an element of $\mathrm{Aut}(G_r)$. For α extends to $(\mathfrak{z}_r + \mathfrak{v}_r) \rtimes H_r$ since $\mathfrak{z}_r$ is an $\mathrm{Ad}(H_r)$–invariant summand of $\Lambda^2_{\mathbb{R}}(\mathfrak{v}_r)$, and that extension of α exponentiates to some $\sigma \in \mathrm{Aut}(G_r)$.

Corollary 2.7. *Let $M_r = G_r/H_r$ be a commutative, connected, simply connected Riemannian nilmanifold, say with $G_r = N_r \rtimes H_r$ where H_r is compact and connected and N_r is nilpotent. Let $M = G/H$ belong to the real form family $\{\{G_r/H_r\}\}$. Then $G = N \rtimes H$ where $\mathfrak{n} = \mathfrak{z} + \mathfrak{v}$, $\mathfrak{z}$ is the center, and each of $\mathfrak{n}$ and $\mathfrak{z}$ has a nondegenerate $Ad(H)$–invariant symmetric bilinear form. In particular $M = G/H$ is a pseudo–Riemannian homogeneous space.*

Proof. By Proposition 1.2 and Theorem 2.5, the pair (G, H) corresponds to an involutive automorphism σ of G_r whose complex extension and restriction to G gives a Cartan involution of H. We may assume that σ preserves the nilradical $\mathfrak{n}_r$ of $\mathfrak{g}_r$, the center $\mathfrak{z}_r$ of $\mathfrak{n}_r$, and the orthocomplement $\mathfrak{v}_r$ of $\mathfrak{z}_r$ in $\mathfrak{n}_r$. Thus, in the Cartan duality construction described in Proposition 1.2, the positive definite $\mathrm{Ad}(H_r)$–invariant inner product on $\mathfrak{n}_r$ (corresponding to the invariant Riemannian metric on

G_r/H_r), gives us a nondegenerate $\mathrm{Ad}(H)$–invariant symmetric bilinear form on $\mathfrak{n}$ for which $\mathfrak{v} = \mathfrak{z}^\perp$. The corollary follows. q.e.d.

Example 2.8. Consider the Heisenberg group case $\mathfrak{n}_r = \mathrm{Im}\,\mathbb{C} + \mathbb{C}^n$ and $\mathfrak{h}_r = \mathfrak{u}(n)$, with $\theta(h) = \overline{h}$. Then θ extends to an involutive automorphism σ of $\mathfrak{g}_r$ by complex conjugation on $\mathfrak{n}_r$. Denote $\widetilde{H_r} = H_r \cup tH_r$ where $t^2 = 1$ and $tht^{-1} = \theta(h)$ for $h \in H_r$. Then $\sigma(th)n = \sigma(tht^{-1} \cdot t)n = \sigma(\theta(h))\sigma(t)n = \sigma(\overline{h})\overline{n} = \sigma(t)(\sigma(h)n)$. Thus in fact σ defines a representation of $\widetilde{H_r}$ given, in this Heisenberg group case, by $\sigma(t) : n \mapsto \overline{n}$.

3. Irreducible commutative nilmanifolds: preliminaries

Recall the definition (2.1) of maximal irreducible commutative Riemannian nilmanifolds. They were classified by Vinberg ([**18**], [**19**]) (or see [**22**, Section 13.4A]), and we are going to extend that classification to the pseudo–Riemannian setting. In order to do that we need some specific results on linear groups and bilinear forms. We work those out in this section, and we extend the Vinberg classification in the next section.

U(1) factors

We first look at the action of θ when H_r has a $U(1)$ factor and see just when θ extends to an involutive automorphism of $\mathfrak{g}_r$, in other words just when we do have a corresponding (G, H) in $\{\{G_r, H_r\}\}$.

Lemma 3.1. *Let (G_r, H_r) be an irreducible commutative Riemannian nilmanifold such that $H_r = U(1) \cdot H_r'$. Suppose that $|U(1) \cap H_r'| \geqq 3$. Let $(G, H) \in \{\{G_r, H_r\}\}$ corresponding to an involutive automorphism θ of H_r. If $\theta|_{H_r'}$ is inner then H has form $U(1) \cdot H'$. If $\theta|_{H_r'}$ is outer then H has form $\mathbb{R}^+ \cdot H'$.*

Lemma 3.2. *Let (G_r, H_r) be an irreducible commutative Riemannian nilmanifold such that $H_r = U(1) \cdot H_r'$. Suppose that $|U(1) \cap H_r'| \leqq 2$. Let θ' be an involutive automorphism of H_r' and H' the corresponding real form of $(H_r')_{\mathbb{C}}$. Then $\{\{G_r, H_r\}\}$ contains both an irreducible commutative pseudo–Riemannian nilmanifold with $H = U(1) \cdot H'$ and an irreducible commutative pseudo–Riemannian nilmanifold with $H = \mathbb{R}^+ \cdot H'$.*

For all G_r/H_r in Vinberg's list (5.1), for which $H_r = (U(1)\cdot)H_r'$, the representation of H_r on $\mathfrak{v}_r$ is not absolutely irreducible. In other words $(\mathfrak{v}_r)_{\mathbb{C}}$ is of the form $\mathfrak{w}_r \oplus \overline{\mathfrak{w}_r}$ in which $\mathfrak{v}_r$ consists of the $(w, \overline{w})$. Thus $u \in U(1)$ acts by $(w, \overline{w}) \mapsto (uw, \overline{uw})$. In consequence,

Lemma 3.3. *If $(G, H) \in \{\{G_r, H_r\}\}$ with $H = \mathbb{R}^+ \cdot H'$ then $\mathfrak{v}_r = \mathfrak{v}_r' \oplus \mathfrak{v}_r''$ direct sum of real vector spaces that are eigenspaces of $\mathbb{R}^+$, in other words by the condition that $e^t \in \mathbb{R}^+$ acts on $\mathfrak{v}_r$ by $v' + v'' \mapsto e^t v' + e^{-t} v''$. In particular $\mathfrak{v}'$ and $\mathfrak{v}''$ are totally isotropic, and paired with each other, for the $Ad(H)$–invariant inner product on $\mathfrak{v}_r$. Consequently* *that invariant inner product has signature* (p,p) *where* $p = \dim_{\mathbb{C}} \mathfrak{v}_r = \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{v}_r$.

Spin representations

Next, we recall signatures of some spin representations for the groups $Spin(k,\ell)$.

Lemma 3.4. ([13, Chapter 13]) *Real forms of* $Spin(7;\mathbb{C})$ *satisfy*

$$Spin(6,1) \subset SO^*(8) \simeq SO(6,2), Spin(5,2) \subset SO^*(8) \simeq SO(6,2)$$
$$\text{and } Spin(4,3) \subset SO(4,4).$$

Real forms of $Spin(9;\mathbb{C})$ *satisfy*

$$Spin(8,1) \subset SO(8,8), Spin(7,2) \subset SO^*(16),$$
$$Spin(6,3) \subset SO^*(16) \text{ and } Spin(5,4) \subset SO(8,8).$$

Real forms of $Spin(10;\mathbb{C})$ *satisfy*

$$Spin(9,1) \subset SL(16;\mathbb{R}), Spin(8,2) \subset SU(8,8),$$
$$Spin(7,3) \subset SL(4;\mathbb{H}) \subset Sp(4,4), Spin(6,4) \subset SU(8,8),$$
$$Spin(5,5) \subset SL(16;\mathbb{R}) \subset SO(16,16),$$
$$\text{and } Spin^*(10) \subset SL(4;\mathbb{H}) \subset Sp(4,4).$$

$\mathbf{E_6}$

Issues involving E_6 are more delicate. If L is a connected reductive Lie group, let $\varphi_{L,b}$ denote the fundamental representation corresponding to the b^{th} simple root in Bourbaki order, $\varphi_{L,0}$ denote the trivial 1–dimensional representation, and write τ for the defining 1–dimensional representation of $U(1)$. Then $\varphi_{E_6,6}|_{C_4} = \varphi_{C_4,2}$, $\varphi_{E_6,6}|_{F_4} = \varphi_{F_4,4} \oplus \varphi_{F_4,0}$ and $\varphi_{E_6,6}|_{A_5A_1} = (\varphi_{A_5,5} \otimes \varphi_{A_1,1}) \oplus (\varphi_{A_5,2} \otimes \varphi_{A_1,0})$. These C_4 and F_4 restrictions are real. As $\varphi_{E_6,6}(H)$ is noncompact, and there are only one or two summands under its maximal compact subgroup, we conclude that $\varphi_{E_6,6}(E_{6,C_4}) \subset SO(27,27)$, $\varphi_{E_6,6}(E_{6,F_4}) \subset SO(26,1)$ and $\varphi_{E_6,6}(E_{6,A_5A_1}) \subset SU(15,12)$.

$\varphi_{E_6,6}|_{D_5T_1}$ has three summands, $(\varphi_{D_5,0}\otimes\tau^{-1})\oplus(\varphi_{D_5,4}\otimes\tau^{-1})\oplus(\varphi_{D_5,1}\otimes\tau^2)$, of respective degrees 1, 16 and 10, so the above argument must be supplemented. For that, we look at $\varphi_{E_6,6}|_{L_r}$ where $L_r \cong SU(3)$ is a certain subgroup of E_6.

Write ξ_b for the b^{th} fundamental highest weight of A_2. Thus A_2 has adjoint representation $\alpha := \varphi_{A_2,\xi_1+\xi_2}$ Denote $\beta := \varphi_{A_2,2\xi_1+2\xi_2}$, so the symmetric square $S^2(\alpha) = \varphi_{A_2,0} \oplus \alpha \oplus \beta$. Then [11, Theorem 16.1] E_6 has a subgroup $L_r \cong SU(3)$ such that $\varphi_{E_6,6}|_{L_r} = \beta$. Further [11, Table 24] L_r is invariant under the outer automorphism of E_6 that interchanges $\varphi_{E_6,6}$ with its dual $\varphi_{E_6,1}$. The representation of H_r on $\mathfrak{v}_r$ treats $\mathfrak{v}_r$ as the unique $(\varphi_{E_6,6} \oplus \varphi_{E_6,1})$–invariant of $\mathfrak{v}_{\mathbb{C}}$, so it is the invariant real form for $\varphi_{E_6,6}(L_r) \oplus \varphi_{E_6,1}(L_r)$. Thus the representation of $H = E_{6,D_5T_1}$ on $\mathfrak{v}$ treats $\mathfrak{v}$ as the invariant real form of $\mathfrak{v}_{\mathbb{C}}$ for $\varphi_{E_6,6}(L) \oplus \varphi_{E_6,1}(L)$ where $L = (L_r)_{\mathbb{C}} \cap H$. L must be one of the real forms $SU(1,2)$ or $SL(3;\mathbb{R})$ of $(L_r)_{\mathbb{C}} = SL(3;\mathbb{C})$. Now $S^2(\alpha)$ has signature $(20,16)$ or $(21,15)$. Subtracting α from $S^2(\alpha)$ leaves signature $(16,12)$, and subtracting $\varphi_{A_2,0}$ (for the Killing form of L) leaves signature $(15,12)$ or $(16,11)$. But this must come from the summands of $\varphi_{E_6,6}|_{D_5T_1}$, which have degrees 1, 16 and 10. Thus $\varphi_{E_6,6}(E_{6,D_5T_1}) \subset SU(16,11)$.

Split quaternion algebra

Another delicate matter concerns the split real quaternion algebra $\mathbb{H}_{sp}$. While $\mathbb{H}_{sp} \cong \mathbb{R}^{2\times 2}$, the conjugation of $\mathbb{H}_{sp}$ over $\mathbb{R}$ is given by $\overline{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Thus $\operatorname{Im}\mathbb{H}_{sp}^{n\times n}$ has real dimension $2n^2+n$ and is isomorphic to the Lie algebra of $Sp(n;\mathbb{R})$, and $\operatorname{Re}\mathbb{H}_{sp}^{n\times n}$ has real dimension $2n^2-n$. In Case 9 of Table 5.2 below, we can have $\mathfrak{v} = \mathbb{H}_{sp}^n$ with $\mathfrak{z} = \operatorname{Re}\mathbb{H}_{sp,0}^{n\times n} \oplus \operatorname{Im}\mathbb{H}_{sp}$, where $H = \{1, U(1), \mathbb{R}^+\}Sp(n;\mathbb{R})$. Then the bracket $\mathfrak{v}\times\mathfrak{v} \to \mathfrak{z}$ has two somewhat different pieces. The obvious one is $\mathfrak{v}\times\mathfrak{v} \to \operatorname{Im}\mathbb{H}_{sp}$, given by $[u,v] = \operatorname{Im} u\overline{v}$. For the more subtle one we note $\mathbb{H}^n \simeq \mathbb{C}^{2n}$ as a $\mathbb{C}^*\cdot Sp(n;\mathbb{C})$–module, $\mathfrak{z} = [\mathfrak{v},\mathfrak{v}] \subset \Lambda^2_{\mathbb{C}}(\mathbb{C}^{2n})$, and $\dim_{\mathbb{R}} \operatorname{Re}\mathbb{H}_{sp}^{n\times n} = 2n^2-n = \dim_{\mathbb{C}}(\Lambda^2_{\mathbb{C}}(\mathbb{C}^{2n}))$, so $\operatorname{Re}\mathbb{H}_{sp}^{n\times n}$ is an $Sp(n;\mathbb{R})$–invariant real form of $\Lambda^2_{\mathbb{C}}(\mathbb{C}^{2n})$. Thus $\operatorname{Re}\mathbb{H}_{sp}^{n\times n} \simeq \Lambda^2_{\mathbb{R}}(\mathbb{R}^{2n})$ as an $Sp(n;\mathbb{R})$–module. If ω denotes the $Sp(n;\mathbb{R})$–invariant antisymmetric bilinear form on $\mathbb{R}^{2n}$ then $\langle u\wedge v, u'\wedge v'\rangle = \omega(u,u')\omega(v,v')$ defines the $Sp(n;\mathbb{R})$–invariant symmetric bilinear form on $\Lambda^2_{\mathbb{R}}(\mathbb{R}^{2n})$. The assertions in Case 9 of Table 5.2 follow.

$\mathbf{SL(n/2;\mathbb{H})}$ and $\mathbf{GL(n/2;\mathbb{H})}$

The real forms $SL(n/2;\mathbb{H})$ of $SU(n)$ and $GL(n/2;\mathbb{H})$ of $U(n)$ can appear or not, in an interesting way.

Lemma 3.5. *Let $(G,H) \in \{\{G_r, H_r\}\}$. Suppose that $H_r = U(n)$ or $H_r = SU(n)$, and that $\mathfrak{v}_r = \mathbb{C}^n$. Then $H \neq GL(n/2;\mathbb{H})$ and $H \neq SL(n/2;\mathbb{H})$.*

Proof. Let H_r be $U(n)$ or $SU(n)$ and $\mathfrak{v}_r = \mathbb{C}^n$. Suppose that H is $GL(n/2;\mathbb{H})$ or $SL(n/2;\mathbb{H})$. Then the corresponding involutive automorphism θ of H_r is given by $\theta(g) = J\overline{g}J^{-1}$ where $J = \operatorname{diag}\{J',\dots,J'\}$ with $J' = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then one extension of θ to $G_r = N_r \rtimes H_r$ is given on $G_r/Z_r \simeq \mathfrak{v}_r \rtimes H_r$ by $\alpha(x,g) = (J\overline{x}, J\overline{g}J^{-1})$. This extension is not involutive: $\sigma^2(x,1) = (-x,1)$. However, since H is $GL(n/2;\mathbb{H})$ or $SL(n/2;\mathbb{H})$, θ has an involutive extension β to G_r. Thus $\beta(x,g) = (B\overline{x}, J\overline{g}J^{-1})$ for some $B \in U(n)$. We compare α and β. Calculate

$$\begin{aligned}\beta(x,g)\beta(x',g') &= (B\overline{x}, J\overline{g}J^{-1})(B\overline{x'}, J\overline{g'}J^{-1})\\ &= (B\overline{x} + J\overline{g}J^{-1}(B\overline{x'}), J\overline{gg'}J^{-1})\end{aligned}$$

and

$$\beta((x,g)(x',g')) = \beta(x+g(x'),gg') = (B\overline{x}+B\overline{gx'}, J\overline{gg'}J^{-1}).$$

Since β is an automorphism this says $J\overline{g}J^{-1}B = B\overline{g}$, in other words $\overline{g}\cdot J^{-1}B = J^{-1}B\cdot\overline{g}$. Thus $J^{-1}B$ is a central element of $U(n)$, in other words $B = cJ$ with $c \in \mathbb{C}, |c| = 1$. As β is involutive we calculate $(x,g) = \beta^2(x,g) = \beta(B\overline{x}, J\overline{g}J^{-1}) = (B\,\overline{B\overline{x}}, J\,\overline{J\overline{g}J^{-1}}\,J^{-1}) = (B\overline{B}x, g)$. Thus $I = B\overline{B} = (cJ)(\overline{cJ}) = |c|^2J^2 = -I$. That contradicts Theorem 2.5, and the Lemma follows. q.e.d.

Remark 3.6. Let $(G,H) \in \{\{G_r,H_r\}\}$ with $H_r = U(n)$ or $SU(n)$. Suppose that H is $GL(n/2;\mathbb{H})$ or $SL(n/2;\mathbb{H})$ as defined by the involutive automorphism θ of H_r. Then $\theta(g) = J\overline{g}J^{-1}$ as noted in the proof of Corollary 3.5. By contrast here, if $\mathfrak{v}_r$ is a subspace of $\bigotimes^2(\mathbb{C}^n)$ then θ does extend to an involutive automorphism of G_r. That extension is given on $\mathfrak{v}_r$ by $\alpha(x\otimes y, g) = (J\overline{x}\otimes J\overline{y}, J\overline{g}J^{-1})$, and on $\mathfrak{z}_r$ as a subspace of $\Lambda^2_{\mathbb{R}}(\mathfrak{v}_r)$. The point here is that $\alpha^2(x\otimes y,g) = (J\overline{J\overline{x}}\otimes J\overline{J\overline{y}}, J\overline{J\overline{g}J^{-1}}J^{-1} = (J^2x\otimes J^2y, g) = ((-x)\otimes(-y), g) = (x\otimes y, g)$.

Remark 3.7. Following the idea of Remark 3.6, suppose that H_r is locally isomorphic to a product, say $H_r = H_r'\cdot H_r''$ with $H_r'' = U(n)$ or $SU(n)$, and that $\mathfrak{v}_r = \mathfrak{v}_r'\otimes\mathfrak{v}_r''$ accordingly with $\mathfrak{v}_r'' = \mathbb{C}^n$. Suppose $\theta = \theta'\otimes\theta''$ so that H splits the same way. Let σ' denote the extension of θ' to $\mathfrak{v}_r'$. If ${\sigma'}^2 = 1$ then H'' cannot be $GL(n/2;\mathbb{H})$ or $SL(n/2;\mathbb{H})$. For example this says that H cannot be $SL(m;\mathbb{R})\times SL(n/2;\mathbb{H})$. But if ${\sigma'}^2 = -1$ and $\mathfrak{v}_r = \mathbb{C}^m\otimes\mathbb{C}^n$, one must consider the possibility that H be $SL(m/2;\mathbb{H})\cdot SL(n/2;\mathbb{H})$.

$\mathbf{U(1)\cdot H_r''}$

Remark 3.8. A small variation the argument of Remark 3.7 has a useful application to some more of the cases $H_r = H_r''\cdot H_r'''$ where H_r'' has form $U(1)\cdot H_r'$. If $\theta|_{H_r''}$ has form $g\mapsto JgJ^{-1}$ with $J^2 = -I$, so that the extension $x\mapsto Jx$ to $\mathfrak{v}_r$ has square $-I$, we can replace J by iJ; then the extension $x\mapsto iJx$ is involutive. If $\theta|_{H_r''}$ has form $g\mapsto J\overline{g}J^{-1}$ with J real and $J^2 = -I$, then this fails, for if $c\in U(1)$ the extension $x\mapsto cJ\overline{x}$ has square -I, thus is not involutive. That could be balanced if extension of $\theta|_{H_r'''}$ also has square $-I$.

$\mathbf{SO^*(2\ell)}$ and $\mathbf{Sp(n;\mathbb{R})}$

The analog of Lemma 3.5 (or at least the analog of the proof) for the groups $SO^*(2\ell)$ is

Lemma 3.9. *Let $(G,H)\in\{\{G_r,H_r\}\}$ with $H_r = SO(2\ell)$ and $\mathfrak{v}_r = \mathbb{R}^{2\ell}$. Then $H \neq SO^*(2\ell)$.*

Proof. Suppose $H = SO^*(2\ell)$. Then it is the centralizer in $SO(2\ell)$ of $J = \left(\begin{smallmatrix}0 & I\\ -I & 0\end{smallmatrix}\right)$, and $\theta(g) = JgJ^{-1}$. Thus θ extends to an automorphism

α of G_r given on $G_r/Z_r \simeq \mathfrak{v}_r \rtimes H_r$ by $\alpha(x,g) = (Jx, JgJ^{-1})$. Note that α^2 is -1 on $\mathfrak{v}_r$. Now let β be an involutive extension of θ, so $\beta(x,g) = (Bx, JgJ^{-1})$ with $B^2 = I$. We compute $\beta(x,g)\beta(x',g') = (Bx + JgJ^{-1}Bx', Jgg'J^{-1})$ and $\beta((x,g)(x',g')) = \beta(x+gx', gg') = (Bx+Bgx', Jgg'J^{-1})$, so $JgJ^{-1}Bx' = Bgx'$ and it follows that $J^{-1}B$ is central in $SO(2\ell)$. Thus either $J^{-1}B = I$ so $J = B$ contradicting $J^2 = -I = -B^2$, or $J^{-1}B = -I$ so $J = -B$ contradicting $J^2 = -I = -B^2$. That contradicts $\beta^2 = 1$, and the Lemma follows. q.e.d.

The arguments of Lemmas 3.5 and 3.9 go through without change for $Sp(n;\mathbb{R})$:

Lemma 3.10. *Let $(G,H) \in \{\{G_r, H_r\}\}$ with $H_r = Sp(n)$ and $\mathfrak{v}_r = \mathbb{C}^{2n}$. Then $H \neq Sp(n;\mathbb{R})$.*

Proof. $Sp(n;\mathbb{R})$ is defined by the involution $\theta : g \mapsto JgJ^{-1}$ of $Sp(n)$ with fixed point set $U(n)$. As before, θ extends to an automorphism α on G_r given on $G_r/Z_r \simeq \mathfrak{v}_r \rtimes H_r$ by $\alpha(x,g) = (Jx, JgJ^{-1})$, where $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, we let β be an involutive extension of θ, and note $\beta(x,g) = (Bx, JgJ^{-1})$ with $B^2 = I$. We do a computation to see that $J^{-1}B$ is central in $Sp(n)$, so it is $\pm I$, and $J = \pm B$. Either choice of sign contradicts $B^2 = I$, and the Lemma follows. q.e.d.

Remark 3.11. Consider cases where the semisimple part of H_r is of the form $H'_r \cdot H''_r$ with $H''_r = Sp(n)$. Let $\theta = \theta' \times \theta''$ define the real form $H'' = Sp(n;\mathbb{R})$ corresponding to θ''. Consider an extension β of θ to G_r, given by $\beta(x' \otimes x'', g) = (B'x' \otimes B''x'', \theta g)$ on $\mathfrak{v}_r \times H_r$. The argument of Lemma 3.10 shows $B''^2 = -I$, so β cannot be involutive unless either $B'^2 = -I$ as well, or H_r has a $U(1)$ factor so that we can replace B' by a scalar multiple with square $-I$. For example, in Table 5.2,

In Case 9: $H \neq Sp(n;\mathbb{R})$ and $H \neq \mathbb{R}^+ \cdot Sp(n;\mathbb{R})$.
In Case 17: $H \neq Sp(1) \times Sp(n;\mathbb{R})$ and $H \neq Sp(1;\mathbb{R}) \otimes Sp(r,s)$.
In Case 18: $H \notin \{Sp(2) \times Sp(n;\mathbb{R}),\ Sp(2;\mathbb{R}) \times Sp(r,s),$
$Sp(1,1) \times Sp(n;\mathbb{R})\}$.
In Case 21: $H \notin \{\{1, \mathbb{R}^+\}(Sp(2;\mathbb{R}) \times SL(n;\mathbb{R})), (Sp(2;\mathbb{R}) \times SU(r,s)\}$.
In Case 22: $H \neq GL(2;\mathbb{R}) \times Sp(n;\mathbb{R})$.
In Case 23: $H \neq GL(3;\mathbb{R}) \times Sp(n;\mathbb{R})$.
Similar methods and restrictions apply to Table 6.1.

Signature of products

Lemma 3.12. *Suppose that H_r is irreducible on $\mathfrak{v}_r$ and that $H_r = H'_r \cdot H''_r$ with $\theta = \theta' \times \theta''$. Suppose further that there is an involutive extension of θ to G_r, resulting in $(G,H) \in \{\{(G_r, H_r\}\}$ with $H = H' \cdot H''$. Further, $\mathfrak{v}$ has form $\mathfrak{v}' \otimes \mathfrak{v}''$ with action of H of the form $\alpha' \otimes \alpha''$, with invariant $\mathbb{R}$–bilinear forms b' and b''. The H–invariant symmetric $\mathbb{R}$–bilinear form on $\mathfrak{v}$ is $b := b' \otimes b''$. If one of b', b'' is antisymmetric, so is the other, and*

b has signature (t,t) where $2t = \dim_{\mathbb{R}}(\mathfrak{v})$. If one of b', b'' has signature of the form (u,u) then also b has signature (t,t) where $2t = \dim_{\mathbb{R}}(\mathfrak{v})$. More generally, if b' is symmetric with signature (k,ℓ) and b'' is symmetric with signature (r,s) then b has signature $(kr+\ell s, ks+\ell r)$.

Proof. Since H_r is irreducible on $\mathfrak{v}_r$ its action there has form $\alpha'_r \otimes \alpha''_r$. Thus the action of H on $\mathfrak{v}$ has form $\alpha' \otimes \alpha''$, and the H–invariant symmetric $\mathbb{R}$–bilinear form on $\mathfrak{v}$ is $b := b' \otimes b''$ as asserted.

If b' is antisymmetric, then b'' is antisymmetric also, because b is symmetric. Then $\mathfrak{v}' = \mathfrak{v}'_1 \oplus \mathfrak{v}'_2$ where $b'(\mathfrak{v}'_i, \mathfrak{v}'_i) = 0$ and b' pairs $\mathfrak{v}'_1$ with $\mathfrak{v}'_2$, and $\mathfrak{v}'' = \mathfrak{v}''_1 \oplus \mathfrak{v}''_2$ similarly. Choose bases $\{e'_{1,i}\}$ of $\mathfrak{v}'_1$, $\{e'_{2,j}\}$ of $\mathfrak{v}'_2$, $\{e''_{1,u}\}$ of $\mathfrak{v}''_1$ and $\{e''_{2,v}\}$ of $\mathfrak{v}''_2$ such that $b'(e'_{1,i}, e'_{2,j}) = \delta_{i,j}$ and $b''(e''_{1,u}, e''_{2,v}) = \delta_{u,v}$. Here of course $\dim \mathfrak{v}'_1 = \dim \mathfrak{v}'_2$ and $\dim \mathfrak{v}''_1 = \dim \mathfrak{v}''_2$. Then $(\mathfrak{v}'_1 \otimes \mathfrak{v}''_1) \oplus (\mathfrak{v}'_2 \otimes \mathfrak{v}''_2)$ is positive definite for b, $(\mathfrak{v}'_1 \otimes \mathfrak{v}''_2) \oplus (\mathfrak{v}'_2 \otimes \mathfrak{v}''_1)$ is negative definite for b, and the two are b–orthogonal and of equal dimension. That proves the first assertion on signature.

Now suppose that b' and b'' are symmetric, that b' has signature of the form (u,u), and that b'' has signature of the form (v,w). Then $\mathfrak{v}' = \mathfrak{v}'_1 \oplus \mathfrak{v}'_2$ into orthogonal positive definite and negative definite summands, similarly $\mathfrak{v}'' = \mathfrak{v}''_1 \oplus \mathfrak{v}''_2$, $\dim \mathfrak{v}'_1 = \dim \mathfrak{v}'_2 = u$, $\dim \mathfrak{v}''_1 = v$ and $\dim \mathfrak{v}''_2 = w$. So the corresponding decomposition of $\mathfrak{v}$ is $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ where $\mathfrak{v}_1 = (\mathfrak{v}'_1 \otimes \mathfrak{v}''_1) + (\mathfrak{v}'_2 \otimes \mathfrak{v}''_2)$ and $\mathfrak{v}_2 = (\mathfrak{v}'_1 \otimes \mathfrak{v}''_2) + (\mathfrak{v}'_2 \otimes \mathfrak{v}''_1)$. Thus $\dim \mathfrak{v}_1 = uv + uw = \dim \mathfrak{v}_2$. That proves the second assertion on signature. The third signature assertion follows by the same calculation. q.e.d.

Lemma 3.13. *We have inclusions*

(i) $Sp(m;\mathbb{R}) \cdot Sp(n;\mathbb{R}) \subset SO(4mn, 4mn)$,
(ii) $Sp(m;\mathbb{R}) \subset SO(2m, 2m)$,
(iii) $Sp(m;\mathbb{R}) \cdot U(r,s) \subset SO(4mn, 4mn), n = r+s$,
(iv) $Sp(m;\mathbb{R}) \cdot SO(r,s) \subset SO(4mn, 4mn), n = r+s$ *and*
(v) $Sp(m;\mathbb{R}) \cdot Sp(r,s) \subset SO(8mn, 8mn), n = r+s$.

Proof. The first of these is immediate from the proof of Lemma 3.12. For (ii) view $Sp(m;\mathbb{R})$ as the diagonal action on two paired real symplectic vector spaces of dimension $2m$, for example on $\mathbb{R}^{4m} = \mathbb{R}^{2m} \oplus \mathbb{R}^{2m}$ or on $\mathbb{R}^{2m} \oplus \sqrt{-1}\mathbb{R}^{2m}$. For (iii), we have the antisymmetric $\mathbb{C}$–bilinear b' on $\mathbb{C}^{2m}$ and the antisymmetric $\mathbb{R}$–bilinear form $b''(u,v) = \operatorname{Im}\langle u, v\rangle$ on $\mathbb{C}^{r,s}$, so $b' \otimes b''$ is a symmetric bilinear form of signature $(4mn, 4mn)$ on the real vector space underlying $\mathbb{C}^{2m} \otimes_{\mathbb{R}} \mathbb{C}^{r,s}$, and the assertion follows as in Lemma 3.12. Then (iv) follows because $SO(r,s) \subset U(r,s)$ and (v) follows because $Sp(r,s) \subset U(2r, 2s)$. q.e.d.

4. General Heisenberg nilmanifolds

We recall the basic facts on commutative nilmanifolds $M_r = G_r/H_r$, where $G_r = N_r \rtimes H_r$ and N_r is the Heisenberg group $\operatorname{Im}\mathbb{C} + \mathbb{C}^n$.

Proposition 4.1. ([**1**, Theorem 4.6] or see [**22**, Theorem 13.2.4]) *Let N_r denote the Heisenberg group* $\mathrm{Im}\,\mathbb{C} + \mathbb{C}^n$ *of dimension $2n+1$, as in* Example 2.8. *Let H_r be a closed connected subgroup of $U(n)$ acting irreducibly on $\mathbb{C}^n$. Then the following are equivalent.*

1. *$M_r = G_r/H_r$ is commutative, where $G_r = N_r \rtimes H_r$.*
2. *The representation of H_r on $\mathbb{C}^n$ is multiplicity free on the ring of polynomials on $\mathbb{C}^n$.*
3. *The representation of H_r on $\mathbb{C}^n$ is equivalent to one of the following.*

	Group H_r	Group $(H_r)_{\mathbb{C}}$	acting on	conditions on n
1	$SU(n)$	$SL(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 2$
2	$U(n)$	$GL(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 1$
3	$Sp(m)$	$Sp(m;\mathbb{C})$	$\mathbb{C}^n$	$n = 2m$
4	$U(1) \cdot Sp(m)$	$\mathbb{C}^* \times Sp(m;\mathbb{C})$	$\mathbb{C}^n$	$n = 2m$
5	$U(1) \cdot SO(n)$	$\mathbb{C}^* \times SO(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 2$
6	$U(m)$	$GL(m;\mathbb{C})$	$S^2(\mathbb{C}^m)$	$m \geqq 2,\ n = \frac{1}{2}m(m+1)$
7	$SU(m)$	$SL(m;\mathbb{C})$	$\Lambda^2(\mathbb{C}^m)$	m odd, $n = \frac{1}{2}m(m-1)$
8	$U(m)$	$GL(m;\mathbb{C})$	$\Lambda^2(\mathbb{C}^m)$	$n = \frac{1}{2}m(m-1)$
9	$SU(\ell) \cdot SU(m)$	$SL(\ell;\mathbb{C}) \times SL(m;\mathbb{C})$	$\mathbb{C}^\ell \otimes \mathbb{C}^m$	$n = \ell m,\ \ell \neq m$
10	$U(\ell) \cdot SU(m)$	$GL(\ell;\mathbb{C}) \times SL(m;\mathbb{C})$	$\mathbb{C}^\ell \otimes \mathbb{C}^m$	$n = \ell m$
11	$U(2) \cdot Sp(m)$	$GL(2;\mathbb{C}) \times Sp(m;\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^{2m}$	$n = 4m$
12	$SU(3) \cdot Sp(m)$	$SL(3;\mathbb{C}) \times Sp(m;\mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^{2m}$	$n = 6m$
13	$U(3) \cdot Sp(m)$	$GL(3;\mathbb{C}) \times Sp(m;\mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^{2m}$	$n = 6m$
14	$U(4) \cdot Sp(4)$	$GL(4;\mathbb{C}) \times Sp(4;\mathbb{C})$	$\mathbb{C}^4 \otimes \mathbb{C}^8$	$n = 32$
15	$SU(m) \cdot Sp(4)$	$SL(m;\mathbb{C}) \times Sp(4;\mathbb{C})$	$\mathbb{C}^m \otimes \mathbb{C}^8$	$n = 8m,\ m \geqq 3$
16	$U(m) \cdot Sp(4)$	$GL(m;\mathbb{C}) \times Sp(4;\mathbb{C})$	$\mathbb{C}^m \otimes \mathbb{C}^8$	$n = 8m,\ m \geqq 3$
17	$U(1) \cdot Spin(7)$	$\mathbb{C}^* \times Spin(7;\mathbb{C})$	$\mathbb{C}^8$	$n = 8$
18	$U(1) \cdot Spin(9)$	$\mathbb{C}^* \times Spin(9;\mathbb{C})$	$\mathbb{C}^{16}$	$n = 16$
19	$Spin(10)$	$Spin(10;\mathbb{C})$	$\mathbb{C}^{16}$	$n = 16$
20	$U(1) \cdot Spin(10)$	$\mathbb{C}^* \times Spin(10;\mathbb{C})$	$\mathbb{C}^{16}$	$n = 16$
21	$U(1) \cdot G_2$	$\mathbb{C}^* \times G_{2,\mathbb{C}}$	$\mathbb{C}^7$	$n = 7$
22	$U(1) \cdot E_6$	$\mathbb{C}^* \times E_{6,\mathbb{C}}$	$\mathbb{C}^{27}$	$n = 27$

In each case, G_r/H_r is a weakly symmetric Riemannian manifolds; see [**22**, Theorem 15.4.7].

Now consider the corresponding real form families. Following Theorems 2.4 and 2.5 we need only enumerate the real forms H of the groups $(H_r)_{\mathbb{C}}$ listed in Proposition 4.1. All of them are weakly symmetric. The only ones of Lorentz signature come from the Riemannian cases ($H = H_r$ compact) by changing the sign of the metric on the center $\mathfrak{z}$ of $\mathfrak{n}$. In all cases H acts trivially on $\mathfrak{z}$, which has dimension 1, and $\mathfrak{v} = \mathfrak{v}_r$ with the action of H given by the restriction of the action of $(H_r)_{\mathbb{C}}$. Note that the action of H on $\mathfrak{v}$ is irreducible except in a few cases, such as $H = SL(n;\mathbb{R})$ in Case 1, where $\mathfrak{v} = \mathbb{R}^n \oplus \mathbb{R}^n$ under the action of H.

In general we need and use the tools from Sections 2 and 3. The discussion of $U(1)$ factors shows that many potential cases do not occur. The discussions of linear groups and signature of product groups also eliminate many potential cases.

Later, in Section 5, we will consider real form families in the non–Heisenberg cases. In view of the length of the classification in the Heisenberg cases, we will limit our considerations in the non–Heisenberg setting to cases where H_r is maximal in the following sense. If $G'_r = N_r \ltimes H'_r$ with G'_r/H'_r weakly symmetric and $H_r \subset H'_r$, then $H_r = H_r$ (and so $G_r = G'_r$).

We run through the real form families corresponding to the entries of the table in Proposition 4.1. We use the notation $k + \ell = m$ and $r + s = n$ where applicable, and if we write e.g. $m/2$ for some case, usually $GL(m/2;\mathbb{H})$, then it is assumed that m is even for that case. The notation $\{L_1, \dots, L_p\}$ means any one of the L_i, as in $\{1, U(1), \mathbb{R}^+\} \cdot H'$. Our convention on possible invariant signatures is that (a, b) represents both possibilities (a, b) and (b, a), that $(a, b) \oplus (c, d)$ represents all four possibilities $(a + c, b + d), (a + d, b + c), (b + d, a + c)$ and $(b + c, a + d)$, etc.

Cases 1 and 2. Lemma 3.5 shows $H \neq SL(n/2;\mathbb{H})$ and $H \neq GL(n/2;\mathbb{H})$. The signatures are obvious.

Case 3. Lemma 3.10 shows $H \neq Sp(m;\mathbb{R})$. The signatures are obvious.

Case 4. Lemma 3.2 covers the other possibilities. The signatures come from Lemma 3.3 and Remark 3.8.

Case 5. The argument of Lemma 3.9 combines with the adjustment described in Remark 3.8 to cover the case $H = U(1) \cdot SO^*(n)$, n even.

Cases 6–8. Remark 3.6 shows that the linear groups do occur here. The signatures for the general linear groups come from Lemma 3.3, and they follow for the special linear groups.

Cases 9–16. Only the linear groups and the groups $Sp(m;\mathbb{R})$ present difficulties here, and for those groups we apply Remark 3.11 and Lemma 3.12.

Cases 17–20. Here we apply Lemma 3.4, Remark 3.7, and the fact that the centers $Z(Spin(7)) \cong \mathbb{Z}_2 \cong Z(Spin(9))$ and $Z(Spin(10)) \cong \mathbb{Z}_4$,

Case 21. This uses the classification of real forms of $G_{2,\mathbb{C}}$ and the fact that the compact simply connected G_2 is centerless and has no outer automorphisms.

Case 22. This uses the classification of real forms of $E_{6,\mathbb{C}}$ and the fact that the compact simply connected E_6 has center $\mathbb{Z}_3$ and that the outer automorphisms of E_6 act on that center by $z \mapsto z^{-1}$.

Omitting the obvious cases $(G_r)_\mathbb{C}/(H_r)_\mathbb{C}$, where $\mathfrak{v}$ and $\mathfrak{z}$ have signatures of the form (n, n) and $(1, 1)$, now

Table 4.2. Irreducible Commutative Heisenberg Nilmanifolds $(N_r \rtimes H_r)/H_r$

	Group H	$\mathfrak{v}$ and signature($\mathfrak{v}$)	$\mathfrak{z}$ and signature($\mathfrak{z}$)
1	$SU(n), n \geqq 2$	$\mathbb{C}^n, \ (2n,0)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$SU(r,s)$	$\mathbb{C}^{r,s}, \ (2r,2s)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$SL(n;\mathbb{R})$	$\mathbb{R}^{n,n}, \ (n,n)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
2	$U(n), n \geqq 1$	$\mathbb{C}^n, \ (2n,0)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$U(r,s)$	$\mathbb{C}^{r,s}, \ (2r,2s)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$GL(n;\mathbb{R})$	$\mathbb{R}^{n,n}, \ (n,n)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
3	$Sp(m)$	$\mathbb{C}^{2m}, \ (4m,0)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$Sp(k,\ell)$	$\mathbb{C}^{2k,2\ell}, \ (4k,4\ell)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
4	$U(1)\cdot Sp(m)$	$\mathbb{C}^{2m}, \ (4m,0)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$\{U(1),\mathbb{R}^+\}\cdot Sp(k,\ell)$	$\mathbb{C}^{2k,2\ell}, \ (4k,4\ell)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$U(1)\cdot Sp(m;\mathbb{R})$	$\mathbb{C}^{m,m}, \ (2m,2m)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
5	$SO(2)\cdot SO(n), n \geqq 2$	$\mathbb{R}^{2\times n}, \ (2n,0)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$SO(2)\cdot SO(r,s)$	$\mathbb{R}^{2\times(r,s)}, \ (2r,2s)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$SO(1,1)\cdot SO(r,s)$	$\mathbb{R}^{(1,1)\times(r,s)}, \ (n,n)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$U(1)\cdot SO^*(n), n$ even	$\mathbb{C}^n \simeq \mathbb{R}^{n,n}, \ (n,n)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
6	$U(m), m \geqq 2$	$S^2_{\mathbb{C}}(\mathbb{C}^m), \ (m^2+m,0)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$U(k,\ell)$	$S^2_{\mathbb{C}}(\mathbb{C}^{k,\ell})$, $(k^2+k+\ell^2+\ell, 2k\ell))$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$GL(m;\mathbb{R})$	$S^2_{\mathbb{C}}(\mathbb{C}^m) \simeq \mathbb{R}^{1,1}\otimes S^2_{\mathbb{R}}(\mathbb{R}^m)$, $(\frac{m^2+m}{2}, \frac{m^2+m}{2})$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$GL(m/2;\mathbb{H})$	$S^2_{\mathbb{C}}(\mathbb{C}^m) \simeq \mathbb{R}^{1,1}\otimes S^2_{\mathbb{R}}(\mathbb{R}^m)$, $(\frac{m^2+m}{2}, \frac{m^2+m}{2})$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
7	$SU(m), m$ odd	$\Lambda^2_{\mathbb{C}}(\mathbb{C}^m), \ (m^2-m,0)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$SU(k,\ell)$	$\Lambda^2_{\mathbb{C}}(\mathbb{C}^{k,\ell})$, $(k^2-k+\ell^2-\ell, 2k\ell)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$SL(m;\mathbb{R})$	$\Lambda^2_{\mathbb{C}}(\mathbb{C}^m) \simeq \mathbb{R}^{1,1}\otimes \Lambda_{\mathbb{R}}(\mathbb{R}^m)$, $(\frac{m^2-m}{2}, \frac{m^2-m}{2})$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
8	$U(m)$	$\Lambda^2_{\mathbb{C}}(\mathbb{C}^m), \ (m^2-m,0)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$U(k,\ell)$	$\Lambda^2_{\mathbb{C}}(\mathbb{C}^{k,\ell})$, $(k^2-k+\ell^2-\ell, 2k\ell))$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$GL(m;\mathbb{R})$	$\Lambda^2_{\mathbb{C}}(\mathbb{C}^m) \simeq \mathbb{R}^{1,1}\otimes \Lambda_{\mathbb{R}}(\mathbb{R}^m)$, $(\frac{m^2-m}{2}, \frac{m^2-m}{2})$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$GL(m/2;\mathbb{H})$	$\Lambda^2_{\mathbb{C}}(\mathbb{C}^m) \simeq \mathbb{R}^{1,1}\otimes \Lambda_{\mathbb{R}}(\mathbb{R}^m)$, $(\frac{m^2-m}{2}, \frac{m^2-m}{2})$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
9	$SU(m)\cdot SU(n)$	$\mathbb{C}^{m\times n}, \ (2mn,0)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$SU(k,\ell)\cdot SU(r,s)$	$\mathbb{C}^{(k,\ell)\times(r,s)}$, $(2kr+2\ell s, 2ks+2\ell r)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$SL(m;\mathbb{R})\cdot SL(n;\mathbb{R})$	$\mathbb{R}^{m\times n}\oplus\mathbb{R}^{m\times n}, \ (mn,mn)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$SL(\frac{m}{2};\mathbb{H})\cdot SL(\frac{n}{2};\mathbb{H})$	$\mathbb{C}^{m\times n}, \ (mn,mn)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
10	$S(U(m)\cdot U(n))$	$\mathbb{C}^{m\times n}, \ (2mn,0)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$
	$S(U(k,\ell)\cdot U(r,s))$	$\mathbb{C}^{(k,\ell)\times(r,s)}$, $(2kr+2\ell s, 2ks+2\ell r)$	$\mathrm{Im}\,\mathbb{C}, \ (1,0)$

... Table 4.2 continued on next page

Table 4.2 continued from previous page ...

	Group H	$\mathfrak{v}$ and signature($\mathfrak{v}$)	$\mathfrak{z}$ and signature($\mathfrak{z}$)
	$S(GL(m;\mathbb{R})\cdot GL(n;\mathbb{R}))$	$\mathbb{R}^{m\times n}\oplus\mathbb{R}^{m\times n},\ \ (mn,mn)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$S(GL(\frac{m}{2};\mathbb{H})\cdot GL(\frac{n}{2};\mathbb{H}))$	$\mathbb{C}^{m\times n},\ \ (mn,mn)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
11	$U(2)\cdot Sp(m)$	$\mathbb{C}^2\otimes_{\mathbb{C}}\mathbb{C}^{2m},\ \ (8m,0)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(a,b)\cdot Sp(k,\ell),\ {\scriptstyle a+b=2 \atop k+\ell=m}$	$\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell}$, $(4ak+4b\ell,4a\ell+4bk)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(a,b)\cdot Sp(m;\mathbb{R}), a+b=2$	$\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2m},\ \ (4m,4m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$GL(2;\mathbb{R})\cdot Sp(k,\ell), k+\ell=m$	$\mathbb{C}^2\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell}\ \ (4m,4m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$GL(1;\mathbb{H})\cdot Sp(m;\mathbb{R})$	$\mathbb{C}^2\otimes_{\mathbb{C}}\mathbb{C}^{2m},\ \ (4m,4m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
12	$SU(3)\cdot Sp(m)$	$\mathbb{C}^3\otimes_{\mathbb{C}}\mathbb{C}^{2m},\ \ (12m,0)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$SU(a,b)\cdot Sp(k,\ell),\ {\scriptstyle a+b=3 \atop k+\ell=m}$	$\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell}$, $(4ak+4b\ell,4a\ell+4bk)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$SL(3;\mathbb{R})\cdot Sp(k,\ell), k+\ell=m$	$\mathbb{C}^3\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell},\ \ (6m,6m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
13	$U(3)\cdot Sp(m)$	$\mathbb{C}^3\otimes_{\mathbb{C}}\mathbb{C}^{2m},\ \ (12m,0)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(a,b)\cdot Sp(k,\ell),\ {\scriptstyle a+b=3 \atop k+\ell=m}$	$\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell}$, $(4ak+4b\ell,4a\ell+4bk)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(a,b)\cdot Sp(m;\mathbb{R}), a+b=3$	$\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2m},\ \ (6m,6m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$GL(3;\mathbb{R})\cdot Sp(k,\ell), k+\ell=m$	$\mathbb{C}^3\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell},\ \ (6m,6m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
14	$U(4)\cdot Sp(4)$	$\mathbb{C}^4\otimes_{\mathbb{C}}\mathbb{C}^8,\ \ (64,0)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(a,b)\cdot Sp(k,\ell),\ {\scriptstyle a+b=4 \atop k+\ell=4}$	$\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell}$, $(4ak+4b\ell,4a\ell+4bk)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(a,b)\cdot Sp(4;\mathbb{R}), a+b=4$	$\mathbb{C}^{a,b}\otimes_{\mathbb{C}}\mathbb{C}^8,\ \ (32,32)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$GL(4;\mathbb{R})\cdot Sp(k,\ell), k+\ell=4$	$\mathbb{C}^4\otimes_{\mathbb{C}}\mathbb{C}^{2k,2\ell},\ \ (32,32)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$GL(2;\mathbb{H})\cdot Sp(4;\mathbb{R})$	$\mathbb{C}^4\otimes_{\mathbb{C}}\mathbb{C}^8,\ \ (32,32)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
15	$SU(m)\cdot Sp(4), m\geqq 3$	$\mathbb{C}^m\otimes_{\mathbb{C}}\mathbb{C}^8,\ \ (16m,0)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$SU(k,\ell)\cdot Sp(r,s),\ {\scriptstyle k+\ell=m \atop r+s=4}$	$\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{C}^{2r,2s}$, $(4kr+4\ell s,4ks+4\ell k)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$SL(m;\mathbb{R})\cdot Sp(r,s), r+s=4$	$\mathbb{C}^m\otimes_{\mathbb{C}}\mathbb{C}^{2r,2s},\ \ (8m,8m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$SL(m/2;\mathbb{H})\cdot Sp(4;\mathbb{R})$	$\mathbb{C}^m\otimes_{\mathbb{C}}\mathbb{C}^8,\ \ (8m,8m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
16	$U(m)\cdot Sp(4), m\geqq 3$	$\mathbb{C}^m\otimes_{\mathbb{C}}\mathbb{C}^8,\ \ (16m,0)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(k,\ell)\cdot Sp(r,s),\ {\scriptstyle k+\ell=m \atop r+s=4}$	$\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{C}^{2r,2s}$, $(4kr+4\ell s,4ks+4\ell r)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(k,\ell)\cdot Sp(4;\mathbb{R}), k+\ell=m$	$\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{C}^8,\ \ (8m,8m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$GL(m;\mathbb{R})\cdot Sp(r,s), r+s=4$	$\mathbb{C}^m\otimes_{\mathbb{C}}\mathbb{C}^{2r,2s},\ \ (8m,8m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$GL(m/2;\mathbb{H})\cdot Sp(4;\mathbb{R})$	$\mathbb{C}^m\otimes_{\mathbb{C}}\mathbb{C}^8,\ \ (8m,8m)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
17	$U(1)\cdot Spin(7)$	$\mathbb{C}^8,\ \ (16,0)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(1)\cdot Spin(6,1)$	$\mathbb{C}^{6,2},\ \ (12,4)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(1)\cdot Spin(5,2)$	$\mathbb{C}^{6,2},\ \ (12,4)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(1)\cdot Spin(4,3)$	$\mathbb{C}^{4,4},\ \ (8,8)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$\mathbb{R}^+\cdot Spin(r,s), r+s=7$	$\mathbb{R}^{8,8},\ \ (8,8)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
18	$U(1)\cdot Spin(9)$	$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^{16},\ \ (32,0)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$U(1)\cdot Spin(r,s), r+s=9$	$\mathbb{C}^{8,8},\ \ (16,16)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$\mathbb{R}^+\cdot Spin(r,s), r+s=9$	$\mathbb{C}^{8,8},\ \ (16,16)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
19	$Spin(10)$	$\mathbb{C}^{16},\ \ (32,0)$	$\mathbb{R},\ \ (1,0)$
	$Spin(9,1)$	$\mathbb{R}^{16,16},\ \ (16,16)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$Spin(8,2)$	$\mathbb{C}^{8,8},\ \ (16,16)$	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$
	$Spin(7,3)$	$\mathbb{H}^{4,4}$, (16,16)	$\mathrm{Im}\,\mathbb{C},\ \ (1,0)$

... Table 4.2 continued on next page

Table 4.2 continued from previous page ...

	Group H	$\mathfrak{v}$ and signature($\mathfrak{v}$)	$\mathfrak{z}$ and signature($\mathfrak{z}$)
	$Spin(6,4)$	$\mathbb{C}^{8,8}$, $(16,16)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$Spin(5,5)$	$\mathbb{R}^{16,16}$, $(16,16)$	$\mathbb{R}$, $(0,1)$
20	$U(1)\cdot Spin(10)$	$\mathbb{C}^{16}$, $(32,0)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\mathbb{R}^+\cdot Spin(9,1)$	$\mathbb{R}^{16,16}$, $(16,16)$	$\mathbb{R}$, $(0,1)$
	$U(1)\cdot Spin(8,2)$	$\mathbb{C}^{8,8}$, $(16,16)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\mathbb{R}^+\cdot Spin(7,3)$	$\mathbb{H}^{4,4}$, (16,16)	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$U(1)\cdot Spin(6,4)$	$\mathbb{C}^{8,8}$, $(16,16)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\mathbb{R}^+\cdot Spin(5,5)$	$\mathbb{R}^{16,16}$, $(16,16)$	$\mathbb{R}$, $(0,1)$
	$U(1)\cdot Spin^*(10)$	$\mathbb{H}^{4,4}$, $(16,16)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
21	$U(1)\cdot G_2$	$\mathbb{C}^7$, $(14,0)$	$\mathrm{Re}\,\mathbb{O}$, $(1,0)$
	$U(1)\cdot G_{2,A_1A_1}$	$\mathbb{C}^{3,4}$, $(6,8)$	$\mathrm{Re}\,\mathbb{O}_{sp}$, $(1,0)$
	$\mathbb{R}^+\cdot G_2$	$\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{R}^7$, $(7,7)$	$\mathrm{Re}\,\mathbb{O}$, $(0,1)$
	$\mathbb{R}^+\cdot G_{2,A_1A_1}$	$\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{R}^{3,4}$, $(7,7)$	$\mathrm{Re}\,\mathbb{O}_{sp}$, $(1,0)$
22	$U(1)\cdot E_6$	$\mathbb{C}^{27}$, $(54,0)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$U(1)\cdot E_{6,A_5A_1}$	$\mathbb{C}^{15,12}$, $(30,24)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$U(1)\cdot E_{6,D_5T_1}$	$\mathbb{C}^{16,11}$, $(32,22)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\mathbb{R}^+\cdot E_{6,C_4}$	$\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{R}^{27}$, $(27,27)$	$\mathbb{R}$, $(0,1)$
	$\mathbb{R}^+\cdot E_{6,F_4}$	$\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{R}^{26,1}$, $(27,27)$	$\mathbb{R}$, $(0,1)$

Certain signatures of pseudo–Riemannian metrics from Table 4.2 are particularly interesting. The Riemannian ones, of course, are just the G_r/H_r, in other words those where H is compact. But every such G_r/H_r also has an invariant Lorentz metric, from the invariant symmetric bilinear form on $\mathfrak{n}_r$ that is positive definite on $\mathfrak{v}_r$ and negative definite on the (one dimensional) center $\mathfrak{z}_r$. But inspection of Table 4.2 shows that there are a few others, where $\mathfrak{v}$ has an invariant bilinear form of Lorentz signature, say $(d,1)$, so that $\mathfrak{n}$ has an invariant bilinear form of Lorentz signature $(d+1,1)$. For each of those, $d=1$ and $H\cong\mathbb{R}^+$, so $\mathfrak{n}$ is the 3–dimensional Heisenberg algebra $\left(\begin{smallmatrix}0&x&z\\0&0&y\\0&0&0\end{smallmatrix}\right)$ where $\mathbb{R}^+$ acts by $t:\left(\begin{smallmatrix}0&x&z\\0&0&y\\0&0&0\end{smallmatrix}\right)\to\left(\begin{smallmatrix}0&tx&z\\0&0&t^{-1}y\\0&0&0\end{smallmatrix}\right)$ and the metric has signature $(2,1)$.

The trans-Lorentz signature is more interesting. Running through the table we see that the only cases there are the following.

Proposition 4.3. *The trans–Lorentz cases in* Table 4.2*, signature of the form* $(p-2,2)$*, all are weakly symmetric. They are*

Case 1. $H=SU(n-1,1)$ *where* G/H *has a* G*–invariant metric of signature* $(2n-1,2)$*, and* $H=SL(2;\mathbb{R})$ *where* G/H *has a* G*–invariant metric of signature* $(3,2)$*.*

Case 2. $H=U(n-1,1)$ *where* G/H *has a* G*–invariant metric of signature* $(2n-1,2)$*, and* $H=GL(2;\mathbb{R})$ *where* G/H *has a* G*–invariant metric of signature* $(3,2)$*.*

Case 4. $H=U(1)\cdot Sp(1;\mathbb{R})$ *where* G/H *has a* G*–invariant metric of signature* $(3,2)$*.*

Case 5. $H = SO(2) \cdot SO(n-1,1)$ *where* G/H *has a* G*–invariant metric of signature* $(2n-1,2)$.

Case 6. $H = U(1,1)$ *where* G/H *has a* G*–invariant metric of signature* $(5,2)$.

Case 7. $H = SU(2,1)$ *where* G/H *has a* G*–invariant metric of signature* $(5,2)$.

Case 8. $H = U(2)$ *and* $H = U(1,1)$, *where* G/H *has a* G*–invariant metric of signature* $(1,2)$; $H = U(2,1)$ *where* G/H *has a* G*–invariant metric of signature* $(5,2)$; $H = GL(2;\mathbb{R})$ *and* $H = GL(1;\mathbb{H})$, *where* G/H *has a* G*–invariant metric of signature* $(1,2)$.

5. Irreducible commutative nilmanifolds: classification

In our notation, Vinberg's classification of maximal irreducible commutative Riemannian nilmanifolds is

Table 5.1. Maximal Irreducible Nilpotent Gelfand Pairs $(N_r \rtimes H_r, H_r)$

	Group H_r	$\mathfrak{v}_r$	$\mathfrak{z}_r$	$U(1)$	max
	Group H_r	$\mathfrak{v}_r$	$\mathfrak{z}_r$	$U(1)$	max
1	$SO(n)$	$\mathbb{R}^n$	$\Lambda\mathbb{R}^{n\times n} = \mathfrak{so}(n)$		
2	$Spin(7)$	$\mathbb{R}^8 = \mathbb{O}$	$\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$		
3	G_2	$\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$	$\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$		
4	$U(1) \cdot SO(n)$	$\mathbb{C}^n$	$\operatorname{Im}\mathbb{C}$		$n \neq 4$
5	$(U(1)\cdot)SU(n)$	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n \oplus \operatorname{Im}\mathbb{C}$	n odd	
6	$SU(n), n$ odd	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n$		
7	$SU(n), n$ odd	$\mathbb{C}^n$	$\operatorname{Im}\mathbb{C}$		
8	$U(n)$	$\mathbb{C}^n$	$\operatorname{Im}\mathbb{C}^{n\times n} = \mathfrak{u}(n)$		
9	$(U(1)\cdot)Sp(n)$	$\mathbb{H}^n$	$\operatorname{Re}\mathbb{H}_0^{n\times n} \oplus \operatorname{Im}\mathbb{H}$		
10	$U(n)$	$S^2(\mathbb{C}^n)$	$\mathbb{R}$		
11	$(U(1)\cdot)SU(n), n \geqq 3$	$\Lambda^2(\mathbb{C}^n)$	$\mathbb{R}$	n even	
12	$U(1) \cdot Spin(7)$	$\mathbb{C}^8$	$\mathbb{R}^7 \oplus \mathbb{R}$		
13	$U(1) \cdot Spin(9)$	$\mathbb{C}^{16}$	$\mathbb{R}$		
14	$(U(1)\cdot)Spin(10)$	$\mathbb{C}^{16}$	$\mathbb{R}$		
15	$U(1) \cdot G_2$	$\mathbb{C}^7$	$\mathbb{R}$		
16	$U(1) \cdot E_6$	$\mathbb{C}^{27}$	$\mathbb{R}$		
17	$Sp(1) \times Sp(n)$	$\mathbb{H}^n$	$\operatorname{Im}\mathbb{H} = \mathfrak{sp}(1)$		$n \geqq 2$
18	$Sp(2) \times Sp(n)$	$\mathbb{C}^{4\times 2n}$	$\operatorname{Im}\mathbb{H}^{2\times 2} = \mathfrak{sp}(2)$		
19	$(U(1)\cdot)SU(m) \times SU(n)$ $m,n \geqq 3$	$\mathbb{C}^m \otimes \mathbb{C}^n$	$\mathbb{R}$	$m = n$	
20	$(U(1)\cdot)SU(2) \times SU(n)$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\operatorname{Im}\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$	$n = 2$	
21	$(U(1)\cdot)Sp(2) \times SU(n)$	$\mathbb{H}^2 \otimes \mathbb{C}^n$	$\mathbb{R}$	$n \leqq 4$	$n \geqq 3$
22	$U(2) \times Sp(n)$	$\mathbb{C}^2 \otimes \mathbb{H}^n$	$\operatorname{Im}\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$		
23	$U(3) \times Sp(n)$	$\mathbb{C}^3 \otimes \mathbb{H}^n$	$\mathbb{R}$		$n \geqq 2$

All groups are real. All spaces $(G_r/H_r, ds^2)$ are weakly symmetric except for entry 9 with $H_r = Sp(n)$; see [**22**, Theorem 15.4.10]. This is due to Lauret [**14**]. For more details see [**22**, Section 15.4]. If a group H_r is denoted $(U(1)\cdot)H'_r$ it can be H'_r or $U(1)\cdot H'_r$. Under certain conditions the only case is $U(1)\cdot H'_r$ then those conditions are noted in the $U(1)$ column. In this section we extend the considerations of Table 5.1 from commutative (including weakly symmetric) Riemannian nilmanifolds to the pseudo–Riemannian setting.

Now we run through the corresponding real form families, omitting the complexifications of the Riemannian forms G_r/H_r. When we write $m/2$ it is implicit that we are in a case where m is even, and similarly $n/2$ assumes that n is even. Further $k+\ell = m$ and $r+s = n$ where applicable. We also use the notation $\{L_1, \dots, L_p\}$ to mean any one of the L_i, as in $\{\{1\}, U(1), \mathbb{R}^+\}\cdot H'$. Finally, our convention on possible invariant signatures is that (a,b) represents both possibilities (a,b) and (b,a), that $(a,b)\oplus(c,d)$ represents all four possibilities $(a+c, b+d), (a+d, b+c), (b+d, a+c)$ and $(b+c, a+d)$, etc.

Case 1. $H \neq SO^*(n)$ by Lemma 3.9. The signature calculations are straightforward.

Case 2. The assertions follow from Lemma 3.4.

Case 3. The assertions are obvious.

Case 4. Lemma 3.9 does not eliminate $H = U(1)\cdot SO^*(2m)$ because $\theta = \mathrm{Ad}(J)$, $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, extends to $\mathbb{C}^{2m}$ as cJ where $c \in U(1)$ with $(cJ)^2 = 1$. However that is necessarily trivial on the $U(1)$ factor, so $H = \mathbb{R}^+\cdot SO^*(2m)$ is eliminatred. The signature calculations are straightforward.

Cases 5a and 5b. $H \neq \{S,G\}L(n/2;\mathbb{H})$ by Lemma 3.5. The signatures for $(S)U(r,s)$ are obvious, and for $\{S,G\}L(n;\mathbb{R})$ they follow from Lemma 3.3.

Cases 6 and 7. The calculations are straightforward.

Case 8. $H \neq GL(n/2;\mathbb{H})$ by Lemma 3.5. The signatures for $U(r,s)$ are obvious, and for $GL(n;\mathbb{R})$ they follow from Lemma 3.3.

Case 9. The $Sp(n;\mathbb{R})$ entry depends on Example 3.12.

Cases 10, 11a and 11b. The calulations are straightforward. Note Remark 3.6 for $H = GL(n/2;\mathbb{H})$.

Cases 12 and 13. The calculations are straightforward.

Case 14. $H = U(1)\cdot Spin^*(10)$ is admissible by lifting $\mathrm{Ad}(J)$, $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, from $SO(10)$ to $\theta = \mathrm{Ad}(c\widetilde{J})$ on $Spin(10)$, where $c \in U(1)$ so that $c\widetilde{J}$ has square 1 on $\mathfrak{v}_r$. But $H \neq Spin^*(10)$ by Lemma 3.9.

Case 15. The assertions are obvious.

Case 16. The assertions follow from the E_6 discussion in Section 3.

Case 17. $H \neq Sp(1)\times Sp(n;\mathbb{R})$ and $H \neq Sp(1;\mathbb{R})\otimes Sp(r,s)$ as noted in Remark 3.11. The signatures are obvious for $H = Sp(1)\times Sp(r,s)$. They follow from Lemma 3.12 for the other two cases.

Case 18. H cannot be $Sp(2) \times Sp(n;\mathbb{R})$, $Sp(2;\mathbb{R}) \times Sp(r,s)$, nor $Sp(1,1) \times Sp(r,s)$, as noted in Remark 3.11. See Lemma 3.13 for the signatures when $H = Sp(2;\mathbb{R}) \times Sp(n;\mathbb{R})$. The other signatures are immediate.

Case 19. Lemma 3.5 eliminates both variations on $\{1,\mathbb{R}^+\}(SL(m;\mathbb{R}) \times SL(n/2;\mathbb{H}))$. The signatures in the other cases are straightforward.

Case 20. H cannot have semisimple part $SU(k,l) \times Sl(n/2;\mathbb{H})$ by Remark 3.7. The signatures are evident in the other four cases.

Case 21–23. Remarks 3.11 and 3.7 eliminate most cases with a real symplectic group and some cases with a quaternion linear group. The signatures are computable.

For the convenience of the reader in using the tables, Table 5.2 repeats some material from Table 4.2.

Table 5.2. Maximal Irreducible Commutative Nilmanifolds $(N_r \rtimes H_r)/H_r$

	Group H	$\mathfrak{v}$ signature($\mathfrak{v}$)	$\mathfrak{z}$ signature($\mathfrak{z}$)
1	$SO(r,s)$	$\mathbb{R}^{r,s}$ (r,s)	$\mathfrak{so}(r,s)$ $(\frac{r(r-1)+s(s-1)}{2}, rs)$
2	$Spin(k,7-k)$ $4 \leqq k \leqq 7$	$\mathbb{R}^{q,8-q}, q = 2[\frac{k+1}{2}]$ $(q,8-q)$	$\mathbb{R}^{k,7-k}$ $(k,7-k)$
3	G_2	$\mathrm{Im}\,\mathbb{O}$, $(7,0)$	$\mathrm{Im}\,\mathbb{O}$, $(7,0)$
	G_{2,A_1A_1}	$\mathrm{Im}\,\mathbb{O}_{sp}$, $(3,4)$	$\mathrm{Im}\,\mathbb{O}_{sp}$, $(3,4)$
4	$U(1) \cdot SO(r,s)$ $n = r+s \neq 4$	$\mathbb{C}^{r,s}$, $(2r,2s)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$U(1) \cdot SO^*(2m)$	$\mathbb{C}^{m,m}$, $(2m,2m)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\mathbb{R}^+ \cdot SO(r,s)$	$\mathbb{R}^{1,1} \otimes_{\mathbb{R}} \mathbb{R}^{r,s}$, (n,n)	$\mathbb{R}$,(0,1)
5a	$SU(r,s)$, $n = r+s$ even	$\mathbb{C}^{r,s}$, $(2r,2s)$	$\Lambda^2_{\mathbb{R}}(\mathbb{C}^{r,s}) \oplus \mathrm{Im}\,\mathbb{C}$ $(2r^2-r+2s^2-s, 4rs)$ $\oplus(1,0)$
	$SL(n;\mathbb{R})$	$\mathbb{R}^{n,n}$, (n,n)	$\Lambda^2_{\mathbb{R}}(\mathbb{R}^{n,n}) \oplus \mathbb{R}$ $(n^2-\frac{n}{2}, n^2-\frac{n}{2})$ $\oplus(1,0)$
5b	$U(r,s), n = r+s$	$\mathbb{C}^{r,s}$, $(2r,2s)$	$\Lambda^2_{\mathbb{R}}(\mathbb{C}^{r,s}) \oplus \mathrm{Im}\,\mathbb{C}$ $(2r^2-r+2s^2-s, 4rs)$ $\oplus(1,0)$
	$GL(n;\mathbb{R})$	$\mathbb{R}^{n,n}$, (n,n)	$\Lambda^2_{\mathbb{R}}(\mathbb{R}^{n,n}) \oplus \mathbb{R}$ $(n^2-\frac{n}{2}, n^2-\frac{n}{2})$ $\oplus(1,0)$
6	$SU(r,s)$ $r+s = n$ odd	$\mathbb{C}^{r,s}$, $(2r,2s)$	$\Lambda^2_{\mathbb{R}}(\mathbb{C}^{r,s})$ $(2r^2-r+2s^2-s, 4rs)$
	$SL(n;\mathbb{R})$	$\mathbb{R}^{n,n}$, (n,n)	$\Lambda^2_{\mathbb{R}}(\mathbb{R}^{n,n})$ $(n^2-\frac{n}{2}, n^2-\frac{n}{2})$
7	$SU(r,s)$ $r+s = n$ odd	$\mathbb{C}^{r,s}$, $(2r,2s)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$

... *Table 5.2 continued on next page*

Table 5.2 continued from previous page ...

	Group H	$\mathfrak{v}$ signature($\mathfrak{v}$)	$\mathfrak{z}$ signature($\mathfrak{z}$)
	$SL(n;\mathbb{R})$	$\mathbb{R}^{n,n}$, (n,n)	$\mathbb{R}$, $(0,1)$
8	$U(r,s)$	$\mathbb{C}^{r,s}$, $(2r,2s)$	$\mathfrak{u}(r,s)$, $(r^2+s^2,2rs)$
	$GL(n;\mathbb{R})$	$\mathbb{R}^{n,n}$, (n,n)	$\mathfrak{gl}(n;\mathbb{R})$ $(\frac{n(n-1)}{2},\frac{n(n+1)}{2})$
9	$\{\{1\},U(1),\mathbb{R}^+\}\cdot Sp(r,s)$	$\mathbb{H}^{r,s}$, $(4r,4s)$	$\mathrm{Re}\,\mathbb{H}_0^{(r,s)\times(r,s)}\oplus\mathrm{Im}\,\mathbb{H}$ $(2n^2$-n-$4rs$-$1,4rs)$ $\oplus(3,0)$
	$U(1)\cdot Sp(n;\mathbb{R})$	$\mathbb{R}^{2n,2n}$, $(2n,2n)$	$\mathrm{Re}\,\mathbb{H}_{sp,0}^{n\times n}\oplus\mathrm{Im}\,\mathbb{H}_{sp}$ (n^2-1,n^2-n) $\oplus(2,1)$
10	$U(r,s)$	$S^2_{\mathbb{C}}(\mathbb{C}^{r,s})$ $(r(r+1)+s(s+1),2rs)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$GL(n;\mathbb{R})$	$\mathbb{R}^{1,1}\otimes_{\mathbb{R}}S^2_{\mathbb{R}}(\mathbb{R}^n)$ $(\frac{n(n+1)}{2},\frac{n(n+1)}{2})$	$\mathbb{R}$, $(0,1)$
	$GL(\frac{n}{2};\mathbb{H})$	$S^2_{\mathbb{C}}(\mathbb{C}^n)$ $(\frac{n(n+1)}{2},\frac{n(n+1)}{2})$	$\mathbb{R}$, $(0,1)$
11a	$SU(r,s)$ $r+s=n>3$ odd	$\Lambda^2_{\mathbb{C}}(\mathbb{C}^{r,s})$ $(r^2-r+s^2-s,2rs)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$SL(n;\mathbb{R})$	$\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\Lambda^2_{\mathbb{R}}(\mathbb{R}^n)$ $(\frac{n(n-1)}{2},\frac{n(n-1)}{2}))$	$\mathbb{R}$, (0,1)
11b	$U(r,s)$ $r+s=n\geqq 3$	$\Lambda^2_{\mathbb{C}}(\mathbb{C}^{r,s})$ $(r^2-r+s^2-s,2rs)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$GL(n;\mathbb{R})$	$\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\Lambda_{\mathbb{R}}(\mathbb{R}^n)$ $(\frac{n(n-1)}{2},\frac{n(n-1)}{2})$	$\mathbb{R}$, (0,1)
	$H=GL(\frac{n}{2};\mathbb{H})$	$\Lambda^2_{\mathbb{C}}(\mathbb{C}^n)$ $(\frac{n(n-1)}{2},\frac{n(n-1)}{2})$	$\mathbb{R}$, $(0,1)$
12	$U(1)\cdot Spin(k,7-k)$ $4\leqq k\leqq 7$, $q=2[\frac{k+1}{2}]$	$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^{q,8-q}$ $(2q,16-2q)$	$\mathbb{R}^{k,7-k}\oplus\mathrm{Im}\,\mathbb{C}$ $(k,7-k)\oplus(1,0)$
	$\mathbb{R}^+\cdot Spin(k,7-k)$ $4\leqq k\leqq 7$, $q=2[\frac{k+1}{2}]$	$\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{R}^{q,8-q}$ $(8,8)$	$\mathbb{R}^{k,7-k}\oplus\mathbb{R}$ $(k,7-k)\oplus(0,1)$
13	$U(1)\cdot Spin(k,9-k)$ $5\leqq k\leqq 9$, $q=2^{1+[\frac{k+3}{4}]}$	$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^{q,16-q}$ $(2q,32-2q)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\mathbb{R}^+\cdot Spin(k,9-k)$ $5\leqq k\leqq 9$, $q=2^{1+[\frac{k+3}{4}]}$	$\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{R}^{q,16-q}$ $(16,16)$	$\mathbb{R}$, $(0,1)$
14	$\{\{1\},U(1)\}\cdot)Spin(10)$	$\mathbb{C}^{16}$, $(32,0)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\{\{1\},\mathbb{R}^+\}\cdot Spin(9,1)$	$\mathbb{R}^{16,16}$, $(16,16)$	$\mathbb{R}$, $(0,1)$
	$\{\{1\},U(1)\}\cdot Spin(8,2)$	$\mathbb{C}^{8,8}$, $(16,16)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\{\{1\},\mathbb{R}^+\}\cdot Spin(7,3)$	$\mathbb{H}^{4,4}$, (16,16)	$\mathbb{R}$, $(0,1)$
	$\{\{1\},U(1)\}\cdot Spin(6,4)$	$\mathbb{C}^{8,8}$, $(16,16)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\{\{1\},\mathbb{R}^+\}\cdot Spin(5,5)$	$\mathbb{R}^{16,16}$, $(16,16)$	$\mathbb{R}$, $(0,1)$
	$U(1)\cdot Spin^*(10)$	$\mathbb{H}^{4,4}$, $(16,16)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
15	$U(1)\cdot G_2$	$\mathbb{C}^7=\mathrm{Im}\,\mathbb{O}_{\mathbb{C}}$, $(14,0)$	$\mathbb{R}=\mathrm{Re}\,\mathbb{O}$, $(0,1)$
	$U(1)\cdot G_{2,A_1A_1}$	$\mathbb{C}^{3,4}$, $(6,8)$	$\mathrm{Re}\,\mathbb{O}_{sp}$, $(1,0)$
	$\mathbb{R}^+\cdot G_2$	$\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{R}^7$, $(7,7)$	$\mathrm{Re}\,\mathbb{O}$, $(0,1)$

... Table 5.2 continued on next page

Table 5.2 continued from previous page ...

	Group H	$\mathfrak{v}$ signature($\mathfrak{v}$)	$\mathfrak{z}$ signature($\mathfrak{z}$)
	$\mathbb{R}^+ \cdot G_{2,A_1A_1}$	$\mathbb{R}^{1,1} \otimes_{\mathbb{R}} \mathbb{R}^{3,4}$, $(7,7)$	$\mathrm{Re}\,\mathbb{O}_{sp}$, $(1,0)$
16	$U(1) \cdot E_6$	$\mathbb{C}^{27}$, $(54,0)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$U(1) \cdot E_{6,A_5A_1}$	$\mathbb{C}^{15,12}$, $(30,24)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$U(1) \cdot E_{6,D_5T_1}$	$\mathbb{C}^{16,11}$, $(32,22)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\mathbb{R}^+ \cdot E_{6,C_4}$	$\mathbb{R}^{1,1} \otimes_{\mathbb{R}} \mathbb{R}^{27}$, $(27,27)$	$\mathbb{R}$, $(0,1)$
	$\mathbb{R}^+ \cdot E_{6,F_4}$	$\mathbb{R}^{1,1} \otimes_{\mathbb{R}} \mathbb{R}^{26,1}$, $(27,27)$	$\mathbb{R}$, $(0,1)$
17	$Sp(1) \cdot Sp(r,s), r+s \geqq 2$	$\mathbb{H}^{r,s}$, $(4r,4s)$	$\mathfrak{sp}(1)$, $(3,0)$
	$Sp(1;\mathbb{R}) \cdot Sp(n;\mathbb{R})$	$\mathbb{R}^{2n,2n}$, $(2n,2n)$	$\mathfrak{sp}(1;\mathbb{R})$, $(1,2)$
18	$Sp(2) \cdot Sp(r,s)$ $r+s=n \geqq 2$	$\mathbb{C}^4 \otimes_{\mathbb{C}} \mathbb{C}^{2r,2s}$, $(16r,16s)$	$\mathfrak{sp}(2)$, $(10,0)$
	$Sp(1,1) \cdot Sp(r,s)$	$\mathbb{C}^{2,2} \otimes_{\mathbb{C}} \mathbb{C}^{2r,2s}$, $(8n,8n)$	$\mathfrak{sp}(1,1)$, $(6,4)$
	$H = Sp(2;\mathbb{C})$ $(n=2)$	$\mathbb{C}^{4\times 4}$, $(16,16)$	$\mathfrak{sp}(2)$, $(10,0)$
	$H = Sp(2;\mathbb{R}) \cdot Sp(n;\mathbb{R})$	$\mathbb{R}^{8n,8n}$, $(8n,8n)$	$\mathfrak{sp}(2;\mathbb{R})$, $(4,6)$
19	$\{\{1\}, U(1)\}\cdot$ $(SU(k,\ell) \cdot SU(r,s))$ $m=k+\ell, n=r+s \geqq 3$ $U(1)$ required if $m=n$	$\mathbb{C}^{(k,\ell)\times(r,s)}$ $(2kr+2\ell s, 2ks+2\ell r)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\mathbb{R}^+ \cdot SL(m;\mathbb{C})$ $(m=n)$	$\mathfrak{gl}(m;\mathbb{C})$ (m^2,m^2)	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\{\{1\}, \mathbb{R}^+\}\cdot$ $(SL(m;\mathbb{R}) \cdot SL(n;\mathbb{R}))$	$\mathbb{R}^{1,1} \otimes_{\mathbb{R}} \mathbb{R}^{m\times n}$ (mn,mn)	$\mathbb{R}$, $(0,1)$
	$\{\{1\}, \mathbb{R}^+\}\cdot$ $(SL(\frac{m}{2};\mathbb{H}) \cdot SL(\frac{n}{2};\mathbb{H}))$	$\mathbb{R}^{1,1} \otimes_{\mathbb{R}} \mathbb{H}^{(m/2)\times(n/2)}$ (mn,mn)	$\mathbb{R}$, $(0,1)$
20	$\{\{1\}, U(1)\}\cdot$ $(SU(k,\ell) \cdot SU(r,s))$ $k+\ell=2, n=r+s \geqq 2$ $U(1)$ required if $n=2$	$\mathbb{C}^{(k,\ell)\times(r,s)}$ $(2kr+2\ell s, 2ks+2\ell r)$	$\mathfrak{u}(k,\ell)$ $(2k,2\ell)$
	$\{\{1\}, \mathbb{R}^+\}\cdot$ $(SL(1;\mathbb{H}) \cdot SL(n/2;\mathbb{H}))$	$\mathbb{C}^{2\times n}$, $(2n,2n)$	$\mathfrak{gl}(1;\mathbb{H})$, $(3,1)$
	$\{\{1\}, \mathbb{R}^+\} \cdot (SL(2;\mathbb{R})\cdot$ $SL(n;\mathbb{R}))$	$\mathbb{R}^{1,1} \otimes_{\mathbb{R}} \mathbb{R}^{2\times n}$ $(2n,2n)$	$\mathfrak{gl}(2;\mathbb{R})$, $(1,3)$
21	$\{\{1\}, U(1)\} \cdot Sp(k,\ell)\cdot$ $SU(r,s)$, $^{k+\ell=2}_{r+s=n\geqq 3}$ $U(1)$ required if $n \leqq 4$	$\mathbb{H}^{k,\ell} \otimes_{\mathbb{R}} \mathbb{C}^{r,s}$ $(8kr+8\ell s, 8ks+8\ell r)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\{\{1\}, \mathbb{R}^+\}$ $\cdot Sp(k,\ell)\cdot$ $SL(n;\mathbb{R}))$, $\mathbb{R}^+$ if $n \leqq 4$	$\mathbb{H}^{k,\ell} \otimes_{\mathbb{R}} \mathbb{R}^{n,n}$ $(8n,8n)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$Sp(2;\mathbb{R}) \cdot U(r,s))$	$\mathbb{R}^{4,4} \otimes_{\mathbb{R}} \mathbb{C}^{r,s}$, $(8n,8n)$	$\mathrm{Im}\,\mathbb{C}$, $(1,0)$
	$\{\{1\}, \mathbb{R}^+\}$ $\cdot (Sp(2;\mathbb{R})\cdot$ $SL(\frac{n}{2};\mathbb{H})$, $\mathbb{R}^+$ if $n \leqq 4$	$\mathbb{R}^{4,4} \otimes_{\mathbb{R}} \mathbb{H}^{n/2}$, $(8n,8n)$	$\mathbb{R}$, $(0,1)$
22	$U(k,\ell) \cdot Sp(r,s)$, $^{k+\ell=2}_{r+s=n}$	$\mathbb{C}^{k,\ell} \otimes_{\mathbb{C}} \mathbb{C}^{2r,2s}$ $(4kr+4\ell s, 4ks+4\ell r)$	$\mathfrak{u}(k,\ell)$ $(2k, 4-2k)$
	$U(k,\ell) \cdot Sp(n;\mathbb{R})$	$\mathbb{C}^{k,\ell} \otimes_{\mathbb{C}} \mathbb{C}^{n,n}$ $(4n,4n)$	$\mathfrak{u}(k,\ell)$ $(2k, 4-2k)$
	$GL(2;\mathbb{R}) \cdot Sp(r,s)$	$\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{H}^{r,s}$, $(4n,4n)$	$\mathfrak{gl}(2;\mathbb{R})$, $(1,3)$
	$GL(1;\mathbb{H}) \cdot Sp(n;\mathbb{R})$	$\mathbb{H} \otimes_{\mathbb{C}} \mathbb{C}^{2n}$, $(4n,4n)$	$\mathfrak{gl}(1;\mathbb{H})$, $(3,1)$

... Table 5.2 continued on next page

Table 5.2 continued from previous page ...

	Group H	$\mathfrak{v}$ signature($\mathfrak{v}$)	$\mathfrak{z}$ signature($\mathfrak{z}$)
23	$U(k,\ell)\cdot Sp(r,s)$ $k+\ell=3, n=r+s\geqq 2$	$\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{C}^{2r,2s}$ $(4kr+4\ell s, 4ks+4\ell r)$	$\operatorname{Im}\mathbb{C},\ (1,0)$
	$U(k,\ell)\cdot Sp(n;\mathbb{R})$	$\mathbb{C}^{k,\ell}\otimes_{\mathbb{C}}\mathbb{C}^{2n},\ (6n,6n)$	$\operatorname{Im}\mathbb{C},\ (1,0)$
	$GL(3;\mathbb{R})\cdot Sp(r,s)$	$\mathbb{C}^3\otimes_{\mathbb{C}}\mathbb{H}^{r,s},\ (6n,6n)$	$\mathbb{R},\ (0,1)$

We now extract special signatures from Table 5.2. In order to avoid redundancy we consider $SO(n)$ only for $n\geqq 3$, $SU(n)$ and $U(n)$ only for $n\geqq 2$, and $Sp(n)$ only for $n\geqq 1$.

Corollary 5.3. *The Lorentz cases, signature of the form* $(p-1,1)$ *in* Table 5.2*, all are weakly symmetric. In addition to their invariant Lorentz metrics, each has invariant weakly symmetric Riemannian metrics:*

Case 4. $H=U(1)\cdot SO(n)$ *with* G*–invariant metric on* G/H *of signature* $(2n,1)$

Case 5. $H=SU(n)$ *and* $H=U(n)$*, each with* G*–invariant metric on* G/H *of signature* $(2n^2+n,1)$

Case 7. $H=SU(n)$ *with* G*–invariant metric on* G/H *of signature* $(2n,1)$

Case 10. $H=U(n)$ *with* G*–invariant metric on* G/H *of signature* $(n^2+n,1)$

Case 11. $H=SU(n)$ *and* $H=U(n)$*, each with* G*–invariant metric on* G/H *of signature* $(n^2-n,1)$

Case 12. $H=U(1)\cdot Spin(7)$ *with* G*–invariant metric on* G/H *of signature* $(23,1)$

Case 13. $H=U(1)\cdot Spin(9)$ *with* G*–invariant metric on* G/H *of signature* $(32,1)$

Case 14. $H=(U(1)\cdot)Spin(10)$ *with* G*–invariant metric on* G/H *of signature* $(32,1)$

Case 15. $H=U(1)\cdot G_2$ *with* G*–invariant metric on* G/H *of signature* $(14,1)$

Case 16. $H=U(1)\cdot E_6$ *with* G*–invariant metric on* G/H *of signature* $(54,1)$

Case 19. $H=(U(1)\cdot)(SU(m)\cdot SU(n))$ *with* G*–invariant metric on* G/H *of signature* $(2mn,1)$

Case 21. $H=(U(1)\cdot)(Sp(2)\cdot SU(n))$ *with* G*–invariant metric on* G/H *of signature* $(16n,1)$

Case 23. $H=U(3)\cdot Sp(n)$ *with* G*–invariant metric on* G/H *of signature* $(12n,1)$

Corollary 5.4. *The complexifications of the Lorentz cases listed in* Corollary 5.3 *all are of trans–Lorentz signature* $(p-2,2)$*. The trans–*

Lorentz cases, signature of the form $(p-2,2)$ *in* Table 5.2*, all are weakly symmetric, and they are given as follows.*

Case 1. $H = SO(2,1)$ *with* G*–invariant metric on* G/H *of signature* $(4,2)$

Case 4. $H = U(1)\cdot SO(n-1,1)$ *with* G*–invariant metric on* G/H *of signature* $(2n-1,2)$

Case 7. $H = SU(n-1,1)$ *with* G*–invariant metric on* G/H *of signature* $(2n-1,2)$

Case 10. $H = U(1,1)$ *with* G*–invariant metric on* G/H *of signature* $(5,2)$

Case 11. $H = SU(2,1)$ *and* $H = U(2,1)$*, each with* G*–invariant metric on* G/H *of signature* $(5,2)$

6. Indecomposable commutative nilmanifolds

In this section we broaden the scope of Table 5.2 from irreducible to indecomposable commutative spaces — subject to a few technical conditions. This is based on a classification of Yakimova ([**24**], [**25**]; or see [**22**]). It settles the case where $(N \rtimes H, H)$ is indecomposable, principal, maximal and $Sp(1)$–saturated.

Since $G = N \rtimes H$ acts almost–effectively on $M = G/H$, the centralizer of N in H is discrete, in other words the representation of H on $\mathfrak{n}$ has finite kernel. (In the notation of [**25**, Section 1.4] this says $H = L = L^\circ$ and $P = \{1\}$.) That simplifies the general definitions [**25**, Definition 6] of *principal* and [**25**, Definition 8] of $Sp(1)$*–saturated*, as follows. Decompose $\mathfrak{v}$ as a sum $\mathfrak{w}_1 \oplus \cdots \oplus \mathfrak{w}_t$ of irreducible $\mathrm{Ad}(H)$–invariant subspaces. Then (G,H) is **principal** if $Z_H^0 = Z_1 \times \cdots \times Z_m$ where $Z_i \subset GL(\mathfrak{w}_i)$, in other words Z_i acts trivially on $\mathfrak{w}_j$ for $j \neq i$. Decompose $H = Z_H^0 \times H_1 \times \cdots \times H_m$ where the H_i are simple. Suppose that whenever some H_i acts nontrivially on some $\mathfrak{w}_j$ and $Z_H^0 \times \prod_{\ell \neq i} H_\ell$ is irreducible on $\mathfrak{w}_j$, it follows that H_i is trivial on $\mathfrak{w}_k$ for all $k \neq j$. Then $H_i \cong Sp(1)$ and we say that (G,H) is $Sp(1)$**–saturated**. The group $Sp(1)$ will be more visible in the definition when we extend the definition to the cases where $H \neq L$.

In the Table 6.1 below, $\mathfrak{h}_{n;\mathbb{F}}$ is the Heisenberg algebra $\mathrm{Im}\,\mathbb{F} + \mathbb{F}^n$ of real dimension $(\dim_{\mathbb{R}} \mathbb{F} - 1) + n \dim_{\mathbb{R}} \mathbb{F}$. Here $\mathbb{F}$ is the real, complex, quaternion or octonion algebra over $\mathbb{R}$, $\mathrm{Im}\,\mathbb{F}$ is its imaginary component, and

$$\mathfrak{h}_{n;\mathbb{F}} = \mathrm{Im}\,\mathbb{F} + \mathbb{F}^n \text{ with product } [(z_1,v_1),(z_2,v_2)] = (\mathrm{Im}\,(v_1 \cdot v_2^*), 0),$$

where the v_i are row vectors and v_2^* denotes the conjugate ($\mathbb{F}$ over $\mathbb{R}$) transpose of v_2. It is the Lie algebra of the (slightly generalized) Heisenberg group $H_{n;\mathbb{F}}$. Also in the table, in the listing for $\mathfrak{n}$ the summands in double parenthesis $((..))$ are the subalgebras $[\mathfrak{w},\mathfrak{w}] + \mathfrak{w}$ where

$\mathfrak{w}$ is an H–irreducible subspace of $\mathfrak{v}$ with $[\mathfrak{w},\mathfrak{w}] \neq 0$, and the summands not in double parentheses are H–invariant subspaces $\mathfrak{w}$ with $[\mathfrak{w},\mathfrak{w}] = 0$. The center $\mathfrak{z} = [\mathfrak{n},\mathfrak{n}] + \mathfrak{u}$ where $\mathfrak{u}$ is the sum of those $\mathfrak{w}$ with $[\mathfrak{w},\mathfrak{w}] = 0$. Thus $\mathfrak{n} = \mathfrak{z} + \mathfrak{v}$ where the center $\mathfrak{z}$ is the sum of $[\mathfrak{n},\mathfrak{n}]$ with those summands listed for $\mathfrak{n}$ that are *not* enclosed in double parenthesis ((..)).

As before, when we write $m/2$ it is assumed that m is even, and similarly $n/2$ requires that n be even. Further $k + \ell = m$ and $r + s = n$ where applicable. In the signatures column we write $\mathfrak{n}'$ for $[\mathfrak{n},\mathfrak{n}]$.

Table 6.1. Maximal Indecomposable Principal Saturated Nilpotent Gelfand Pairs $(N \rtimes H, H)$ for N Nonabelian, Where the Action of H on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ is Reducible

	Group H, Algebra $\mathfrak{n}$	Signatures
1	$U(r,s)$ $((\mathfrak{h}_{r+s;\mathbb{C}})) \oplus \mathfrak{su}(r,s)$	$\mathfrak{v} : (2r,2s)$ $\mathfrak{u} : (r^2+s^2-1, 2rs)$ $\mathfrak{n}' : (1,0)$
	$GL(n;\mathbb{R})$ $((\mathfrak{h}_{n;\mathbb{C}})) \oplus \mathfrak{sl}(n;\mathbb{R})$	$\mathfrak{v} : (n,n)$ $\mathfrak{u} : (n(n-1)/2, n(n+1)/2-1)$ $\mathfrak{n}' : (0,1)$
2	$U(k,\ell)$, $(k,\ell) = (4,0)$ or $(2,2)$ $((\mathrm{Im}\,\mathbb{C} + \Lambda^2(\mathbb{C}^{k,\ell}) + \mathbb{C}^{k,\ell})) \oplus \Lambda^2(\mathbb{R}^{k,\ell})$	$\mathfrak{v} : (2k,2\ell)$ $\mathfrak{u} : (k(k-1)/2 + \ell(\ell-1)/2, k\ell)$ $\mathfrak{n}' : (1,0) \oplus ((12,0)$ or $(4,8))$
	$GL(4;\mathbb{R})$ $((\mathbb{R} + \Lambda^2(\mathbb{C}^{2,2}) + \mathbb{C}^{2,2})) \oplus \mathbb{R}^{3,3}$	$\mathfrak{v} : (4,4)$ $\mathfrak{u} : (3,3)$ and $\mathfrak{n}' : (0,1) \oplus (6,6)$
3	$U(1) \cdot SU(r,s) \cdot U(1)$, $r+s=n$ $((\mathfrak{h}_{n;\mathbb{C}})) \oplus \mathfrak{h}_{n(n-1)/2;\mathbb{C}}))$	$\mathfrak{v} : (2r,2s) \oplus (r^2-r+s^2-s, 2rs)$ $\mathfrak{u} : 0$ and $\mathfrak{n}' : (1,0) \oplus (1,0)$
	$\mathbb{R}^+ \cdot SL(n;\mathbb{R}) \cdot \mathbb{R}^+$ $((\mathfrak{h}_{n;\mathbb{C}})) \oplus +((\mathfrak{h}_{n(n-1)/2;\mathbb{C}}))$	$\mathfrak{v} : (n,n) \oplus (\frac{n(n-1)}{2}, \frac{n(n-1)}{2})$ $\mathfrak{u} : 0$ and $\mathfrak{n}' : (0,1) \oplus (0,1)$
4	$SU(2k,2\ell)$, $(k,\ell) = (2,0)$ or $(1,1)$ $((\mathrm{Im}\,\mathbb{C} + \mathrm{Re}\,\mathbb{H}^{(k,\ell)\times(k,\ell)} + \mathbb{C}^{2k,2\ell}))$ $\oplus\mathbb{R}^{k(2k-1)+\ell(2\ell-1),4k\ell}$	$\mathfrak{v} : (4k,4\ell)$ $\mathfrak{u} : (k(2k-1)+\ell(2\ell-1), 4k\ell)$ $\mathfrak{n}' : (1,0) \oplus ((6,0)$ or $(2,4))$
	$SL(4;\mathbb{R})$ $((\mathbb{R} + \mathrm{Re}\,\mathbb{H}^{2\times 2}_{sp} + \mathbb{R}^{4,4})) \oplus \mathbb{R}^{3,3}$	$\mathfrak{v} : (4,4)$ $\mathfrak{u} : (3,3)$ and $\mathfrak{n}' : (0,1) \oplus (4,2)$
5	$U(k,\ell) \times U(2r,2s)$, $\substack{k+\ell=2 \\ r+s=2}$ $((\mathfrak{u}(k,\ell) + \mathbb{C}^{(k,\ell)\times(2r,2s)}))$ $\oplus\mathbb{R}^{r(2r-1)+s(2s-1),4rs}$	$\mathfrak{v} : (2kr+2\ell s, 2ks+2\ell r)$ $\mathfrak{u} : (r(2r-1)+s(2s-1), 4rs)$ $\mathfrak{n}' : (2r,2s)$
	$GL(2;\mathbb{R}) \cdot GL(4;\mathbb{R})$ $((\mathfrak{gl}(2;\mathbb{R}) + \mathbb{R}^{1,1} \otimes \mathbb{R}^{2\times 4})) \oplus \mathbb{R}^{3,3}$	$\mathfrak{v} : (8,8)$ $\mathfrak{u} : (3,3)$ and $\mathfrak{n}' : (1,3)$
	$GL(1;\mathbb{H}) \cdot GL(2;\mathbb{H})$ $((\mathfrak{gl}(1;\mathbb{H}) + \mathbb{R}^{1,1} \otimes_{\mathbb{R}} \mathbb{H}^2)) \oplus \mathbb{R}^{5,1}$	$\mathfrak{v} : (8,8)$ $\mathfrak{u} : (5,1)$ and $\mathfrak{n}' : (3,1)$
6	$S(U(2k,2\ell) \times U(r,s))$, $\substack{k+\ell=2 \\ r+s=n}$ $((\mathfrak{h}_{4n;\mathbb{C}})) \oplus \mathbb{R}^{k(2k-1)+\ell(2\ell-1),4k\ell}$	$\mathfrak{v} : (8r,8s)$ $\mathfrak{u} : (k(2k-1)+\ell(2\ell-1), 4k\ell)$ $\mathfrak{n}' : (1,0)$
	$S(GL(4;\mathbb{R}) \cdot GL(n;\mathbb{R}))$ $((\mathbb{R} + \mathbb{R}^{1,1} \otimes \mathbb{R}^{4\times n})) \oplus \mathbb{R}^{3,3}$	$\mathfrak{v} : (4n,4n)$ $\mathfrak{u} : (3,3)$ and $\mathfrak{n}' : (0,1)$
	$S(GL(2;\mathbb{H}) \cdot GL(n/2;\mathbb{H}))$ $((\mathbb{R} + \mathbb{R}^{1,1} \otimes_{\mathbb{R}} \mathbb{H}^{2\times n/2})) \oplus \mathbb{R}^{5,1}$	$\mathfrak{v} : (4n,4n)$ $\mathfrak{u} : (5,1)$ and $\mathfrak{n}' : (0,1)$
7	$U(k,\ell) \cdot U(r,s)$, $\substack{k+\ell=m \\ r+s=n}$ $((\mathfrak{h}_{mn;\mathbb{C}})) \oplus ((\mathfrak{h}_{m;\mathbb{C}}))$	$\mathfrak{v} : (2kr+2\ell s, 2ks+2\ell r) \oplus (2k,2\ell)$ $\mathfrak{u} : 0$ and $\mathfrak{n}' : (1,0) \oplus (1,0)$
	$GL(m;\mathbb{R}) \cdot GL(n;\mathbb{R})$ $((\mathbb{R} + \mathbb{R}^{1,1} \otimes \mathbb{R}^{m\times n})) \oplus ((\mathbb{R} + \mathbb{R}^{m,m}))$	$\mathfrak{v} : (mn,mn) \oplus (m,m)$ $\mathfrak{u} : 0$ and $\mathfrak{n}' : (0,1) \oplus (0,1)$
8	$U(1) \cdot Sp(r,s) \cdot U(1)$, $r+s=n$ $((\mathfrak{h}_{2n;\mathbb{C}})) \oplus ((\mathfrak{h}_{2n;\mathbb{C}}))$	$\mathfrak{v} : (4r,4s) \oplus (4r,4s)$ $\mathfrak{u} : 0$ and $\mathfrak{n}' : (1,0) \oplus (1,0)$

... *Table 6.1 continued on next page*

Table 6.1 continued from previous page ...

	Group H, Algebra $\mathfrak{n}$	Signatures
	$\mathbb{R}^+ \cdot Sp(r,s) \cdot U(1)$, $r+s=n$ $((\mathbb{R}+\mathbb{R}^{2n,2n})) \oplus ((\mathfrak{h}_{2n;\mathbb{C}}))$	$\mathfrak{v}:(4r,4s)\oplus(4r,4s)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(0,1)\oplus(1,0)$
	$\mathbb{R}^+ \cdot Sp(r,s) \cdot \mathbb{R}^+$, $r+s=n$ $((\mathbb{R}+\mathbb{R}^{2n,2n})) \oplus ((\mathbb{R}+\mathbb{R}^{2n,2n}))$	$\mathfrak{v}:(4r,4s)\oplus(4r,4s)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,0)\oplus(1,0)$
	$U(1)\cdot Sp(n;\mathbb{R})\cdot U(1)$ $((\mathfrak{h}_{2n;\mathbb{C}}))\oplus((\mathfrak{h}_{2n;\mathbb{C}}))$	$\mathfrak{v}:(2n,2n)\oplus(2n,2n)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,0)\oplus(1,0)$
9	$Sp(1)\cdot Sp(r,s)\cdot\{U(1),\,\mathbb{R}^+\}$ $((\mathfrak{h}_{n;\mathbb{H}}))\oplus((\mathfrak{h}_{2n;\mathbb{C}}))$, $r+s=n$	$\mathfrak{v}:(4r,4s)\oplus(4r,4s)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(3,0)\oplus(1,0)$
	$Sp(1;\mathbb{R})\cdot Sp(n;\mathbb{R})\cdot U(1)$ $((\mathfrak{h}_{n;\mathbb{H}}))\oplus((\mathfrak{h}_{2n;\mathbb{C}}))$	$\mathfrak{v}:(2n,2n)\oplus(2n,2n)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,2)\oplus(1,0)$
10	$H_r = Sp(1)\cdot Sp(r,s)\cdot Sp(1)$ $((\mathfrak{h}_{n;\mathbb{H}}))+((\mathfrak{h}_{n;\mathbb{H}}))$, $r+s=n$	$\mathfrak{v}:(4r,4s)\oplus(4r,4s)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(3,0)\oplus(3,0)$
	$Sp(1;\mathbb{R})\cdot Sp(n;\mathbb{R})\cdot Sp(1;\mathbb{R})$ $((\mathfrak{h}_{n;\mathbb{H}}))\oplus((\mathfrak{h}_{n;\mathbb{H}}))$	$\mathfrak{v}:(2n,2n)\oplus(2n,2n)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,2)\oplus(1,2)$
11	$Sp(k,\ell)\cdot\{Sp(1),U(1),\{1\}\}\cdot Sp(r,s)$ $((\mathfrak{h}_{n;\mathbb{H}}))\oplus\mathbb{H}^{(k,\ell)\times(r,s)}$, ${}_{\substack{k+\ell=m\\ r+s=m}}$	$\mathfrak{v}:(4k,4\ell)$ $\mathfrak{u}:(4kr+4\ell s,4ks+4\ell r)$ $\mathfrak{n}':(3,0)$
	$Sp(m;\mathbb{R})\cdot\{Sp(1;\mathbb{R}),U(1)\}\cdot Sp(n;\mathbb{R})$ $((\mathfrak{h}_{m;\mathbb{H}}))+\mathbb{H}^{m\times n}$	$\mathfrak{v}:(2m,2m)$ $\mathfrak{u}:(2mn,2mn)$ and $\mathfrak{n}':(2,1)$
12	$Sp(k,\ell)\cdot\{Sp(1),U(1),\{1\}\}$, $k+\ell=m$ $((\mathfrak{h}_{m;\mathbb{H}}))\oplus\mathrm{Re}\,\mathbb{H}_0^{(k,\ell)\times(k,\ell)}$	$\mathfrak{v}:(4k,4\ell)$ $\mathfrak{u}:(2m^2-m-1-4k\ell,4k\ell)$ $\mathfrak{n}':(3,0)$
	$H=Sp(m;\mathbb{R})\cdot\{Sp(1;\mathbb{R}),U(1)\}$ $((\mathfrak{h}_{m;\mathbb{H}}))\oplus\mathrm{Re}\,\mathbb{H}_{sp,0}^{m\times m}$	$\mathfrak{v}:(2m,2m)$ $\mathfrak{u}:(m^2-1,m^2-m)$ $\mathfrak{n}':(2,1)$
13	$Spin(k,\ell)\cdot\{\{1\},\,SO(r,s)\}$ $((\mathfrak{h}_{1;\mathbb{O}}))\oplus\mathbb{R}^{(k,\ell)\times(r,s)}$ $k+\ell=7, \ell\leqq k, (r,s)=(2,0)$ or $(1,1)$	$\mathfrak{v}:(q,8-q)$, $q=2[\frac{k+1}{2}]$ $\mathfrak{u}:(rk+s\ell,r\ell+sk)$ $\mathfrak{n}':(k,\ell)$
14	$U(1)\cdot Spin(k,\ell)$, $k+\ell=7,\ell\leqq k$ $((\mathfrak{h}_{7;\mathbb{C}}))\oplus\mathbb{R}^{q,8-q}$, $q=2[\frac{k+1}{2}]$	$\mathfrak{v}:(2k,2\ell)$ $\mathfrak{u}:(q,8-q)$ and $\mathfrak{n}':(1,0)$
	$\mathbb{R}^+\cdot Spin(k,\ell)$, $k+\ell=7,\ell\leqq k$ $((\mathbb{R}+\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{R}^{k,\ell}))\oplus\mathbb{R}^{q,8-q}$, $q=2[\frac{k+1}{2}]$	$\mathfrak{v}:(7,7)$ $\mathfrak{u}:(q,8-q)$ and $\mathfrak{n}':(0,1)$
15	$U(1)\cdot Spin(k,\ell)$, $k+\ell=7,\ell\leqq k$ $((\mathfrak{h}_{8;\mathbb{C}}))\oplus\mathbb{R}^{k,\ell}$	$\mathfrak{v}:(2q,16-2q)$, $q=2[\frac{k+1}{2}]$ $\mathfrak{u}:(k,\ell)$ and $\mathfrak{n}':(1,0)$
	$\mathbb{R}^+\cdot Spin(k,\ell)$, $k+\ell=7,\ell\leqq k$ $((\mathbb{R}+\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{R}^{q,8-q}))\oplus\mathbb{R}^{k,\ell}$, $q=2[\frac{k+1}{2}]$	$\mathfrak{v}:(8,8)$ $\mathfrak{u}:(k,\ell)$ and $\mathfrak{n}':(0,1)$
16	$U(1)\cdot Spin(k,\ell)\cdot U(1)$, $k+\ell=8,\ell\leqq k$, $((\mathfrak{h}_{8;\mathbb{C}}))\oplus((\mathfrak{h}_{8;\mathbb{C}}))$	$\mathfrak{v}:(2k,2\ell)\oplus(2k,2\ell)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,0)\oplus(1,0)$
	$\mathbb{R}^+\cdot Spin(k,\ell)\cdot U(1)$, $k+\ell=8,\ell\leqq k$, $((\mathbb{R}+\mathbb{R}^{1,1}\otimes\mathbb{R}^{k,\ell}))\oplus((\mathfrak{h}_{8;\mathbb{C}}))$	$\mathfrak{v}:(8,8)\oplus(2k,2\ell)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(0,1)\oplus(1,0)$
	$\mathbb{R}^+\cdot Spin(k,\ell)\cdot\mathbb{R}^+$, $k+\ell=8,\ell\leqq k$, $((\mathbb{R}+\mathbb{R}^{1,1}\otimes\mathbb{R}^{k,\ell}))\oplus((\mathbb{R}+\mathbb{R}^{1,1}\otimes\mathbb{R}^{k,\ell}))$	$\mathfrak{v}:(8,8)\oplus(8,8)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(0,1)\oplus(0,1)$
	$U(1)\cdot SO^*(8)\cdot U(1)$ $((\mathfrak{h}_{8;\mathbb{C}}))\oplus((\mathfrak{h}_{8;\mathbb{C}}))$	$\mathfrak{v}:(8,8)\oplus(8,8)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,0)\oplus(1,0)$
17	$U(1)\cdot Spin(2k,2\ell)$, $k=3,4,5;\ \ell=5-k$ $((\mathfrak{h}_{16;\mathbb{C}}))\oplus\mathbb{R}^{2k,2\ell}$	$\mathfrak{v}:(q,16-q), q=2^{[\frac{k+1}{2}]+2}$ $\mathfrak{u}:(2k,2\ell)$ and $\mathfrak{n}':(1,0)$
	$\mathbb{R}^+\cdot Spin(2k-1,2\ell+1)$, ${}_{k=3,4,5;\ \ell=5-k}$ $((\mathbb{R}+\mathbb{R}^{16,16}))\oplus\mathbb{R}^{2k-1,2\ell+1}$	$\mathfrak{v}:(16,16)$ $\mathfrak{u}:(2k-1,2\ell+1)$ $\mathfrak{n}':(0,1)$
	$U(1)\cdot Spin^*(10)$ $((\mathfrak{h}_{16;\mathbb{C}}))\oplus\mathbb{C}^5$	$\mathfrak{v}:(16,16)$ $\mathfrak{u}:(10,0)$ and $\mathfrak{n}':(1,0)$
18	$\{SU(k,\ell),U(k,\ell),U(1)Sp(\frac{m}{2})\}\cdot SU(r,s)$ $((\mathfrak{h}_{2m;\mathbb{C}}))+\mathfrak{su}(r,s)$, $k+\ell=m, r+s=2$	$\mathfrak{v}:(2kr+2\ell s,2ks+2\ell r)$ $\mathfrak{u}:(3-2rs,2rs)$ and $\mathfrak{n}':(1,0)$
	$\{SL(m;\mathbb{R}),GL(m;\mathbb{R})\}\cdot SL(2;\mathbb{R})$ $((\mathbb{R}+\mathbb{R}^{1,1}\otimes\mathbb{R}^{m\times 2}))\oplus\mathfrak{sl}(2;\mathbb{R})$	$\mathfrak{v}:(2m,2m)$ $\mathfrak{u}:(1,2)$ and $\mathfrak{n}':(0,1)$
	$\{SL(m/2;\mathbb{H}),GL(m/2;\mathbb{H})\}\cdot SL(1;\mathbb{H})$ $((\mathbb{R}+\mathbb{H}^{m/2,m/2}))\oplus\mathfrak{sl}(1;\mathbb{H})$	$\mathfrak{v}:(2m,2m)$ $\mathfrak{u}:(3,0)$ and $\mathfrak{n}':(0,1)$
	$Sp(k/2,\ell/2)\cdot GL(2;\mathbb{R})$ $((\mathbb{R}+\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{H}^{k/2,\ell/2}))\oplus\mathfrak{sl}(2;\mathbb{R})$	$\mathfrak{v}:(2m,2m)$ $\mathfrak{u}:(1,3)$ and $\mathfrak{n}':(0,1)$

... Table 6.1 continued on next page

Table 6.1 continued from previous page ...

	Group H, Algebra $\mathfrak{n}$	Signatures
	$Sp(m/2;\mathbb{R})\cdot GL(1;\mathbb{H})$ $((\mathfrak{h}_{2m;\mathbb{C}}))\oplus\mathfrak{sl}(2;\mathbb{R})$	$\mathfrak{v}:(2m,2m)$ $\mathfrak{u}:(1,2)$ and $\mathfrak{n}':(0,1)$
19	$\{SU(k,\ell),U(k,\ell),U(1)Sp(k/2,\ell/2)\}\cdot$ $\cdot U(r,s)$, $k+\ell=m, r+s=2$ $((\mathfrak{h}_{2m;\mathbb{C}}))\oplus((\mathfrak{h}_{2;\mathbb{C}}))$	$\mathfrak{v}:(2kr+2\ell s,2ks+2\ell r)\oplus(2r,2s)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,0)\oplus(1,0)$
	$\{SL(m;\mathbb{R}),GL(m;\mathbb{R})\}\cdot GL(2;\mathbb{R})$ $((\mathbb{R}+\mathbb{R}^{2m,2m}))\oplus((\mathbb{R}+\mathbb{R}^{2,2}))$	$\mathfrak{v}:(2m,2m)\oplus(2,2)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,0)\oplus(1,0)$
	$\mathbb{R}^+\cdot Sp(k/2,\ell/2)\cdot GL(2;\mathbb{R})$ $((\operatorname{Im}\mathbb{C}+\mathbb{C}^{2k,2\ell}\otimes_{\mathbb{R}}\mathbb{R}^{1,1}))\oplus((\operatorname{Im}\mathbb{C}+\mathbb{R}^{2,2}))$	$\mathfrak{v}:(2m,2m)\oplus(2,2)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,0)\oplus(1,0)$
	$U(1)Sp(m/2;\mathbb{R})\cdot U(r,s)$ $((\mathfrak{h}_{2m;\mathbb{C}}))\oplus((\mathfrak{h}_{2;\mathbb{C}}))$	$\mathfrak{v}:(2m,2m)\oplus(2r,2s)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,0)\oplus(1,0)$
20	$\{SU(k,\ell),U(k,\ell),U(1)Sp(\frac{k}{2},\frac{\ell}{2})\}\cdot SU(a,b)\cdot$ $\cdot\{SU(r,s),U(r,s),U(1)Sp(\frac{r}{2},\frac{s}{2})\}$ $((\mathfrak{h}_{2m;\mathbb{C}}))\oplus((\mathfrak{h}_{2n;\mathbb{C}}))$, $k+\ell=m,a+b=2,r+s=n$	$\mathfrak{v}:(2(ak+b\ell),2(a\ell+bk))\oplus$ $(2(ar+bs),2(as+br))$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,0)\oplus(1,0)$
	$\{SU(k,\ell),U(k,\ell),U(1)Sp(\frac{k}{2},\frac{\ell}{2})\}\cdot SU(a,b)\cdot$ $\cdot U(1)Sp(\frac{n}{2};\mathbb{R})\}$ $((\mathfrak{h}_{2m;\mathbb{C}}))\oplus((\mathfrak{h}_{2n;\mathbb{C}}))$	$\mathfrak{v}:(2(ak+b\ell),2(a\ell+bk))\oplus(2n,2n)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(1,0)\oplus(0,1)$
	$U(1)Sp(\frac{m}{2};\mathbb{R})\cdot SU(a,b)\cdot U(1)Sp(\frac{n}{2};\mathbb{R})$ $((\mathfrak{h}_{2m;\mathbb{C}}))\oplus((\mathfrak{h}_{2n;\mathbb{C}}))$	$\mathfrak{v}:(2m,2m)\oplus(2n,2n)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(0,1)\oplus(0,1)$
	$\{SL(m;\mathbb{R}),GL(m;\mathbb{R})\}\cdot SL(2;\mathbb{R})\cdot$ $\cdot\{SL(n;\mathbb{R}),GL(n;\mathbb{R}),\mathbb{R}^+Sp(\frac{r}{2},\frac{s}{2})\}$ $((\mathfrak{h}_{2m;\mathbb{C}}))\oplus((\mathfrak{h}_{2n;\mathbb{C}}))$	$\mathfrak{v}:(2m,2m)\oplus(2n,2n)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(0,1)\oplus(0,1)$
	$\mathbb{R}^+Sp(\frac{k}{2},\frac{\ell}{2})\cdot SL(2;\mathbb{R})\cdot\mathbb{R}^+Sp(\frac{r}{2},\frac{s}{2})$ $((\mathfrak{h}_{2m;\mathbb{C}}))\oplus((\mathfrak{h}_{2n;\mathbb{C}}))$	$\mathfrak{v}:(2m,2m)\oplus(2n,2n)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(0,1)\oplus(0,1)$
	$\{SL(m/2;\mathbb{H}),GL(\frac{m}{2};\mathbb{H})\}\cdot SL(1;\mathbb{H})\cdot$ $\cdot\{SL(\frac{n}{2};\mathbb{H}),GL(\frac{n}{2};\mathbb{H}),\mathbb{R}^+Sp(\frac{n}{2};\mathbb{R})\}$ $((\mathbb{R}+\mathbb{H}^{\frac{m}{2}}\otimes_{\mathbb{R}}\mathbb{R}^{1,1}))\oplus((\mathbb{R}+\mathbb{R}^{1,1}\otimes_{\mathbb{R}}\mathbb{H}^{\frac{n}{2}}))$	$\mathfrak{v}:(2m,2m)\oplus(2n,2n)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(0,1)\oplus(0,1)$
	$\mathbb{R}^+Sp(\frac{m}{2};\mathbb{R})\cdot SL(1;\mathbb{H})\cdot\mathbb{R}^+Sp(\frac{n}{2};\mathbb{R})$ $((\mathbb{R}+\mathbb{R}^{2m,2m}))\oplus((\mathbb{R}+\mathbb{R}^{2n,2n}))$	$\mathfrak{v}:(2m,2m)\oplus(2n,2n)$ $\mathfrak{u}:0$ and $\mathfrak{n}':(0,1)\oplus(0,1)$
21	$\{SU(k,\ell),U(k,\ell),U(1)Sp(\frac{k}{2},\frac{\ell}{2})\}\cdot SU(a,b)\cdot$ $\cdot U(r,s)$, $k+\ell=m,a+b=2,(r,s)=(4,0)$ or $(2,2)$ $((\mathfrak{h}_{2m;\mathbb{C}}))\oplus((\mathfrak{h}_{8;\mathbb{C}}))\oplus\mathbb{R}^{2r-2+s,s}$	$\mathfrak{v}:(4k,4\ell)\oplus$ $\oplus(2ar+2bs,2as+2br)$ $\mathfrak{u}:(2r-2+s,s)$ $\mathfrak{n}':(1,0\oplus(1,0)$
	$\{SL(m;\mathbb{R}),GL(m;\mathbb{R})\}\cdot SL(2;\mathbb{R})\cdot GL(4;\mathbb{R})$ $((\mathbb{R}+\mathbb{R}^{2m,2m}))\oplus((\mathbb{R}+\mathbb{R}^{8,8}))\oplus\mathbb{R}^{3,3}$	$\mathfrak{v}:(2m,2m)\oplus(8,8)$ $\mathfrak{u}:(3,3)$ and $\mathfrak{n}':(0,1)\oplus(0,1)$
	$\{SL(\frac{m}{2};\mathbb{H}),GL(\frac{m}{2};\mathbb{H})\}\cdot SL(1;\mathbb{H})\cdot GL(2;\mathbb{H})$ $((\mathbb{R}+\mathbb{H}^{\frac{m}{2},\frac{m}{2}}))\oplus((\mathbb{R}+\mathbb{H}^{2,2}))\oplus\mathbb{R}^{5,1}$	$\mathfrak{v}:(2m,2m)\oplus(8,8)$ $\mathfrak{u}:(5,1)$ and $\mathfrak{n}':(0,1)\oplus(0,1)$
	$Sp(k/2,\ell/2)\cdot GL(2;\mathbb{R})\cdot GL(4;\mathbb{R})$ $((\mathfrak{h}_{2m;\mathbb{C}}))\oplus((\mathbb{R}+\mathbb{R}^{8,8}))\oplus\mathbb{R}^{3,3}$	$\mathfrak{v}:(4k,4\ell)\oplus(8,8)$ $\mathfrak{u}:(3,3)$ and $\mathfrak{n}':(0,1)\oplus(0,1)$
	$Sp(\frac{m}{2};\mathbb{R})\cdot U(a,b)\cdot U(r,s)$ $((\mathbb{R}+\mathbb{R}^{2m,2m}))\oplus((\mathfrak{h}_{8;\mathbb{C}}))\oplus\mathbb{R}^{2r-2+s,s}$	$\mathfrak{v}:(2m,2m)\oplus$ $\oplus(2ar+2bs,2as+2br)$ $\mathfrak{u}:(2r-2+s,s)$ $\mathfrak{n}':(0,1)\oplus(1,0)$
	$Sp(\frac{m}{2};\mathbb{R})\cdot GL(1;\mathbb{H})\cdot GL(2;\mathbb{H})$ $((\mathbb{R}+\mathbb{R}^{2m,2m}))\oplus((\mathbb{R}+\mathbb{H}^{2,2}))\oplus\mathbb{R}^{5,1}$	$\mathfrak{v}:(2m,2m)\oplus(8,8)$ $\mathfrak{u}:(5,1)$ and $\mathfrak{n}':(0,1)\oplus(0,1)$
22	$U(a,b)\cdot U(r,s)$, $(a,b)=(2,0)$ or $(1,1)$, $(r,s)=(4,0)$ or $(2,2)$ $((\mathfrak{h}_{8;\mathbb{C}}))+\mathbb{R}^{2r-2+s,s}+\mathfrak{su}(a,b)$	$\mathfrak{v}:(ar+bs,as+br)$ $\mathfrak{u}:(2r-2+s,s)\oplus(2a-1,2b)$ $\mathfrak{n}':(1,0)$
	$GL(2;\mathbb{R})\cdot GL(4;\mathbb{R})$ $((\mathbb{R}+\mathbb{R}^{8,8}))\oplus\mathbb{R}^{3,3}\oplus\mathfrak{sl}(2;\mathbb{R})$	$\mathfrak{v}:(8,8)$ $\mathfrak{u}:(3,3)\oplus(1,2)$ and $\mathfrak{n}':(0,1)$
	$GL(1;\mathbb{H})\cdot GL(2;\mathbb{H})$ $((\mathbb{R}+\mathbb{H}^{2,2}))\oplus\mathbb{R}^{5,1}\oplus\mathfrak{sl}(1;\mathbb{H})$	$\mathfrak{v}:(8,8)$ $\mathfrak{u}:(5,1)\oplus(3,0)$ and $\mathfrak{n}':(0,1)$
23	$U(k,\ell)\cdot U(a,b)\cdot U(r,s)$ $(k,\ell),(r,s)=(4,0)$ or $(2,2)$; and $(a,b)=(2,0)$ or $(1,1)$ $\mathbb{R}^{2k-2+\ell,\ell}\oplus((\mathfrak{h}_{8;\mathbb{C}}))\oplus((\mathfrak{h}_{8;\mathbb{C}}))\oplus\mathbb{R}^{2r-2+s,s}$	$\mathfrak{v}:(ak+b\ell,a\ell+bk)\oplus$ $\oplus(ar+bs,as+br)$ $\mathfrak{u}:(2k-2+\ell,\ell)\oplus(2r-2+s,s)$ $\mathfrak{n}':(1,0)\oplus(1,0)$
	$GL(4;\mathbb{R})\cdot GL(2;\mathbb{R})\cdot GL(4;\mathbb{R})$ $\mathbb{R}^{3,3}\oplus((\mathbb{R}+\mathbb{R}^{8,8}))\oplus((\mathbb{R}+\mathbb{R}^{8,8}))\oplus\mathbb{R}^{3,3}$	$\mathfrak{v}:(8,8)\oplus(8,8)$ $\mathfrak{u}:(3,3)\oplus(3,3)$ $\mathfrak{n}':(0,1)\oplus(0,1)$
	$GL(2;\mathbb{H})\cdot GL(1;\mathbb{H})\cdot GL(2;\mathbb{H})$ $\mathbb{R}^{5,1}\oplus((\mathbb{R}+\mathbb{H}^{2,2}))\oplus((\mathbb{R}+\mathbb{H}^{2,2}))\oplus\mathbb{R}^{5,1}$	$\mathfrak{v}:(8,8)\oplus(8,8)$ $\mathfrak{u}:(5,1)\oplus(5,1)$ $\mathfrak{n}':(0,1)\oplus(0,1)$

... Table 6.1 continued on next page

Table 6.1 continued from previous page ...

	Group H, Algebra $\mathfrak{n}$	Signatures
24	$U(1) \cdot SU(k,\ell) \cdot U(1)$, $(k,\ell)=(4,0)$ or $(2,2)$ $((\mathfrak{h}_{4;\mathbb{C}})) \oplus ((\mathfrak{h}_{4;\mathbb{C}})) \oplus \mathbb{R}^{2k-2+\ell,\ell}$	$\mathfrak{v} : (2k, 2\ell) \oplus (2k, 2\ell)$ $\mathfrak{u} : (2k-2+\ell, \ell)$ $\mathfrak{n}' : (1,0) \oplus (1,0)$
	$\mathbb{R}^+ \cdot SL(4;\mathbb{R}) \cdot \mathbb{R}^+$ $((\mathbb{R} + \mathbb{R}^{4,4})) \oplus ((\mathbb{R} + \mathbb{R}^{4,4})) \oplus \mathbb{R}^{3,3}$	$\mathfrak{v} : (4,4) \oplus (4,4)$ $\mathfrak{u} : (3,3)$ and $\mathfrak{n}' : (0,1) \oplus (0,1)$
25	$\{\{1\}, U(1)\} \cdot SU(k,\ell) \cdot \{\{1\}, U(1)\}$ $((\mathfrak{h}_{4;\mathbb{C}})) + \mathbb{R}^{k(k-1)+\ell(\ell-1),2k\ell}$, $k+\ell=4$	$\mathfrak{v} : (2k, 2\ell)$ $\mathfrak{u} : (k(k-1)+\ell(\ell-1), 2k\ell)$ $\mathfrak{n}' : (1,0)$
	$\{\{1\}, \mathbb{R}^+\} \cdot SL(4;\mathbb{R}) \cdot \{\{1\}, \mathbb{R}^+\}$ $((\mathbb{R} + \mathbb{R}^{4,4})) \oplus \mathbb{R}^{6,6}$	$\mathfrak{v} : (4,4)$ $\mathfrak{u} : (6,6)$ and $\mathfrak{n}' : (0,1)$

All the spaces $G_r/H_r = (N_r \rtimes H_r)/H_r$, corresponding to entries of Table 5.2, are weakly symmetric Riemannian manifolds except entry 11 with $H_r = Sp(m) \times Sp(n)$, entry 12 with $H_r = Sp(m)$, entry 13 with $H_r = Spin(7) \times (\{1\}$ or $SO(2)$, and entry 25 with $H_r = (\{1\}$ or $U(1))$. In those four cases G_r/H_r is not weakly symmetric. See [**22**, Theorem 15.4.12].

We now extract special signatures from Table 6.1. In order to avoid redundancy we consider $SO(n)$ only for $n \geqq 3$, $SU(n)$ and $U(n)$ only for $n \geqq 2$, and $Sp(n)$ only for $n \geqq 1$.

Corollary 6.2. *The Lorentz cases, signature of the form* $(p-1, 1)$ *in* Table 6.1*, all are weakly symmetric. In addition to their invariant Lorentz metrics, all except* $H = GL(1;\mathbb{R})$ *in Case 1 and* $H = \mathbb{R}^+ \cdot SL(2;\mathbb{R}) \cdot \mathbb{R}^+$ *in Case 3 have invariant weakly symmetric Riemannian metrics. They are*

Case 1. $H = U(n)$ *with metric on* G/H *of signature* $(n^2+2n-1, 1)$, $H = GL(1;\mathbb{R})$ *with metric on* G/H *of signature* $(2,1)$.

Case 2. $H = U(4)$ *with metric on* G/H *of signature* $(26, 1)$.

Case 3. $H = U(1) \cdot SU(n) \cdot U(1)$ *with metric on* G/H *of signature* $(n^2+n+1, 1)$, $H = \mathbb{R}^+ \cdot SL(2;\mathbb{R}) \cdot \mathbb{R}^+$ *with metric on* G/H *of signature* $(3,1)$.

Case 4. $H = SU(4)$ *with metric on* G/H *of signature* $(20, 1)$.

Case 6. $H = S(U(4) \times U(n))$ *with metric on* G/H *of signature* $(8n+6, 1)$.

Case 8. $H = U(1) \cdot Sp(n) \cdot U(1)$ *with metric on* G/H *of signature* $(8n+1, 1)$

Case 9. $H = Sp(1) \cdot Sp(n) \cdot U(1)$ *with metric on* G/H *of signature* $(8n+3, 1)$

Case 14. $H = U(1) \cdot Spin(7)$ *with metric on* G/H *of signature* $(22, 1)$

Case 15. $H = U(1) \cdot Spin(7)$ *with metric on* G/H *of signature* $(23, 1)$

Case 16. $H = U(1) \cdot Spin(8) \cdot U(1)$ *with metric on* G/H *of signature* $(33, 1)$

Case 17. $H = U(1) \cdot Spin(10)$ *with metric on* G/H *of signature* $(42, 1)$

Case 18. $H = \{SU(m), U(m), U(1)Sp(m/2)\} \cdot SU(2)$ *with metric on* G/H *of signature* $(4m+3, 1)$

Case 19. $H = \{SU(m), U(m), U(1)Sp(m/2)\} \cdot U(2)$ *with metric on* G/H *of signature* $(4m + 5, 1)$

Case 20. $H = \{SU(m), U(m), U(1)Sp(m/2)\} \cdot SU(2) \cdot \{SU(n), U(n), U(1)Sp(n/2)\}$ *with metric on* G/H *of signature* $(4m + 4n + 1, 1)$

Case 21. $H = \{SU(n), U(n), U(1)Sp(\frac{n}{2})\} \cdot SU(2) \cdot U(4)$ *with metric on* G/H *of signature* $(4n + 23, 1)$.

Case 22. $H = U(2) \cdot U(4)$ *with metric on* G/H *of signature* $(25, 1)$

Case 23. $H = U(4) \cdot U(2) \cdot U(4)$ *with metric on* G/H *of signature* $(45, 1)$

Case 24. $H = U(1) \cdot SU(4) \cdot U(1)$ *with metric on* G/H *of signature* $(23, 1)$

Case 25. $H = (U(1)\cdot)SU(4)(\cdot SO(2))$ *with metric on* G/H *of signature* $(20, 1)$

Corollary 6.3. *The complexifications of the Lorentz cases listed in* Corollary 6.2 *all are of trans–Lorentz signature* $(p - 2, 2)$. *The trans–Lorentz cases, signature of the form* $(p-2, 2)$ *in* Table 6.1*, all are weakly symmetric and are as follows.*

Case 1. $H = GL(1; \mathbb{R})$ *with metric on* G/H *of signature* $(1, 2)$.

Case 3. $H = U(1) \cdot SU(n) \cdot U(1)$ *with metric on* G/H *of signature* $(n^2 + n, 2)$*,* $H = U(1) \cdot SU(1, 1) \cdot U(1)$ *with metric on* G/H *of signature* $(6, 2)$*,* $H = \mathbb{R}^+ \cdot SL(2; \mathbb{R}) \cdot \mathbb{R}^+$ *with metric on* G/H *of signature* $(2, 2)$

Case 8. $H = U(1) \cdot Sp(n) \cdot U(1)$ *with metric on* G/H *of signature* $(8n, 2)$

Case 16. $H = U(1) \cdot Spin(8) \cdot U(1)$ *with metric on* G/H *of signature* $(32, 2)$

Case 19. $H = \{SU(m), U(m), U(1)Sp(m/2)\} \cdot U(2)$ *with metric on* G/H *of signature* $(4m + 4, 2)$

Case 20. $H = \begin{array}{l} \{SU(m), U(m), U(1)Sp(m/2)\} \cdot SU(2)\cdot \\ \cdot\{SU(n), U(n), U(1)Sp(n/2)\} \end{array}$ *with metric on* G/H *of signature* $(4m + 4n, 2)$

Case 21. $H = \{SU(n), U(n), U(1)Sp(\frac{n}{2})\} \cdot SU(2) \cdot U(4)$ *with metric on* G/H *of signature* $(4n + 22, 2)$.

Case 23. $H = U(4) \cdot U(2) \cdot U(4)$*, metric on* G/H *of signature* $(44, 2)$

Case 24. $H = U(1) \cdot SU(4) \cdot U(1)$*, metric on* G/H *of signature* $(22, 2)$

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