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Representations on partially holomorphic cohomology spaces, revisited

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Abstract. This is a semi–expository update and rewrite of my 1974 AMS AMS Memoir describing Plancherel formulae and partial Dolbeault cohomology realizations for standard tempered representations for general real reductive Lie groups. Even after so many years, much of that Memoir is up to date, but of course there have been a number of refinements, advances and new developments, most of which have applied to smaller classes of real reductive Lie groups. Here we rewrite that AMS Memoir in in view of these advances and indicate the ties with some of the more recent (or at least less classical) approaches to geometric realization of unitary representations.

0. Introduction

In the 1960's I initiated the study of the orbit structure for the action of a general real semisimple Lie group G on a complex flag manifold $X = G_{\mathbb{C}}/Q$ [**50**]. Here Q is a parabolic subgroup of the complexification $G_{\mathbb{C}}$ of G. In the early 1970's I associated the various series of standard tempered representations of G to certain G–orbits in X. I showed how to construct the standard tempered representations as the natural action of G on certain square integrable partially holomorphic cohomology spaces, corresponding to the CR structure of the orbit and to partially holomorphic complex vector bundles over the orbit [**51**]. At the time these geometric realizations required the infinitesimal characters of the standard tempered representations be sufficiently nonsingular [**40**]; but that condition is no longer needed [**41**]. This simplifies the theory and clarifies our exposition.

The advances in the theory include (i) getting rid of the "sufficiently nonsingular" condition, as mentioned above; (ii) a complete description of the irreducible constituents of the standard tempered representations ([**33**], [**31**], [**32**]) and their analytic continuation ([**47**], [**48**]); (iii) a better approach to the Plancherel Formula ([**27**], [**28**]); and (iv) the Atlas project [**1**]. Most of the advances (ii), (iii) and (iv) apply to smaller classes of groups than the ones we consider here, specifically to groups of Harish–Chandra class or even the smaller class of real reductive linear algebraic groups. In this paper we go over the material of the original AMS Memoir, indicating the principal changes in view of those developments. We also fix one error and a few trivial typos.

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Semisimple representation theory has benefited from increased use of algebraic methods, with more emphasis on Harish–Chandra modules than on representations of the group itself. This has led to better results in many cases, such as the advances (i) and (ii) mentioned above, but has tended to sever the connection with differential geometry. Papers that bridge the gap to some extent include [**26**], [**42**], [**43**], [**53**] and [**54**]. While one must be aware of those results we do not attempt to describe them here, first because of limitations of space, and second because many of them have not yet been extended to general real reductive Lie groups.

0.1. In this paper we work with the class of *general real reductive Lie groups*. Those are the Lie groups G whose Lie algebra $\mathfrak{g}$ is reductive and such that (a) if $g \in G$ then $\mathrm{Ad}(g)$ is an inner automorphism of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and (b) G has a closed normal abelian subgroup Z such that (i) Z centralizes the identity component G^0, (ii) $|G/ZG^0| < \infty$ and (iii) $Z \cap G^0$ is co-compact in the center Z_{G^0} of G^0. These conditions are inherited by reductive components of parabolic subgroups, and the class of groups that satisfy them includes both Harish–Chandra's class $\mathcal{H}$ and all connected real semisimple Lie groups. See [**51**, §0.3] for a discussion. We work out geometric realizations for all classes in the unitary dual $\widehat{G}$ except for a set of Plancherel measure zero, and we express the Plancherel formula in terms of them.

The first part is the construction and analysis of standard tempered representations of general real reductive Lie groups. Roughly speaking this is a matter of extending Harish–Chandra's work from groups of class $\mathcal{H}$, in other word from the case where $|G/G^0| < \infty$ and $[G^0, G^0]$ has finite center. We construct a series of representations for each conjugacy class of Cartan subgroups of G, describe the characters of those representations, and use them for a Plancherel formula. This material is in §§2 through 5.

The second part is the geometric realization of the representations just constructed. Given a Cartan subgroup $H \subset G$ we construct partially complex homogeneous spaces Y of G, partially holomorphic G–homogeneous vector bundles $\mathbb{E} \to Y$, and Hilbert spaces $\mathcal{H}_2^{0,q}(Y;\mathbb{E})$ of square integrable partially harmonic $\mathbb{E}$–valued $(0,q)$–forms on Y. If $[\pi] \in \widehat{G}$ belongs to the series for the G–conjugacy class of H then it is realized by a representation of G by its natural action on one of the $\mathcal{H}_2^{0,q}(Y;\mathbb{E})$. If G/Z is compact and connected this reduces to the Bott–Borel–Weil Theorem. If H/Z is compact this includes the discrete series realizations of Schmid and Narasimhan–Okamoto. This material is in §§6 through 8.

In §1 we illustrate the geometric realization procedure by carrying it out for the "principal series", corresponding to the conjugacy class of minimal parabolic subgroups $B \subset G$. We construct an Iwasawa decomposition $G = KAN$ where $Z \subset K$, and $B = MAN$ where M is the centralizer of A in K. We construct certain subgroups $U \subset M$ that contain Cartan subgroups of M, partially complex homogeneous spaces $Y = G/UAN$ and G–equivariant fibrations $Y \to Y/M = G/B$ whose fibers are maximal complex subvarieties of Y. Given a unitary equivalence class $[\mu] \in \widehat{U}$ and a functional $\sigma \in \mathfrak{a}^*$ we construct a G–homogeneous bundle $\mathbb{E}_{\mu,\sigma} \to Y$ that is holomorphic over the fibers of $Y \to Y/M$. We extend the Bott–Borel–Weil Theorem to groups that are compact modulo their center and identify the sheaf cohomology representation η_μ^q of M on $H^q(M/U; \mathcal{O}(\mathbb{E}_{\mu,\sigma}|_{M/U}))$. Given a unitary equivalence class $[\eta] \in \widehat{M}$ we find all $[\mu] \in \widehat{U}$ and all $q \geqq 0$ such that

$\eta^q_\mu \in [\eta]$. That done we show that the principal series class

$$[\pi_{\eta,\sigma}] = [\mathrm{Ind}\,{}^G_B(\eta \otimes e^{i\sigma})], \text{ where } (\eta \otimes e^{i\sigma})(man) = e^{i\sigma}(a)\eta(m)$$

is realized the natural action of G on a certain square integrable partially holomorphic cohomology space $\mathcal{H}^{0,q}_2(Y;\mathbb{E}_{\mu,\sigma})$. The representations of U and M come out of the Peter–Weyl Theorem and are characterized by Cartan's highest weight theory. In the general case we substitute an extension of Harish–Chandra's theory of the discrete series, based on results of Schmid. In §1 the η^q are identified by a mild extension of the Bott–Borel–Weil Theorem. In the general case the matter is delicate and we need detailed information on distribution characters of representations induced from parabolic subgroups.

0.2. Here is a more detailed description of the contents of this paper and a comparison with the earlier AMS Memoir, except §1, which is described above.

In §2 we record the basic facts on square integrable representations of a locally compact unimodular group G relative to a unitary character $\zeta \in \widehat{Z}$, Here Z is a closed normal abelian subgroup. Denote

$$L_2(G/Z,\zeta) = \{f : G \to \mathbb{C} \mid f(gz) = \zeta(z)^{-1} \text{ and } \int_{G/Z} |f(g)|^2 d(gZ) < \infty\}$$

and $\widehat{G}_\zeta = \{[\pi] \in \widehat{G} \mid \pi(gz) = \zeta(z)\pi(g)\}$. Then $\widehat{G} = \bigcup_{\widehat{Z}} \widehat{G}_\zeta$ and $L_2(G) = \int_{\widehat{Z}} L_2(G/Z,\zeta)d\zeta = \int_{\widehat{G}_\zeta} \mathcal{H}_\pi \widehat{\otimes} \mathcal{H}^*_\pi \, d_\zeta[\pi]$ where $d_\zeta[\pi]$ is the Plancherel measure on $\widehat{G}_\zeta$. We say *ζ–discrete* for the classes in $\widehat{G}_{\zeta-disc}$. The set of all these classes forms the *ζ–discrete series* $\widehat{G}_{\zeta-disc}$ of G, and the *relative discrete series* is the union $\widehat{G}_{disc} = \bigcup \widehat{G}_{\zeta-disc}$. If $[\pi] \in \widehat{G}_\zeta$ then $[\pi] \in \widehat{G}_{\zeta-disc}$ if and only if its coefficient functions $\phi_{u,v}(g) = \langle u, \pi(g)v\rangle$ satisfy $|\phi_{u,v}| \in L_2(G/Z)$. When Z is compact, $\widehat{G}_{disc}$ is the usual discrete series of G.

Let U be a closed subgroup of G with $Z \subset U$ and U/Z compact. We give a short proof that $\widehat{U} = \widehat{U}_{disc}$ and that every class in $\widehat{U}$ is finite dimensional. Then we write down the relative Plancherel formula for the $L_2(G/Z,\zeta)$ and the absolute Plancherel formula for $L_2(G)$.

In §3 we see how Harish–Chandra's theory of the discrete series for his class $\mathcal{H}$ extends to our class of general real reductive Lie groups. This was originally done in [**51**, §3] by looking at central extensions $1 \to S \to G[\zeta] \to ZG^0/Z \to 1$ where S is the circle group $\{s \in \mathbb{C} \mid |s| = 1\}$ and $G[\zeta] = \{S \times ZG^0\}/\{\zeta(z)^{-1}, z) \mid z \in Z\}$. That led to a bijection $\widehat{G[\zeta]}_1 \to (\widehat{ZG^0})_\zeta$. We showed that $G[\zeta]$ is a connected reductive Lie group with compact center and verified that Harish–Chandra's discrete series theory applies to such groups. Here we go directly and apply results of R. Herb and the author ([**27**], [**28**]). In particular we see that $\widehat{G}_{disc}$ is nonempty if and only if G/Z has a compact Cartan subgroup, and in that case one has the expected infinitesimal and global character formulae.

In §4 we construct a series of unitary representations of G for every conjugacy class of Cartan subgroups $H \subset G$. For lack of a better term we continue to refer to the series for $\{\mathrm{Ad}(g)H \mid g \in G\}$ as the "H–series" of G. If H/Z is compact then the H–series is the relative discrete series. If H/Z is maximally noncompact it is the principal series. If G/G^0 and the center of $[G^0, G^0]$ are finite, in other words if G is of class $\mathcal{H}$, then the various H–series are just the standard tempered series constructed by Harish–Chandra and used by him to decompose $L_2(G)$. Our constructions and

results first came as straightforward extensions of results of Harish–Chandra, Hirai and Lipsman, but now they follow directly from Herb and the author [**27**].

If H is a Cartan subgroup of G we construct a "Cartan involution" θ of G that leaves H stable. Thus $\theta^2 = 1$, $\theta(H) = H$, $K = \{g \in G \mid \theta(g) = g\}$ contains Z, and K/Z is a maximal compact subgroup of G/Z. Further $H = T \times A$ where $T = H \cap K$ and $A = \exp(\mathfrak{a})$, $\mathfrak{a} = \{\xi \in \mathfrak{h} \mid \theta(\xi) = -\xi\}$. Choose a positive $\mathfrak{a}$–root system $\Sigma_{\mathfrak{a}}^+$ on $\mathfrak{g}$, define $\mathfrak{n} = \sum_{\phi \in \Sigma_{\mathfrak{a}}^+} \mathfrak{g}^{-\phi}$, and let N be the analytic subgroup of G for $\mathfrak{n}$. All this defines

$$P = \{g \in G \mid \mathrm{Ad}(g)N = N\}, \text{ cuspidal parabolic subgroup of } G.$$

The centralizer $Z_G(A) = M \times A$ with $\theta(M) = M$, $P = MAN$, N is the unipotent radical of P, and MA is the reductive part of P. The groups M and MA inherit our working conditions from G: they belong to the class of general real reductive Lie groups. Further, T is a Cartan subgroup of M, and T/Z is compact, so $\widehat{M}_{disc}$ is not empty. If $[\eta] \in \widehat{M}$ and $\sigma \in \mathfrak{a}^*$ then $(\eta \otimes e^{i\sigma})(ma) = e^{i\sigma}(a)\eta(m)$ defines an irreducible unitary representation of P. The H–*series* of G consists of the unitary equivalence classes $[\pi_{\eta,\sigma}] = [\mathrm{Ind}_P^G(\eta \otimes e^{i\sigma})]$ where $[\eta] \in \widehat{M}_{disc}$ and $\sigma \in \mathfrak{a}^*$.

We compute central, infinitesimal and distribution characters of the classes $[\mathrm{Ind}_P^G(\eta \otimes e^{i\sigma})]$. In particular we see that the H–series classes are finite sums of irreducibles, and that the H–series of G depends only on the conjugacy class of H, independent of the choice of $\Sigma_{\mathfrak{a}}^+$. We also see that if H_1 and H_2 are non–conjugate Cartan subgroups of H then the H_1–series and the H_2–series are disjoint.

In §5 we describe the Plancherel formula for G. There is a sharp improvement ([**27**] and [**28**]) on our original argument [**51**, §5]. It gives a precise description of the character formula and shows that the densities $m_{j,\zeta,\nu}$ in the Plancherel formula are restrictions of meromorphic functions. However that proof is rather technical, involving machinery that takes some space to describe, and we do not need the meromorphicity properties of the $m_{j,\zeta,\nu}$. For that reason we simply state the result as proved in [**51**, §5]. That ends the first part of this paper.

§6 starts the second part of this paper, the geometric realization of the representations involved in the Plancherel formula, based on the action of G on complex flag manifolds $X = \overline{G}_{\mathbb{C}}/Q$. Here $\overline{G} = G/Z_G(G^0)$ is the adjoint group, $\overline{G}_{\mathbb{C}}$ is its complexification, and Q is a parabolic subgroup of $\overline{G}_{\mathbb{C}}$. Then G acts naturally on X, as in [**50**], through the lift of the action of $\overline{G}$. The main points for us are the concept of holomorphic arc component and measurable orbit. The orbits over which we realize a general H–series representation of G are measurable, and the relative discrete series representations of M are constructed over holomorphic arc components of measurable orbits. We end §6 with a classification and structure theory for the orbits $G(x) \subset X$ over which we have geometric realizations of H–series representations of G.

In §7 we work out the partially holomorphic cohomology realizations of relative discrete series representations of G. Suppose that G has a Cartan subgroup H with H/Z compact. Then we have complex flag manifolds $X = \overline{G}_{\mathbb{C}}/Q$ with orbits $Y = G(x) \subset X$ such that $Y = G/U$ and $H \subset U$ with U/Z compact, and all such G–orbits on X are open. They are defined by a choice of positive $\mathfrak{h}_{\mathbb{C}}$–root system Σ^+ and a subset Φ of the corresponding simple root system, such that Φ is the simple root system for $\mathfrak{u}_{\mathbb{C}}$. For each $[\mu] \in \widehat{U}$ we have a G–homogeneous hermitian holomorphic vector bundle $\mathbb{E} \to Y$. Let $q \geqq 0$. Then we have the Hilbert

space $\mathcal{H}_2^{0,q}(Y;\mathbb{E})$ of $\mathbb{E}$–valued square integrable harmonic $(0,q)$–forms on Y, and G acts on $\mathcal{H}_2^{0,q}(Y;\mathbb{E})$ by a unitary representation π_μ^q. Write $\Theta_{\pi_\mu^q,disc}$ for the sum of the distribution characters of the irreducible summands of π_μ^q. Let ρ denote half the sum of the positive roots. Thus $\pi_\mu^q = \pi_{\chi,\lambda+\rho}$ in the notation of §3 where $[\chi] \in \widehat{Z_G(G^0)}_\zeta$ and $\lambda+\rho \in i\mathfrak{h}^*$ integrates to a unitary character on H^0 that agrees with ζ on Z_{G^0}. We can (and do) arrange this in such a way that λ is $\mathfrak{u}$–dominant, i.e. $\langle\lambda,\phi\rangle \geqq 0$ for every $\phi \in \Phi$. Note $U = Z_G(G^0)U^0$ so $[\mu] = [\chi \otimes \mu^0]$.

$$q(\lambda+\rho) = \#\{\alpha \text{ compact } \mid \langle\lambda+\rho,\alpha\rangle < 0\} + \#\{\alpha \text{ noncompact } \mid \langle\lambda+\rho,\alpha\rangle > 0\}.$$

We prove

(i) $\sum_{q\geqq 0}(-1)^q\Theta_{\pi_\mu^q,disc} = (-1)^{|\Sigma^+|+q(\lambda+\rho)}\Theta_{\chi,\lambda+\chi}$,

(ii) If $q \neq q(\lambda+\rho)$ then $\mathcal{H}_2^{0,q}(Y;\mathbb{E}) = 0$, and (iii) $[\pi_\mu^{q(\lambda+\rho)}] = [\pi_{\chi,\lambda+\rho}] \in \widehat{G}_{disc}$. This improvement over [**51**, §7] is mostly due to the improvement of [**41**] over [**40**], but it still relies on the Plancherel formula.

In §8 we work out the geometric realization for all standard tempered representations. Fix a Cartan subgroup $H = T \times A$ and a corresponding cuspidal parabolic subgroup $P = MAN \subset G$. The H–series classes are realized over measurable orbits $Y = G(x) \subset \overline{G}_{\mathbb{C}}/Q = X$ such that (i) The G–normalizer $N_{[x]}$ of the holomorphic arc component $S_{[x]}$ through x is an open subgroup of P and (ii) $T \subset U = \{m \in M \mid m(x) = x\}$ with U/Z compact. (Such pairs always exist and were classified at the end of §6.) Then G has isotropy subgroup UAN at x, $N_{[x]} = M^\dagger AN$ where $M^\dagger = Z_M(M^0)M^0$, and $U = Z_M(M^0)U^0$ with $U \cap M^0 = U^0$.

Let $\rho_{\mathfrak{a}} = \frac{1}{2}\sum_{\phi\in\Sigma_{\mathfrak{a}}^+}(\dim \mathfrak{g}^\phi)\phi$ where $\mathfrak{m} = \sum_{\phi\in\Sigma_{\mathfrak{a}}^+}\mathfrak{g}^{-\phi}$. Given $[\mu] \in \widehat{U}$ and $\sigma \in \mathfrak{a}^*$ we have the G–homogeneous vector bundle $\mathbb{E}_{\mu,\sigma} \to G/UAN = Y$ associated to $[\mu \otimes e^{\rho_{\mathfrak{a}}+i\sigma}] \in \widehat{UAN}$. $\mathbb{E}_{\mu,\sigma}$ carries a K–invariant hermitian metric and it is holomorphic over every holomorphic arc component $S_{[x]}$ of Y. That leads to the Hilbert spaces $\mathcal{H}_2^{0,q}(Y;\mathbb{E}_{\mu,\sigma})$ of all square integrable partially harmonic $\mathbb{E}_{\mu,\sigma}$–valued $(0,q)$–forms on Y. G acts there by a unitary representation $\pi_{\mu,\sigma}^q$, and we prove $[\pi_{\mu,\sigma}^q] = [\mathrm{Ind}_{N_{[x]}}^G(\eta_\mu^q \otimes e^{i\sigma})]$ where η_μ^q is the representation of $M^\dagger$ on $\mathcal{H}_2^{0,q}(S_{[x]};\mathbb{E}_{\mu,\sigma}|_{S_{[x]}})$.

Decompose $[\mu] = [\chi \otimes \mu^0]$ where $[\chi] \in \widehat{Z_M(M^0)}$. Let ν be the highest weight of $[\mu^0]$. Then $\chi = e^\nu$ on the center of M^0. Our main result, Theorem 8.3.2, is

(i) The H–series constituents of $\pi_{\mu,\sigma}^q$ are just its irreducible subrepresentations, and $\pi_{\mu,\sigma}^q$ has well defined distribution character $\Theta_{\pi_{\mu,\sigma}^q}$.

(ii) $\sum_{q\geqq 0}(-1)^q\Theta_{\pi_{\mu,\sigma}^q} = (-1)^{|\Sigma_{\mathfrak{t}}^+|+q_M(\nu+\rho_{\mathfrak{t}})}\Theta_{\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}}$ where $[\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}]$ is the H–series class as denoted in §4.

(iii) If $q \neq q_M(\nu+\rho_{\mathfrak{t}})$ then $\mathcal{H}_2^{0,q}(Y;\mathbb{E}_{\mu,\sigma}) = 0$

(iv) $[\pi_{\mu,\sigma}^{q_M(\nu+\rho_{\mathfrak{t}})}]$ is the H–series representation class $[\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}]$.

That gives explicit geometric realizations for the H–series classes of unitary representations of G. The improvement over [**51**, §8] comes from the improvement of §7 over [**51**, §7].

1. The Principal Series

The first example of geometric realization for general real reductive Lie groups is given by the "principal series", combining the Bott–Borel–Weil Theorem with Mackey's theory of unitary induction.

1.1. Let M be a reductive Lie group, i.e. $\mathfrak{m} = \mathfrak{z}_{\mathfrak{m}} \oplus [\mathfrak{m}, \mathfrak{m}]$ where $\mathfrak{z}_{\mathfrak{m}}$ is the center of $\mathfrak{m}$. We assume that M has a closed normal abelian subgroup Z such that (i) M/Z is compact and (ii) if $m \in M$ then $\mathrm{Ad}(m)$ is an inner automorphism on $\mathfrak{g}_{\mathbb{C}}$. From (ii), ZM^0 has finite index in M and $Z \cap M^0$ is co-compact in the center Z_{M^0} of M^0.

As usual, if η is a unitary representation of M then $[\eta]$ is its unitary equivalence class. The set of all such equivalence classes $[\eta]$, with η irreducible, is the *unitary dual* $\widehat{M}$. If E is a close central subgroup of M and $\xi \in \widehat{E}$ then $\widehat{M}_\xi := \{[\eta] \in \widehat{M} \mid \eta|_E$ is a multiple of $\xi\}$.

Proposition 1.1.1. *Let $Z_M(M^0)$ denote the centralizer of M^0 in M. Then*

1. *$Z_M(M^0) \cap M^0 = Z_{M^0}$ and $M = Z_M(M^0)M^0$.*
2. *If $[\eta] \in \widehat{M}$ then η is finite dimensional.*
3. *If $[\eta] \in \widehat{M}$ then there exist unique $[\xi] \in \widehat{Z_{M^0}}$, $[\chi] \in \widehat{Z_M(M^0)}_\xi$ and $[\eta^0] \in (\widehat{M^0})_\xi$ such that $[\eta] = [\chi \otimes \eta^0]$.*
4. *Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}$, $\Sigma_{\mathfrak{t}}^+$ a positive root system, $\rho_{\mathfrak{t}}$ half the sum of the positive roots, $T^0 = \exp(\mathfrak{t})$, and*

$$L_{\mathfrak{m}}^+ = \{\nu \in i\mathfrak{t}^* \mid e^{\nu - \rho_{\mathfrak{t}}} \in \widehat{T^0} \text{ is well defined and } \langle \nu, \phi \rangle > 0 \text{ for every } \phi \in \Sigma_{\mathfrak{t}}^+\}.$$

Then there is a bijection $\nu \mapsto [\eta_\nu^0]$ of $L_{\mathfrak{m}}^+$ onto $\widehat{M^0}$ where $\nu - \rho_{\mathfrak{t}}$ is the highest weight of η_ν^0. Further, $[\eta_\nu^0] \in (\widehat{M^0})_\xi$ where $\xi = e^{\nu - \rho_{\mathfrak{t}}}|_{Z_{M^0}}$.

Proof. If $m \in M$ then $\mathrm{Ad}(m)$ is an inner automorphism on M^0, so $M = Z_M(M^0)M^0$, in particular Z_{M^0} is central in M.

Let $[\eta] \in \widehat{M}$. Then $\eta|_{Z_{M^0}}$ is a multiple of a unitary character ξ on Z_{M^0}. As M^0/Z_{M^0} and $Z_M(M^0)/Z_{M^0}$ are compact, M^0 and $Z_M(M^0)$ are of type I with every irreducible representation finite dimensional; see §2.4 below. Now $[\eta|_{Z_M(M^0)}]$ is a multiple of a finite dimensional class $[\chi] \in \widehat{Z_M(M^0)}_\xi$ and $[\eta|_{M^0}]$ is a multiple of a finite dimensional class $[\eta^0] \in \widetilde{M^0}_\xi$. Assertions (2) and (3) follow, and assertion (4) boils down to the highest weight theory of $\mathfrak{m}$. □

The Bott–Borel–Weil Theorem extends from compact connected Lie groups to give geometric realizations of the classes in $\widehat{M}$.

The homogeneous Kähler manifolds of M are the manifolds S_Φ given by

$$\begin{aligned} &\Pi_{\mathfrak{t}} : \text{ simple } \mathfrak{t}_{\mathbb{C}}\text{–root system on } \mathfrak{m}_C \text{ for } \Sigma_{\mathfrak{t}}^+, \\ &\Phi : \text{ arbitrary subset of } \Pi_{\mathfrak{t}} \\ &\mathfrak{z}_\Phi = \{x \in \mathfrak{t} \mid \Phi(x) = 0\} \text{ and } Z_\Phi^0 = \exp(\mathfrak{z}_\Phi) \subset T^0 \\ &U_\Phi \text{ is the } M\text{–centralizer of } Z_\Phi^0 \text{ and } S_\Phi = M/U_\Phi \end{aligned} \tag{1.1.2}$$

Note that Φ is a simple $\mathfrak{t}_{\mathbb{C}}$–root system for $\mathfrak{u}_\Phi$. Using Proposition 1.1.1(1), construct

$$\begin{aligned} &\overline{M} := M/Z_M(M^0) = M^0/Z_{M^0} \text{ compact connected Lie group.} \\ &\overline{T} := T/Z_M(M^0) = T^0/Z_{M^0} \text{ maximal torus in } \overline{M}. \end{aligned} \tag{1.1.3}$$

This leads to the homogeneous Kähler structure on S_Φ. $\overline{M}$ has complexification $\overline{M}_{\mathbb{C}} = \mathrm{Int}(\mathfrak{m}_{\mathbb{C}})$, inner automorphism group of $\mathfrak{m}_{\mathbb{C}}$. The group $\overline{U}_\Phi := U_\Phi / Z_M(M^0)$ is connected, and we define

$$\begin{aligned} &\mathfrak{r}_\Phi := \overline{\mathfrak{u}}_{\Phi,\mathbb{C}} + \sum\nolimits_{\phi\in\Sigma_{\mathfrak{t}}^+} \overline{\mathfrak{m}}^{-\phi} \text{ subalgebra of } \overline{\mathfrak{m}}_{\mathbb{C}}\,, \\ &R_\Phi \text{ is the complex analytic subgroup of } \overline{M}_{\mathbb{C}} \text{ for } \mathfrak{r}_\Phi\,. \end{aligned} \tag{1.1.4}$$

Then R_Φ is closed in $\overline{M}_{\mathbb{C}}$, and M acts on $\overline{M}_{\mathbb{C}}/R_\Phi$ by the projection $m \mapsto \overline{m}$ of M onto $\overline{M}$. The orbit $M(1R_\Phi)$ is closed because $\overline{M}$ is compact, and one checks that it has the same real dimension as $\overline{M}_{\mathbb{C}}/R_\Phi$. Thus M is transitive on $\overline{M}_{\mathbb{C}}/R_\Phi$, and $mU_\Phi \mapsto \overline{m}R_\Phi$ is a covering space projection. But $\overline{M}_{\mathbb{C}}/R_\Phi$ is simply connected.

Lemma 1.1.5. *The map $mU_\Phi \mapsto \overline{m}R_\Phi$ is an M–equivariant bijection of $S_\Phi = M/U_\Phi$ onto $\overline{M}_{\mathbb{C}}/R_\Phi$.*

Now the complex presentation $S_\Phi = \overline{M}_{\mathbb{C}}/R_\Phi$ defines an M–homogeneous complex structure on S_Φ, and the coboundary of any $\mathfrak{u}_\Phi$–regular element of $\mathfrak{z}_\Phi^*$ is an M–homogeneous Kähler metric on S_Φ.

The irreducible M–homogeneous holomorphic vector bundles $\mathbb{E}_\mu \to S_\Phi$ are constructed as follows. Let $[\mu] \in \widehat{U}_\Phi$ and let E_μ denote its representation space. Let $\mathbb{E}_\mu \to S_\Phi$ denote the associated complex vector bundle. The group $M^{(1)} = [M^0, M^0]$ is compact, connected and semisimple, so the projection. $M \to \overline{M}$ restricts to a finite covering $M^{(1)} \to \overline{M}$, and that complexifies to a finite holomorphic covering $p : M_{\mathbb{C}}^{(1)} \to \overline{M}_{\mathbb{C}}$. Denote $R_\Phi^{(1)} = p^{-1}(R_\Phi)$. It is connected. Now $\mu^{(1)}$ has a unique completely reducible holomorphic extension $\mu^{(1)}$ to $R_\Phi^{(1)}$. That defines an $M_{\mathbb{C}}^{(1)}$–homogeneous holomorphic vector bundle $\mathbb{E}_\mu = \mathbb{E}_{\mu_{\mathbb{C}}^{(1)}} \to M_{\mathbb{C}}^{(1)}/R_\Phi^{(1)} = \overline{M}_{\mathbb{C}}/R_\Phi = S_\Phi$. That bundle structure is stable under the action of M, proving

Lemma 1.1.6. *$\mathbb{E}_\mu \to S_\Phi$ is an M–homogeneous holomorphic vector bundle.*

The sheaf $\mathcal{O}(\mathbb{E}_\mu) \to S_\Phi$ of germs of holomorphic sections of $\mathbb{E}_\mu \to S_\Phi$ defines

$$\begin{aligned} &H_2^{0,q}(S_\Phi, \mathbb{E}_\mu) : L_2 \text{ harmonic } (0,q)\text{–forms on } S_\Phi \text{ with values in } \mathbb{E}_\Phi\,, \\ &\eta_\mu^q : \text{ representation of } M \text{ on } H^q(S_\Phi; \mathcal{O}(\mathbb{E}_\mu)) \text{ and on } H_2^{0,q}(S_\Phi, \mathbb{E}_\mu) \end{aligned} \tag{1.1.7}$$

Simple connectivity of S_Φ implies $U_\Phi \cap M_\Phi^0 = U_\Phi^0$. With Proposition 1.1.1,

$$\begin{aligned} &\widehat{U_\Phi} = \bigcup\nolimits_{\xi\in\widehat{Z_{M^0}}} (\widehat{U_\Phi})_\xi \text{ where} \\ &(\widehat{U_\Phi})_\xi = \{[\chi\otimes\mu^0] \mid [\chi] \in \widehat{Z_M(M^0)}_\xi \text{ and } [\mu^0] \in (\widehat{U_\Phi^0})_\xi\}. \end{aligned} \tag{1.1.8}$$

The Bott–Borel–Weil Theorem [**3**] extends to our setting as follows.

Proposition 1.1.9. *Let $[\mu] = [\chi\otimes\mu^0] \in \widehat{U}_\Phi$ as in* (1.1.8) *and let β be the highest weight of μ^0.*

1. *If $\langle \beta + \rho_{\mathfrak{t}}, \phi\rangle = 0$ for some $\phi \in \Sigma_{\mathfrak{t}}^+$ then $H^q(S_\Phi; \mathcal{O}(\mathbb{E}_\mu)) = 0$ for all integers q.*

2. *If $\langle \beta + \rho_{\mathfrak{t}}, \phi\rangle \neq 0$ for every $\phi \in \Sigma_{\mathfrak{t}}^+$ define q_0 to be the number of roots $\phi \in \Sigma_{\mathfrak{t}}^+$ for which $\langle \beta + \rho_{\mathfrak{t}}, \phi\rangle < 0$, and let $\nu \in L_{\mathfrak{m}}^+$ that is $W(M^0, T^0)$–conjugate to $\beta + \rho_{\mathfrak{t}}$. Then $[\eta_\mu^{q_0}] = [\chi\otimes\eta_\nu^0]$ and $H^q(S_\Phi; \mathcal{O}(\mathbb{E}_\mu)) = 0$ for every integer $q \neq q_0$.*

PROOF. $[\mu] = [\chi \otimes \mu^0]$ has representation space $E_\mu = E_\chi \otimes E_{\mu^0}$. As $Z_M(M^0)$ acts trivially on S_Φ the associated bundle $\mathbb{E}_\mu = E_\chi \otimes \mathbb{E}_{\mu^0}$. This reduces the proof to the case where M is connected.

Now that M is connected, $M = Z_M^0\, M^{(1)}$ where $M^{(1)} = [M, M]$ is compact, connected and semisimple and where $F := Z_M^0 \cap M^{(1)}$ is finite. Now $U_\Phi = Z_M^0\, U_\Phi^{(1)}$ where $U_\Phi^{(1)} = U_\Phi \cap M^{(1)}$ and $Z_M^0 \cap U^{(1)} = F$. Split $[\mu] = [\varepsilon \otimes \mu^{(1)}]$ where $[\varepsilon] \in \widehat{Z_M^0}$ and $[\mu^{(1)}] \in \widehat{U_\Phi^{(1)}}$ give the same character $\varepsilon|_F$ on F. As above $[\eta_\mu^q] = [\varepsilon \otimes \eta_{\mu^{(1)}}^q]$ where $\eta_{\mu^{(1)}}^q$ induces the representation of $M^{(1)}$ on $H^q(S_\Phi; \mathcal{O}(\mathbb{E}_\mu))$. We have reduced the proof to the case where M is compact, connected and semisimple, which is [**3**]. $\square$

1.2. Let G be a general real reductive Lie group. So $\mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{c}$ is the center and the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ is semisimple. We recall the global conditions:

$$\begin{aligned} &\text{(i) if } m \in M \text{ then } \mathrm{Ad}(g) \text{ is an inner automorphism on } \mathfrak{g}_{\mathbb{C}}\,, \text{ and} \\ &\text{(ii)}\quad G \text{ has a closed normal abelian subgroup } Z \text{ such that} \\ &\qquad Z \text{ centralizes } G^0\,, |G/ZG^0| < \infty, \text{ and } Z \cap G^0 \text{ is co-compact in } Z_{G^0} \end{aligned} \tag{1.2.1}$$

Fix a Cartan involution θ of G. Its fixed point set $K = G^\theta$ contains $Z_G(G^0)$ and

$$K/Z_G(G^0) \text{ is a maximal compact subgroup of } G/Z_G(G^0)\,. \tag{1.2.2}$$

Let $\mathfrak{a}$ be a maximal abelian subspace of $\{x \in \mathfrak{g} \mid \theta(x) = -x\}$. Any two choices are $\mathrm{Ad}_G(K)$–conjugate, $\mathrm{Ad}(\mathfrak{a})$ is diagonalizable on $\mathfrak{g}$, and $\mathfrak{g}$ is the direct sum the joint eigenspaces $\mathfrak{g}^\phi = \{x \in \mathfrak{g} \mid [a, x] = \phi(a)x \text{ for all } a \in \mathfrak{a}\}$. The $\mathfrak{a}$–*root system* of $\mathfrak{g}$ is $\Sigma_\mathfrak{a} := \{\text{joint eigenvalues } \phi \mid \mathfrak{g}^\phi \neq 0\}$. Choose a positive subsystem $\Sigma_\mathfrak{a}^+$; any two choices are conjugate by an element of the K–normalizer of $\mathfrak{a}$. The pair $(\mathfrak{a}, \Sigma_\mathfrak{a}^+)$ specifies

$$\begin{aligned} &N: \text{ analytic subgroup of } G \text{ for } \mathfrak{n} = \textstyle\sum_{\alpha \in \Sigma_\mathfrak{a}^+} \mathfrak{g}^{-\alpha}\,, \\ &A = \exp(\mathfrak{a}), M = Z_K(A), \text{ and } B = \{g \in G \mid Ad(g)N = N\}. \end{aligned} \tag{1.2.3}$$

Then $MA = M \times A = Z_G(A)$, $B = MAN$ is a *minimal parabolic* subgroup of G and $G = KAN$ is the *Iwasawa decomposition*. Further, M and MA satisfy (1.2.1).

G has a Cartan subgroup $H = T \times A$ where T is a Cartan subgroup of M. The corresponding positive root system Σ^+ satisfies $\Sigma_\mathfrak{a}^+ = \{\gamma|_\mathfrak{a} \mid \gamma \in \Sigma^+ \text{ and } \gamma|_\mathfrak{a} \neq 0\}$ and $\Sigma_\mathfrak{t}^+ = \{\gamma|_\mathfrak{t} \mid \gamma \in \Sigma^+ \text{ and } \gamma|_\mathfrak{a} = 0\}$.

Let $[\chi \otimes \eta_\nu^0] \in \widehat{M}$ and $\sigma \in \mathfrak{a}^*$. Together they specify

$$[\alpha_{\chi,\nu,\sigma}] \in \widehat{B} \text{ by } \alpha_{\chi,\nu,\sigma}(man) = (\chi \otimes \eta_\nu^0)(m)e^{i\sigma}(a). \tag{1.2.4}$$

The corresponding *principal series representation* of G is

$$\pi_{\chi,\nu,\sigma} = \mathrm{Ind}\,_B^G(\alpha_{\chi,\nu,\sigma}), \text{ unitarily induced representation.} \tag{1.2.5}$$

We construct partially holomorphic cohomology spaces Y_Φ and use them for geometric constructions of the principal series representations $[\pi_{\chi,\nu,\sigma}]$. Let Π be the simple $(\mathfrak{t} + \mathfrak{a})$ root system for Σ^+ Then $\Phi \subset \Pi_\mathfrak{t} \subset \Pi$. Denote

$$\overline{G} = G/Z_G(G^0), \text{ so } \overline{G}_{\mathbb{C}} = \mathrm{Int}(\mathfrak{g}_{\mathbb{C}}) \text{ and } \overline{\mathfrak{g}} = \mathfrak{g}/\mathfrak{c}. \tag{1.2.6}$$

$\mathfrak{q}_\Phi = (\mathfrak{u}_\Phi/\mathfrak{c}) + \mathfrak{a}_{\mathbb{C}} + \sum_{\gamma \in \Sigma^+} \mathfrak{g}^{-\gamma}$, is a parabolic subalgebra of $\overline{\mathfrak{g}}_{\mathbb{C}}$. Let Q_Φ denote the corresponding parabolic subgroup of $\overline{G}_{\mathbb{C}}$; it is the $\overline{G}_{\mathbb{C}}$–normalizer of N.

That defines the complex homogeneous projective variety,

$$X_\Phi = \overline{G}_{\mathbb{C}}/Q_\Phi\,, \text{ complex flag manifold.} \tag{1.2.7}$$

As Q_Φ is its own normalizer in $\overline{G}_{\mathbb{C}}$ we can identify X_Φ with the set of all $\overline{G}_{\mathbb{C}}$–conjugates of Q_Φ. As discussed in §1.1, G acts on X_Φ by holomorphic diffeomorphisms, through the homomorphism $G \to \overline{G}$. Denote

$$x_\Phi = 1 \cdot Q_\Phi \in X_\Phi \text{ and } Y_\Phi = G(x_\Phi).$$

Then, as in Lemma 1.1.5,

$$U_\Phi AN = \{g \in G \mid g(x_\Phi) = x_\Phi\}, \text{ so } Y_\Phi = U_\Phi AN. \tag{1.2.8}$$

In particular Y_Φ contains

$$S_\Phi := M(x_\Phi) = (MAN)(x_\Phi), \text{ compact complex submanifold of } X_\Phi\,.$$

Since the minimal parabolic subgroup B is its own G–normalizer one can look on the Lie algebra level to see that $B = \{g \in G \mid gS_\Phi = S_\Phi\}$.

LEMMA 1.2.9. *If $g \in G$ then gS_Φ is a complex submanifold of X_Φ contained in Y_Φ. If $S \subset Y_\Phi$ is a connected complex submanifold of X_Φ then S is contained in one of the gS_Φ, $Y_\Phi \to G/B = K/M$ is a well defined equivariant fibration, and gS_Φ is the fiber over gB.*

PROOF. In the terminology of §6.4 the topological component of x_Φ in S_Φ is the holomorphic arc component of Y_Φ through x_Φ. Let τ denote complex conjugation. Then $\mathfrak{q}_\Phi + \tau\mathfrak{q}_\Phi = (\mathfrak{m}/\mathfrak{c})_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}} = (\mathfrak{b}/\mathfrak{c})_{\mathbb{C}}$ subalgebra of $\overline{\mathfrak{g}}_{\mathbb{C}}$. In other words Y_Φ is integrable in the sense of §6.4. Also, $\mathfrak{q}_\Phi$ has τ–stable Levi component $(\mathfrak{u}_\Phi/\mathfrak{c})_{\mathbb{C}}$. Now the assertion is a special case of some results from §6.4. □

We now construct the partially holomorphic bundles and cohomology spaces for the principal series. Let $[\mu] \in \widehat{U_\Phi}$, $\sigma \in \mathfrak{a}^*$ and $\rho_{\mathfrak{a}} = \frac{1}{2}\sum_{\phi\in\Sigma_{\mathfrak{a}}^+}(\dim \mathfrak{g}^{-\phi})\phi \in \mathfrak{a}^*$. Then we have a representation $\gamma_{\mu,\sigma}$ of $U_\Phi AN$ on E_μ, and the associated G–homogeneous vector bundle, given by

$$\gamma_{\mu,\sigma}(uan) = e^{\rho_{\mathfrak{a}}+i\sigma}(a)\mu(u) \text{ and } \mathbb{E}_{\mu,\sigma} \to G/U_\Phi AN = Y_\Phi\,. \tag{1.2.10}$$

Note that $\mathbb{E}_{\mu,\sigma}|_{S_\Phi} = \mathbb{E}_\mu$. If $g \in G$ then $\mathbb{E}_{\mu,\sigma}|_{gS_\Phi} \to gS_\Phi$ is an $\mathrm{Ad}(g)B$–homogeneous holomorphic vector bundle. As $[\mu]$ is unitary and K permutes $\{gS_\Phi\}$ transitively, $\mathbb{E}_{\mu,\sigma}$ carries a K–invariant hermitian metric, which we will use without comment.

If $y \in Y_\Phi$, say $y = g(x_\Phi)$, we view the holomorphic tangent space to gS_Φ at $g(x_\Phi)$ as a subspace $T_{g(x_\Phi)}$ of the complexified tangent space of Y_Φ. These subspaces define a sub-bundle $\mathbb{T} \to Y_\Phi$ of the complexified tangent bundle of Y_Φ. Note that $\mathbb{T} \to Y_\Phi$ is G–homogeneous and is holomorphic over every gS_Φ. Then

$$A^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}) = \{C^\infty \text{ sections of } \mathbb{E}_{\mu,\sigma} \otimes \Lambda^q(\overline{\mathbb{T}}^*) \to Y_\Phi\} \tag{1.2.11}$$

is the space of C^∞ *partially* $(0,q)$*–forms on* Y_Φ *with values in* $\mathbb{E}_{\mu,\sigma}$. From any K–invariant hermitian metric on $\mathbb{T}$, thus also on $\mathbb{E}_{\mu,\sigma} \otimes \Lambda^q(\overline{\mathbb{T}}^*)$, we have Kodaira–Hodge operators

$$\Lambda^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}) \xrightarrow{\#} \Lambda^{n,n-q}(Y_\Phi, \mathbb{E}^*_{\mu,\sigma}) \xrightarrow{\widetilde{\#}} \Lambda^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}) \tag{1.2.12}$$

where $n = \dim_{\mathbb{C}} S_\Phi$. The $\overline{\partial}$ operator of X_Φ induces the $\overline{\partial}$ on each gS_Φ. They fit together to define operators $\overline{\partial} : A^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}) \to A^{0,q+1}(Y_\Phi, \mathbb{E}_{\mu,\sigma})$. Each $A^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma})$ is a pre–Hilbert space with inner product

$$\langle \alpha, \beta \rangle = \int_{K/M} d(kM) \int_{kS_\Phi} \alpha \,\bar{\wedge}\, \#\beta \tag{1.2.13}$$

where $\bar{\wedge}$ signifies exterior product followed by contraction $E_\mu \otimes E_\mu^* \to \mathbb{C}$. Define

$$L_2^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}) : \text{ Hilbert space completion of } A^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}). \tag{1.2.14}$$

Then $\overline{\partial}$ has formal adjoint $\overline{\partial}^* = -\widetilde{\#}\overline{\partial}\#$ there, and that leads to the essentially self adjoint partial Kodaira–Hodge operators

$$\Box = (\overline{\partial} + \overline{\partial}^*)^2 = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \text{ on } L_2^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}). \tag{1.2.15}$$

The *partially harmonic* $(0,q)$*–forms on* Y_Φ *with values in* $\mathbb{E}_{\mu,\sigma}$ are the elements of

$$H^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}) = \{\omega \in L_2^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}) \mid \Box\,\omega = 0\}. \tag{1.2.16}$$

Note that $H^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma})$ consists of all Borel–measurable sections ω of $\mathbb{E}_{\mu,\sigma} \otimes \Lambda^q(\overline{\mathbb{T}}^*) \to Y_\Phi$ such that (i) for almost all $k \in K$, $\omega|_{kS_\Phi}$ is harmonic in the ordinary sense and (ii) the L_2 norms satisfy $\int_{K/M} ||\omega|^2_{kS_\Phi} d(kM) < \infty$. In particular $H^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma})$ is a closed subspace of $L_2^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma})$ and the natural action of G on $H^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma})$ is a continuous representation $\pi^q_{\mu,\sigma}$.

THEOREM 1.2.17. *The representation* $\pi^q_{\mu,\sigma}$ *of* G *on* $H^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma})$ *is unitary. Let* $[\mu] = [\chi \otimes \mu^0]$ *as in* (1.1.8) *and let* β *be the highest weight of* μ^0.

1. *If* $\langle \beta + \rho_{\mathfrak{t}}, \phi \rangle = 0$ *for some* $\phi \in \Sigma_{\mathfrak{t}}^+$ *then* $H^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}) = 0$ *for all* q.

2. *If* $\langle \beta + \rho_{\mathfrak{t}}, \phi \rangle \neq 0$ *for every* $\phi \in \Sigma_{\mathfrak{t}}^+$, *let* q_0 *be the number of roots* $\phi \in \Sigma_{\mathfrak{t}}^+$ *for which* $\langle \beta + \rho_{\mathfrak{t}}, \phi \rangle < 0$, *and let* ν *be the unique element of* $L_{\mathfrak{m}}^+$ *conjugate to* $\beta + \rho_{\mathfrak{t}}$ *by an element of the Weyl group* $W(M^0, T^0)$. *Then* $[\pi^{q_0}_{\mu,\sigma}] = [\pi_{\chi,\nu,\sigma}]$, *principal series class, and* $H^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma}) = 0$ *for every* $q \neq q_0$.

3. *In particular, given a principal series class* $[\pi_{\chi,\nu,\sigma}]$, *we can realize it on* $H^{0,0}(Y_\Phi, \mathbb{E}_{\mu,\sigma})$ *where* $\mu = [\chi \otimes \mu^0]$ *and* μ^0 *has highest weight* $\nu - \rho_{\mathfrak{t}}$;.

PROOF. Let $\tilde{\pi}^q_{\mu,\sigma}$ denote the representation of G on $L_2^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma})$. Factor $\gamma_{\mu,\sigma} = {}'\gamma_{\mu,\sigma} \cdot e^{\rho_{\mathfrak{a}}}$ where ${}'\gamma_{\mu,\sigma}(uan) = e^{i\sigma}(a)\mu(u)$ is unitary and e^{ρ_a} compensates non–unimodularity of $U_\Phi AN$. Thus $\tilde{\pi}^q_{\mu,\sigma} = \operatorname{Ind}_{U_\Phi AN}^G(\gamma_{\mu,\sigma})$ is unitary, so its subrepresentation $\pi^q_{\mu,\sigma} = \operatorname{Ind}_{U_\Phi AN}^G(\gamma_{\mu,\sigma})$ is unitary.

Write E^q_μ for $E_\mu \otimes \Lambda^q(\overline{T}^*_{x_\Phi})$, so $L_2^{0,q}(Y_\Phi, \mathbb{E}_{\mu,\sigma})$ is the space of all measurable $f : G \to E^q_\mu$ such that $f(guan) = (\gamma_{\mu,\sigma}(uan)^{-1} \otimes \Lambda^q \operatorname{Ad}(uan)^{-1}_{\mathfrak{n}})f(g)$ and $\int_{K/U_\Phi} ||f(k)||^2 d(kU_\Phi) < \infty$. Note $\tilde{\pi}^q_{\mu,\sigma} = \operatorname{Ind}_B^G(\tilde{\psi})$ where $\tilde{\psi} = \operatorname{Ind} U_\Phi AN^B({}'\gamma_{\mu,\sigma})$. The representation space elements annihilated by the partial Laplacian $\Box$ correspond to the subspace of the representation space of $\tilde{\psi}$ annihilated by the full Laplacian of $\mathbb{E}_{\mu,\sigma}|_{S_\Phi}$, thus $\pi^q_{\mu,\sigma} = \operatorname{Ind}_B^G(\psi)$ where ψ represents B on $H^{0,q}(S_\Phi, \mathbb{E}_{\mu,\sigma}|_{S_\Phi})$. Denote the representation of M on $H^{0,q}(S_\Phi, \mathbb{E}_\mu)$ by η^q_μ. Now $\psi(man) = \eta^q_\mu(m)e^{i\sigma}(a)$. The theorem now follows from Proposition 1.1.9. $\Box$

2. General Notion of Relative Discrete Series

We recall some basic facts on the relative discrete series of a locally compact group G. Left Haar measure is denoted dg, and the left regular representation of G on $L_2(G)$ is $\ell(g)f](x) = f(g^{-1}x)$. If π is an irreducible unitary representation of G then $\mathcal{H}_\pi$ is its representation space and $[\pi]$ its unitary equivalence class. Its coefficients are the functions $\phi_{u,v}(g) = \langle u, \pi(g)v\rangle$. The set of all equivalence classes of irreducible unitary representations is $\widehat{G}$.

2.1. Let Z be a closed normal abelian subgroup of G. It has left regular representation $\ell^Z = \int_{\widehat{Z}} \zeta\, d\zeta$. The left regular representation of G decomposes as

$$\ell = \operatorname{Ind}_{\{1\}}^G(1) = \operatorname{Ind}_Z^G(\ell_Z) = \int_{\widehat{Z}} \ell_\zeta d\zeta = \int_{\widehat{Z}} \operatorname{Ind}_Z^G(\zeta) d\zeta \tag{2.1.1}$$

where $\ell_\zeta := \operatorname{Ind}_Z^G(\zeta)$ is the left regular representation of G on the Hilbert space

$$L_2(G/Z, \zeta) = \{f : G \to \mathbb{C} \mid f(gz) = \zeta(z)^{-1} f(g) \text{ and } \int_{G/Z} |f(g)|^2 dg < \infty\}. \tag{2.1.2}$$

Thus $L_2(G) = \int_{\widehat{Z}} L_2(G/Z, \zeta)$.

Given $\zeta \in \widehat{Z}$ let $\widehat{G}_\zeta = \{[\pi] \in \widehat{G} \mid \zeta \text{ is a subrepresentation of } \pi|_Z\}$. Thus ℓ_ζ is a direct integral over $\widehat{G}_\zeta$. We say that $[\pi] \in \widehat{G}$ is *ζ–discrete* if π is a subrepresentation of ℓ_ζ. The ζ–discrete classes form the *ζ–discrete series* $\widehat{G}_{\zeta-disc}$. The relative (to Z) discrete series is $\widehat{G}_{disc} = \bigcup \widehat{G}_{\zeta-disc}$.

2.2. Let the closed normal abelian subgroup Z be central in G. If $[\pi] \in \widehat{G}$ there is a character $\zeta_\pi \in \widehat{Z}$ such that $\pi|_Z$ is a multiple of ζ_π. So $\widehat{G}_{disc} = \bigcup \widehat{G}_{\zeta-disc}$ is disjoint.

If $[\pi] \in \widehat{G}$ and $\zeta \in \widehat{Z}$ the following are equivalent: (i) there exists $0 \neq u \in \mathcal{H}_\pi$ with $\phi_{u,u} \in L_2(G/Z, \zeta)$, (ii) if $u, v \in \mathcal{H}_\pi$ then $\phi_{u,v} \in L_2(G/Z, \zeta)$, (iii) $[\pi] \in \widehat{G}_{\zeta-disc}$. Under these conditions there is a number $\deg(\pi) > 0$, the *formal degree* of $[\pi]$, such that

$$\begin{aligned} &\langle \phi_{u,v}, \phi_{u',v'}\rangle = \tfrac{1}{\deg(\pi)} \langle u, u'\rangle \overline{\langle v, v'\rangle} \text{ for } u, u', v, v' \in \mathcal{H}_\pi \\ &\langle \phi_{u,v}, \phi_{u',v'}\rangle = 0 \text{ for } u, v \in \mathcal{H}_\pi \text{ and } u', v' \in \mathcal{H}_{\pi'} \text{ with } [\pi] \neq [\pi]. \end{aligned} \tag{2.2.1}$$

If G is compact, $Z = \{1\}$ and we normalize $\int_G dg = 1$, then $\widehat{G} = \widehat{G}_{disc}$, $\deg \pi$ is the degree in the usual sense, and (2.2.1) reduces to the Frobenius–Schur Relations.

Define $L_p(G/Z, \zeta)$ in the same way, integration over G/Z, for $1 \leqq p < \infty$. Since Z is central $(f * h)(x) = \int_{G/Z} f(g)h(g^{-1}x)dx$ gives a well defined convolution product $L_1(G/Z, \zeta) \times L_p(G/Z, \zeta) \to L_p(G/Z, \zeta)$. The first line of (2.2.1) can be expressed $\phi_{u,v} * \phi_{u',v'} = \frac{1}{\deg(\pi)} \langle u, v'\rangle \phi_{u',v} \in L_2(G/Z, \zeta)$.

These results are due to Godement [**7**] for compact Z, to Harish–Chandra [**15**] for semisimple G. See Dixmier's exposition [**6**, §14] of Godement or apply Rieffel's results [**38**] to the convolution algebra $L_1(G/Z, \zeta) \cap L_2(G/Z, \zeta)$.

2.3. We can relax the condition that Z be central in G, requiring only that G have a finite index subgroup J that centralizes Z. Then we may assume $Z \subset J$ and J normal in G. If $\zeta \in \widehat{Z}$ denote $\ell_\zeta^J = \operatorname{Ind}_Z^J(\zeta)$, so $\ell_\zeta = \operatorname{Ind}_J^G(\ell_\zeta^J)$. Since $|G/J| < \infty$

the restriction $\ell_\zeta|_J = \sum_{xJ\in G/J} \ell^J_{\zeta_x}$, finite sum, where $\zeta_x(z) = \zeta(x^{-1}zx)$. Now

$$\begin{aligned} &\widehat{G}_{disc} = \{[\pi] \in \widehat{G} \mid \pi|_J \text{ has a subrepresentation } [\psi] \in \widehat{J}_{disc}\} \text{ and} \\ &\widehat{J}_{disc} = \{[\psi] \in \widehat{J} \mid \psi \text{ is a subrepresentation of } \pi|_J \text{ for some } [\pi] \in \widehat{G}_{disc}\} \end{aligned} \tag{2.3.1}$$

In our applications, if $[\psi] \in \widehat{J}_{disc}$ then $\operatorname{Ind}_J^G(\psi)|_J$ is a finite sum of mutually inequivalent representations, so $[\psi] \mapsto [\operatorname{Ind}_J^G(\psi)]$ maps $\widehat{J}_{disc}$ onto $\widehat{G}_{disc}$.

2.4. Later we will construct the relative discrete series $\widehat{G}_{disc}$ on homogeneous vector bundles $\mathbb{E} \to G/U$ where $Z \subset U$ and U/Z is compact. For that we will need a mild extension of the Peter–Weyl Theorem.

LEMMA 2.4.1. *Let U be a locally compact group and Z a closed central subgroup such that U/Z is compact. Then every class $[\chi] \in \widehat{U}$ is finite dimensional.*

PROOF. Let S be the circle group $\{s \in \mathbb{C} \mid |s| = 1\}$ and $1_S \in \widehat{S}$ the character $1_S(s) = s$. Given $\zeta \in \widehat{Z}$ define $U[\zeta] = \{S \times U\}/\{(\zeta(z)^{-1}, z) \mid z \in Z\}$. If $[\chi] \in \widehat{U}_\zeta$ now $[1_S \otimes \chi] \in \widehat{S \times U}$ factors through $U[\zeta]$. But $U[\zeta]$ is an extension $1 \to S \to U[\zeta] \to U/Z \to 1$ of a compact group by a compact group, so it is compact. Thus $1_S \otimes \chi$ is finite dimensional. □

LEMMA 2.4.2. *If $\zeta \in \widehat{Z}$ then $\widehat{U}_\zeta = \widehat{U}_{\zeta-disc}$.*

PROOF. Coefficients of $[\chi] \in \widehat{U}_\zeta$ are in $L_2(U/Z, \zeta)$ because U/Z is compact. □

LEMMA 2.4.3. *Let $\zeta \in \widehat{Z}$, $[\chi] \in \widehat{U}_\zeta$, E_χ the representation space of χ, and L_χ the space of coefficients. Then $L_2(U/Z, \zeta)$ is the orthogonal direct sum $\sum_{\widehat{U}_\zeta} L_\chi = \sum_{\widehat{U}_\zeta} E_\chi \otimes E_\chi^*$ where $U \times U$ acts on $L_\chi = E_\chi \otimes E_\chi^*$ by $(\chi \otimes 1) \boxtimes (1 \otimes \chi^*)$.*

PROOF. Let $q : U \to U[\zeta]$ be the restriction to U of the projection $S \times U \to U[\zeta]$ in the proof of Lemma 2.4.1. The assertions follow from the part concerning $\widehat{U[\zeta]}_{1_S}$ in the standard Peter–Weyl Theorem for $U[\zeta]$ □

Normalize Haar measures by $\int_U du = \int_Z dz \int_{U/Z} d(uZ)$ and $\int_{U/Z} d(uZ) = 1$. $C_c(U)$ denotes the space of continuous compactly supported functions on U. If $f \in C_c(U)$ then $\chi(f) : E_\chi \to E_\chi$ by $\chi(f)v = \int_U f(u)\chi(u)v\, du$.

PROPOSITION 2.4.4. *Let $\zeta \in \widehat{Z}$, $[\chi] \in \widehat{U}_\zeta$. If $f \in C_c(U)$ then* trace $\chi(f) = \int_U f(u)$ trace $\chi(u)\, du$ *and* $\dim \chi (= \dim E_\chi)$ *is the formal degree* $\deg \chi$. *Further, orthogonal projection $L_2(U/Z, \zeta) \to L_\chi$ is given by right (or left) convolution with* $(\dim \chi)$ trace $\overline{\chi}$.

PROOF. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of E_χ and $\phi_{i,j} = \langle v_i, \chi(v_j)\rangle$. Then trace $\chi(f) = \sum \langle \chi(f)v_i, v_i\rangle = \sum \int_U \langle f(u)\chi(u)v_i, v_i\rangle du = \int_U f(u)$ trace $\chi(u)du$. That is the first assertion. For the second, use $q : U \to U[\zeta]$ as in the proof of Lemma 2.4.3 to see $\dim \chi = \dim(1_S \otimes \chi) = \deg(1_S \otimes \chi) = \deg \chi$. For the last, use $\phi_{u,v} * \phi_{u',v'} = \frac{1}{\deg(\pi)}\langle u, v'\rangle \phi_{u',v} \in L_2(G/Z, \zeta)$ (from the first line of (2.2.1)) to calculate $1 = \langle \text{trace } \chi, \text{trace } \chi\rangle = \sum_{i,j} \langle \phi_{i,i}, \phi_{j,j}\rangle = n/\deg(\chi)$. □

Combining (2.1.2), Lemma 2.4.3 and Proposition 2.4.4 we have the Plancherel Formula for U. If $f \in C_c(U)$ and $\zeta \in \widehat{Z}$ we denote $f_\zeta(u) = \int_Z f(uz)\zeta(z)dz$.

PROPOSITION 2.4.5. *$L_2(U) = \int_{\widehat{Z}} \big(\sum_{\widehat{U}_\zeta} E_\chi \otimes E_\chi^*\big)\, d\zeta$. If $f \in C_c(U)$ and $u \in U$ then $f(u) = \int_{\widehat{Z}} \big(\sum_{\widehat{U}_\zeta} \operatorname{trace}\ \chi((r(u)f)_\zeta) \dim \chi\big) d\zeta$ where $r(u)f(u') = f(uu')$.*

3. Relative Discrete Series for Reductive Groups

Harish–Chandra's theory of the discrete series ([**18**], [**19**]) extends from his class $\mathcal{H}$ to our class of reductive Lie groups that contains all connected reductive groups and has certain hereditary properties. We describe that class in §3.1. Certain points of Harish–Chandra's general character theory are recalled in §3.2. In §3.3 we show how to reduce certain questions from connected reductive groups in general to the case of compact center. We use that in §3.4 to extend Harish–Chandra's theory of the discrete series to all connected reductive Lie groups, in particular to all connected semisimple Lie groups. Then in §3.5 we obtain the discrete series for our class of groups described in §3.1.

3.1. From now on, G is a reductive Lie group in the sense that its Lie algebra is reductive: $\mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{c}$ is the center and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is semisimple.

DEFINITION 3.1.1. Suppose that G satisfies the condition

$$\text{if } g \in G \text{ then } \operatorname{Ad}(g) \text{ is an inner automorphism on } \mathfrak{g}_{\mathbb{C}}$$

and that G has a closed normal abelian subgroup Z such that

$$Z \text{ centralizes } G^0\,,\ G/ZG^0 \text{ is finite, and } ZG^0/(Z \cap G^0) \text{ is compact.}$$

Then we say that G is a *general real reductive Lie group*, i.e., that G of class $\widetilde{\mathcal{H}}$.

If G/G^0 is finite then Z_{G^0} satisfies the conditions for Z in the definition. The Harish–Chandra class $\mathcal{H}$ consists of the groups $G \in \widetilde{\mathcal{H}}$ for which Z is finite. In any case, the discrete series relative to Z is independent of choice of group Z that satisfies the conditions of Definition 3.1.1. We use the notation

$$G^\dagger = \{g \in G \mid \operatorname{Ad}(g) \text{ is an inner automorphism on } G^0. \tag{3.1.2}$$

Then $G^\dagger = Z_G(G^0)G^0$, $Z_G(G^0)$ is compact and $G^\dagger/ZG^0$ is finite.

3.2. Using the second condition of Definition(3.1.1) we choose a maximal compact subgroup K/Z of G/Z. Then $K^0 = K \cap G^0$, K meets every topological component of G, and $Z_{G^0} \subset K$. The basis of Harish–Chandra's character theory is

LEMMA 3.2.1. *There is an integer $n_G \geqq 1$ such that, if $[\kappa] \in \widehat{K}$ and $[\pi] \in \widehat{G}$, then the multiplicity $mult(\kappa, \pi|_K) \leqq n_G \dim(\kappa) < \infty$.*

PROOF. This was proved by Harish–Chandra for connected reductive Lie groups [**11**], in particular for G^0. If $[\kappa_1] \in \widehat{ZK^0}$ and $[\pi_1] \in \widehat{ZG^0}$ we have $\zeta, \zeta' \in \widehat{Z}, [\kappa_0] \in \widehat{K^0}$ and $[\pi_0] \in \widehat{G^0}$ such that $\kappa_1 = \zeta \otimes \kappa_0$ and $\pi_1 = \zeta' \otimes \pi_o$. Now $mult(\kappa_1, \pi_1|_{ZK^0}) \leqq mult(\kappa_0, \pi_0|_{K^0}) \leqq n_{G^0} \dim(\kappa_0) = n_{G^0} \dim(\kappa)$, so the lemma follows for ZG^0. Finally, if $[\kappa] \in \widehat{K}$ and $[\pi] \in \widehat{G}$, we decompose $\kappa|_{ZK^0} = \sum \kappa_i$ and $\pi|_{ZG^0} = \sum \pi_j$ into irreducible constituents, and

$$\begin{aligned} mult(\kappa, \pi|_K) &\leqq \sum_{i,j} mult(\kappa_i, \pi_j|_{ZK^0}) \leqq \sum_{i,j} n_{G^0} \dim(\kappa_i) \\ &= \sum_i n_{G^0} \dim(\kappa) \leqq (n_{G^0} |G/ZG^0|) \dim(\kappa). \end{aligned}$$

□

REMARK 3.2.2. *Using Casselman–Miličić* [**5**] *one can see that* $n_G \leqq |G/ZG^0|$.

The first consequence of Lemma 3.2.1 is that the group G is CCR. In other words, if $[\pi] \in \widehat{G}$ and $f \in L_1(G)$ then $\pi(f) = \int_G f(g)\pi(g)dg$ is a compact operator on $\mathcal{H}_\pi$. In particular G is of type I. The second consequence is that $\pi(f)$ is of trace class for $f \in C_c^\infty(G)$ and that

$$\Theta_\pi : C_c^\infty \to \mathbb{C} \text{ by } \Theta_\pi(f) = \text{trace } \pi(f) \text{ is a Schwartz distribution on } G. \tag{3.2.3}$$

Θ_π is the *global character* or *distribution character* of π. Classes $[\pi] = [\pi']$ if and only if $\Theta_\pi = \Theta_{\pi'}$.

A differential operator z on G has *transpose* given by $\int_G^t z(f)(g)h(g)dg = \int_G f(g)z(h)(g)dg$. The operator z acts on distributions by $(z\Theta)(f) = \Theta({}^t z(f))$. Given Θ now $z \mapsto z\Theta$ is linear in z. A distribution on G is *invariant* if $\Theta(f) = \Theta(f \cdot \mathrm{Ad}(g))$ for $f \in C_c^\infty(G)$ and $g \in G$.

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the associative algebra of all left–invariant differential operators on G. The center $\mathcal{Z}(\mathfrak{g})$ consists of the bi–invariant operators. That uses (3.1.1). A distribution Θ on G is an *eigendistribution* if $\dim \mathcal{Z}(\mathfrak{g})(\Theta) \leqq 1$. In that case, using commutativity of $\mathcal{Z}(\mathfrak{g})$, we have an algebra homomorphism $\chi_\Theta : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ defined by $z\Theta = \chi_\Theta(z)\Theta$. If $[\pi] \in \widehat{G}$ its distribution character Θ_π is an invariant eigendistribution on G, and the associated homomorphism is

$$\chi_\pi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}\,,\ z\Theta_\pi = \chi_\pi(z)\Theta_\pi\,, \text{ is the } \textit{infinitesimal character} \text{ of } [\pi]. \tag{3.2.4}$$

Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $I(\mathfrak{h}_\mathbb{C})$ denote the algebra of all polynomials on $\mathfrak{h}_\mathbb{C}^*$ invariant under the Weyl group $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let $\gamma : \mathcal{Z}(\mathfrak{g}) \to I(\mathfrak{h}_\mathbb{C})$ denote the Harish–Chandra homomorphism [**16**]. If $\lambda \in \mathfrak{h}_\mathbb{C}^*$ then

$$\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C} \text{ by } \chi_\lambda(z) = [\gamma(z)](\lambda) \tag{3.2.5}$$

is a homomorphism, every homomorphism $\mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ is a χ_λ, and $\chi_\lambda = \chi_{\lambda'}$ if and only if λ and λ' are $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$–equivalent.

Harish–Chandra [**16**] used the equations $z\Theta_\pi = \chi_\pi(z)\Theta_\pi$ and the description (3.2.5) of χ_π to show that Θ_π is a locally L_1 function that is analytic on the *regular set* G'. Here G' is the dense open subset of all $g \in G$ for which $\{\xi \in \mathfrak{g} \mid \mathrm{Ad}(g)\xi = \xi\}$ is a Cartan subalgebra. The complement of G' has measure zero in G.

The differential equations also show that at most finitely many classes in $\widehat{G}$ can have the same infinitesimal character.

3.3. We describe a method for reducing questions of harmonic analysis on ZG^0 and G^0 to the same questions on connected reductive groups with compact center. With that we extends some of Harish–Chandra's discrete series results [**19**] to G^0 in §3.4 and then to G in §3.5. This uses the Mackey central extension $1 \to S \to G[\zeta] \to ZG^0/Z \to 1$ for $\delta\zeta$ as a normalized multiplier on ZG^0. Thus $G[\zeta] = \{S \times ZG^0\}/\{(\zeta(z)^{-1}, z) | z \in Z\}$ and S is the circle group $\{s \in \mathbb{C} \mid |s| = 1\}$. $G[\zeta]$ is a connected reductive group with Lie algebra $\mathfrak{s} \oplus (\mathfrak{g}/\mathfrak{z})$ and compact center.

LEMMA 3.3.1. *Let* $p : ZG^0 \to G[\zeta]$ *be the restriction of the projection* $S \times ZG^0 \to G[\zeta]$. *Then* $f \mapsto f \cdot p$ *gives a* G*–equivariant isometry of* $L_2(G[\zeta]/S, 1_S)$ *onto* $L_2(ZG^0/Z, \zeta)$.

PROOF. View $f \in L_2(G[\zeta]/S, 1_S)$ as a function on $S \times ZG^0$. If $g \in ZG^0$ and $z \in Z$ then $(f \cdot p)(gz) = f(1, gz) = f(\zeta(z), g) = \zeta(z)^{-1} f(1, g) = \zeta(z)^{-1}[f \cdot p](g)$, and

$\int_{ZG^0/Z} |(f\cdot p)(g)|^2 d(gZ) = \int_{(S\times ZG^0)/(S\times Z)} |f(s,g)|^2 d(sS\times gZ) = \int_{G[\zeta]/S} |f(\overline{g})|^2 d(\overline{g}S)$. Thus $f \mapsto f\cdot p$ is an isometric injection of $L_2(G[\zeta]/S, 1_S)$ into $L_2(ZG^0/Z,\zeta)$. It is surjective because every $f' \in L_2(ZG^0/Z,\zeta)$ has form $f\cdot p$ with f defined on $S\times ZG^0$ by $f(s,g) = s^{-1}f'(g)$. □

THEOREM 3.3.2. *$\varepsilon[\psi] = [\psi\cdot f]$ defines a bijection $\varepsilon = \varepsilon_\zeta : \widehat{G[\zeta]}_{1_S} \to (\widehat{ZG^0})_\zeta$. It maps $\widehat{G[\zeta]}_{1_S-disc}$ onto $\widehat{ZG^0}_{\zeta-disc}$ and carries Plancherel measure of $\widehat{G[\zeta]}_{1_S}$ to Plancherel measure of $(\widehat{ZG^0})_\zeta$. Distribution characters satisfy $\Theta_{\varepsilon[\psi]} = \Theta_{[\psi]}\cdot p$.*

PROOF. View $[\psi] \in \widehat{G[\zeta]}_{1_S}$ as a representation of $S\times ZG^0$. If $z\in Z$ and $g\in ZG^0$ then $\psi\cdot p)(gz) = \psi(1,gz) = \psi(\zeta(z),g) = \zeta(z)\psi(1,g) = \zeta(z)[\psi\cdot p](g)$. Thus the central character of $\psi\cdot p$ restricts to ζ on Z, and $[\psi\cdot p]\in(\widehat{ZG^0})_\zeta$.

Let $[\psi],[\psi'] \in \widehat{G[\zeta]}_{1_S}$ and $b : \mathcal{H}_\psi \to \mathcal{H}_{\psi'}$ an isometry. If $\psi' = b\psi b^{-1}$ the above calculation shows $(\psi' p) = b(\psi p)b^{-1}$. If b intertwines ψp with $\psi' p$ the same calculation shows that it intertwines $\psi|_{ZG^0}$ with $\psi'|_{ZG^0}$. Then it intertwines ψ and ψ' because $\psi'(s,g) = s\psi'(1,g) = sb\psi(1,g)b^{-1} = b\psi(s,g)b^{-1}$. Now $\varepsilon_\zeta : \widehat{G[\zeta]}_{1_S} \to (\widehat{ZG^0})_\zeta$ is a well defined bijection.

We reduce the proof of Theorem 3.3.2 to the case where $Z\subset G^0$, i.e. where ZG^0 is connected. Let $\zeta^0 = \zeta|_{ZG^0}$. Then $G^0 \hookrightarrow G$ induces a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & S & \longrightarrow & G^0[\zeta^0] & \longrightarrow & G^0/(Z\cap G^0) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle a} & & \downarrow & & \\
1 & \longrightarrow & S & \longrightarrow & G[\zeta] & \longrightarrow & ZG^0/Z & \longrightarrow & 1
\end{array}$$

where a is an isomorphism because $S\to S$ and $G^0/(Z\cap G^0)\to ZG^0/Z$ are. That results in a bijection ε that is the composition $\widehat{G[\zeta]}_{1_S} \xrightarrow{a^*} \widehat{G^0[\zeta^0]}_{1_S} \xrightarrow{\varepsilon^0} (\widehat{G^0})_{\zeta^0} \xrightarrow{b} (\widehat{ZG^0})_\zeta$ where a^* is induced by a and $b[\pi^0] = [\zeta\otimes\pi^0]$. Plancherel measure and relative discrete series are transported by b. If $[\pi^0]\in(\widetilde{G^0})_{\zeta^0}$, $z\in Z$ and $g\in G^0$ then $\Theta_{b[\pi^0]}(zg) = \zeta(z)\Theta_{\pi^0}(g)$. Thus if $\varepsilon : \widehat{G^0[\zeta^0]}_{1_S} \to (\widehat{G^0})_{\zeta^0}$ has the properties asserted in Theorem 3.3.2 those properties pass over to $\varepsilon : \widehat{G[\zeta]}_{1_S} \to (\widehat{ZG^0})_\zeta$.

Now we assume $Z\subset G^0$, and we further reduce the proof to the case where G^0 is simply connected. Let $\tau : \widetilde{G}\to G^0$ be the universal cover, $\widetilde{Z} = \tau^{-1}(Z)$, and $\widetilde{\zeta}$ the corresponding lift of ζ. Then we have

$$\begin{array}{ccc}
\widehat{\widetilde{G[\widetilde{\zeta}]}}_{1_S} & \xrightarrow{\widetilde{\varepsilon}} & \widehat{\widetilde{G}}_{\widetilde{\zeta}} \\
{\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle j} \\
\widehat{G[\zeta]}_{1_S} & \xrightarrow{\varepsilon} & (\widehat{G^0})_{\zeta_0}
\end{array}$$

linking ε to $\widetilde{\varepsilon}$. Everything is preserved by i^*, and j transports Plancherel measure and relative discrete series.

We check that $\Theta_{j[\psi]} = \Theta_{[\psi]}\tau$ for $[\psi]\in(\widehat{G^0})_\zeta$. It suffices to test this on functions $f\in C_c^\infty(\widetilde{G})$ in a component U of $\tau^{-1}(U_1)$ where U_1 is an open subset of G^0 admissible for the covering. Now f is the lift to U of $f_1\in C_c^\infty(U_1)$ and we calculate $\Theta_{j[\psi]}(f) = \text{trace}\int_U f(x)(\psi\cdot\tau)(x)dx = \text{trace}\int_U f_1(\tau x)dx = \text{trace}\int_{U_1} f_1(x_1)\psi(x_1)dx_1 = \Theta_{[\psi]}(f_1)$. That shows $\Theta_{j[\psi]} = \Theta_{[\psi]}\tau$.

We have shown that if $\widetilde{\varepsilon} : \widehat{\widetilde{G[\widetilde{\zeta}]}}_{1_S} \to \widehat{\widetilde{G}}_{\widetilde{\zeta}}$ satisfies Theorem 3.3.2 then the same is true for $\varepsilon : \widehat{G[\zeta]}_{1_S} \to \widehat{G^0}_\zeta$. This reduces the proof of Theorem 3.3.2 to the case where $Z\subset G^0$ and G^0 is simply connected. There $G^0 = V\times G_{ss}$ where V is a vector group and G_{ss} is semisimple. We can enlarge Z to VZ and assume $Z = V\times D$

where D is a subgroup of finite index in the center Z_{ss} of G_{ss}. That done, $\zeta = \nu \boxtimes \delta$ accordingly, and $G_{ss} \hookrightarrow G^0$ gives an isomorphism $a : G_{ss}[\delta] \cong G^0[\zeta]$. This results in a commutative diagram $\begin{array}{ccc} \widehat{G_{ss}[\delta]}_{1_S} & \xrightarrow{\varepsilon_{ss}} & (\widehat{G_{ss}})_\delta \\ {\scriptstyle a^*}\uparrow & & \downarrow{\scriptstyle b} \\ \widehat{G^0[\zeta]}_{1_S} & \xrightarrow{\varepsilon} & (\widehat{G^0})_{\zeta_0} \end{array}$ where $b[\pi_{ss}] = [\nu \otimes \pi_{ss}]$. As in the reduction to $Z \subset G^0$, if Theorem 3.3.2 holds for G_{ss} it holds for G.

We are finally reduced to the case where $Z \subset G^0$ and G^0 is semisimple. Then Z is discrete, so $S \times G^0 \to G^0[\zeta]$ is a Lie group covering. The method of reduction to simply connected G^0 completes the proof of Theorem 3.3.2. □

3.4. We extend Harish–Chandra's description [**19**] of the discrete series of a connected semisimple Lie group. In fact his analysis extends without change to connected reductive Lie groups with compact center, as in the case of G^0 when $Z \cap G^0$ is compact. We formulate the results. Recall that G^0 is an arbitrary connected reductive Lie group and that $Z \cap G^0$ is co-compact in Z_{G^0}.

THEOREM 3.4.1. *G^0 has a relative discrete series representation if and only if $G^0/(Z \cap G^0)$ has a compact Cartan subgroup.*

The compact Cartan subgroups of $G^0/(Z \cap G^0)$ have form $H^0/(Z \cap G^0)$. The unitary characters on H^0 are in bijective correspondence with

$$L = \{\lambda \in i\mathfrak{h}^* \mid e^\lambda \text{ is well defined on } H^0. \tag{3.4.2}$$

Choose a positive root system Σ^+ and use the standard notation $\rho = \frac{1}{2}\sum_{\phi \in \Sigma^+} \phi$, $\varpi(\lambda) = \prod_{\phi \in \Sigma^+} \langle \phi, \lambda \rangle$ and $\Delta = \prod_{\phi \in \Sigma^+} (e^{\phi/2} - e^{-\phi/2})$ where $\langle \cdot, \cdot \rangle$ comes from the Killing form. Then $\Sigma^+ \subset L$ so $2\rho \in L$. Passing to a 2–sheeted covering group of $G^0/(Z \cap G^0)$, if necessary, we may assume $\rho \in L$ and $e^\rho(Z \cap G^0) = 1$. Δ is an analytic function on H^0, well defined on $H^0/(Z \cap G^0)$. The *regular set* is $L' = \{\lambda \in L \mid \varpi(\lambda) \neq 0\}$. Note $\rho \in L'$. If $\lambda \in L'$ we define

$$q(\lambda) = |\{\text{compact } \phi \in \Sigma^+ \mid \langle \phi, \lambda \rangle < 0\}| + |\{\text{ noncompact } \phi \in \Sigma^+ \mid \langle \phi, \lambda \rangle > 0\}|. \tag{3.4.3}$$

Thus $(-1)^{q(\lambda)} = (-1)^q \operatorname{sign} \varpi(\lambda)$.

THEOREM 3.4.4. *If $\lambda \in L'$ there is a unique class $[\pi_\lambda] \in \widehat{G^0}_{disc}$ whose distribution character satisfies $\Theta_{\pi_\lambda}|_{H^0 \cap G'} = (-1)^{q(\lambda)} \frac{1}{\Delta} \sum_{w \in W_{G^0}} \det(w) e^{w\lambda}$. Every class in $\widehat{G^0}_{disc}$ is one of these $[\pi_\lambda]$. Classes $[\pi_\lambda] = [\pi_{\lambda'}]$ precisely when $\lambda' \in W_{G^0}(\lambda)$. With a certain normalization of Haar measure on G^0, $[\pi_\lambda]$ has formal degree $|\varpi(\lambda)|$.*

COROLLARY 3.4.5. *$[\pi_\lambda] \in (\widehat{G^0})_{disc}$ has dual $[\pi_\lambda^*] = [\pi_{-\lambda}]$, central character $e^{\lambda - \rho}|_{Z_{G^0}}$ and infinitesimal character χ_λ as in* (3.2.5)*; $\chi_\lambda(Casimir) = ||\lambda||^2 - ||\rho||^2$.*

When Z_{G^0} is compact, Theorems 3.4.1 and 3.4.4 reduce to Harish–Chandra's celebrated results [**19**, Theorems 13 and 16]. We describe the reduction.

Let $[\pi] \in (\widehat{G^0})_\zeta$. By Theorem 3.3.2 there exists $[\psi] \in \widehat{G^0[\zeta]}_{1_S}$ such that $\varepsilon_\zeta[\psi] = [\pi]$. In particular [**19**, Theorem 13] $G^0[\zeta]$ has a compact Cartan subgroup, and it must have form $H^0[\zeta]$ where H^0 is a Cartan subgroup of G^0. Since $H^0[\zeta]$ is compact, so is $H^0/(Z \cap G^0)$. That proves the "only if" part of Theorem 3.4.1.

Conversely, let $H^0/(Z \cap G^0)$ be a compact Cartan subgroup of $G^0/(Z \cap G^0)$ and $\zeta \in \widehat{Z \cap G^0}$, so $H^0[\zeta]$ is a compact Cartan subgroup of $G^0[\zeta]$. Denote

$L[\zeta] = \{\nu \in i\mathfrak{h}[\zeta]^* \mid e^\nu \text{ is well defined on } H^0[\zeta]\}$,

$L[\zeta]'_{1_S} = \{\nu \in L[\zeta] \mid \varpi(\nu) \neq 0 \text{ and } e^\nu|_S = 1_S\}$, and
$L'_\zeta = \{\lambda \in L \mid \varpi(\lambda) \neq 0 \text{ and } e^\lambda|_{Z\cap G^0} = \zeta\}$.

Since $G^0[\zeta]$ is a connected reductive Lie group with compact center, and $e^\rho(S) = 1$, [**19**, Theorem 16] gives a map $\omega_1 : L[\zeta]'_{1_S} \to \widehat{G^0[\zeta]}_{1-disc}$, by $\nu \mapsto [\psi_\nu]$, that satisfies the assertions of Theorem 3.4.4. We construct the corresponding $\omega_\zeta : L'_\zeta \to \widehat{G^0}_{\zeta-disc}$ by $\omega_\zeta \cdot \delta = \varepsilon \cdot \omega_1$ where ε is the bijection of Theorem 3.3.2 and $\delta : L[\zeta]'_{1_S} \to L'_\zeta$ is defined as follows.

Let $\nu \in L[\zeta]'_{1_S}$. The distribution character of $\varepsilon\omega_1(\nu) = \omega_\zeta\delta(\nu)$ must have $(H^0 \cap G')$–restriction $(-1)^{q(\delta\nu)}\frac{1}{\Delta}\sum \det(w)e^{w\delta\nu} = (-1)^{q(\nu)}\frac{1}{\Delta}\sum \det(w)e^{w\nu} \cdot p$. For that, define δ by $e^{\delta\nu} = e^\nu \cdot p$, i.e. $\delta\nu = p^*\nu$ under $p : \mathfrak{g} \to \mathfrak{g}[\zeta]$. Since p restricts to an isomorphism of derived algebras, δ bijects $L[\zeta]'_{1_S}$ to L'_ζ equivariantly for W. Our assertions now go over from ω_1 to ω_ζ.

That completes the derivation of Theorems 3.4.1 and 3.4.4 from [**19**].

3.5. We extend the description of the relative discrete series from connected reductive groups to the class of real reductive Lie groups specified in §3.1.

LEMMA 3.5.1. *ZZ_{G^0} has finite index in $Z_G(G^0)$. Every class $[\chi] \in \widehat{Z_G(G^0)}$ has dimension* $\dim\chi \leqq |Z_G(G^0)/ZZ_{G^0}|$.

PROOF. The second condition of Definition 3.1.1 shows that every $[\chi] \in \widehat{Z_G(G^0)}$ is a summand of $\operatorname{Ind}_{\widetilde{Z}}^{Z_G(G^0)}(\beta)$ for some $\beta \in \widehat{\widetilde{Z}}$. □

PROPOSITION 3.5.2. *$\widehat{G^\dagger}$ is the disjoint union of the sets*

$$(\widehat{G^\dagger})_\xi = \{[\chi \otimes \pi] \mid [\chi] \in \widehat{Z_G(G^0)}_\xi \text{ and } [\pi] \in (\widehat{G^0})_\xi \text{ where } \xi \in \widehat{Z_{G^0}}\}. \tag{3.5.3}$$

Here $[\chi \otimes \pi]$ has the same infinitesimal character χ_π and $[\pi]$ and has distribution character $\Theta_{\chi\otimes\pi}(zg) = (\text{trace } \chi(z))\Theta_\pi(g)$ for $z \in Z_G(G^0)$ and $g \in G^0$. Further, $[\chi \otimes \pi] \in (\widehat{G^\dagger})_{disc}$ if and only if $[\pi] \in (\widehat{G^0})_{disc}$.

PROOF. $\widehat{G^\dagger}$ is the disjoint union of the $(\widehat{G^\dagger})_\xi$, $\xi \in \widehat{Z_{G^0}}$, because Z_{G^0} is central in $G^\dagger$. Now fix $\xi \in \widehat{Z_{G^0}}$, $[\chi] \in \widehat{Z_G(G^0)}_\xi$ and $[\pi] \in (\widehat{G^0})_\xi$. Note $[\chi\otimes\pi] \in (\widehat{G^\dagger})_\xi$, and $(\chi \otimes \pi)|_{G^0} = m\pi$ where $m = \dim\chi < \infty$. Thus $\chi \otimes \pi$ has infinitesimal character χ_π, and is discrete relative to Z exactly when π is discrete relative to $Z \cap G^0$.

To prove the formula for $\Theta_{\chi\otimes\pi}$ we need only consider test functions $f \in C_c^\infty(G^\dagger)$ supported in a single coset z_0G^0, and there we compute
trace $(\chi\otimes\pi)(f) = \text{trace } \int_{G^\dagger} f(zg)\chi(z) \otimes \pi(g)dg = \text{trace } \int_{G^0} f(z_0g)\chi(z_0) \otimes \pi(g)dg$
$= \text{trace } \big(\chi(z_0) \otimes \int_{G^0} f(z_0g)\pi(g)dg\big) = (\text{trace } \chi(z_0))\big(\text{trace } \int_{G^0} f(z_0g)\pi(g)dg\big)$
$= (\text{trace } \chi(z_0)) \int_{G^0} f(z_0g)\Theta_\pi(g)dg = \int_{G^0} f(z_0g)(\text{trace } \chi(z_0))\Theta_\pi(g)dg$
$= \int_{G^\dagger} f(zg)(\text{trace } \chi(z))\Theta_\pi(g)d(zg)$, so $\Theta_{\chi\otimes\pi}(zg) = (\text{trace } \chi(z))\Theta_\pi(g)$, as asserted.

Finally let $[\gamma] \in (\widehat{G^\dagger})_\xi$. Since $Z_G(G^0)$ acts trivially on $\widehat{G^0}$ and G^0 is of type I, now $\gamma|_{G^0} = mn$ where $[\chi] \in \widehat{Z_G(G^0)}_\xi$. Thus $[\gamma] = [\chi \otimes \pi]$ because $[\chi \otimes \pi]$ is a subrepresentation. That proves (3.5.3). Proposition 3.5.2 is proved. □

Proposition 3.5.2 gives the relative discrete series of $G^\dagger$ in terms of those of $Z_G(G^0)$ and G^0. The following lets us go on to G.

PROPOSITION 3.5.4. *Let* $[\gamma] = [\chi \otimes \pi] \in \widehat{G^\dagger}$ *and define* $\psi = \mathrm{Ind}_{G^\dagger}^G(\psi)$. *Then* (1) $[\psi]$ *has the same infinitesimal character* χ_π *as* $[\pi]$ *and* (2) $[\pi]$ *has distribution character that is a locally* L_1 *function supported in* $G^\dagger$ *and given there by*

$$\Theta_\psi(zg) = \sum\nolimits_{xG^\dagger \in G/G^\dagger} (\mathrm{trace}\ \chi(x^{-1}zx))\Theta_\pi(x^{-1}gx) \tag{3.5.5}$$

for $z \in Z_G(G^0)$ *and* $g \in G^0$. *In particular* Θ_ψ *is analytic on the regular set* G' *and satisfies* $\Theta_\psi|_{G^\dagger} = \Theta|_{\psi|_{G^\dagger}}$. *Further* (3) *if* $[\pi] \in (\widehat{G^0})_{disc}$ *then* $[\psi] \in \widehat{G}_{disc}$, *and every class in* $\widehat{G}_{disc}$ *is obtained this way.*

PROOF. We follow an argument [**49**, Lemma 4.3.3] of Frobenius for (1) and (2). As $G^\dagger$ is normal and has finite index in G, Θ_ψ exists and is supported in $G^\dagger$, where $\Theta_\psi|_{G^\dagger} = \Theta|_{\psi|_{G^\dagger}}$. Note $\psi|_{G^\dagger} = \sum_{G/G^\dagger} \gamma \cdot \mathrm{Ad}(x^{-1})$. If $z \in Z_G(G^0)$ and $g \in G^0$ now $\Theta_\psi(zg) = \sum \Theta_{\gamma\cdot\mathrm{Ad}(x^{-1})}(zg) = \sum \Theta_\gamma(z^{-1}zx \cdot x^{-1}gx)$. Assertion (2) follows from Proposition 3.5.2.

If $x \in G$ then $\mathrm{Ad}(x)$ is an inner automorphism on $\mathfrak{g}_\mathbb{C}$, hence trivial on $\mathcal{Z}(\mathfrak{g})$, so all the $\gamma \cdot \mathrm{Ad}(x^{-1})$ are the same on $\mathcal{Z}(\mathfrak{g})$. Now ψ has infinitesimal character $\chi_\psi = \chi_\gamma = \chi_\pi$.

Every class in $\widehat{G}_{disc}$ is a subrepresentation of an $[\mathrm{Ind}_{\widehat{G^\dagger}}^G(\gamma)]$, $[\gamma] \in (\widehat{G^\dagger})_{disc}$, because $|G/G^\dagger| < \infty$. If $[\gamma] = [\chi \otimes \pi]$ as in Proposition 3.5.2 then $[\gamma] \in (\widehat{G^\dagger})_{disc}$ is equivalent to $[\pi] \in (\widehat{G^0})_{disc}$. To prove (3) now we need only check that $\psi = \mathrm{Ind}_{G^\dagger}^G(\gamma)$ is irreducible whenever $[\pi] \in (\widehat{G^0})_{disc}$.

Choose a Cartan subgroup $H^0 \subset G^0$ with $H^0/(Z \cap G^0)$ compact. The corresponding Cartan subgroup of G is the centralizer H of $\mathfrak{h}$. Hypothesis (3.1.1) says that the Weyl group W_G is a subgroup of the complex Weyl group $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. As any two compact Cartan subgroups of $G^0/(Z \cap G^0)$ are conjugate we have a system $\{x_1, \dots, x_r\}$ of representatives of G modulo $G^\dagger$ such that each $\mathrm{Ad}(x_i)\mathfrak{h} = \mathfrak{h}$. Now $W_G = \bigcup (x_j H) W_{G^0} \subset W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let $[\gamma] = [\chi \otimes \pi]$ with $[\pi] \in (\widehat{G^0})_{disc}$. Express $[\pi] = [\pi_\lambda]$ with $\lambda \in L'$. Then $[\pi \cdot \mathrm{Ad}(x_j^{-1})] = [\pi_{\lambda_j}]$ where $\lambda_j = \mathrm{Ad}(x_j^{-1})^*(\lambda)$. Since $\lambda \in L'$ the λ_j are distinct modulo the action of W_{G^0}. Theorem 3.4.4 now says that the $\pi \cdot \mathrm{Ad}(x_j^{-1})$ are mutually inequivalent. It follows that ψ is irreducible. □

We formulate the extensions of Theorems 3.4.1 and 3.4.4 from G^0 to G.

THEOREM 3.5.6. *G has a relative discrete series representation if and only if G/Z has a compact Cartan subgroup.*

Let H/Z be a compact Cartan subgroup of G/Z. Retain the notation of §3.4 for $(\widehat{G^0})_{disc}$ and the notation $G = \bigcup x_j G^\dagger$ where the x_j normalize H^0. Write w_j for the element of W_G represented by x_j.

THEOREM 3.5.7. *Let* $\lambda \in L'$ *and* $[\chi] \in \widehat{Z_G(G^0)}_\xi$ *where* $\xi = e^{\lambda-\rho}|_{Z_{G^0}}$. *Let* $[\pi] \in (\widehat{G^0})_{disc}$ *as in* Theorem 3.4.4. *Then* $[\pi_{\chi,\lambda}] := [\mathrm{Ind}_{G^\dagger}^G(\chi \otimes \pi_\lambda)]$ *is the unique class in* $\widehat{G}_{disc}$ *whose distribution character satisfies*

$$\Theta_{\pi_{\chi,\lambda}}(zh) = \sum_{1 \leqq j \leqq r} (-1)^{q(w_j\lambda)} \mathrm{trace}\ \chi(x_j^{-1}zx_j) \cdot \sum_{w \in W_{G^0}} \det(ww_j) e^{ww_j\lambda}(h) \tag{3.5.8}$$

for $z \in Z_G(G^0)$ *and* $h \in H^0 \cap G'$. *Every class in* $\widehat{G}_{disc}$ *is one of these* $[\pi_{\chi,\lambda}]$. *Classes* $[\pi_{\chi,\lambda}] = [\pi_{\chi',\lambda'}]$ *precisely when* $([\chi'], \lambda') \in W_G([\chi], \lambda)$. *For appropriate*

normalizations of Haar measures the formal degree $\deg(\pi_{\chi,\lambda}) = r \cdot \dim(\chi) \cdot |\varpi(\lambda)|$. *Finally,* $[\pi_{\chi,\lambda}]$ *has dual* $[\pi_{\chi^*,-\lambda}]$ *and has infinitesimal character* χ_λ *as in* (3.2.5), *so in particular* $\chi_{[\pi_{\chi,\lambda}]}(Casimir) = ||\lambda||^2 - ||\rho||$.

PROOF. If $\widehat{G}_{disc}$ is nonempty then Propositions 3.5.2 and 3.5.4 show that $(\widehat{G^0})_{disc}$ is not empty, so $G^0/(Z \cap G^0)$ has a compact Cartan subgroup by Theorem 3.4.1. As $G^0/(Z \cap G^0)$ has finite index in G/Z the latter also has a compact Cartan subgroup. If H/Z is a compact Cartan subgroup of G/Z then Theorem 3.5.7 follows directly from Theorem 3.4.4 and Propositions 3.5.2 and 3.5.4. □

4. Tempered Series Representations of Reductive Lie Groups

G is a reductive Lie group of the class described in §3.1. In §3 we used the conjugacy class of Cartan subgroups H of G, with H/Z compact, to construct the relative discrete series $\widehat{G}_{disc}$. Here we construct a series of unitary representations for every conjugacy class of Cartan subgroups of G.

In §4.1 and 4.2 we work out the relation between Cartan involutions θ of G, Cartan subgroups H of G, and cuspidal parabolic subgroups $P = MAN$ of G. Here $H = T \times A$, T/Z compact and A split/$\mathbb{R}$, and $Z_G(A) = M \times A$ where M is in the class of §3.2 and T is a Cartan subgroup of M. Then $H \mapsto P$ gives a bijection from the set of all conjugacy classes of Cartan subgroups of G to the set of all "association classes" of cuspidal parabolic subgroups of G.

In §4.3 we describe these representations, calculating infinitesimal and distribution characters. They form the "H–series" $\widehat{G}_H$. Then in §4.5 we examine the correspondence $\widehat{M}_{disc} \to \widehat{G}_H$. Finally in §4.5 we look at questions of irreducibility.

4.1. G is a real reductive Lie group as in §3.1, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and $H = Z_G(\mathfrak{h})$ is the corresponding Cartan subgroup of G. If G^0 is a linear group, or if H/Z is compact, then $H \cap G^0$ is commutative. In general one only knows that $H^0 = \exp(\mathfrak{h})$ is commutative. We collect some information.

LEMMA 4.1.1. *If K/Z is a maximal compact subgroup of G/Z then there is a unique involutive automorphism θ of G with fixed point set K. These automorphisms θ are the "Cartan involutions" of G, and any two are* $\mathrm{Ad}(G^0)$*–conjugate. Every Cartan subgroup of G is stable under a Cartan involution.*

LEMMA 4.1.2. *If K/Z is a maximal compact subgroup of G/Z then $K^0 = K \cap G^0$, K meets every component of G, and $K = \{g \in G \mid \mathrm{Ad}(g)K^0 = K^0\}$.*

These lemmas are standard when $Z = \{1\}$ and G is either linear or semisimple.

PROOF. $Z_{G^0} \subset K$ and $(K \cap G^0)/(Z_{G^0})^0$ is connected, is its own $G^0/(Z_{G^0})^0$–normalizer, and is unique up to conjugacy. The same follows for $K \cap G^0$ in G^0. Let $E = \{g \in G \mid \mathrm{Ad}(g)K^0 = K^0\}$; now $E \cap G^0 = K^0 = K \cap G^0$. If $g \in G$ now some $g' \in G^0$ send $\mathrm{Ad}(g)K^0$ to K^0, so E meets gG^0. Now $K \subset E \subset K$ and Lemma 4.1.2 follows. For Lemma 4.1.1 each simple ideal $\mathfrak{g}_i \subset \mathfrak{g}$ has a unique involution θ_i with fixed point set $\mathfrak{g}_i \cap \mathfrak{k}$, and we define θ as their sum with the identity map on the center of $\mathfrak{g}$. Then θ extends uniquely to the universal cover of G^0, and there its fixed point set $\exp(\mathfrak{k})$ contains the center, so θ extends uniquely to G^0 with fixed point set K^0. Now θ extends uniquely to $G = KG^0$ with fixed point set K, using Lemma 4.1.2. As any two choices of K/Z are $\mathrm{Ad}(G^0)$–conjugate, that completes

the proof of the first statement of Lemma 4.1.1. For the second just note that any two choices of $\mathfrak{k}$ are $\mathrm{Ad}(G^0)$–conjugate. $\square$

Now fix the data: a Cartan subgroup H of G, a Cartan involution θ of G with $\theta(H) = H$, and $K = G^\theta$ fixed point set of θ. We decompose

$$\begin{aligned} &\mathfrak{h} = \mathfrak{t} + \mathfrak{a} \text{ into } (\pm 1)\text{–eigenspaces of } \theta|_{\mathfrak{h}} \\ &H = T \times A \text{ where } T = H \cap K \text{ and } A = \exp(\mathfrak{a}). \end{aligned} \tag{4.1.3}$$

The $\mathfrak{a}$–*root spaces* of $\mathfrak{g}$ are the $\mathfrak{g}^\phi = \{\xi \in \mathfrak{g} \mid [\alpha, \xi] = \phi(\alpha)\xi \text{ for } \alpha \in \mathfrak{a}\}$ with $0 \neq \phi \in \mathfrak{a}^*$ and $\mathfrak{g}^\phi \neq 0$. The $\mathfrak{a}$–roots are these functionals ϕ, and $\Sigma_\mathfrak{a}$ denotes the set of all $\mathfrak{a}$–roots. The corresponding $\mathfrak{a}$–root decomposition is $\mathfrak{g} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) + \sum_{\phi \in \Sigma_\mathfrak{a}} \mathfrak{g}^\phi$ where $\mathfrak{z}_\mathfrak{g}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Then it is not too difficult to see that the centralizer $Z_G(A)$ has a unique splitting $Z_G(A) = M \times A$ with $\theta(M) = M$. In particular $\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \sum_{\phi \in \Sigma_\mathfrak{a}} \mathfrak{g}^\phi$ with $\theta(\mathfrak{m}) = \mathfrak{m}$. The hereditary properties of §3.1 pass down from G to M as follows.

PROPOSITION 4.1.4. *M inherits the conditions of* §3.1 *from G: every* $\mathrm{Ad}(m)$ *is inner on $\mathfrak{m}_\mathbb{C}$, Z centralizes M^0, $|M/ZM^0| < \infty$, and $Z_{M^0}/(Z \cap M^0)$ is compact. Further, T/Z is a compact Cartan subgroup of M/Z.*

The proof of Proposition 4.1.4 requires some information on the $\mathfrak{a}$–root system.

Every $\phi \in \Sigma_\mathfrak{a}$ defines $\phi^\perp := \{\alpha \in \mathfrak{a} \mid \phi(\alpha) = 0\}$. The complement $\mathfrak{a} \setminus \bigcup_{\Sigma_\mathfrak{a}} \phi^\perp$ is a finite union of convex open cones, its topological components, the *Weyl chambers*. A Weyl chamber $\mathfrak{d} \subset \mathfrak{a}$ defines a *positive root system* $\Sigma_\mathfrak{a}^+ = \{\phi \in \Sigma_\mathfrak{a} \mid \phi(\mathfrak{d}) \subset \mathbb{R}^+\}$.

LEMMA 4.1.5. *If $\Sigma_\mathfrak{a}^+$ is a positive $\mathfrak{a}$–root system on $\mathfrak{g}$ and $\Sigma_\mathfrak{t}^+$ is a positive $\mathfrak{t}_\mathbb{C}$–root system on $\mathfrak{m}_\mathbb{C}$ then there is a unique positive $\mathfrak{h}_\mathbb{C}$–root system Σ^+ on $\mathfrak{g}_\mathbb{C}$ such that $\Sigma_\mathfrak{a}^+ = \{\gamma|_\mathfrak{a} \mid \gamma \in \Sigma^+ \text{ and } \gamma|_\mathfrak{a} \neq 0\}$ and $\Sigma_\mathfrak{t}^+ = \{\gamma|_\mathfrak{t} \mid \gamma \in \Sigma^+ \text{ and } \gamma|_\mathfrak{a} = 0\}$.*

PROOF. Choose ordered bases $\beta_\mathfrak{a}$ of $\mathfrak{a}^*$ and $\beta_\mathfrak{t}$ of $i\mathfrak{t}^*$ whose associated lexicographic orders give $\Sigma_\mathfrak{a}^+$ and $\Sigma_\mathfrak{t}^+$. Then the ordered basis $\{\beta_\mathfrak{a}, \beta_\mathfrak{t}\}$ of $\mathfrak{a}^* + i\mathfrak{t}^*$ gives a lexicographic order whose associated positive $\mathfrak{h}_\mathbb{C}$–root system Σ^+ has the required properties. Uniqueness of Σ^+ is similarly straightforward. $\square$

PROOF OF PROPOSITION 4.1.4. $H \subset M \times A$ is a Cartan subgroup of G, hence also of $M \times A$, so T is a Cartan subgroup of M. T/Z is compact because K/Z is compact.

Let $m \in M$ with $\mathrm{Ad}(m)$ outer on $\mathfrak{m}_\mathbb{C}$. We may move m within mM^0 and assume $\mathrm{Ad}(m)\mathfrak{t} = \mathfrak{t}$. As the Weyl group of (M^0, T^0) is simply transitive on the Weyl chambers in $i\mathfrak{t}$, $\mathrm{Ad}(m)$ preserves and acts nontrivially on some $\Sigma_\mathfrak{t}^+$. Choose $\Sigma_\mathfrak{a}^+$; now $\mathrm{Ad}(m)$ preserves and acts nontrivially on the corresponding positive $\mathfrak{h}_\mathbb{C}$–root system of $\mathfrak{g}_\mathbb{C}$, in other words $\mathrm{Ad}(m)$ is outer on $\mathfrak{g}_\mathbb{C}$. That contradicts (3.1.1). Thus $\mathrm{Ad}(m)$ is inner on $\mathfrak{m}_\mathbb{C}$. In particular $M^\dagger := \{m \in M \mid \mathrm{Ad}(m) \text{ inner on } M^0\}$ has finite index in M and $M^\dagger = TM^0$. Now T/Z is a compact subgroup of M/Z such that $M/T = (M/Z)/(T/Z)$ has only finitely any components. Thus $|M/ZM^0| < \infty$. The center of $M^0/(Z \cap M^0)$ is compact because it is a closed subgroup in the torus $T^0/(Z \cap M^0)$, so that center is $Z_{M^0}/(Z \cap M^0)$. $\square$

4.2. We apply the considerations of §4.1 to study cuspidal parabolic subgroups.

Retain the splittings (4.1.3) and $Z_G(A) = M \times A$. A positive $\mathfrak{a}$–root system $\Sigma_{\mathfrak{a}}^+$ on $\mathfrak{g}$ defines

$$\mathfrak{n} = \sum_{\phi \in \Sigma_{\mathfrak{a}}^+} \mathfrak{g}^{-\phi} \text{ nilpotent subalgebra of } \mathfrak{g}\,, N = \exp(\mathfrak{n}), P = \{g \in G \mid \mathrm{Ad}(g)N = N\}.$$

LEMMA 4.2.1. *P is a real parabolic subgroup of G. It has unipotent radical $P^{unip} = N$ and Levi (reductive) part $P^{red} = M \times A$. Thus $(m, a, n) \mapsto man$ is a real analytic diffeomorphism of MAN onto P.*

PROOF. We follow the idea of the proof for the case where $G = G^\dagger$ and G is linear, which follows from the complex case. Let $\pi : G \to \overline{G} = G/ZZ_{G^0}$. That is a real linear algebraic group, so $\pi(P)$ is a parabolic subgroup of $\overline{G}$, normalizer of $\pi(N)$, and $\pi(N) = \pi(P)^{unip}$.

Note $ZZ_{G^0} \subset MA \subset P$. Thus (i) we can choose $\pi(P)^{red}$ to contain $\pi(MA)$ and (ii) $P = P^{red} \cdot N$ semidirect where $P^{red} = \pi^{-1}(\pi(P)^{red})$. Now $MA \subset P^{red}$ and, by dimension, $M^0A = (P^{red})^0$. A is normal in P^{red} by uniqueness from (4.1.3).

Let $V = \{w \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \mid w(\mathfrak{a}) = \mathfrak{a}\,, w|_{\mathfrak{a}} \neq 1\}$. Choose $x \in \mathfrak{a} \setminus \bigcup_{v \in V}\{y \in \mathfrak{a} \mid v(y) = y\}$. Let $g \in P^{red}$. Then $x, \mathrm{Ad}(g)x$ are conjugate by an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$, thus [**39**, Theorem 2.1] conjugate by an inner automorphism on $\mathfrak{g}$. Now $\mathrm{Ad}(g')x = x$ for some $g' \in gG^0$, and $\mathrm{Ad}(g')M = M$. We may assume $\mathrm{Ad}(g')H = H$. Now $g' \in MA$. Thus $P^{red} = MA(P^{red} \cap G^0)$. As $P^{red} \cap G^0 \in Z_G(A)$ now $P^{red} = MA$ as asserted. □

We say that two parabolics in G are *associated* if their reductive parts are G–conjugate. Thus the association class of $P = MAN$ is independent of N, i.e. is independent of $\Sigma_{\mathfrak{a}}^+$. We say that a parabolic $Q \subset G$ is *cuspidal* if $[(Q^{red})^0, (Q^{red})^0]$ has a Cartan subgroup E such that $E/(E \cap Z_{G^0})$ is compact.

PROPOSITION 4.2.2. *Let Q be a parabolic subgroup of G. Then the following are equivalent.*

(i) *Q is a cuspidal parabolic subgroup of G.*

(ii) *There exist a Cartan subgroup $H = T \times A$ of G and a positive $\mathfrak{a}$–root system $\Sigma_{\mathfrak{a}}^+$ such that Q is the group $P = MAN$ of* Lemma 4.2.1.

(iii) *Q^{red} has a relative discrete series representation.*

(iv) *$(Q^{red})^0$ has a relative discrete series representation.*

In particular, the construction $H \mapsto P = MAN$ induces a bijection from the set of all conjugacy classes of Cartan subgroups of G onto the set of all association classes of cuspidal parabolic subgroups of G.

PROOF. Let $\pi : G \to \overline{G} = G/ZZ_{G^0}$ as before. Note $ZZ_{G^0} \subset Q^{red}$. Thus (i), (ii), (iii) each holds for Q exactly when it holds for $\pi(Q)$. Also, (iii) and (iv) are equivalent because Q^{red}/ZZ_{G^0} has only finitely many components. Thus we need only prove the equivalence of (i), (ii) and (iv) when G is a connected centerless semisimple group.

Decompose $Q^{red} = M_Q \times A_Q$, stable under θ, where A_Q is the $\mathbb{R}$–split component of the center of Q^{red}. Thus Q is cuspidal if and only if M_Q has a compact Cartan subgroup T_Q. That is the case just when G has a Cartan subgroup $H = T_Q \times A_Q$ from which Q is constructed as in Lemma 4.2.1. Thus (i) and (ii) are equivalent. Apply Theorem 3.4.1 to $(Q^{red})^0$. Then (ii) implies (iv) by Proposition 4.1.4 and (iv) implies (i) directly. Now (i), (ii), (iii) and (iv) are equivalent, and the bijection statement follows. □

Two Cartan subalgebras of $\mathfrak{g}$ are conjugate by an inner automorphism of $\mathfrak{g}$ precisely when they are conjugate by an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$ [**39**, Corollary 2.4]. By Proposition 4.2.2 the same holds for association classes of cuspidal parabolic subgoups of G. Thus we could use G^0–conjugacy, G^0–association, or both, in the bijection of Proposition 4.2.2.

4.3. We define a series of unitary representations of G for each conjugacy class of Cartan subgroups. Then we work out some generalities on the character theory for that series. The precise character theory is in §4.4.

Retain the notation of §§4.1 and 4.2, including $H = T \times A$, $Z_G(A) = M \times A$ and $P = MAN$. The general unitary equivalence class in $\widehat{P^{red}} = \widehat{M} \times A$ has form $[\eta \otimes e^{i\sigma}]$ where $[\eta] \in \widehat{M}$ and $\sigma \in \mathfrak{a}^*$. That extends to a class $[\eta \otimes e^{i\sigma}] \in \widehat{P}$ that annihilates N: $(\eta \otimes e^{i\sigma})(man) = e^{i\sigma}(a)\eta(m)$. Then we have the unitarily induced representation

$$\pi_{\eta,\sigma} = \operatorname{Ind}_P^G(\eta \otimes e^{i\sigma}) \tag{4.3.1}$$

of G. The H–*series* of G is $\{[\pi_{\eta,\sigma}] \mid [\eta] \in \widehat{M}_{disc} \text{ and } \sigma \in \mathfrak{a}^*\}$. If H/Z is compact then $M = G$ and the H–series is just the relative discrete series $\widehat{G}_{disc}$. If H/ZZ_{G^0} is maximally $\mathbb{R}$–split, i.e. if P is a minimal parabolic subgroups of G, then the H–series is the principal series. We refer to any H–series as a *nondegenerate series*. Later we will see that the H–series depends only on the conjugacy class of H.

Given $\Sigma_{\mathfrak{a}}^+$ define $\rho_{\mathfrak{a}} = \frac{1}{2}\sum_{\phi \in \Sigma_{\mathfrak{a}}^+} (\dim \mathfrak{g}^\phi)\phi$. Then $\mathfrak{a}$ acts (under the adjoint representation of $\mathfrak{g}$) on $\mathfrak{n}$ and on $\mathfrak{p}$ with trace $-2\rho_{\mathfrak{a}}$. Thus $P = MAN$ has modular function δ_P, $\int_P f(xy^{-1})dx = \delta_P(y)\int_P f(x)dx$, given by

$$\delta_P(man) = e^{2\rho_a}(a) \text{ for } m \in M,\ a \in A,\ n \in N. \tag{4.3.2}$$

Let $[\eta] \in \widehat{M}$ with representation space E_η, and let $\sigma \in \mathfrak{a}^*$. Then we have the Hilbert space bundle $\mathbb{E}_{\eta,\sigma} \to G/P = K/(K \cap M)$ associated to the non–unitary representation $\eta \otimes e^{\rho_{\mathfrak{a}} + i\sigma}$ of P. Here G acts on the bundle but the hermitian metric is invariant only under K. We have the K–invariant probability measure $d(kZ)$ on $G/P = K/(K \cap Z)$. Thus we have a well defined space of square integrable sections of $\mathbb{E}_{\eta,\sigma} \to G/P$ given by

$$\begin{aligned} L_2(G/P;\mathbb{E}_{\eta,\sigma}) = {} & \text{ all Borel–measurable } f : G \to E_\eta \text{ such that} \\ & f(gp) = (\eta \otimes e^{\rho_{\mathfrak{a}} + i\sigma})(p)^{-1} f(g) \text{ and } \int_{K/Z} ||f(k)||^2 d(kZ) < \infty. \end{aligned} \tag{4.3.3}$$

It is a Hilbert space with inner product $\langle f, f' \rangle = \int_{K/Z} \langle f(k), f'(k) \rangle d(kZ)$, and G acts unitarily on it by the representation $(\pi_{\eta,\sigma}(g)(f))(g') = f(g^{-1}g')$ of (4.3.1).

4.4. We now describe the distribution character $\Theta_{\pi_{\eta,\sigma}}$ of $\pi_{\eta,\sigma}$ in terms of the character Ψ_η of η. This is based on a minor variation $C_c^\infty(G) \to C_c^\infty(MA)$ of the Harish–Chandra transform.

Let J be a Cartan subgroup of G. Then these are equivalent: (i) $J \subset MA$, (ii) J is a Cartan subgroup of MA, and (iii) $J = J_M \times A$ where $J_M = J \cap M$ is a Cartan subgroup of M. Without loss of generality we may assume J_M stable under the Cartan involution $\theta|_M$ of M. Choose a positive $(\mathfrak{j}_M)_{\mathbb{C}}$–root system $\Sigma_{j_M}^+$ on $\mathfrak{m}_{\mathbb{C}}$. As in Lemma 4.1.5 there is a unique positive $\mathfrak{j}_{\mathbb{C}}$–root system $\Sigma_{\mathfrak{j}}^+$ on $\mathfrak{g}_{\mathbb{C}}$ such that

$\Sigma^+_{\mathfrak{a}} = \{\gamma|_{\mathfrak{a}} \mid \gamma \in \Sigma^+_{\mathfrak{j}}$ and $\gamma|_{\mathfrak{a}} \neq 0\}$ and $\Sigma^+_{\mathfrak{j}_m} = \{\gamma|_{\mathfrak{j}_M} \mid \gamma \in \Sigma^+_{\mathfrak{j}}$ and $\gamma|_{\mathfrak{a}} = 0\}$. Then let $\rho_{\mathfrak{j}} = \frac{1}{2}\sum_{\gamma\in\Sigma^+_{\mathfrak{j}}} \gamma$; so $\rho_{\mathfrak{a}} = \rho_{\mathfrak{j}}|_{\mathfrak{a}}$. Now define

$$\Delta_{G,J} = \prod\nolimits_{\gamma\in\Sigma^+_{\mathfrak{a}}}(e^{\gamma/2} - e^{-\gamma/2}) \text{ and } \Delta_{M,J_M} = \prod\nolimits_{\phi\in\Sigma^+_{\mathfrak{j}_M}}(e^{\phi/2} - e^{-\phi/2}). \tag{4.4.1}$$

LEMMA 4.4.2. *If γ is a $\mathfrak{j}_{\mathbb{C}}$ root then e^{γ} is well defined on J, unitary on $J\cap K$, and $e^{\gamma}(Z_G(G^0)) = 1$. If ϕ is a $(\mathfrak{j}_M)_{\mathbb{C}}$–root, say $\phi = \gamma|_{\mathfrak{j}_M}$, then $e^{\phi} = e^{\gamma}|_{J\cap M}$.*

LEMMA 4.4.3. *We can replace Z by a subgroup of index $\leqq 2$, or replace G by a $\mathbb{Z}_2$ extension, so that the following holds. If L is any Cartan subgroup of G then for any positive $\mathfrak{l}_{\mathbb{C}}$–root system (i) $e^{\rho_{\mathfrak{l}}}$ is well defined on L with $e^{\rho_{\mathfrak{l}}}(Z) = 1$, and (ii) $\Delta_{G,L}$ is a well defined analytic function on L. In particular, then, $e^{\rho_{\mathfrak{j}}}$ and $\Delta_{G,J}$ are well defined on J, so $e^{\rho_{\mathfrak{j}_M}}$ and Δ_{M,J_M} are well defined on J_M.*

PROOF OF LEMMAS. If γ is a $\mathfrak{j}_{\mathbb{C}}$ root then e^{γ} is well defined on the Cartan subgroup $(J/Z_G(G^0))_{\mathbb{C}}$ of the inner automorphism group $\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$, because that Cartan is connected. Lemma 4.4.2 follows.

With the adjustments of Lemma 4.4.3 we can factor $\mathrm{Ad} : G \to \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ as $G \to G/Z \to Q \to \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ where $q : Q \to \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ is a 1– or 2–sheeted covering with $e^{\rho_{\mathfrak{j}}}$ well defined on $q^{-1}(J/Z_G(G^0))_{\mathbb{C}}$ and $e^{\rho_{\mathfrak{j}}}(Z) = 1$. As any two Cartan subgroups of $\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ are conjugate we have $\overline{x} \in \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ such that $\mathrm{Ad}(\overline{x})(L/Z_G(G^0))_{\mathbb{C}} = (J/Z_G(G^0))_{\mathbb{C}}$ and $\mathrm{Ad}(\overline{x})^*\rho_{\mathfrak{j}} = \rho_{\mathfrak{l}}$. Now $e^{\rho_{\mathfrak{l}}}$ is well defined on $q^{-1}(L/Z_G(G^0))_{\mathbb{C}}$, thus is well defined on L with Z in its kernel. So also $\Delta_{G,L} = e^{-\rho_{\mathfrak{l}}} \cdot \prod_{\Sigma^+_{\mathfrak{l}}}(e^{\gamma} - 1)$ is well defined on L.

Q was defined so that it has a faithful irreducible holomorphic representation ψ of highest weight $\rho_{\mathfrak{j}}$ relative to $(\mathfrak{j}, \Sigma^+_{\mathfrak{j}})$. Realize ψ as a subrepresentation of the left multiplication action λ of the Clifford algebra on the Lie algebra of $\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$. The Clifford subalgebra for M is stable under $\lambda(q^{-1}(\mathrm{Ad}_{\mathfrak{g}}(M)))$, and the corresponding representation of M has an irreducible summand of highest weight $\rho_{\mathfrak{j}_M}$. Now $e^{\rho_{\mathfrak{j}_M}}$ and Δ_{M,J_M} are well defined on J_M. □

G' denotes the G–regular set in G, $M''A$ is the MA–regular set in MA, and

$$\begin{aligned} &\mathrm{Car}(G): \text{ the } G\text{–conjugacy classes of Cartan subgroups of } G \text{ and} \\ &\mathrm{Car}(MA): \text{ the } MA\text{–conjugacy classes of Cartan subgroups of } MA. \end{aligned} \tag{4.4.4}$$

Then

$$\begin{aligned} &G' = \bigcup\nolimits_{L\in\mathrm{Car}(G)} G'_L \text{ where } G'_L = \bigcup\nolimits_{g\in G} \mathrm{Ad}(g)(L\cap G') \text{ and} \\ &M''A = \bigcup\nolimits_{J\in\mathrm{Car}(MA)} M''_J A \text{ where } M''_J A = \bigcup\nolimits_{m\in M} (\mathrm{Ad}(m)(J\cap M''))A. \end{aligned} \tag{4.4.5}$$

The following theorem unifies and extends various results of Bruhat [**4**, Ch. III], Harish–Chandra ([**21**, p. 544] and [**22**, §11]), Hirai [**29**, Theorems 1, 2] and Lipsman [**34**, Theorem 9.1]. We assume the adjustment of Lemma 4.4.3. The specialization to H–series is in §4.5.

THEOREM 4.4.6. *Let $\zeta \in \widehat{Z}$, $[\eta] \in \widehat{M_\zeta}$ and $\sigma \in \mathfrak{a}^*$. Let χ_ν be the infinitesimal character of $[\eta]$ relative to $\mathfrak{t}$ and let Ψ_η denote the distribution character of $[\eta]$.*

1. *$[\pi_{\eta,\sigma}]$ has infinitesimal character $\chi_{\nu+i\sigma}$ relative to $\mathfrak{h}$.*

2. *$[\pi_{\eta,\sigma}]$ is a finite sum of classes from $\widehat{G}_\zeta$. In particular $[\pi_{\eta,\sigma}]$ has distribution character $\Theta_{\pi_{\eta,\sigma}}$ that is a locally summable function analytic on the regular set G'.*

3. *$\Theta_{\pi_{\eta,\sigma}}$ has support in the closure of $\bigcup G'_J$ where J runs over a system of representatives of the G–conjugacy classes of Cartan subgroups of MA.*

4. *Let $J \in \mathrm{Car}(MA)$ and $\Xi(J)$ consist of all G–conjugates of J contained in $\mathrm{Car}(MA)$. Enumerate $\Xi(J) = \{J_1, \dots, J_\ell\}$ with $J_i = \mathrm{Ad}(g_i)J$. If $h \in J \cap G'$ denote $h_i = \mathrm{Ad}(g_i)h$. Then*

$$\Theta_{\pi_{\eta,\sigma}}(h) = \frac{1}{|\Delta_{G,J}(h)|} \sum_{J_i \in \Xi(J)} \sum_{h' \in N_G(J_i)h_i} \frac{|\Delta_{MA,J_i}(h')|}{|N_{MA}(J_i)h'|} \Psi_\eta(h'_M) e^{i\sigma}(h'_A).$$

{*Note: If $h \in J^0$ then the second sum runs over the Weyl group orbit $W_{G,J_i}(h_i)$.* }

Corollary 4.4.7. *The class $[\pi_{\eta,\sigma}]$ is independent of the choice of parabolic subgroup $P = MAN$ associated to the Cartan subgroup $H = T \times A$ of G.*

The proof of Theorem 4.4.6 is based on the following minor variation of the Harish–Chandra transform $C_c^\infty(G) \to C_c^\infty(MA)$.

Proposition 4.4.8. *Let $b \in C_c^\infty(G)$ and define*

$$b_P(ma) = e^{-\rho_{\mathfrak{a}}}(a) \int_{K/Z} \left(\int_N b(kmank^{-1}) dn \right) d(kZ). \tag{4.4.9}$$

Then $b_P(C_c^\infty(MA)$, $\pi_{\eta,\sigma}(b)$ is of trace class, and

$$\textit{trace } \pi_{\eta,\sigma}(b) = \int_{MA} b_P(ma) \Psi_\eta(m) e^{i\sigma}(a)\, dm\, da. \tag{4.4.10}$$

Proof. Let K_1 denote the image of a Borel section of $K \to K/Z$. If $f \in L_2(G/P; \mathbb{E}_{eta,\sigma})$ is continuous it is determined by $f|_{K_1}$. Compute

$$\begin{aligned}(\pi_{\eta,\sigma}(b)f)(k') &= \int_G b(g) f(g^{-1}k') dg = \int_G b(k'g) g(g^{-1} dg \\ &= \int_{K/K\cap M} d(kM) \int_{MAN} b(k'mank^{-1}) f(k(man)^{-1}) e^{-2\rho_{\mathfrak{a}}}(a) dm\, da\, dn \\ &= \int_{K/K\cap M} \left(\int_{MAN} b(k'mank^{-1}) e^{-2\rho_{\mathfrak{a}}}(a) \eta(m) dm\, da\, dn \right) f(k) d(kM).\end{aligned}$$

Define $\Phi_b(k', k) = \int_{MAN} b(k'mank^{-1}) e^{-\rho_{\mathfrak{a}} + i\sigma}(a) \eta(m) dm\, da\, dn : E_\eta \to E_\eta$. Then $\Phi_b(k', km_1) f(km_1) = \Phi_b(k', k) f(k)$ for $m_1 \in M \cap K$. Thus $(\pi_{\eta,\sigma}(b)f)(k') = \int_{K/K\cap M} \Phi_b(k', k) f(k) d(kM)$. As $Z \subset K \cap M$ and K/Z is compact we write this as $(\pi_{\eta,\sigma}(b)f)(k') = \int_{K/Z} \Phi_b(k', k) f(k) d(kZ)$; so

$$\mathrm{trace}\, \pi_{\eta,\sigma}(b) = \int_{K/Z} \mathrm{trace}\, \Phi_b(k, k) d(kZ).$$

Set $\varphi_b(k,m) = \int_{NA} e^{-\rho_{\mathfrak{a}}+i\sigma} b(kmank^{-1})dn\,da$. Then $\varphi_b \in C_c^\infty((K/Z)\times M)$. Noting that we always have absolute convergence we calculate

$$\begin{aligned}\int_{K/Z} \operatorname{trace}\Phi_b(k,k)d(kZ) &= \int_{K/Z}\left(\operatorname{trace}\int_M \Phi_b(k,m)\eta(m)dm\right)d(kz)\\ &= \int_{K/Z}\left(\int_M \Phi_b(k,m)\Psi_\eta(m)dm\right)d(kZ)\\ &= \int_{K/Z}\left(\int_M\left(\int_{NA} e^{-\rho_{\mathfrak{a}}+i\sigma}(a)b(kmank^1)dn\,da\right)\Psi_\eta(m)dm\right)d(kZ)\\ &= \int_M\left(\int_A b_P(ma)e^{i\sigma}(a)\Psi_\eta(m)da\right)dm.\end{aligned}$$

That completes the proof of Proposition 4.4.8 □

Let $\mathcal{Z}(\mathfrak{g})$ and $\mathcal{Z}(\mathfrak{m}+\mathfrak{a})$ denote the respective centers of the enveloping algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{m}+\mathfrak{a})$. Recall the canonical homomorphisms γ_G and γ_{MA} to Weyl group invariant polynomials. Then [**17**, §12] $\mu_{MA} = \gamma_{MA}^{-1}\cdot\Gamma_G\mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{m}+\mathfrak{a})$ has the property that $\mathcal{Z}(\mathfrak{m}+\mathfrak{a})$ has finite rank over its subalgebra $\mu_{MA}(\mathcal{Z}(\mathfrak{g}))$.

Proposition 4.4.8 says $\operatorname{trace}\pi_{\eta,\sigma}(b) = \operatorname{trace}(\eta\otimes e^{i\sigma})(b_P)$ for $b\in C_c^\infty(G)$. Now Harish–Chandra's [**17**, Lemma 52] says $\operatorname{trace}\pi_{\eta,\sigma}(zb) = \operatorname{trace}(\eta\otimes e^{i\sigma})(\mu_{MA}(z)b_P)$ for $z\in\mathcal{Z}(\mathfrak{g})$. From that, the infinitesimal character $\chi_{\pi_{\eta,\sigma}}(z) = \chi_{\eta\otimes e^{i\sigma}}(\mu_{MA}(z)) = \chi_{\nu+i\sigma}(\mu_{MA}(z))$ (χ for $((\mathfrak{m}+\mathfrak{a})_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$) $= \gamma_{MA}(\mu_{MA}(z))(\nu+i\sigma) = \gamma_G(z)(\nu+i\sigma) = \chi_{\nu+i\sigma}(z)$ (for $(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$). That proves the first assertion of Theorem 4.4.6.

To see that $[\pi_{\eta,\sigma}]$ is a direct integral over $\widehat{G}_\zeta$ we set $G^1 = ZG^0$ and $M^1 = M\cap G^1$. Then $[\eta]\in\widehat{M}_\zeta$ gives $[\eta^1]\in\widehat{M^1}_\zeta$ such that $[\eta]$ is a subrepresentation of $[\operatorname{Ind}_{M^1}^1(\eta^1)]$. Thus $\pi_{\eta,\sigma}$ is a subrepresentation of $\operatorname{Ind}_P^G((\operatorname{Ind}_{M^1}^M(\eta^1))\otimes e^{i\sigma}) = \operatorname{Ind}_{G^1}^G(\operatorname{Ind}_{M^1AN}^{G^1}(\eta^1\otimes e^{i\sigma}))$. Here $[\eta^1]\in\widetilde{M^1}_\zeta$, so $\operatorname{Ind}_{M^1AN}^{G^1}(\eta^1\otimes e^{i\sigma})|_Z$ is a multiple of ζ. Then $[\pi_{\eta,\sigma}]$ is a direct integral over $\widehat{G}_\zeta$.

We now prove that there is an integer $n>0$ such that

$$\pi_{\eta,\sigma}|_K = \sum\nolimits_{\widehat{K}_\zeta} m_\kappa\,\kappa \text{ where } 0\leqq m_\kappa \leqq n\cdot\dim\kappa<\infty. \tag{4.4.11}$$

It suffices to prove this for the finite index subgroup ZG^0, so we may assume Z central in G. Then (4.3.3) and the discussion just above give us a K–equivariant injective isometry $r_K : L_2(G/P;\mathbb{E}_{\eta,\sigma})\to L_2(K/Z;\zeta)$ by $r_K(f)\ f|_K$. As $L_2(K/Z;\zeta) = \sum_{\widehat{K}_\zeta} V_\kappa\otimes V_\kappa^*$ the multiplicity of V_κ here is $\dim((V_\kappa^*\otimes E_\eta)^{M\cap K})$. But $\eta|_{M\cap K} = \sum_{(\widehat{M\cap K})_\zeta} m_i\mu_i$ where $0\leqq m_i\leqq_M \dim\mu_i<\infty$. If $\kappa\in\widehat{K}_\zeta$ then $\dim\kappa<\infty$ so $\kappa_{M\cap K}$ is a finite sum $\sum_{\widehat{M\cap K}_\zeta} m_{\kappa,i}\mu_i$. Now

$$\dim((V_\kappa^*\otimes E_\eta)^{M\cap K}) = \sum m_{\kappa,i}\mu_i \leqq n_M\sum m_{\kappa,i}\dim\mu_i = n_M\dim\kappa<\infty,$$

proving (4.4.11). Note from the proof that $n\leqq n_M|G/ZG^0|$.

We show that $[\pi_{\eta,\sigma}]$ is a finite sum from $\widehat{G}_\zeta$, following Harish–Chandra. The discussion of (3.2.1) shows that we need only consider the case where G is connected. Then Z is central so $[\pi_{\eta,\sigma}]$ has central character ζ and infinitesimal character $\chi_{\nu+i\sigma}$. By (4.4.11) $\pi_{\eta,\sigma}$ has no nontrivial subrepresentation of infinite multiplicity. Thus it is quasi–simple in the sense of Harish–Chandra [**16**, p. 145]. Consequently it has distribution character $\Theta_{\pi_{\eta,\sigma}}$ that is a locally summable function analytic on the regular set G' [**16**, Theorem 6], and $\pi_{\eta,\sigma} = \sum\pi_j$ discrete direct sum of irreducibles

[**12**, Lemma 2]. Each $[\pi_j] \in \widehat{G}_\zeta$ and each $\chi_{\pi_j} = \chi_{\nu+i\sigma}$. Further, the differential equations $z\Theta = \chi_{\nu+i\sigma}(z)\Theta$ $(z \in \mathcal{Z}(\mathfrak{g}))$ constrain the Θ_{π_j} to a finite dimensional space of functions on G. Inequivalent classes in $\widehat{G}_\zeta$ have linearly independent distribution characters, so $\pi_{\eta,\sigma} = \sum \pi_j$ involves only finitely many classes from $\widehat{G}_\zeta$. Since the multiplicities $m(\pi_j, \pi_{\eta,\sigma}) < \infty$, $\pi_{\eta,\sigma}$ is a finite sum from $\widehat{G}_\zeta$. That is the second assertion of Theorem 4.4.6.

We now calculate $\Theta_{\pi,\sigma}$ by extending Lipsman's argument [**34**, Theorem 9.1] to our more general situation. Recall (4.4.5) and the definition (4.4.9) of b_P.

Lemma 4.4.12. *Let $L \in \mathrm{Car}(G)$ not conjugate to a Cartan subgroup of MA and $b \in C_c^\infty(G'_L)$. Then $b_P = 0$. On the other hand, if $J \in \mathrm{Car}(MA)$ and $b \in C_c^\infty(G'_J)$ then $b_P \in C_c^\infty(MA \cap G'_J) \subset C_c^\infty((MA)''_J)$.*

Proof. If $ma \in MA$ and $(\mathrm{Ad}(ma)-1)^{-1}$ is nonsingular on N then [**20**, Lemma 11] gives us $\int_N b(kmank^{-1})dn = |\det(\mathrm{Ad}(ma)^{-1} - 1)_{\mathfrak{n}}| \int_N b(knman^{-1}k^{-1})dn$. If $b \in C_c^\infty(G'_L)$ where G'_L doesn't meet MA then $b(knman^{-1}k^{-1})$ is identically zero, so $b_P = 0$. If $b \in C_c^\infty(G'_J)$ where $J \in \mathrm{Car}(MA)$ there is a compact set $S \subset G'_J$ such that, if $\int_N b(kmank^{-1})dn \neq 0$ for some $k \in K$ then $ma \in S$. Thus b_P is supported in $S \cap MA \subset G'_J \cap MA \subset (MA)''_J$. □

Let $L \in Car(G) \setminus \mathrm{Car}(MA)$. Let $b \in C_c^\infty(G'_L)$. Combine Proposition 4.4.8 and Lemma 4.4.12 to see $\Theta_{\pi_{\eta,\sigma}}(b) = \int_{MA} b_P(ma)\Psi_\eta(m)e^{i\sigma}(a)dmda = 0$. Thus $\Theta_{\pi_{\eta,\sigma}}|_{G'_L} = 0$. That is the third assertion of Theorem 4.4.6.

Fix $J = J_M \times A \in \mathrm{Car}(MA)$. To compute $\Theta_{\pi_{\eta,\sigma}}|_{G'_J}$ we need a variation on the Weyl Integration Formula. The center Z_J of J is open in J so it inherits Haar measure dh. Normalize measure on G/Z_J by $\int_G f(g)dg = \int_{G/Z_J}\big(\int_{Z_J} f(gh)dh\big)d(gZ_J)$ and on MA/Z_J by $\int_{MA} F(x)dx = \int_{MA/Z_J}\big(\int_{Z_J} F(xh)dh\big)d(xZ_J)$. Extending Harish–Chandra's extension [**19**] of Weyl's argument,

Lemma 4.4.13. *If $b \in C_c(G'_J)$ and $B \in C_c((MA)''_J)$ then*
$\int_G b(g)dg = \int_{J\cap G'} |N_G(J)(h)|^{-1}\big(\int_{G/Z_J} b(ghg^{-1})d(gZ_J)\big)|\Delta_{G,J}(h)|^2 dh$ *and*
$\int_{MA} B(x)dx = \int_{J\cap(MA)''} |N_{MA}(J)(h)|^{-1}\big(\int_{MA/Z_J} b(xhx^{-1})d(xZ_J)\big)|\Delta_{MA,J}(h)|^2 dh.$

Proof. $(G/Z_J) \times (J \cap G') \to G'_J$, by $(gZ_J, h) \mapsto ghg^{-1}$, is regular, surjective, and $|N_G(J)(h)|$ to one with Jacobian determinant $|\det(\mathrm{Ad}(h)^{-1} - 1)_{\mathfrak{g}/\mathfrak{j}}| = |\prod_{\gamma\in\Sigma_{\mathfrak{j}}}(e^\gamma - 1)(h)|$ at (gZ_J, h). But $\prod_{\gamma\in\Sigma_{\mathfrak{j}}}(e^\gamma - 1)(h)$ is the product over $\Sigma_{\mathfrak{j}}^+$ times the product over $-\Sigma_{\mathfrak{j}}^+$, which is $(-1)^n\Delta_{G,J}(h)^2$ where $n = |\Sigma_{\mathfrak{j}}^+|$, so that Jacobian is $|\Delta_{G,J}(h)|^2$. That proves the first equation; the second is similar. □

Given $b \in C_c^\infty(G'_J)$, $\Theta_{\pi_{\eta,\sigma}}(b) = \int_{MA} b_P(ma)\Psi_\eta(m)e^{i\sigma}(a)dmda$ by Proposition 4.4.8. Lemma 4.4.12 ensures convergence. Now, by Lemma 4.4.13,

$$\Theta_{\pi_{\eta,\sigma}}(b) = \int_{J\cap G'} |N_{MA}(J)(h)|^{-1} \Big(\int_{MA/Z_J} b_P(xhx^{-1}d(xZ_J)\Big)\Psi_\eta(h_M)e^{i\sigma}(h_A)|\Delta_{MA,J}(h)|^2 dh \tag{4.4.14}$$

where $h = h_A h_M$ along $J = J_M \times A$. As A is central in MA,

$$\int_{MA/Z_J} b_P(xhx^{-1})d(xZ_J)$$
$$= e^{-\rho_{\mathfrak{a}}(h_A)} \int_{MA/Z_J} d(xZ_J) \cdot \int_{K/Z} d(kZ) \cdot \int_N b(k.xhx^{-1} \cdot n \cdot k^{-1})dn.$$

Unimodularity of N and [**19**, Lemma 11] say

$$\int_N b(k \cdot maha^{-1}m^{-1} \cdot n \cdot k^{-1})dn = |\det(\mathrm{Ad}(h^{-1}) - 1)_{\mathfrak{n}}| \int_N b(knma \cdot h \cdot (knma)^{-1}dn.$$

We modify Harish–Chandra's evaluation [**19**, Lemma 12] of $|\det(\mathrm{Ad}(h^{-1})-1)_{\mathfrak{n}}|$ for the case $J = H$. Choose $c \in \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ with $c(\mathfrak{g}_{\mathbb{C}}) = \mathfrak{h}_{\mathbb{C}}$, $c(x) = x$ for $x \in \mathfrak{a}$, and $c^*\Sigma^+ = \Sigma^+_{\mathfrak{j}}$, so c also preserves $\mathfrak{m}_{\mathbb{C}}$ and $\mathfrak{n}_{\mathbb{C}}$. Then $\det(\mathrm{Ad}(h^{-1}) - 1)_{\mathfrak{n}}$ $= \det(\mathrm{Ad}(c(h^{-1})) - 1)_{\mathfrak{n}} = \prod_{\gamma \in \Sigma^+ \setminus \Sigma^+_{\mathfrak{t}}} (e^{\gamma}(c(h^{-1})) - 1 = \prod_{\beta \in \Sigma^+_{\mathfrak{j}} \setminus \Sigma^+_{\mathfrak{j}_M}} (e^{\beta}(h)) - 1)$ $= e^{\rho_{\mathfrak{j}} - \rho_{\mathfrak{j}_M}}(h) \frac{\Delta_{G,J}(h)}{\Delta_{MA,J}(h)} = e^{\rho_{\mathfrak{a}}}(h_A) \frac{\Delta_{G,J}(h)}{\Delta_{MA,J}(h)}$, so $\int_{MA/Z_J} b_P(xhx^{-1})d(xZ_J)$ is equal to

$$\int_{MA/Z_J} d(xZ_J) \cdot \int_{K/Z} d(kZ) \cdot |\Delta_{G,J}(h)/\Delta_{MA,J}(h)| \int_N b(\mathrm{Ad}(knma)h)dn$$
$$= |\tfrac{\Delta_{G,J}(h)}{\Delta_{MA,J}(h)}| \int_{K/K\cap M} \Big(\int_{MNA/Z_J} b(\mathrm{Ad}(kmna)h)dm\, da\, dn \Big) d(kM)$$
$$= |\Delta_{G,J}(h)/\Delta_{MA,J}(h)| \int_{G/Z_J} b(ghg^{-1})d(gZ_J).$$

Substituting this into (4.4.14),

$$\Theta_{\pi_{\eta,\sigma}}(b) = \int_{J\cap G'} |N_{MA}(J)(h)|^{-1} \Big(\int_{G/Z_J} b(ghg^{-1})d(gZ_J) \Big) \Psi_\eta(h_M) e^{i\sigma}(h_A) |\Delta_{G,J}(h)\Delta_{MA,J}(h)|^2 dh \tag{4.4.15}$$

We extend $\Phi_{\eta,\sigma,J}(h) := \frac{1}{|\Delta_{G,J}(h)|} \sum_{h' \in N_G(J)h} \frac{|\Delta_{MA,J}(h')|}{|N_{MA}(J)h'|} \Psi_\eta(h'_M) e^{i\sigma}(h'_A)$ to a class function on G'_J and substitute that into (4.4.15). Thus $\Theta_{\pi_{\eta,\sigma}}(b)$ is

$$\int_{J\cap G'} |N_G(J)(h)|^{-1}|^{-1} Bigl(\int_{G/Z_J} b(ghg^{-1})\Phi_{\eta,\sigma,J}(ghg^{-1})d(gZ_J) \Big) |\Delta_{G,J}(h)|^2 dh.$$

From Lemma 4.4.13 we see that $\Theta_{\pi_{\eta,\sigma}}|_{G'_J}$ is given by $\Phi_{\eta,\sigma,J}$. That proves the character formula and completes the proof of Theorem 4.4.6.

4.5. We specialize the results of §4.4 to the H–series of G, where $[\eta] \in \widehat{M}_{disc}$. The Cartan subgroup $H = T \times A$ and the associated cuspidal parabolic subgroup $P = MAN$ are fixed. The two principal simplifications here are (1) $\Xi(H) = \{H\}$ and (2) the character formulae for H–series representations are explicit [**27**].

The choice of H and P specifies $\Sigma^+_{\mathfrak{a}}$ with $\mathfrak{n} = \sum_{\alpha \in \Sigma^+_{\mathfrak{a}}} \mathfrak{g}^{-\alpha}$. Choose $\Sigma^+_{\mathfrak{t}}$ and specify Σ^+ as in Lemma 4.1.5. We have ρ, $\rho_{\mathfrak{a}}$, $\rho_{\mathfrak{t}}$, $\Delta_{G,H}$ and $\Delta_{M,T}$ as in (4.4.1). Make the adjustment of Lemma 4.4.3 if needed, so that $e^{\rho} \in \widehat{H}$ and $e^{\rho_{\mathfrak{t}}} \in \widehat{T}$ are well defined, and $e^{\rho}(Z) = e^{\rho_{\mathfrak{t}}}(Z) = 1$. Then $\Delta_{G,H}$ is well defined on H and $\Delta_{M,T}$ is well defined on M. Proposition 4.1.4 says that M has relative discrete series as described in §§3.4 and 3.5. It comes out as follows. Let $\varpi_{\mathfrak{t}}(\nu) = \prod_{\phi \in \Sigma^+_{\mathfrak{t}}} \langle \phi, \nu \rangle$ for

$\nu \in \mathfrak{t}^*_{\mathbb{C}}$ and $L''_{\mathfrak{t}} = \{\nu \in i\mathfrak{t}^* \mid e^\nu \in \widehat{T^0}$ and $\varpi_{\mathfrak{t}}(\nu) \neq 0\}$. Every $\nu \in L''_{\mathfrak{t}}$ specifies a class $[\eta_\nu] \in (\widehat{M^0})_{disc}$ whose distribution character satisfies

$$\Psi_{\eta_\nu}|_{T^0 \cap M''} = (-1)^{q_M(\nu)} \tfrac{1}{\Delta_{M,T}} \sum\nolimits_{w \in W(M^0,T^0)} \det(w) e^{w\nu}$$

with q_M defined on $L''_{\mathfrak{t}}$ as in (3.4.3). Every class in $(\widehat{M^0})_{disc}$ is one of these $[\eta_\nu]$. Classes $[\eta_\nu] = [\eta_{\nu'}]$ if and only if $\eta' \in W(M^0, T^0)(\nu)$. Finally, $[\eta_\nu]$ has central character $e^{\nu - \rho_{\mathfrak{t}}}|_{Z^0_M}$ and infinitesimal character χ_ν relative to $\mathfrak{t}$.

If $\nu \in L''_{\mathfrak{t}}$ and $[\chi] \in \widehat{Z_M(M^0)}_\xi$ where $\xi = e^{\nu-\rho_{\mathfrak{t}}}|_{Z^0_M}$ then we have

$$\eta_{\chi,\nu} = \operatorname{Ind}_{M^\dagger}^{M}(\chi \otimes \eta_\nu) \in \widehat{M}_{disc}. \tag{4.5.1}$$

Here recall $M^\dagger := Z_M(M^0)M^0$. Also, $\eta_{\chi,\nu} \in \widehat{M}_{disc}$, and it is the only class there with distribution character given on $Z_M(M^0) \cdot (T^0 \cap M'')$ by

$$\begin{aligned} &\Psi_{\eta_{\chi,\nu}}(zt) = \\ &= \sum_{1 \leqq j \leqq r} (-1)^{q_M(w_j\nu)} \operatorname{trace}\ \chi(x_j^{-1} z x_j) \cdot \frac{1}{\Delta_{M,T}} \sum_{w \in W_{M^0,T^0}} \det(ww_j) e^{ww_j\nu}(t) \end{aligned} \tag{4.5.2}$$

where there the $w_j = \operatorname{Ad}(x_j T)_{\mathfrak{t}}$ are representatives of $W_{M,T}$ modulo W_{M^0,T^0}. Every class in $\widehat{M}_{disc}$ is one of the $[\eta_{\chi,\nu}]$. Classes $[\eta_{\chi,\nu}] = [\eta_{\chi',\nu'}]$ exactly when $([\chi'], \nu') \in W_{M,T}([\chi], \nu)$. Finally, $[\eta_{\chi,\nu}]$ has infinitesimal character χ_ν relative to $\mathfrak{t}$. Now we combine this description with Theorem 4.4.6. Recall that the normalizers $N_{MA}(H) = N_M(T) \times A$ and $N_G(H)$ have all orbits finite on $H \cap G'$.

THEOREM 4.5.3. *Let $\nu \in L''_{\mathfrak{t}}$, $\sigma \in \mathfrak{a}^*$ and $[\chi] \in \widehat{Z_M(M^0)}_\xi$ where $\xi = e^{\nu-\rho_{\mathfrak{t}}}|_{Z^0_M}$. Define $\eta_{\chi,\nu}$ and $\Psi_{\eta_{\chi,\nu}}$ by (4.5.1) and (4.5.2). Then $[\pi_{\chi,\nu,\sigma}] := [\operatorname{Ind}_P^G(\eta_{\chi,\nu} \otimes e^{i\sigma})]$ is the unique H–series representation class on G whose distribution character satisfies*

$$\Theta_{\pi_{\chi,\nu,\sigma}}(ta) = \frac{|\Delta_{M,T}(t)|}{|\Delta_{G,H}(ta)|} \sum_{w(ta) \in N_G(H)(ta)} |N_M(T)(wt)|^{-1} \Psi_{\eta_{\chi,\nu}}(wt) e^{i\sigma}(wa)$$

for $t \in T$, $a \in A$ and $ta \in G'$. Every H–series class on G is one of the $[\pi_{\chi,\nu,\sigma}]$, and classes $[\pi_{\chi,\nu,\sigma}] = [\pi_{\chi',\nu',\sigma'}]$ if and only if $(\chi', \nu', \sigma') \in W_{G,H}(\chi, \nu, \sigma)$. $[\pi_{\chi',\nu',\sigma'}]$ is a finite sum from $\widehat{G}_\zeta$ where $[\eta_{\chi,\nu}] \in \widehat{M}_\zeta$. The dual $[\pi^*_{\chi,\nu,\sigma}] = [\pi_{\bar{\chi},-\nu,-\sigma}]$. The infinitesimal character is $\chi_{\nu+i\sigma}$ relative to $\mathfrak{h}$, so $[\pi_{\chi,\nu,\sigma}]$ sends the Casimir element of $\mathcal{U}(\mathfrak{g})$ to $||\nu||^2 + ||\sigma||^2 - ||\rho||^2$.

PROOF. First note that $\Xi(H) = \{H\}$ because any two fundamental Cartan subgroups of MA are $\operatorname{Ad}(M^0)$–conjugate. That eliminates the sum over $\Xi(H)$ expected from Theorem 4.4.6. Now we need only show that $\Theta_{\pi_{\chi,\nu,\sigma}}|_{H \cap G'}$ determines (χ, ν, σ) modulo $W_{G,H}$. Let $\Theta_{\pi_{\chi,\nu,\sigma}}|_{H\cap G'} = \Theta_{\pi_{\chi',\nu',\sigma'}}|_{H \cap G'}$. By linear independence of characters $e^{i\sigma''}$ on A we may replace σ' by any element of $N_G(H)\sigma'$ and assume $\sigma' = \sigma$. Thus, on $H \cap G'$, $\sum |N_M(T)(wt)|^{-1} e^{i\sigma}(wa)\big(\Psi_{\eta_{\chi,\eta}}(wt) - \Psi_{\eta_{\chi',\eta'}}(wt)\big) = 0$. Here $|N_M(T)(wt)|$ is locally constant on $T \cap M''$ and the functions $\Psi_{\eta_{\chi'',\eta''}}$ are linearly independent on $T \cap M''$. Thus $\Psi_{\eta_{\chi,\eta}} = \Psi_{\eta_{\chi',\eta'}}$, so $(\chi, \eta) = (\chi', \eta')$, and thus $(\chi', \eta') \in W_{M,T}(\chi, \eta)$. $\square$

COROLLARY 4.5.4. *The H–series classes $[\pi_{\chi,\nu,\sigma}]$ are independent of the choice of cuspidal parabolic subgroups P associated to H*

The support of $\Theta_{\pi_{\chi,\nu,\sigma}}$ meets the interior of G'_H, and by Theorem 4.4.6(3) it determines the conjugacy class of H. A stronger result, due to Lipsman [**34**, Theorem 11.1] for connected semisimple groups with finite center, is

THEOREM 4.5.5. *Let H and $'H$ be non–conjugate Cartan subgroups of G. Let $[\pi] \in \widehat{G}$ be H–series and let $['\pi] \in \widehat{G}$ be $'H$–series. Then the infinitesimal characters $\chi_\pi \neq \chi_{'\pi}$, and $[\pi]$ and $['\pi]$ are disjoint (no composition factors in common).*

PROOF. Take both Cartans are θ–stable, $H = T \times A$ and $'H = {'T} \times {'A}$. Express $[\pi] = [\pi_{\chi,\nu,\sigma}]$ using H and $['\pi] = [\pi_{'\chi,'\nu,'\sigma}]$ using $'H$. Then $\chi_\pi = \chi_{\nu+i\sigma}$ using $\mathfrak{h}$ and $\chi_{'\pi} = \chi_{'\nu+i'\sigma}$ using $'\mathfrak{h}$.

If $\chi_\pi = \chi_{'\pi}$ there exists $\beta \in \mathrm{Int}(\mathfrak{g}_\mathbb{C})$ such that $\beta(\mathfrak{h}_\mathbb{C}) = {'\mathfrak{h}_\mathbb{C}}$ and $\beta^*('\nu + '\sigma) = (\nu + i\sigma)$. β^* sends real span of roots to real span of roots, so $\beta^*('\nu) = \nu$ and $\beta^*('\sigma) = \sigma$. Further, we may suppose $\beta^*('\Sigma^+) = \Sigma^+$. It follows that $\beta(\mathfrak{h}) = '\mathfrak{h}$. Consequently [**39**, Corollary 2.4] there is an inner automorphism of $\mathfrak{g}$ that sends $\mathfrak{h}$ to $'\mathfrak{h}$, contradicting nonconjugacy of H and $'H$. Thus $\chi_\pi \neq \chi_{'\pi}$. Now $[\pi]$ and $['\pi]$ are disjoint because common factors would have the same infinitesimal character. □

4.6. We discuss irreducibility for H–series representations. As before fox $H = T \times A$ and $P = MAN$. Let $[\eta] \in \widehat{M}$ have infinitesimal character $\chi_n u$ relative to $\mathfrak{t}_\mathbb{C}$. We say that $[\eta]$ has *real infinitesimal character* if $\langle \phi, \nu \rangle$ is real for every $\phi \in \Sigma_\mathfrak{t}^+$. The classes in $\widehat{M}_{disc}$ have real infinitesimal character.

An element $\sigma \in \mathfrak{a}^*$ is $(\mathfrak{g}, \mathfrak{a})$–*regular* if $\langle \psi, \sigma \rangle \neq 0$ for all $\psi \in \Sigma_\mathfrak{a}$. Choose a minimal parabolic subgroup $P_0 = M_0 A_0 N_0$ of G with $A \subset A_0 = \theta A_0$. The $\mathfrak{a}$–roots are just the nonzero restrictions of the $\mathfrak{h}_\mathbb{C}$–roots, and so they are the nonzero restrictions of the $\mathfrak{a}_0$–roots. If $w \in W(\mathfrak{g}, \mathfrak{a}_0)$ and if $\sigma \in \mathfrak{a}^*$ is $(\mathfrak{g}, \mathfrak{a})$–regular, then $\mathfrak{a}$ is central in the centralizer $\mathfrak{g}^\sigma$ and $w \in W(\mathfrak{g}^\sigma, \mathfrak{a}_0)$, so w is generated by reflections in roots that annihilate $\mathfrak{a}$, forcing $w|_\mathfrak{a}$ to be trivial. In summary,

LEMMA 4.6.1. *If $\sigma \in \mathfrak{a}^*$ then the following conditions are equivalent:* (i) *σ is $(\mathfrak{g}, \mathfrak{a})$–regular,* (ii) *If $\phi \in \Sigma^+$ and $\phi|_\mathfrak{a} \neq 0$ then $\langle \phi, \sigma \rangle \neq 0$,* (iii) *If ψ_0 is an $\mathfrak{a}_0$–root of $\mathfrak{g}$ and $\psi_0|_\mathfrak{a} \neq 0$ then $\langle \psi_0, \sigma \rangle \neq 0$,* (iv) *If $w \in W(\mathfrak{g}, \mathfrak{a}_0)$ and $w|_\mathfrak{a} \neq 1$ then $w(\sigma) \neq \sigma$.*

The following theorem was proved by Harish–Chandra (unpublished):

THEOREM 4.6.2. *Let $[\eta] \in \widehat{M}$ have real infinitesimal character and let $\sigma \in \mathfrak{a}^*$ be $(\mathfrak{g}, \mathfrak{a})$–regular. Then $[\pi_{\eta,\sigma}] = [\mathrm{Ind}_P^G(\eta \otimes e^{i\sigma})]$ is irreducible.*

COROLLARY 4.6.3. *If $\sigma \in \mathfrak{a}^*$ is $(\mathfrak{g}, \mathfrak{a})$–regular then every H–series class $[\pi_{\chi,\nu.\sigma}]$ is irreducible.*

After that, irreducibility were settled by Knapp and Zuckerman ([**31**], [**32**]) for connected reductive real linear algebraic groups (the case where G is connected and is isomorphic to a closed subgroup of some general linear group $GL(n; \mathbb{R})$). In view of Langlands theorem [**33**], that completed the classification of irreducible admissible representations for reductive real linear algebraic groups. For those groups, and more generally for groups of class $\mathcal{H}$, Vogan's treatment of the Kazhdan–Lusztig conjecture and construction and analysis of the KLV polynomials ([**47**], [**48**]) includes a complete analysis of the composition series of any H–series representation $[\pi_{\chi,\nu.\sigma}]$. Finally, the *Atlas* software, `http://www.liegroups.org/software/`, allows explicit computation of those composition factors; see [**1**].

5. The Plancherel Formula for General Real Reductive Lie Groups

As before, G is a real reductive Lie group that satisfies (1.2.1). The Harish–Chandra class $\mathcal{H}$ consists of all such groups for which G/G^0 is finite and the derived group $[G^0, G^0]$ has finite center. We fix a Cartan involution θ of G and a system $\mathrm{Car}(G) = \{H_1, \dots, H_\ell\}$ of θ–stable representatives of the conjugacy classes of Cartan subgroups of G. Harish–Chandra's announced [**22**, §11] a Plancherel formula for groups of class $\mathcal{H}$: there are unique continuous functions $m_{j,\eta}$ on $\mathfrak{a}_j^*$, meromorphic on $(\mathfrak{a}_j^*)_{\mathbb{C}}$, invariant under the Weyl group $W(G, H_j)$, such that

$$f(x) = \sum_{1 \leqq j \leqq \ell} \sum_{[\eta] \in (\widehat{M_j})_{disc}} \deg(\eta) \int_{\mathfrak{a}_j^*} \Theta_{\pi_{\eta,\sigma}} r_x(f) m_{j,\eta}(\sigma) d\sigma,$$

absolutely convergent for $x \in G'$ and $f \in C_c^\infty(G)$. This was extended to our class in [**51**] without consideration of meromorphicity. Later Harish–Chandra published a complete treatment for G of class $\mathcal{H}$ and f in the Harish–Chandra Schwartz space $\mathcal{S}(G)$ ([**23**],[**24**],[**25**]). Still later Herb and I extended those results to general real reductive groups, including explicit formulae for the various constants and functions that enter into the Plancherel measure ([**27**], [**28**]).

Here, for lack of space or necessity, I'll only indicate the results from [**51**], because that is all that is needed in §§7 and 8 below.

5.1. As above, we have G, θ, $K = G^\theta$, $\mathrm{Car}(G) = \{H_1, \dots, H_\ell\}$, $H_j = T_j \times A_j$, $\Sigma_{\mathfrak{a}_j}^+$ and $P_j = M_j A_j N_j$ with $M_j \times A_j = Z_G(A_j)$. As in §4, $L_j = \{\nu \in i\mathfrak{t}_j^* \mid e^\nu \in \widehat{T_j^0}\}$ and L_j'' is its M_j–regular set. Fix the $\Sigma_{\mathfrak{t}_j}^+$ and set $\varpi_{\mathfrak{t}_j}(\nu) = \prod_{\phi \in \Sigma_{\mathfrak{t}_j}^+} \langle \phi, \nu \rangle$, so $L_j'' = \{\nu \in L_j \mid \varpi_{\mathfrak{t}_j}(\nu) \neq 0\}$.

If $\zeta \in \widehat{Z}$ then $L_{j,\zeta} = \{\nu \in L_j \mid e^{\nu - \rho_{\mathfrak{t}_j}}|_{Z \cap M_j^0} = \zeta|_{Z \cap M_j^0}\}$ and $L_j'' = L_{j,\zeta} \cap L_j''$. Write ξ_ν for $e^{\nu - \rho_{\mathfrak{t}_j}}$. Since $ZZ_{M_j^0}$ has finite index in $Z_{M_j}(M_j^0)$ we define finite subsets $S(\nu, \zeta) \subset \widehat{Z_{M_j}(M_j^0)}$ by $S(\nu, \zeta) = \widehat{Z_{M_j}(M_j^0)}_{\zeta \otimes \xi_\nu}$ if $\xi|_{Z \cap M_j^0} = \xi_\nu|_{Z \cap M_j^0}$, $S(\nu, \zeta) = \emptyset$ otherwise. When $\nu \in L_j''$ and $\sigma \in \mathfrak{a}_j^*$ the H_j–series classes $[\pi_{\chi,\nu,\sigma}]$ that transform by ζ are just the ones with $[\chi] \in S(\nu, \zeta)$. Thus we have finite sums

$$\pi_{j,\zeta,\nu+i\sigma} = \sum\nolimits_{S(\nu,\zeta)} (\dim \chi) \pi_{\chi,\nu,\sigma} \text{ and } \Theta_{j,\zeta,\nu+i\sigma} = \sum\nolimits_{S(\nu,\zeta)} (\dim \chi) \Theta_{\pi_{\chi,\nu,\sigma}} .$$

If $\zeta|_{Z \cap M_j^0} \neq \xi_\nu|_{Z \cap M_j^0}$, in other words if $\nu \notin L_{j,\zeta}$, then $\Theta_{j,\zeta,\nu+i\sigma} = 0$. Here is the extension [**51**] of the Harish–Chandra Plancherel Formula ([**21**], [**22**]) to the $\widehat{G}_\zeta$.

THEOREM 5.1.1. *Let G be a general real reductive Lie group* (1.2.1) *and $\zeta \in \widehat{Z}$. Then there are unique Borel–measurable functions $m_{j,\zeta,\nu}$ on $\mathfrak{a}_j^*$, $1 \leqq j \leqq \ell$, defining the* Plancherel measure on $\widehat{G}_\zeta$, *as follows.*

1. *The $m_{j,\zeta,\nu}$ are W_{G,H_j}–equivariant: $w^* m_{j,\zeta,\nu}(\sigma) = m_{j,w^*\zeta,w^*\nu}(w^*\sigma)$.*
2. *If $\nu \notin L_{j,\zeta}$ then $m_{j,\zeta,\nu} = 0$.*

3. *Let $f \in L_2(G/Z, \zeta)$ be C^∞ with support compact modulo Z. If $x \in G$ define $(r_x f)(g) = f(gx)$. Then*

$$\begin{aligned} &\sum_{1 \leqq j \leqq \ell} \sum_{\nu \in L''_{j,\zeta}} |\varpi_{\mathfrak{t}_j}(\nu)| \int_{\mathfrak{a}_j^*} |\Theta_{j,\zeta,\nu+i\sigma}(r_x f) m_{j,\zeta,\nu}(\sigma)| d\sigma < \infty \text{ and} \\ &f(x) = \sum_{1 \leqq j \leqq \ell} \sum_{\nu \in L''_{j,\zeta}} |\varpi_{\mathfrak{t}_j}(\nu)| \int_{\mathfrak{a}_j^*} \Theta_{j,\zeta,\nu+i\sigma}(r_x f) m_{j,\zeta,\nu}(\sigma) d\sigma. \end{aligned} \tag{5.1.2}$$

The following corollary is used for realization of H–series representations on spaces of partially harmonic spinors [**52**].

Corollary 5.1.3. *Let $\omega \in \mathcal{U}(\mathfrak{g})$ be the Casimir element. If $c \in \mathbb{R}$ and $\zeta \in \widehat{Z}$ then $\{[\pi] \in \widehat{G}_\zeta \setminus \widehat{G}_{\zeta-disc} \mid \chi_\pi(\omega) = c\}$ has Plancherel measure 0 on $\widehat{G}_\zeta$.*

Corollary 5.1.5 below is needed when we consider spaces of square integrable partially harmonic $(0, q)$–forms in §§7 and 8. It follows from Theorems 4.6.2 and 5.1.1; or one can also derive it from

Lemma 5.1.4. *Let $[\pi]$ be an irreducible constituent of an H–series class $[\pi_{\chi,\nu,\sigma}]$ where $\nu + i\sigma \in \mathfrak{h}^*$ is $\mathfrak{g}$–regular. If G has relative discrete series representations, and if H/Z is noncompact, then $\Theta_\pi|_{K \cap G'} = 0$.*

Corollary 5.1.5. *If G has relative discrete series representations and if $\zeta \in \widehat{Z}$ then $\{[\pi] \in \widehat{G}_\zeta \setminus \widehat{G}_{\zeta-disc} \mid \Theta_\pi|_{K \cap G'} \neq 0\}$ has Plancherel measure 0 on $\widehat{G}_\zeta$.*

Corollary 5.1.6. *Fix $\zeta \in \widehat{Z}$. Let $\widehat{G}_{H_j,\zeta}$ denote the set of all H_j–series classes $[\pi_{\chi,\nu,\sigma}]$ for ζ such that σ is $(\mathfrak{g}, \mathfrak{a}_j)$–regular. Then each $\widehat{G}_{H_j,\zeta} \subset \widehat{G}_\zeta$ and the Plancherel measure on $\widehat{G}_\zeta$ is concentrated on $\bigcup_{1 \leqq j \leqq \ell} \widehat{G}_{H_j,\zeta}$.*

The "absolute" version of Theorem 5.1.1 derives from $f(x) := \int_{\widehat{Z}} f_\zeta(x)\zeta(z)d\zeta$ where $f_\zeta(x) = \int_Z f(xz)\zeta(z)dz$. Given $f \in C_c^\infty(G)$ we apply Theorem 5.1.1 to each f_ζ and sum over $\widehat{Z}$. The same holds for the corollaries.

6. Real Groups and Complex Flags

While G is a general real reductive Lie group (1.2.1) the adjoint representation takes G to a real reductive semisimple Lie group $\overline{G} := G/Z_G(G^0)$. That group has complexification $\overline{G}_\mathbb{C} = \mathrm{Int}(\mathfrak{g}_\mathbb{C})$, the group of inner automorphisms of $\mathfrak{g}_\mathbb{C}$. Notice that $\overline{G}_\mathbb{C}$ is connected. Now G acts on all complex flag manifolds $X = \overline{G}_\mathbb{C}/Q$. Here we recall the part of [**50**] needed for geometric realization of standard induced representations, extending them from $\overline{G}^0$ to G as needed. We discuss holomorphic arc components of G–orbits; consider measurable, integrable and flag type orbits; and give a complete analysis of the orbits on which our representations are realized in §§7 and 8.

Notation: Q is used for a (complex) parabolic subgroup of $\overline{G}_\mathbb{C}$ and P is reserved for cuspidal parabolics in G. Roots are ordered so that $X = \overline{G}_\mathbb{C}/Q$ has holomorphic tangent space spanned by positive root spaces.

6.1. It is standard that the following are equivalent for a closed complex subgroup $Q \subset \overline{G}_{\mathbb{C}}$: (i) $X = \overline{G}_{\mathbb{C}}$ is compact, (ii) X is a compact simply connected Kähler manifold, (iii) X is a $\overline{G}_{\mathbb{C}}$–homogeneous projective algebraic variety, (iv) X is a closed $\overline{G}_{\mathbb{C}}$–orbit in a (finite dimensional) projective representation, and (v) Q contains a Borel subgroup of $\overline{G}_{\mathbb{C}}$. Under these conditions we say that (1) Q is a *parabolic subgroup* of $\overline{G}_{\mathbb{C}}$, (2) $\mathfrak{q}$ is a *parabolic subalgebra* of $\overline{\mathfrak{g}}_{\mathbb{C}}$, and (3) $X = \overline{G}_{\mathbb{C}}/Q$ is a *complex flag manifold* of $\overline{\mathfrak{g}}_{\mathbb{C}}$. Given (i) through (v) Q is the analytic subgroup of $\overline{G}_{\mathbb{C}}$ for $\overline{\mathfrak{q}}_{\mathbb{C}}$, in fact is the $\overline{G}_{\mathbb{C}}$–normalizer of $\overline{\mathfrak{q}}_{\mathbb{C}}$.

Recall the structure. Choose a Cartan subalgebra $\overline{\mathfrak{h}}_{\mathbb{C}}$ of $\overline{\mathfrak{g}}_{\mathbb{C}}$ and a system Π of simple $\overline{\mathfrak{h}}_{\mathbb{C}}$–roots on $\overline{\mathfrak{g}}_{\mathbb{C}}$. Any subset $\Phi \subset \Pi$ specifies

- Φ^r: all roots that are linear combinations of elements of Φ;
- Φ^u: all negative roots not contained in Φ^r;
- $\mathfrak{q}^r_\Phi = \overline{\mathfrak{h}}_{\mathbb{C}} + \sum_{\Phi^r} \overline{\mathfrak{g}}^\phi$, $\mathfrak{q}^u_\Phi = \sum_{\Phi^u} \overline{\mathfrak{g}}^\phi$ and $\mathfrak{q}_\Phi = \mathfrak{q}^r_\Phi + \mathfrak{q}^u_\Phi$.

Then $\overline{G}_{\mathbb{C}}$ has analytic subgroups Q^r_Φ for $\mathfrak{q}^r_\Phi$, Q^u_Φ for $\mathfrak{q}^u_\Phi$ and $Q_\Phi = Q^u_\Phi \rtimes Q^r_\Phi$ for $\mathfrak{q}_\Phi$. Q^u_Φ and $\mathfrak{q}^u_\Phi$ are the nilradicals, and Q^r_Φ and $\mathfrak{q}^r_\Phi$ are the Levi (reductive) components. Φ is a simple $\overline{\mathfrak{h}}_{\mathbb{C}}$–root system for $\mathfrak{q}^r_\Phi$. Q_Φ is a parabolic subgroup of $\overline{G}_{\mathbb{C}}$ and every parabolic subgroup of $\overline{G}_{\mathbb{C}}$ is conjugate to exactly one of the Q_Φ. Any parabolic Q_Φ is its own normalizer in $\overline{G}_{\mathbb{C}}$, so the complex flag manifold $X = \overline{G}_{\mathbb{C}}/Q$ is in one-one correspondence $x \leftrightarrow Q_x$ with the set of $\overline{G}_{\mathbb{C}}$–conjugates of Q, by $Q_x = \{\overline{g} \in \overline{G}_{\mathbb{C}} \mid \overline{g}(x) = x\}$. We will make constant use of this identification.

6.2. Let $\overline{G}$ be an open subgroup of a real form $\overline{G}_{\mathbb{R}}$ of $\overline{G}_{\mathbb{C}}$, so $\overline{G}^0$ is the real analytic subgroup of $\overline{G}_{\mathbb{C}}$ for $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_{\mathbb{R}}$. Denote

$$\tau : \text{ complex conjugation of } \overline{G}_{\mathbb{C}} \text{ over } \overline{G}_{\mathbb{R}} \text{ and of } \overline{\mathfrak{g}}_{\mathbb{C}} \text{ over } \overline{\mathfrak{g}}_{\mathbb{R}}. \tag{6.2.1}$$

The isotropy subgroup of $\overline{G}$ at $x \in X$ is $\overline{G} \cap Q_x$. The latter has Lie algebra $\overline{\mathfrak{g}} \cap \mathfrak{q}_x$ which is a real form of $\mathfrak{q}_x \cap \tau\mathfrak{q}_x$. The intersection of any two Borel subgroups contains a Cartan, and using care it follows that we have

$$\begin{aligned} &\text{a Cartan subalgebra } \overline{\mathfrak{h}} \subset \overline{\mathfrak{g}} \cap \mathfrak{q}_x \text{ of } \overline{\mathfrak{g}}, \text{ a system } \Pi \text{ of simple } \overline{\mathfrak{h}}_{\mathbb{C}}\text{–roots} \\ &\text{of } \overline{\mathfrak{g}}_{\mathbb{C}}, \text{ and a subset } \Phi \subset \Pi \text{ such that } \mathfrak{q}_x = \mathfrak{q}_\Phi. \end{aligned} \tag{6.2.2}$$

Then we have the key decomposition to understanding G–orbits on X:

$$\begin{aligned} &\mathfrak{q}_x \cap \tau\mathfrak{q}_x = (\mathfrak{q}_x \cap \tau\mathfrak{q}_x)^r + (\mathfrak{q}_x \cap \tau\mathfrak{q}_x)^u \text{ where } (\mathfrak{q}_x \cap \tau\mathfrak{q}_x)^r = \overline{\mathfrak{h}}_{\mathbb{C}} + \sum_{\Phi^r \cap \tau\Phi^r} \overline{\mathfrak{g}}^\phi \\ &\text{and } (\mathfrak{q}_x \cap \tau\mathfrak{q}_x)^u = \sum_{\Phi^r \cap \tau\Phi^u} \overline{\mathfrak{g}}^\phi + \sum_{\Phi^u \cap \tau\Phi^r} \overline{\mathfrak{g}}^\phi + \sum_{\Phi^u \cap \tau\Phi^u} \overline{\mathfrak{g}}^\phi. \end{aligned} \tag{6.2.3}$$

This shows that $\overline{G}(x)$ has real codimension $|\Phi^u \cap \tau\Phi^u|$ in X, in particular that $\overline{G}(x)$ is open in X if and only if $\Phi^u \cap \tau\Phi^u$ is empty, and also that there are only finitely many $\overline{G}$–orbits on X. This last shows that $\overline{G}$ has both open and closed orbits.

Recall $\overline{G} = G/Z_G(G^0) = \mathrm{Int}(\mathfrak{g})_{\mathbb{C}}$, so G acts on $X = \overline{G}/Q$ through $G \to \overline{G}$, specifically by $Q_{g(x)} = \mathrm{Ad}(g)Q_x$. Thus $G(x) = \overline{G}(x)$. Now the results on orbits and isotropy of $\overline{G}$ and $\overline{\mathfrak{g}}$ on X, apply as well to orbits and isotropy of G and $\mathfrak{g}$.

6.3. Let H be a Cartan subgroup of G and θ a Cartan involution with $\theta(H) = H$. Let K be the fixed point set G^θ. As in (4.1.3) $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ and $H = T \times A$ under the action of θ. Thus [**50**, Theorem 4.5] the following conditions are equivalent: (i) T is a Cartan subgroup of K, (ii) $\mathfrak{t}$ contains a regular element of $\mathfrak{g}$, and (iii) some simple system Π of $\mathfrak{h}_{\mathbb{C}}$–roots satisfies $\tau\Pi = -\Pi$. Then those conditions hold, one

says that $\mathfrak{h}$ is a *fundamental Cartan subalgebra* of $\mathfrak{g}$ and H is a *fundamental Cartan subgroup* of G. Equivalently, $\mathfrak{h}$ is a *maximally compact Cartan subalgebra* of $\mathfrak{g}$ and H is a *maximally compact Cartan subgroup* of G. From (6.2.3),

LEMMA 6.3.1. *$G(x)$ is open in X if and only if there exist a maximally compact Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a simple $\mathfrak{h}_{\mathbb{C}}$–root system Π with $\tau\Pi = -\pi$, and a subset $\Phi \subset \Pi$, such that $\mathfrak{q}_x = \mathfrak{q}_\Phi$.*

An open orbit $G(x)$ carries an invariant Radon measure if and only if the isotropy subgroup at x is reductive, i.e., if and only if the choices in Lemma 6.3.1 can be made so that $\tau\Phi^r = \Phi^r$ and $\tau\Phi^u = -\Phi^u$. Thus [**50**, Theorem 6.3] these conditions are equivalent: (i) $G(x)$ is open in X and has a G–invariant positive Radon measure, (ii) $G(x)$ has a G–invariant possibly–indefinite Kähler metric, (iii) $\mathfrak{q}_x \cap \tau\mathfrak{q}_x$ is reductive, i.e., $\mathfrak{q}_x \cap \tau\mathfrak{q}_x = \mathfrak{q}_x^r \cap \tau\mathfrak{q}_x^r$ and (iv) $\tau\Phi^r = \Phi^r$ and $\tau\Phi^u = -\Phi^u$. Under those conditions we say that the open orbit $G(x)$ is *measurable*.

A closely related set of equivalent conditions [**50**, Theorem 6.7] is (a) some open G–orbit on X is measurable, (b) every open G–orbit on X is measurable and (c) if $\mathfrak{q} = \mathfrak{q}_\Phi$ then $\tau\mathfrak{q}$ is conjugate to the opposite parabolic $\mathfrak{q}^r + \mathfrak{q}^{-u}$ where $\mathfrak{q}^{-u} = \sum_{\Phi^u} \overline{\mathfrak{g}}^{-\phi}$. These conditions are automatic [**50**, Corollary 6.8] if rank K = rank G, i.e., if G has relative discrete series representations. In that regard we will need

LEMMA 6.3.2. *Let U be the isotropy subgroup of G at $x \in X$. Suppose that $\mathfrak{q}$ does not contain any nonzero ideal of $\overline{\mathfrak{g}}_{\mathbb{C}}$. Then the following are equivalent.*

1. U acts on the tangent space to $G(x)$ as a compact group.

2. $G(x)$ has a G–invariant positive definite hermitian metric.

3. $\overline{\mathfrak{g}} \cap \mathfrak{q}_x$ is contained in the fixed point set of a Cartan involution of $\overline{\mathfrak{g}}$

Under these conditions, $G(x)$ is open in X and the maximal compact subgroups $\overline{K} \subset \overline{G}$ satisfy rank $\overline{K}$ = rank $\overline{G}$.

Suppose rank K = rank G. Let $G(x)$ be an open orbit, so $\overline{\mathfrak{g}}$ has a Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{k}} \cap \mathfrak{q}_x$ where $\overline{H} = H/Z_G(G^0)$ and $\overline{K} = K/Z_G(G^0)$. Let W_K , $W_{G_{\mathbb{C}}}$ and $W_{Q_x^r}$ denote Weyl groups relative to $\overline{\mathfrak{h}}$. Then [**50**, Theorem 4.9] the open G–orbits on X are enumerated by the double coset space $W_K \backslash W_{G_{\mathbb{C}}} / W_{Q_x^r}$.

6.4. We look at the maximal complex analytic pieces of a G–orbit on X.

Let V be a complex analytic space and $D \subset V$. By *holomorphic arc* in D we mean a holomorphic map $f : \{z \in C \mid |z| < 1\} \to V$ with image in D. A *chain of holomorphic arcs* in D is a finite sequence $\{f_1, \ldots, f_m\}$ of holomorphic arcs in D such that the image of f_{k-1} meets the image of f_k . A *holomorphic arc component* of D is an equivalence class of elements of D under $u \sim v$ if there is a chain $\{f_1, \ldots, f_m\}$ of holomorphic arcs in D with u in the image of f_1 and v in the image of f_m . If g is a holomorphic diffeomorphism of V, $g(D) = D$, and S is a holomorphic arc component of D then $g(S)$ is holomorphic arc component of D.

Let L be a group of holomorphic diffeomorphisms of V that preserve D. If S is a holomorphic arc component of D denote its L–normalizer $\{\alpha \in L| \mid \alpha(S) = S\}$ by $N_L(S)$. If $\alpha \in L$ and $\alpha(S)$ meets S then $\alpha(S) = S$. So if D is an L–orbit then S is an $N_L(S)$–orbit. It can happen that D is a real submanifold of V but is not a complex submanifold; see [**50**, Example 8.12].

Now we turn to holomorphic arc components of an orbit $G(x) \subset X = \overline{G}_{\mathbb{C}}/Q$. It is a finite union of G^0–orbits, which are its topological components. So we have

$$S_{[x]} : \text{ holomorphic arc component of } G(x) \text{ through } x.$$

in the topological component of x in $G(x)$, and its G– and $\overline{G}$–normalizers

$$N_{[x]} = \{g \in G \mid gS_{[x]} = S_{[x]}\} \text{ and } \overline{N}_{[x]} = N_{[x]}/Z_G(G^0).$$

The main general fact concerning these groups and their Lie algebras [**50**, Theorems 8.5 and 8.15] is that $\mathfrak{n}_{[x],\mathbb{C}}$ is a τ–stable parabolic subalgebra of $\overline{\mathfrak{g}}_{\mathbb{C}}$, so $N_{[x],\mathbb{C}}$ is a parabolic subgroup of $\overline{G}$, and $\mathfrak{n}_{[x]} = \overline{\mathfrak{g}} \cap \mathfrak{n}_{[x],\mathbb{C}}$ is a parabolic subalgebra of $\overline{\mathfrak{g}}$, so $N_{[x]}$ is a subgroup of finite index in a parabolic subgroup of $\overline{G}$. This ensures $G = KN_{[x]}$ and $\overline{G} = \overline{KN}_{[x]}$. In other words, K and $\overline{K}$ are transitive on the space $G/N_{[x]} = \overline{G}/\overline{N}_{[x]}$ of all holomorphic arc components of $G(x)$.

With $x \in X$ fixed and $\mathfrak{q}_x = \mathfrak{q}_\Phi$ and in (6.2.3) we consider the real linear form $\delta_x = \sum_{\Phi^u \cap \tau\Phi^u} \phi : \overline{\mathfrak{h}} \to \mathbb{R}$. That defines $\mathfrak{q}_{[x]} = \overline{\mathfrak{h}}_{\mathbb{C}} + \sum_{\langle\phi,\delta_x\rangle \geqq 0} \mathfrak{g}^\phi$, a τ–stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Then $(\mathfrak{q}_x^u \cap \tau\mathfrak{q}_x^u) \subset \mathfrak{q}_{[x]} \subset \{\overline{\mathfrak{n}}_{[x],\mathbb{C}} \cap (\mathfrak{q}_x + \tau\mathfrak{q}_x)\}$. Let $\Gamma = \{\phi \in \Delta_{\mathfrak{g}} \mid \langle\phi,\delta_x\rangle < 0, -\phi \notin \Phi^u \cap \tau\Phi^u, \phi + \tau\phi \text{ not a root.}\}$ and define $\mathfrak{m}_{[x]} = \mathfrak{q}_{[x]} + \sum_\Gamma \overline{\mathfrak{g}}^\phi$. Then [**50**, Theorem 8.9] $\mathfrak{m}_{[x]} \subset \{\overline{\mathfrak{n}}_{[x],\mathbb{C}} \cap (\mathfrak{q}_x + \tau\mathfrak{q}_x)\}$ and the following are equivalent: (i) The holomorphic arc components of $G(x)$ are complex submanifolds of X, (ii) $\overline{\mathfrak{n}}_{[x],\mathbb{C}} \subset (\mathfrak{q}_x + \tau\mathfrak{q}_x)$, (iii) $\overline{\mathfrak{n}}_{[x],\mathbb{C}} = \mathfrak{m}_{[x]}$, and (iv) $\mathfrak{m}_{[x]}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$. When these hold, we say that the orbit $G(x)$ is *partially complex.*

We will need stronger conditions. An orbit $G(x)$ is *of flag type* if the Zariski closure $\overline{N}_{[x],\mathbb{C}}$ of $S_{[x]}$ is a complex flag manifold, *measurable* if the holomorphic arc components carry positive Radon measures invariant under their normalizers, *integrable* if $(\mathfrak{q}_x + \tau\mathfrak{q}_x)$ is a subalgebra of $\overline{\mathfrak{g}}_{\mathbb{C}}$. Given $\mathfrak{q}_x = \mathfrak{q}_\Phi$ we denote

$$\mathfrak{v}_x^- = \sum_{\Phi^u \cap -\tau\Phi^u} \overline{\mathfrak{g}}^\phi,\ \mathfrak{v}_x^+ = \sum_{-\Phi^u \cap -\tau\Phi^u} \overline{\mathfrak{g}}^\phi,\ \mathfrak{v}_x = \mathfrak{v}_x^- + \mathfrak{v}_x^+ .$$

Then [**50**, Theorem 9.2] $G(x)$ is measurable if and only if $\overline{\mathfrak{n}}_{[x],\mathbb{C}} = (\mathfrak{q}_x \cap \tau\mathfrak{q}_x) + \mathfrak{v}_x$. It follows that, in $G(x)$ is measurable, then (i) the invariant measure on $S_{[x]}$ comes from an $N_{[x]}$–invariant possibly–indefinite Kähler metric, (ii) $G(x)$ is partially complex and of flag type, and (ii) $G(x)$ is integrable if and only if $\tau\mathfrak{q}_x^r = \mathfrak{q}_x^r$.

On the other hand, if $\tau\mathfrak{q}_x^r = \mathfrak{q}_x^r$ then [**50**, Theorem 9.19] $G(x)$ is measurable $\Leftrightarrow G(x)$ is integrable $\Leftrightarrow G(x)$ is partially complex and of flag type, and under those conditions $\overline{\mathfrak{n}}_{[x],\mathbb{C}} = (\mathfrak{q}_x \cap \tau\mathfrak{q}_x) + \mathfrak{v}_x = \mathfrak{q}_{[x]} = \mathfrak{q}_x + \tau\mathfrak{q}_x$. Open orbits are obviously integrable, partially complex and of flag type. Closed orbits are another matter. There is just one closed G–orbit on X, every maximal compact subgroup of $\overline{G}$ is transitive on it, and it is connected. There is a problem with [**50**, Theorem 9.12], where it was asserted that the closed orbit always is measurable, hence partially complex (consider [**50**, Example 8.12] applied to $SU(m,m)$). But if Q is a Borel subgroup of $\overline{G}$, then the closed orbit is measurable, hence partially complex and of flag type, and in that case is integrable.

6.5. We now describe a class of orbits that plays a key role in the geometric realization of the various nondegenerate series of representations of G. Fix a Cartan subgroup $H = T \times A$ of G and an associated cuspidal parabolic $P = MAN$. We need complex flag manifolds $X = \overline{G}_{\mathbb{C}}/Q$ and measurable integrable orbits $Y = G(x) \subset X$ such that the G–normalizers of the holomorphic arc components of Y satisfy

$$N_{[x]} = \{g \in G \mid gS_{[x]} = S_{[x]} \text{ has Lie algebra } \mathfrak{p}\}. \tag{6.5.1}$$

As Y is to be measurable $S_{[x]}$ will be an open M–orbit in the sub–flag $\overline{M}_{\mathbb{C}}(x)$. So AN will act trivially on $S_{[x]}$ and the isotropy subgroup of G at x will have form

UAN with $T \subset U \subset M$. Finally, we need the condition that

$$U/Z_G(G^0) = \{m \in M \mid m(x) = x\}/Z_G(G^0) \text{ is compact.} \tag{6.5.2}$$

We look at some consequences of (6.5.1) and (6.5.2). Write $\mathfrak{z}_\mathfrak{g}$ for the center of $\mathfrak{g}$, so $\mathfrak{g} \cong \mathfrak{z}_\mathfrak{g} \oplus \overline{\mathfrak{g}}$. Since Y is measurable and integrable, §6.4 would lead to

$$\begin{aligned} &\mathfrak{p}_\mathbb{C} = \mathfrak{z}_{\mathfrak{g},\mathbb{C}} + \mathfrak{q}_x + \tau\mathfrak{q}_x\,,\ \tau\mathfrak{q}_x^r = \mathfrak{q}_x^r\,,\ \mathfrak{n}_\mathbb{C} = \mathfrak{p}_\mathbb{C}^u = \mathfrak{q}_x^u \cap \tau\mathfrak{q}_x^u\,, \\ &(\mathfrak{m}+\mathfrak{a})_\mathbb{C} = \mathfrak{p}_\mathbb{C}^r = (\mathfrak{z}_{\mathfrak{g},\mathbb{C}} + \mathfrak{q}_x^r) + (\mathfrak{q}_x^{-u} \cap \tau\mathfrak{q}_x^u) + (\mathfrak{q}_x^u \cap \tau\mathfrak{q}_x^{-u})\,, \\ &(\mathfrak{u}+\mathfrak{a})_\mathbb{C} = \mathfrak{z}_{\mathfrak{g},\mathbb{C}} + \mathfrak{q}_x^r \text{ and } \mathfrak{m}_\mathbb{C} = \mathfrak{u}_\mathbb{C} + (\mathfrak{q}_x^{-u} \cap \tau\mathfrak{q}_x^u) + (\mathfrak{q}_x^u \cap \tau\mathfrak{q}_x^{-u}). \end{aligned} \tag{6.5.3}$$

Since $S_{[x]}$ a measurable open M^0–orbit in the flag $\overline{M}_\mathbb{C}(x)$, (6.5.3) ensures that

$$\mathfrak{r} := \mathfrak{m}_\mathbb{C} + (\mathfrak{z}_{\mathfrak{g},\mathbb{C}} + \mathfrak{q}_x) \text{ is parabolic in } \mathfrak{m}_\mathbb{C} \text{ with } \mathfrak{r}^r = \mathfrak{u}_\mathbb{C} \text{ and } \mathfrak{r}^u = \mathfrak{q}_x^u \cap \tau\mathfrak{q}_x^{-u}\,. \tag{6.5.4}$$

The following Proposition shows that (6.5.3) and (6.5.4) give us the parabolics that we need for our geometric realizations.

PROPOSITION 6.5.5. *Let G be a real reductive Lie group in the class $\widetilde{\mathcal{H}}$ of* (1.2.1)*, $H = T \times A$, and $P = MAN$ an associated cuspidal parabolic subgroup of G. Suppose that* (i) *$\mathfrak{u} \subset \mathfrak{m}$ is the $\mathfrak{m}$–centralizer of a subalgebra of $\mathfrak{t}$ such that $U^0/(U^0 \cap Z_G(G^0))$ is compact,* (ii) *$\mathfrak{r} \subset \mathfrak{m}_\mathbb{C}$ is a parabolic subalgebra with $\mathfrak{r}^r \subset \mathfrak{u}_\mathbb{C}$,* (iii) *$\mathfrak{q}$ is the $\overline{\mathfrak{g}}_\mathbb{C}$–normalizer of $\mathfrak{r}^u + \mathfrak{n}_\mathbb{C}$ and Q is the corresponding analytic subgroup of $\overline{G}_\mathbb{C}$, and* (iv) *$X = \overline{G}_\mathbb{C}/Q$ and $x = 1Q \in X$. Then Q is a parabolic subgroup of $\overline{G}_\mathbb{C}$, $\mathfrak{q}^u = \mathfrak{r}^u + \mathfrak{n}_\mathbb{C}$, and $G(x)$ is a measurable integrable orbit, and (X, x) satisfies* (6.5.1) *and* (6.5.2)*. Conversely every pair (X, x) satisfying* (6.5.1) *and* (6.5.2)*, $G(x)$ measurable and integrable, $U^0/(U^0 \cap Z_G(G^0))$ compact, is constructed as above.*

PROOF. Denote $\overline{M} = M/Z_G(G^0)$. Let $\overline{R}$ be the analytic subgroup of $\overline{M}_\mathbb{C}$ for $\overline{\mathfrak{r}} := \mathfrak{r}/\mathfrak{z}_{\mathfrak{g},\mathbb{C}}$ and $S = \overline{M}_\mathbb{C}/\overline{R}$. Then S is a complex flag manifold of $\overline{M}_\mathbb{C}$ by (ii). The isotropy subalgebra of $\mathfrak{m}$ at $s = 1\overline{R} \in S$ is $\mathfrak{m} \cap \mathfrak{r}$. It has reductive part $\mathfrak{u}$ by (ii). As $\mathfrak{t} \subset \mathfrak{u}$ by (i), and as $M(s)$ is measurable and open in S by Lemmas 6.3.1 and 6.3.2, we have $\mathfrak{m} \cap \mathfrak{r} = \mathfrak{u}$ and $\mathfrak{r} \cap \tau\mathfrak{r} = \mathfrak{u}_\mathbb{C}$.

Define $\mathfrak{q}$ and Q as in (iii). The contribution to $\mathfrak{q}$ from $\overline{\mathfrak{m}}_\mathbb{C}$ is $\overline{\mathfrak{r}}$, all of $(\mathfrak{a}+\mathfrak{n})_\mathbb{C}$ from $(\mathfrak{a}+\mathfrak{n})_\mathbb{C}$, and 0 from $\mathfrak{n}_\mathbb{C}^-$. So $\mathfrak{q} = (\overline{\mathfrak{u}}_\mathbb{C} + \mathfrak{a}_\mathbb{C}) + (\mathfrak{r}^u + \mathfrak{n}_\mathbb{C})$, thus is parabolic in $\overline{\mathfrak{g}}_\mathbb{C}$. Now $X = \overline{G}_\mathbb{C}/Q$ is a complex flag manifold, $\mathfrak{z}_{\mathfrak{g},\mathbb{C}} + \mathfrak{q}^r = \mathfrak{u}_\mathbb{C} + \mathfrak{a}_\mathbb{C}$, and $\mathfrak{q}^u = \mathfrak{r}^u + \mathfrak{n}_\mathbb{C}$. In particular, $\mathfrak{q} + \tau\mathfrak{q} = \mathfrak{p}_\mathbb{C}/\mathfrak{z}_\mathfrak{g}$ is a subalgebra of $\mathfrak{g}_\mathbb{C}$, so the orbit $G(x) \subset X$ is integrable, and $\tau\mathfrak{q}^r = \mathfrak{r}^r$ so $G(x)$ is measurable with $\overline{\mathfrak{n}}_{[x],\mathbb{C}} = \mathfrak{q} + \tau\mathfrak{q} = \mathfrak{p}_\mathbb{C}/\mathfrak{z}_{\mathfrak{g},\mathbb{C}}$. We conclude $\overline{\mathfrak{n}}_{[x]} = \mathfrak{p}/\mathfrak{z}_\mathfrak{g}$ and so $\mathfrak{n}_{[x]} = \mathfrak{p}$. We have shown that $G(x)$ is a measurable integrable orbit, and (X, x) satisfies (6.5.1) and (6.5.2).

For the converse compare (6.5.1) and (6.5.2) with the construction. □

We enumerate the (X, x) of Proposition 6.5.5. Let $\Pi_\mathfrak{t}$ be a simple $\mathfrak{t}_\mathbb{C}$–roots system on $\mathfrak{m}_\mathbb{C}$ and $\Phi_\mathfrak{t}$ a subset of $\Pi_\mathfrak{t}$. Let Π be the simple $\mathfrak{h}_\mathbb{C}$–root system on $\mathfrak{g}_\mathbb{C}$ that contains $\Pi_\mathfrak{t}$ and induces the positive $\mathfrak{a}$–root system used for construction of $P = MAN$. Define $\Phi = \Phi_\mathfrak{t} \cup (\Pi \setminus \Pi_\mathfrak{t})$. The parabolic subalgebras $\mathfrak{q} \subset \overline{\mathfrak{g}}_\mathbb{C}$ of Proposition 6.5.5 are just the $\mathfrak{q}_\Phi$.

COROLLARY 6.5.6. *Given $G(x) \in X$ as in* Proposition 6.5.5*, $M^\dagger$ is the stabilizer $\{m \in M \mid mS_{[x]} = S_{[x]}\}$ of $S_{[x]}$ in M. Thus $U \subset M^\dagger$ and $N_{[x]} = M^\dagger AN$.*

PROOF. Let $M^1 = \{m \in M \mid mS_{[x]} = S_{[x]}\}$. Then $M^\dagger \subset M^1$ because $M^\dagger = Z_M(M^0)M^0$ and $Z_M(M^0)$ acts trivially on $S_{[x]}$. The isotropy subgroup U of M at x is in M^1 and M^0 is transitive on $S_{[x]}$, so $M^1 = UM^0$. Let $u \in U$. All Cartan

subalgebras and all Weyl chambers of $\mathfrak{u}$ are $\mathrm{Ad}(U^0)$–conjugate, so we choose a Weyl chamber $\mathfrak{d} \subset i\mathfrak{t}^*$ for $\mathfrak{u}$ and replace u within uU^0 so that $\mathrm{Ad}(u)$ preserves $\mathfrak{t}$ and $\mathfrak{d}$. Thus $\mathrm{Ad}(u)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$ that is the identity on $\mathfrak{t}$, so $u \in T \subset M^\dagger$. We have shown $M^1 \subset M^\dagger$. As $M^\dagger \subset M^1$ now $M^1 = M^\dagger$. □

COROLLARY 6.5.7. *Given $G(x) \in X$ as in* Proposition 6.5.5 *and $u \in U$,* $\mathrm{Ad}(u)$ *is an inner automorphism on U^0.*

7. Open Orbits and Discrete Series

Let G be a reductive Lie group of our class specified in §3.1. We consider complex flag manifolds $X = \overline{G}_C/Q$ and open orbits $Y = G(x) \subset X$ such that $U = \{g \in G : g(x) = x\}$ is compact modulo Z. In §7.1 we see that these pairs (X, x) exist precisely when G has relative discrete series representations, that $U = Z_G(G^0)U^0$ with $U^0 = U \cap G^0$, and that Y has $|G/G^\dagger|$ topological components. If $[\mu] \in \widehat{U}$ we show that the associated G–homogeneous hermitian vector bundle $\mathcal{V}_\mu \to Y$ has a unique G–homogeneous holomorphic vector bundle structure. That allows us to construct the Hilbert spaces $H_2^{0,q}(Y; \mathcal{V}_\mu)$ of square integrable harmonic $(0,q)$–forms on Y with values in $\mathcal{V}_\mu$, and unitary representations π_μ^q of G on $H_2^{0,q}(\mathcal{V}_\mu)$. The remainder of §7 shows that the $[\pi_\mu^q]$, $q \geqq 0$ and $[\mu] \in \widehat{U}$, are the relative discrete series classes in $\widehat{G}$.

Section 7.2 is the formulation and history of our main result, Theorem 7.2.3. Let $[\mu] \in \widehat{U}$. Then $[\mu] = [\chi \otimes \mu^0]$ where $[\chi] \in \widehat{Z_G(G^0)}$ and $[\mu^0] \in \widehat{U^0}$. Let $\Theta^{disc}_{\pi_\mu^q}$ denote the character of the discrete part of π_μ^q. We prove

$$\sum_{q \geqq 0} (-1)^q \Theta^{disc}_{\pi_\mu^q} = (-1)^{n+q(\lambda+\rho)} \Theta_{\pi_{\chi,\lambda+\rho}}$$

where λ is the highest weight of μ^0, n is the number of positive roots, and ρ is half the sum of the positive roots. We note that $H_2^{0,q}(\mathcal{V}_\mu) = 0$ for $q \neq q(\lambda+\rho)$, and we show that $[\pi_\mu^{q(\lambda+\rho)}] = [\pi_{\chi,\lambda+\rho}] \in \widehat{G}_{disc}$. Theorem 7.2.3 is proved in §§7.3 through 7.7.

We reduce the proof of Theorem 7.2.3 to the case $G = G^\dagger$ in §7.3, to the case $G = G^0$ in §7.4, and then further to the case where Q is a Borel subgroup of $\overline{G}_C$ in §7.5. In §7.6 we use results of Harish–Chandra and a method of W. Schmid to prove the alternating sum formula for the $\Theta^{disc}_{\pi_\mu^q}$. The vanishing statement comes out of work of Schmid [**41**] cited above. It combines with the alternating sum formula to identify $[\pi_{\chi,\lambda+\rho}] \in \widehat{G}_{disc}$ as the discrete part of $[\pi_\mu^{q(\lambda+\rho)}]$. This trick is due to Narasimhan and Okamoto. Finally we use Corollary 5.7.2 of our Plancherel Theorem to show that $[\pi_\mu^{q(\lambda+\rho)}]$ has no nondiscrete part, so $[\pi_\mu^{q(\lambda+\rho)}] = [\pi_{\chi,\lambda+\rho}]$, completing our proof in §7.7.

7.1. G is a Lie group of the class $\widetilde{\mathcal{H}}$ of general real reductive Lie groups defined in §3.1. As explained in §6.2, $\overline{G} = G/Z_G(G^0)$ has complexification $\overline{G}_C = \mathrm{Int}(\mathfrak{g}_C)$, and G acts on the complex flag manifolds of $\overline{G}_C$. To realize the relative discrete series of G we work with

$$\begin{aligned} &X = \overline{G}_C/Q \text{ complex flag manifold of } \overline{G}_C \\ (7.1.1) \qquad &Y = G(x) \subset X \text{ open } G\text{–orbit such that} \\ &\qquad\qquad \text{the isotropy subgroup } U \text{ of } G \text{ at } x \text{ is compact modulo } Z. \end{aligned}$$

We collect some immediate consequences of (7.1.1).

LEMMA 7.1.2. *Suppose* (X, x) *is given as in* (7.1.1). *Then* U/Z *contains a compact Cartan subgroup* H/Z *of* G/Z, *so* G *has relative discrete series representations. Further, the open orbit* $Y = G(x) \subset X$ *is measurable and integrable, and* (X, x) *is the case* $P = G$ *of* (6.7.1). *Finally,* $U = Z_G(U^0)U^0$, $U \cap G^0 = U^0$, $UG^0 = G^\dagger$, *and* $G/G^\dagger$ *enumerates the topological components of* Y.

Remark. As a consequence of the second assertion, all possibilities for (7.1.1) are enumerated in the paragraph following Proposition 6.5.5.

PROOF. Isotropy subgroups of G on X all contain Cartan subgroups of G by (6.2.2). Now the first assertion follows from (7.1.1) and Theorem 3.5.6.

U acts on the tangent space at x as $U/Z_G(G^0)$, which is compact by (7.1.1). Thus the orbit $Y = G(x)$ is measurable by Lemma 6.3.2. As open orbits are integrable with $\mathfrak{q}_x + \tau\mathfrak{q}_x = \overline{\mathfrak{g}}_C$. Now we have (6.5.1) and (6.5.2) with $P = M = G = N_{[x]}$.

Let $u \in U$. Corollary 6.5.7 says that $\mathrm{Ad}(u)$ is trivial on some Cartan subalgebra of $\mathfrak{u}$, thus on a Cartan subalgebra of $\mathfrak{g}$. Now $\mathrm{Ad}(u)$ is an inner automorphism of G^0, i.e. $u \in G^\dagger$. We have just seen $U \subset G^\dagger = Z_G(G^0)G^0$. On the other hand, open orbits are simply connected, so $U \cap G^0 = U^0$, Thus $U = Z_G(G^0)U^0$ and $UG^0 = G^\dagger$. Since UG^0 is the G–normalizer of $G^0(x)$, now $G/G^\dagger$ parameterizes the components of $G(x)$. □

The facts about U in Lemma 7.1.2 tell us

$$\begin{aligned} &\widehat{U} = \{[\chi \otimes \mu^0] : [\chi] \in \widehat{Z_G(G^0)} \text{ and } [\mu^0] \in \widehat{U^0}\}; \text{ so} \\ &\text{if } [\mu] \in \widehat{U} \text{ then its representation space } E_\mu \text{ has } \dim E_\mu < \infty \\ &\text{and we have } \mathbb{E}_\mu \to Y, \text{ a } G\text{–homogeneous hermitian vector bundle.} \end{aligned} \tag{7.1.3}$$

LEMMA 7.1.4. *There is a unique complex structure on* $\mathbb{E}_\mu$ *such that* $\mathbb{E}_\mu \to Y$ *is a* G*–homogeneous holomorphic vector bundle.*

PROOF. The action of G^0 on X maps $\mathfrak{g}_\mathbb{C}$ to a Lie algebra of holomorphic vector fields. Define

$$\mathfrak{l} = \{\xi \in \mathfrak{g}_\mathbb{C} \mid \xi_x = 0\},$$

isotropy subalgebra at x. The homomorphism $G^0 \to \overline{G}$ induces a homomorphism α of $\mathfrak{g}_C$ onto $\overline{\mathfrak{g}}_C$, and $\mathfrak{l} = \alpha^{-1}(\mathfrak{q}_x)$. Note that $\mathfrak{u}_C = \alpha^{-1}(\mathfrak{q}_x^r)$ reductive subalgebra of $\mathfrak{l}$. Choose a linear algebraic group with Lie algebra $\mathfrak{g}_\mathbb{C}$ and observe that α is a homomorphism of algebraic Lie algebras. Thus $\mathfrak{u}_\mathbb{C}$ is a maximal reductive subalgebra of $\mathfrak{l}$, and there is a nilpotent ideal $\mathfrak{l}^-$ such that $\mathfrak{l} = \mathfrak{u}_\mathbb{C} + \mathfrak{l}^-$ semidirect sum. Observe that $\mathrm{Ad}(u)\mathfrak{l}^- = \mathfrak{l}^-$ for all $u \in U$.

By extension of μ from U to $\mathfrak{l}$, we mean a (complex linear) representation λ of $\mathfrak{l}$ on V_μ such that

$$\lambda|_\mathfrak{u} = \mu\,, \text{ i.e., } \lambda(\xi) = \mu(\xi) \text{ for all } \xi \in \mathfrak{u},$$

and

$$\mu(u)\lambda(\xi)\mu(u)^{-1} = \lambda(\mathrm{Ad}(u)\xi) \text{ for all } u \in U \text{ and } \xi \in \mathfrak{l}.$$

Let λ be such an extension. Then $\lambda(\mathfrak{l}^-)$ consists of nilpotent linear transformations because λ is an algebraic representation of $\mathfrak{l}$, and that implies $\lambda(\mathfrak{l}^-) = 0$ because μ is irreducible. Thus there is just one extension of μ from U to $\mathfrak{l}$; it is given by $\lambda(\xi_1 + i\xi_2 + \eta) = \mu(\xi_1) + i\mu(\xi_2)$ where $\xi_1, \xi_2 \in \mathfrak{u}$ and $\eta \in \mathfrak{l}^-$.

Our lemma now follows from the fact [**45**, Theorem 3.6] that the G–homogeneous holomorphic vector bundle structures on $\mathbb{E}_\mu \to Y$ are in bijective correspondence with the extensions of μ from U to $\mathfrak{l}$. $\square$

Using (7.1.1) we fix a G–invariant hermitian metric on the complex manifold Y. The unitary structure of E_μ specifies a G–invariant hermitian metric on the fibers of $\mathbb{E}_\mu \to Y$. Denote

$$\begin{aligned} &A^{p,q}(Y;\mathbb{E}_\mu) = \{C^\infty(p,q)\text{–forms on } Y \text{ with values in } \mathbb{E}_\mu\} \text{ so we have} \\ &\text{Hodge–Kodaira maps } A^{p,q}(Y;\mathbb{E}_\mu) \xrightarrow{\#} A^{n-p,n-q}(Y;\mathbb{E}_\mu^*) \xrightarrow{\widetilde{\#}} A^{p,q}(Y;\mathbb{E}_\mu) \end{aligned} \tag{7.1.5}$$

Here $n = \dim_{\mathbb{C}} Y$ and $\mathbb{E}_\mu^* = \mathbb{E}_{\mu^*}$ is the dual bundle. If $\alpha,\ \beta \in A^{p,q}(Y;\mathbb{E}_\mu)$ then $\alpha \wedge \#\beta \in A^{n,n}(Y;\mathbb{E}_\mu \otimes \mathbb{E}_\mu^*)$. The pairing $E_\mu \otimes E_\mu^* \to \mathbb{C}$ sends $\alpha \wedge \#\beta$ to an ordinary (n,n)–form on Y that we denote $\alpha \bar{\wedge} \#\beta$. This gives us a pre Hilbert space

$$A_2^{p,q}(\mathcal{V}_\mu) = \left\{ \alpha \in A^{p,q}(Y;\mathbb{E}_\mu) \,\middle|\, \int_Y \alpha \bar{\wedge} \#\alpha < \infty \right\},\ \langle \alpha, \beta \rangle = \int_Y \alpha \bar{\wedge} \#\beta. \tag{7.1.6}$$

The space of *square integrable* (p,q)*–forms* on Y with values in $\mathbb{E}_\mu$ is

$$L_2^{p,q}(Y;\mathbb{E}_\mu) : \text{ Hilbert space completion of } A_2^{p,q}(Y;\mathbb{E}_\mu). \tag{7.1.7}$$

The operator $\overline{\partial} : A^{p,q}(Y;\mathbb{E}_\mu) \to A^{p,q+1}(Y;\mathbb{E}_\mu)$ is densely defined on $L_2^{p,q}(Y;\mathbb{E}_\mu)$ with formal adjoint $\overline{\partial}^* = -\widetilde{\#}\,\overline{\partial}\,\#$. That gives us a second order elliptic operator

$$\square = (\overline{\partial} + \overline{\partial}^*)^2 = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} : \text{ Kodaira–Hodge–Laplacian.} \tag{7.1.8}$$

The hermitian metric on Y is complete by homogeneity, so the work of Andreotti and Vesentini [**2**] applies. First, it says that $\square$, with domain consisting of the compactly supported forms in $A^{p,q}(Y;\mathbb{E}_\mu)$, is essentially self adjoint on $L_2^{p,q}(Y;\mathbb{E}_\mu)$. We also write $\square$ for the unique self adjoint extension, which coincides both with the adjoint and the closure. Its kernel

$$H_2^{p,q}(Y;\mathbb{E}_\mu) = \{\omega \in L_2^{p,q}(Y;\mathbb{E}_\mu) \mid \square\,\omega = 0\} \tag{7.1.9}$$

consists of the *square integrable harmonic* (p,q)*–forms* on Y with values in $\mathbb{E}_\mu$. $H_2^{p,q}(Y;\mathbb{E}_\mu)$ is a closed subspace of $L_2^{p,q}(Y;\mathbb{E}_\mu)$. It is contained in $A_2^{p,q}(Y;\mathbb{E}_\mu)$ by ellipticity of $\square$. Write $c\ell$ for closure. The Andreotti–Vesentini work shows that $\overline{\partial}^*$ has closed range and gives us an orthogonal direct sum

$$L_2^{p,q}(Y;\mathbb{E}_\mu) = c\ell\,\overline{\partial} L_2^{p,q-1}(\mathcal{V}_\mu) \oplus \overline{\partial}^* L_2^{p,q+1}(\mathcal{V}_\mu) \oplus H_2^{p,q}(\mathcal{V}_\mu) \tag{7.1.10}$$

Here $\overline{\partial}$ has kernel $c\ell\,\overline{\partial} L_2^{p,q-1}(Y;\mathbb{E}_\mu) \oplus H_2^{p,q}(Y;\mathbb{E}_\mu)$ and the kernel of $\overline{\partial}^*$ is the closed subspace $\overline{\partial}^* L_2^{p,q+1}(Y;\mathbb{E}_\mu) \oplus H_2^{p,q}(Y;\mathbb{E}_\mu)$. Thus $H_2^{p,q}(Y;\mathbb{E}_\mu)$ is a square integrable Dolbeault cohomology group.

The metrics and complex structures on Y and $\mathbb{E}_\mu$ are invariant under the action of G. Thus G acts on $L_2^{p,q}(Y;\mathbb{E}_\mu)$ by a unitary representation $\widetilde{\pi}_\mu^{p,q}$ that has a subrepresentation

$$\pi_\mu^{p,q} : \text{ unitary representation of } G \text{ on } H_2^{p,q}(\mathbb{E}_\mu). \tag{7.1.11}$$

For convenience we also denote $\pi_\mu^q = \pi_\mu^{0,q}$. The program here in §7 is to represent the classes in $\widehat{G}_{disc}$ by the various π_μ^q.

7.2. Fix a compact Cartan subgroup H/Z of G/Z with $H \subset U$ as in Lemma 7.1.2. Choose a system Π of simple $\mathfrak{h}_{\mathbb{C}}$–roots of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{q}_x = \mathfrak{q}_\Phi$ where $\Phi \subset \Pi$. Let Σ^+ denote the corresponding positive root system. As usual we pass to a $\mathbb{Z}_2$ extension if necessary so that e^ρ and Δ are well defined on H^0 and define

$$\rho = \frac{1}{2}\sum_{\phi\in\Sigma^+}\phi\,,\quad \Delta = \prod_{\phi\in\Sigma^+}(e^{\phi/2}-e^{-\phi/2})\,,\quad \text{and}\ \ \widetilde{\omega}(\cdot) = \prod_{\phi\in\Sigma^+}\langle\cdot,\phi\rangle, \tag{7.2.1}$$
$$L = \{\lambda \in i\mathfrak{h}^* : e^\lambda \text{ is well defined on } H^0\} \text{ and } L' = \{\lambda \in L : \widetilde{\omega}(\lambda) \neq 0\}.$$

Let θ be the (unique) Cartan involution under which H is stable and $K = G^\theta$, so $H \subset U \subset K$. Now $\Sigma^+ = \Sigma_k^+ \cup \Sigma_m^+$ (disjoint) where Σ_k^+ consists of the compact positive roots ($\overline{\mathfrak{g}}^\phi \subset \mathfrak{k}_{\mathbb{C}}$) and Σ_m^+ consists of the noncompact positive roots ($\overline{\mathfrak{g}}^\phi \not\subset \mathfrak{k}_{\mathbb{C}}$). If $\lambda \in L'$ we have

$$q(\lambda) = |\{\phi \in \Sigma_k^+ : \langle\lambda,\phi\rangle < 0\}| + |\{\phi \in \Sigma_m^+ : \langle\lambda,\phi\rangle > 0\}|. \tag{7.2.2}$$

Recall the statement of Theorem 3.5.9. The main result of §7 is

THEOREM 7.2.3. *Let $[\mu] \in \widehat{U}$, say $[\mu] = [\chi \otimes \mu^0]$ as in (7.1.3). Let λ be the highest weight of μ^0 for the positive $\mathfrak{h}_{\mathbb{C}}$–root system $\Sigma^+ \cap \Phi^r$ of $\mathfrak{u}_{\mathbb{C}}$. Then $\lambda \in L$ and $[\mu] \in \widehat{U}_\zeta$ where $\zeta \in \widehat{Z}$ coincides with e^λ on $Z \cap G^0$ and $[\chi] \in \widehat{Z_G(G^0)}_\zeta$. Assume $\lambda + \rho \in L'$. Then $H_2^{0,q}(Y;\mathbb{E}_\mu) = 0$ whenever $q \neq q(\lambda+\rho)$, and the natural action of G on $H_2^{0,q(\lambda+\rho)}(Y;\mathbb{E}_\mu)$ is the ζ–discrete series class $[\pi_{\chi,\lambda+\rho}]$.*

Theorem 7.2.3 gives a number of explicit geometric realizations of the relative discrete series representations of G. The case where G is a connected semisimple Lie group with finite center and $U = H$ is due to W. Schmid ([**40**], [**41**]); to some extent we follow his ideas. The case where G is a connected semisimple Lie group with finite center and $Y = G(x)$ is a hermitian symmetric space was proved by M. S. Narasimhan and K. Okamoto [**35**]. Some results for groups with possibly infinite center were proved by Harish–Chandra [**15**] and J. A. Tirao [**46**]. Also, W. Schmid (unpublished) and R. Parthasarathy ([**36**], [**37**]) obtained realizations on spaces of square integrable harmonic spinors. Finally, R. Hotta [**30**] realized discrete series representations of connected semisimple groups of finite center on certain eigenspaces of the Casimir operator.

We carry out the proof of Theorem 7.2.3 in §§7.3 through 7.7.

7.3. We reduce Theorem 7.2.3 to the case $G = G^\dagger$.

Choose a system $\{g_1, \cdots, g_r\}$ of coset representatives of G modulo $G^\dagger$. According to Lemma 7.2.1, the topological components of $Y = G(x)$ are the $Y_i = G^\dagger(g_i x)$. Let ${}^i\pi_\mu^q$ denote the representation of $G^\dagger$ on

$$H_2^{0,q}(Y;\mathbb{E}_\mu|_{Y_i}) = \{\omega \in H_2^{0,q}(Y;\mathbb{E}_\mu) : \omega \text{ is supported in } Y_i\}.$$

Evidently $H_2^{0,q}(Y;\mathbb{E}_\mu) = H_2^{0,q}(Y;\mathbb{E}_\mu|_{Y_1}) \oplus \cdots \oplus H_2^{0,q}(Y;\mathbb{E}_\mu|_{Y_r})$ as orthogonal direct sum. Thus $\pi_\mu^{0,q} = {}^1\pi_\mu^{0,q} \oplus \cdots \oplus {}^r\pi_\mu^{0,q}$. Also, $\pi_\mu^q(g_i)$ sends $H_2^{0,q}(Y;\mathbb{E}_\mu|_{Y_j})$ to $H_2^{0,q}(Y;\mathbb{E}_\mu|_{Y_k})$ where $g_iY_k = Y_j$, i.e., where $g_i^{-1}g_j \in g_kG^\dagger$. In summary,

LEMMA 7.3.1. $\pi_\mu^q = \operatorname{Ind}_{G^\dagger}^G({}^i\pi_\mu^{0,q})$ *for* $1 \leqq i \leqq r$.

We know from Theorem 3.5.7 that $\widehat{G}_{disc}$ consists of the classes $[\pi] = [\operatorname{Ind}_{G^\dagger}^G(\pi^\dagger)]$ where $[\pi^\dagger] \in \widehat{G^\dagger}_{disc}$. Further, Θ_π is supported in $G^\dagger$, where it coincides with $\Theta_{\pi_{G^\dagger}}$.

Now Lemma 7.3.1 tells us that, if Theorem 7.2.3 holds for $G^\dagger$ with each of the $\mathbb{E}_{\mu|_{Y_i}}$. Then Theorem 7.2.3 is valid for G with $\mathbb{E}_\mu$. In summary,

LEMMA 7.3.2. *In the proof of* Theorem 7.2.3 *we may assume* $G = G^\dagger$.

7.4. We reduce Theorem 7.2.3 to the case where G is connected. Using Lemma 7.3.2 we assume $G = G^\dagger$. Thus $G = Z_G(G^0)G^0$ and $Y = G(x)$ is connected. Recall $[\mu] = [\chi \otimes \mu^0]$ with $[\chi] \in \widehat{Z_G(G^0)}$ and $[\mu^0] \in \widehat{U^0}$, so $E_\mu = E_\chi \otimes E_{\mu^0}$. Now $[\mu^0]$ specifies a G^0–homogeneous holomorphic vector bundle $\mathbb{E}_{\mu^0} \to Y$. Let $\pi^q_{\mu^0}$ denote the representation of G^0 on $H^{0,q}_2(Y; \mathbb{E}_{\mu^0})$.

LEMMA 7.4.1. $\pi^q_\mu = \chi \otimes \pi^q_{\mu^0}$ *for all* $q \geqq 0$.

PROOF. $Z_G(G^0)$ acts trivially on X, so it acts trivially on the bundle of ordinary $(0,q)$–forms over the orbit $Y \subset X$. Thus $Z_G(G^0)$ acts on $L^{0,q}_2(Y; \mathbb{E}_\mu)$ as a type I primary representation ω_χ. In particular $\pi^q_\mu|_{Z_G(G^0)}$ is a multiple of χ. But $\mu|_{U^0} = (\dim \chi)\mu^0$ so $\pi^q_\mu|_{G^0} = (\dim \chi)\pi^q_{\mu^0}$. We conclude $\pi^q_\mu = \chi \otimes \pi^q_{\mu^0}$. □

We know from Proposition 3.5.2 that $\widehat{G^\dagger}_{disc}$ consists of the $[\chi \otimes \pi^0]$ where $[\chi] \in \widehat{Z_G(G^0)}$ and $[\pi^0] \in \widehat{G^0}_{disc}$ agree on Z_{G^0}. The distribution character $\Theta_{\chi\otimes\pi^0} = (\text{trace } \chi)\Theta_{\pi^0}$. If Theorem 7.2.3 holds for G^0 with $\mathbb{E}_{\mu^0}$ now Lemma 7.4.1 ensures the result for G with $\mathbb{E}_\mu$. In summary

LEMMA 7.4.2. *In the proof of* Theorem 7.2.3 *we may assume* G *is connected.*

7.5. We reduce Theorem 7.2.3 to the case where $G = G^0$ and $U = H^0$.

Choose a Borel subgroup $B \subset Q$ of $\overline{G}_\mathbb{C}$. Denote $X' = \overline{G}_\mathbb{C}/B$ and consider the G-equivariant projection $r : X' \to X$ defined by $r(\bar{g}B) = \bar{g}Q$. Now choose a base point $x' \in r^{-1}(x)$ defined by $\mathfrak{b}_{x'} = \mathfrak{q}_\emptyset$ relative to $(\overline{\mathfrak{h}}_\mathbb{C}, \Pi)$. Since $\tau\phi = -\phi$ for every $\mathfrak{h}$-root, the isotropy subalgebra of $\mathfrak{g}$ at x' is just $\mathfrak{h}$. Now $Y' = G(x')$ is open in X' and $H = \{g \in G : g(x') = x'\}$, and $r : Y' \to Y$ is G-equivariant and holomorphic.

Following Lemma 7.4.2, we assume G connected, so U and H are connected. Now λ is the highest weight of μ and $e^\lambda \in \widehat{H}$ specifies

$$\begin{aligned} &\mathbb{L}_\lambda \to Y' : \ G\text{–homogeneous holomorphic line bundle.} \\ &\pi^q_\lambda : \ \text{ representation of } G \text{ on } H^{0,q}_2(Y; \mathbb{L}_\lambda). \end{aligned} \tag{7.5.1}$$

LEMMA 7.5.2. $[\pi^q_\lambda] = [\pi^q_\mu]$.

PROOF. This is a Leray spectral sequence argument. Let $\mathcal{O}(Y'; \mathbb{L}_\lambda)$ denote the sheaf of germs of holomorphic sections of $\mathbb{L}_\lambda \to Y'$. Each integer $s \geqq 0$ gives a sheaf $\mathcal{R}^s(Y; \mathbb{L}_\lambda) \to Y$, associated to the presheaf that assigns the sheaf cohomology group $H^s(Y' \cap r^{-1}D; \mathcal{O}(Y'; \mathbb{L}_\lambda))$ to an open set $D \subset Y$. Since $r : Y' \to Y$ is a holomorphic fiber bundle, $\mathcal{R}^s(\mathbb{L}_\lambda)$ is the sheaf of germs of holomorphic sections of the holomorphic vector bundle over Y whose fiber at $y \in Y$ is $H^s(Y' \cap r^{-1}(y); \mathcal{O}(\mathbb{L}_\lambda))$.

Recall our Borel-Weil Theorem from Proposition 1.1.9 with $q_0 = 0$, and apply it to $Y' \cap r^{-1}(y) = U(x') \cong U/H$. That says $H^0(Y' \cap r^{-1}(y); \mathcal{O}(\mathbb{L}_\lambda)) = E_\mu$ as U-module and $H^s(Y' \cap r^{-1}(y); \mathcal{O}(\mathbb{L}_\lambda)) = 0$ for $s > 0$. Now $\mathcal{R}^0(\mathbb{L}_\lambda) = \mathcal{O}(\mathbb{E}_\mu)$ and $H^s(Y; \mathbb{L}_\lambda) = 0$ for $s > 0$.

Our analysis of the direct image sheaves $\mathcal{R}^s(\mathbb{L}_\lambda)$ shows that the Leray spectral sequence collapses for $r : Y' \to Y$, so each $H^q(Y'; \mathcal{O}(\mathbb{L}_\lambda)) = H^q(Y; \mathcal{O}(\mathbb{E}_\mu))$ as G-modules. More to the point, we carry the spectral sequence over from sheaf cohomology to Dolbeault cohomology and use the Andreotti–Vesentini theory ((7.1.9)

and (7.1.10)) to restrict considerations to square integrable forms. Then the resultant square integrable Leray spectral sequence collapses and we conclude that each $H_2^{0,q}(Y';\mathbb{L}_\lambda) = H_2^{0,q}(Y;\mathbb{E}_\mu)$ as G-modules. □

As immediate consequence of Lemmas 7.4.2 and 7.5.2 we have

LEMMA 7.5.3. *In the proof of* Theorem 7.2.3 *we may assume that G is connected, that Q is a Borel subgroup of $\overline{G}_\mathbb{C}$ and that $U = H$.*

7.6. Next, we prove the formula $\sum_{q\geqq 0}\Theta^{disc}_{\pi^q_\mu} = (-1)^{|\Sigma^+|+q(\lambda+\rho)}\Theta_{\pi_{\chi,\lambda+\rho}}$. By Lemma 7.5.3 we may assume that G is connected, that Q is a Borel subgroup of $\overline{G}_\mathbb{C}$ and that $U = H$. K/Z is the maximal compact subgroup of G/Z that contains the compact Cartan subgroup H/Z. If $[\pi] \in \widehat{G}_\zeta$ Lemma 3.2.1, and an argument [**11**, §5] of Harish–Chandra say that

$$\begin{aligned}
&\pi|_K = \sum_{\widehat{K}_\zeta} m_\kappa \cdot \kappa \text{ where } 0 \leqq m_\kappa \leqq n_G(\dim\kappa)\\
&(\pi|_K)(f) = \int_K f(k)\pi(k)dk, f \in C_c^\infty(K), \text{ is of trace class, and}\\
&T_\pi : C_c^\infty(K) \to \mathbb{C} \text{ defined by } f \mapsto \text{trace } (\pi|_K)(f) \text{ is a distribution on } K.
\end{aligned} \tag{7.6.1}$$

Harish–Chandra's argument [**16**, §12] now shows that $T_\pi|_{K\cap G'}$ is a real analytic function on $K \cap G'$ and that $T_\pi|_{K\cap G'} = \Theta_\pi|_{K\cap G'}$.

Recall the Cartan involution θ of G with fixed point set K. Fix a nondegenerate invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}_\mathbb{C}$ that restricts to the Killing form on the derived algebra and is negative definite on $\mathfrak{k} = (\mathfrak{k} \cap [\mathfrak{g},\mathfrak{g}]) \oplus \mathfrak{c}$. That gives us a positive definite $\mathrm{Ad}(K)$-invariant hermitian inner product $(u,v) = -\langle u, \theta\tau v\rangle$ on $\mathfrak{g}_\mathbb{C}$ where τ is complex conjugation of $\mathfrak{g}_\mathbb{C}$ over $\mathfrak{g}$.

Consider the nilpotent algebra $\mathfrak{n} = \sum_{\phi\in\Sigma^+}\overline{\mathfrak{g}}^{-\phi} = \mathfrak{q}_x^u \subset \mathfrak{g}_\mathbb{C}$. Denote

$$\Lambda(\mathrm{Ad}^*) = \sum_{j\geqq 0}\Lambda^j(\mathrm{Ad}^*) : \text{ representation of } \mathfrak{q}_x \text{ on } \Lambda\mathfrak{n}^* = \sum_{j\geqq 0}\Lambda^j\mathfrak{n}^*. \tag{7.6.2}$$

The inner product $(\ ,\ )$ gives $\mathfrak{n}$, thus $\mathfrak{n}^*$, thus also $\Lambda\mathfrak{n}^*$, a Hilbert space structure; and $\mathrm{Ad}^*(\mathfrak{h})$ acts by skew-hermitian transformations.

Fix $[\pi] \in \widehat{G}_\zeta$ and write $\mathcal{H}_\pi$ for its representation space. Let $\mathcal{H}_\pi^0$ denote the space of K-finite vectors in $\mathcal{H}_\pi$. It is dense and consists of analytic vectors, by (7.6.1). Now $\mathfrak{h}$ acts on $\mathcal{H}_\pi^0 \otimes \Lambda\mathfrak{n}^*$ by $\pi\otimes\mathrm{Ad}^*$ by skew–hermitian transformations. Let $\{y_1,\cdots,y_n\}$ be a basis of $\mathfrak{n}$, let $\{\omega^j\}$ be the dual basis of $\mathfrak{n}^*$, and $e(\omega^j) : \Lambda\mathfrak{n}^* \to \Lambda\mathfrak{n}^*$ the exterior product. Then $\delta := \sum\{(\pi(\omega_j)\otimes e(\omega^j)) + \frac{1}{2}1 \otimes (e(\omega_j)\,\mathrm{Ad}^*(y_j))\}$ is the coboundary operator $\mathcal{H}_\pi^0 \to \mathcal{H}_\pi^0$ of the Lie algebra cohomology for the action of $\mathfrak{h}$. It has formal adjoint $\delta^* = \sum\{(-\pi(\tau y_j)\otimes i(\omega^j)) + (\frac{1}{2}1 \otimes \mathrm{Ad}^*(y_j)^* i(\omega^j))\}$ where $i(\omega^j)$ denotes interior product. Now $\delta + \delta^*$ is a densely defined symmetric operator on $\mathcal{H}_\pi \otimes \Lambda\mathfrak{n}^*$.

Choose a basis $\{z_i\}$ of $\mathfrak{k}_\mathbb{C}$ that is orthonormal relative to $(\ ,\ )$. Then $\Omega_K = \sum z_i z_i \in \mathcal{U}(\mathfrak{k})$ is independent of choice of the basis $\{z_i\}$. In particular Ω_K is a linear combination, positive coefficients, of the Casimir operators of the simple ideals of $\mathfrak{k}$ plus the Laplacian on the center of $\mathfrak{k}$. Thus (7.6.1) $\pi(\Omega_K)$ is symmetric non-negative on $\mathcal{H}_\pi^0$ and has a unique self adjoint extension $\pi(\Omega_K)$ to $\mathcal{H}_\pi$. Further $\mathcal{H}_\pi$ is the discrete direct sum of the (all non–negative) eigenspaces of $\pi(\Omega_K)$. As

$$\{[\kappa] \in \widehat{K}_\zeta : \kappa(\text{ Casimir element of } \mathcal{U}(\mathfrak{k})) \leqq c\}$$

is finite for every real c, (7.6.1) also says that the sum of the eigenspaces of $\pi(\Omega_K)$ for eigenvalues $\leqq c$ has finite dimension. Thus $(1+\pi(\Omega_K))^{-1}$ is a self adjoint compact operator on $\mathcal{H}_\pi$. With this preparation, Wilfried Schmid's arguments [**40**, §3]. are valid in our situation. We state the result.

LEMMA 7.6.3. *The closure of $\delta+\delta^*$ from the domain $\mathcal{H}^0_\pi \otimes \Lambda\mathfrak{n}^*$ is the unique self adjoint extension of $\delta+\delta^*$ on $\mathcal{H}_\pi \otimes \Lambda\mathfrak{n}^*$. Each*

$$\mathcal{H}^q(\pi): \text{ kernel of } \delta+\delta^* \text{ on } \mathcal{H}_\pi \otimes \Lambda^q\mathfrak{n}^*$$

is a finite dimensional H-module. Define $f_\pi = \sum(-1)^q(\text{character of } H \text{ on } \mathcal{H}^q(\pi))$. Let Δ and ρ be as in (7.2.1) and $n = \dim_{\mathbb{C}}\mathfrak{n} = |\Sigma^+|$. Then

$$f_\pi|_{H\cap G'} = (-1)^n \Delta e^\rho \cdot T_\pi|_{H\cap G'}.$$

Let $d\pi$ denote Plancherel measure on $\widehat{G}_\zeta$, so $L_2(G/Z,\zeta) = \int_{\widehat{G}_\zeta} \mathcal{H}_\pi \widehat{\otimes} \mathcal{H}^*_\pi \, d\pi$. We have the unitary G-module structure $L^{0,q}_2(Y;\mathbb{L}_\lambda) = \int_{\widehat{G}_\zeta} \mathcal{H}_\pi \widehat{\otimes} \{\mathcal{H}^*_\pi \otimes \Lambda^q\mathfrak{n}^* \otimes L_\lambda\}^H d\pi$ where L_λ is the representation space of e^λ, where H acts on $\mathcal{H}^*_\pi \otimes \Lambda^q\mathfrak{n}^* \otimes L_\lambda$ by $\pi^* \otimes \mathrm{Ad}^* \otimes e^\lambda$, and where $(\cdot)^H$ denotes the fixed points of H there. Now $\overline{\partial}: A^{0,q}(Y;\mathbb{L}_\lambda) \to A^{0,q+1}(Y;\mathbb{L}_\lambda)$ and its formal adjoint $\overline{\partial}^*$ act by

$\overline{\partial}(f\cdot\omega^J\cdot\ell) = \sum_{1\le k\le n}(y_k(f)\cdot e(\omega^k)\omega^J\cdot\ell) + \frac{1}{2}\sum_{1\le k\le n}(f\cdot e(\omega^k)\,\mathrm{Ad}^*(y_k)\omega^J\cdot\ell)$ and

$\overline{\partial}^*(f\cdot\omega^I\cdot\ell) = -\sum_{1\le k\le n}(\tau(y_k)f\cdot i(\omega^k)\omega^I\cdot\ell) + \frac{1}{2}\sum_{1\le k\le n}(f\cdot\mathrm{Ad}^*(y_k)^* i(\omega^k)\omega^I\cdot\ell)$ where I and J are multi-indices and $\ell \in L_\lambda$. These correspond to the formulae for δ and δ^*. The argument of [**40**, Lemmas 5 and 6] shows that $[\pi] \mapsto \{\mathcal{H}^q(\pi^*)\otimes L_\lambda\}^H$ is a measurable assignment of Hilbert spaces on $\widehat{G}_\zeta$, and that

$$\mathcal{H}^{0,q}_2(Y;\mathbb{L}_\lambda) = \int_{\widehat{G}_\zeta} \mathcal{H}_\pi \otimes \{\mathcal{H}^q(\pi^*)\otimes L_\lambda\}^H d\pi, \text{ unitary } G\text{–module; i.e.,}$$

$$\pi^q_\lambda = \int_{\widehat{G}_\zeta} \dim(\mathcal{H}^q(\pi^*)\otimes L_\lambda)^H \cdot \pi \, d\pi, \text{ which has discrete part} \tag{7.6.4}$$

$${}^0\pi^q_\lambda = \sum\nolimits_{\widehat{G}_{\zeta-disc}} \dim(\mathcal{H}^q(\pi^*)\otimes L_\lambda)^H \cdot \pi.$$

The equation for ${}^0\pi^q_\lambda$ is summation over the discrete set $W_G\backslash\{\nu \in L' : e^{\nu-\rho}|_Z = \zeta\}$.

Write $\Theta^{disc}_{\pi^q_\lambda}$ for the formal sum of the characters of the irreducible (in this case, ζ-discrete) subrepresentations of π^q_λ. Define

$$\begin{aligned} F_\lambda =& \sum\nolimits_{q\geqq 0}(-1)^q \dim(\mathcal{H}^q(\pi^*)\otimes L_\lambda)^H \cdot \Theta_\pi \\ =& \sum\nolimits_{\pi\in\widehat{G}_{\zeta\text{-disc}}}(\sum_{q\geqq 0}(-1)^q \dim(\mathcal{H}^q(\pi^*)\otimes L_\lambda)^H \cdot \Theta_\pi \\ =& \sum\nolimits_{\widehat{G}_{\zeta-disc}}(\text{coefficient of } e^{-\lambda} \text{ in } f_{\pi^*})\,\Theta_\pi \end{aligned}$$

because $\dim(\mathcal{H}^q(\pi^*)\otimes L_\lambda)^H$ is the multiplicity of $e^{-\lambda}$ for H on $\mathcal{H}^q(\pi^*)$. Let $\nu \in L'$ with $e^{\nu-\rho}|_Z = \zeta$, so $[\pi_\nu] \in \widehat{G}_{\zeta-disc}$. By (7.6.1),

$$\Delta\, T_{\pi^*_\nu|_{H\cap G'}} = \Delta\, T_{\pi_{-\nu}}|_{H\cap G'} = (-1)^{q(\nu)} \sum_{w\in W_G} \det(w) e^{-w(\nu)}.$$

Thus Lemma 7.6.4 says $f_{\pi^*_\nu} = (-1)^{n+q(\nu)}\sum_{w\in W_G}\det(w)e^{-w(\nu)}$. In particular the coefficient of $e^{-\lambda}$ in $f_{\pi^*_\nu}$ is 0 if $\lambda+\rho \notin W_G(\nu)$, is $(-1)^{n+q(\nu)}\det(w)$ if $w(\nu) = \lambda+\rho$

for some $w \in W_G$. Now $F_\lambda = (-1)^{n+q(\nu)}\Theta_{\pi_{\lambda+\rho}}$. In view of Lemma 7.5.3, we have just proved the alternating sum formula

$$\sum\nolimits_{q\geqq 0} \Theta^{disc}_{\pi^q_\mu} = (-1)^{|\Sigma^+|+q(\lambda+\rho)}\Theta_{\pi_{\chi,\lambda+\rho}}\,. \tag{7.6.5}$$

This is a key step in the proof of Theorem 7.2.3.

7.7. We complete the proof of Theorem 7.2.3. The crux of the matter is the vanishing statement,

$$\text{if } \lambda+\rho \in L' \text{ then } H_2^{0,q}(Y;\mathbb{E}_\mu) = 0 \text{ for } q \neq q(\lambda+\rho)$$

combined with the alternating sum formula (7.6.5).

The vanishing statement was proved by Griffiths and Schmid [**8**, Theorem 7.8] for the case where G is a connected semisimple Lie group with finite center, Q is a Borel subgroup of $\overline{G}_{\mathbb{C}}$, and $\lambda+\rho$ is "sufficiently" far from the walls of the Weyl chamber that contains it. Then the requirement of "sufficiently" far from the wall was eliminated by Schmid [**41**] using methods not available earlier. Both proofs go through without change in our case.

Now we have $H_2^{0,q}(Y;\mathbb{L}_\lambda) = 0$ for $q \neq q(\lambda+\rho)$. Using the alternating sum formula (7.6.5) and linear independence of the Θ_π for $[\pi] \in \widehat{G}_{\zeta-disc}$, we see that

$$[\pi_{\lambda+\rho}] \text{ is the discrete part } [{}^0\pi_\lambda^{q(\lambda+\rho)}] \text{ of } [\pi_\lambda^{q(\lambda+\rho)}]. \tag{7.7.1}$$

Corollary 5.1.5 applied to $\overline{\zeta}$, with (7.6.1), tells us that

$$\{[\pi] \in \widehat{G}_\zeta \setminus \widehat{G}_{\zeta-disc} \mid T_\pi \neq 0\} \text{ has Plancherel measure zero in } \widehat{G}_\zeta\,.$$

Lemma 7.6.3 says $f_{\pi^*} = 0$ for almost all $[\pi] \in \widehat{G}_\zeta \setminus \widehat{G}_{\zeta-disc}$. If $q \neq q(\lambda+\rho)$ now (7.6.4) and $H_2^{0,q}(Y;\mathbb{E}_\mu) = 0$ force $(\mathcal{H}^q(\pi^*)\otimes L_\lambda)^H = 0$, so $e^{-\lambda}$ has multiplicity 0 in the representation of H on $\mathcal{H}^q(\pi^*)$. If $f_{\pi^*} = 0$ then also $e^{-\lambda}$ has multiplicity 0 in the representation of H on $\mathcal{H}^{q(\lambda+\rho)}(\pi^*)$, so $(\mathcal{H}^{q(\lambda+\rho)}(\pi^*)\otimes L_\lambda)^H = 0$. In summary,

$$(\mathcal{H}^{q(\lambda+\rho)}(\pi^*)\otimes L_\lambda)^H = 0 \text{ for almost all } [\pi] \in \widehat{G}_\zeta \setminus \widehat{G}_{\zeta-disc}\,. \tag{7.7.2}$$

The measure $(\dim(\mathcal{H}^{q(\lambda+\rho)}(\pi^*)\otimes L_\lambda)^H d\pi$ on $\widehat{G}_\zeta$ is concentrated on $\widehat{G}_{\zeta-disc}$ by (7.7.2). Now (7.6.4) says that $[\pi_\lambda^{q(\lambda+\rho)}] = [{}^0\pi_\lambda^{q(\lambda+\rho)}]$, so $[\pi_\lambda^{q(\lambda+\rho)}] = [\pi_{\lambda+\rho}]$ by (7.7.1). That completes the proof of Theorem 7.2.3.

8. Measurable Orbits and Nondegenerate Series

Let G be a reductive Lie group from the class $\widetilde{\mathcal{H}}$ of general real reductive Lie groups defined in §3.1. If $\zeta \in \widehat{Z}$ then Theorem 5.1.1 shows that Plancherel measure on $\widehat{G}_\zeta$ is supported by the constituents of H–series classes that transform by ζ, as H runs over the conjugacy classes of Cartan subgroups of G. Here we work out geometric realizations for all these H–series classes. Our method is a reduction to the special case of the relative discrete series (H/Z compact) that we studied in §7.

Fix a Cartan subgroup $H = T\times A$ in G and an associated cuspidal parabolic subgroup $P = MAN$ of G. We work over measurable orbits $Y = G(x) \subset X = \overline{G}_{\mathbb{C}}/Q$ such that (i) the G–normalizer $N_{[x]}$ of the holomorphic arc $S_{[x]}$ is open in P and (ii) $U = \{m \in M : m(x) = x\}$ is compact modulo Z. In §8.1 we first check that G has isotropy subgroup UAN at x, $U = Z_M(M^0)U^0$, and that $N_{[x]} = M^\dagger AN$. If $[\mu] \in \widehat{U}$ and $\sigma \in \mathfrak{a}^*$ we show that the G–homogeneous complex vector bundle

$$p : \mathbb{E}_{\mu,\sigma} \to G/UAN = Y \text{ associated to } \mu\otimes e^{\rho_{\mathfrak{a}}+i\sigma}$$

is holomorphic over each holomorphic arc component of Y, in an essentially unique manner.

Let K be the fixed point set of a Cartan involution that preserves H. Since μ is unitary we get a K–invariant hermitian metric on $\mathbb{E}_{\mu,\sigma}$. Since U/Z is compact we get a K–invariant assignment of hermitian metrics on the holomorphic arc components of Y. This results in Hilbert spaces $\mathcal{H}_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$ of "square integrable partially harmonic (p,q)–forms" on Y with values in $\mathbb{E}_{\mu,\sigma}$: measurable ω such that (i) $\omega|_{S_{[kx]}}$ is a harmonic (p,q)–form on $S_{[kx]}$ with values in $\mathbb{E}_{\mu,\sigma}|_{S_{[kx]}}$ and L_2 norm $||\omega|_{S_{[kx]}}|| < \infty$ for almost all $k \in K$ and (ii) $\int_{K/Z} ||\omega|_{S_{[kx]}}||^2 d(kZ) < \infty$. We end §8.1 by showing that the natural action of G on $\mathcal{H}_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$ is a unitary representation.

The representation of G on $\mathcal{H}_2^{0,q}(Y;\mathbb{E}_{\mu,\sigma})$ is denoted $\pi^q_{\mu,\sigma}$. Let η^q_μ denote the representation of $M^\dagger$ on $\mathcal{H}_2^{0,q}(S_{[x]};\mathbb{E}_{\mu,\sigma}|_{S_{[x]}})$; we studied these in Section 7. Now we have a representation

$$\eta^q_{\mu,\sigma}(man) = e^{i\sigma}(a)\eta^q_\mu(m) \text{ of } N_{[x]} = M^\dagger AN.$$

In §8.2 we prove $[\pi^q_{\mu,\sigma}] = [\mathrm{Ind}\,^G_{N_{[x]}}(\eta^q_{\mu,\sigma})]$.

Our main result is Theorem 8.3.2. Split $[\mu] = [\chi \otimes \mu^0]$ where $[\chi] \in \widehat{Z_M(M^0)}$ and $[\mu^0] \in \widehat{U^0}$, where $[\mu^0]$ has highest weight ν such that $\nu + \rho_{\mathfrak{t}}$ is $\mathfrak{m}$–regular. Then the H–series constituents of $\pi^q_{\mu,\sigma}$ are just its irreducible subrepresentations. Their sum ${}^H\pi^q_{\mu,\sigma}$ has distribution character $\Theta^H_{\pi^q_{\mu,\sigma}}$. Further

$$\sum\nolimits_{q\geqq 0}(-1)^q\Theta^H_{\pi^q_{\mu,\sigma}} = (-1)^{|\Sigma^+_{\mathfrak{t}}|+q_M(\nu+\rho_{\mathfrak{t}})}\Theta_{\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}}.$$

Also, if $q \neq q_M(\nu+\rho_{\mathfrak{t}})$ then $\mathcal{H}_2^{0,q}(S_{[x]};\mathbb{E}_{\mu,\sigma}) = 0$. This combines with the alternating sum formula and some consequences of the Plancherel Theorem, yielding

$$[\pi^{q_M(\nu+\rho_{\mathfrak{t}})}_{\mu,\sigma}] = [\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}],\ H\text{–series class.}$$

The proof is a matter of applying the results from Section 7 to every holomorphic arc component of Y and combining those results by means of the induced representation theorem of §8.2.

8.1. G is a general real reductive Lie group from our class $\widetilde{\mathcal{H}}$ defined in §3.1. As noted at the end of §6.2, $\overline{G} = G/Z_G(G^0)$ is a linear semisimple group with complexification $\overline{G}_{\mathbb{C}} = \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$, and G acts on the complex flag manifolds of $\overline{G}_{\mathbb{C}}$.

For the remainder of Section 8 we fix a Cartan subgroup $H = T \times A$ of G and an associated cuspidal parabolic subgroups $P = MAN$. In order to realize the H–series of G we work with a complex flag manifold $X = \overline{G}_{\mathbb{C}}/Q$ and a measurable G–orbit $Y = G(x) \subset X$ such that the G–normalizers of the holomorphic arc components of Y in X have the property

$$N_{[x]} = \{g \in G : gS_{[x]} = S_{[x]} \text{ has Lie algebra } \mathfrak{p}. \tag{8.1.1}$$

Since the orbit $Y = G(x)$ is measurable, it is partially complex and of flag type. Thus $S_{[x]}$ is an open M^0–orbit on the smaller flag manifold $\overline{M}_{\mathbb{C}}(x)$ where $\overline{M} = M/Z_G(G^0)$ and AN acts trivially on $S_{[x]}$. The isotropy group of G at x is UAN where $T \subset U \subset M$. We require that

$$U/Z_G(G^0) = \{m \in M : m(x) = x\}/Z_G(G^0) \text{ is compact.} \tag{8.1.2}$$

The G–orbits discussed studied in Theorem 6.5.5 form the special case in which the orbit is integrable We obtain a number of examples of that class from the construction in the paragraph after the proof of Theorem 6.5.5.

LEMMA 8.1.3. *Suppose that $G(x) \subset X$ is a measurable orbit, that $N_{[x]} = \{g \in G : gS_{[x]} = S_{[x]}$ has Lie algebra $\mathfrak{p}$, and that the isotropy group of G at x is UAN with $U/Z_G(G^0)$ compact. Then the open orbit $M(x) \subset \overline{M}_{\mathbb{C}}(x)$ is measurable and integrable. Further $U = Z_M(M^0)U^0$, $U \cap M^0 = U^0$, $UM^0 = M^\dagger$, and $M/M^\dagger$ generates the topological components of $M(x)$. Finally, $N_{[x]} = M^\dagger AN$, and $G/M^\dagger G^0$ enumerates the topological components of $Y = G(x)$.*

PROOF. The open orbit $M(x) \subset \overline{M}_{\mathbb{C}}(x) = \overline{M}_{\mathbb{C}}/(Q \cap \overline{M}_{\mathbb{C}})$ satisfies (7.1.1). Applying Lemma 7.1.2 to it, we get the first two assertions. For the third, $N_{[x]} = UN^0_{[x]} = UM^0AN = M^\dagger AN$, and the G–normalizer of $G^0(x)$ is $UG^0 = UM^0G^0 = M^\dagger G^0$. □

REMARK 8.1.4. $G^\dagger \subset M^\dagger G^0$ in general, but one can have $G^\dagger \neq M^\dagger G^0$. For example let $G = SL(2,R) \cup \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) SL(2,R)$ and $\mathfrak{h} = \{\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) : a \text{ real }\}$. Then $M^\dagger = M = \{\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\pm\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$ so $M^\dagger G^0 = G \neq G^\dagger = G^0$.

Fix $[\mu] \in \widehat{U}$ and $\sigma \in \mathfrak{a}^*$, so $[\mu \otimes e^{i\sigma}] \in \widehat{U \times A}$. As usual, $\Sigma^+_{\mathfrak{a}}$ is the positive $\mathfrak{a}$–root system on $\mathfrak{g}$ such that $\mathfrak{n}$ is the sum of the negative $\mathfrak{a}$–root spaces, and $\rho_{\mathfrak{a}} = \frac{1}{2}\sum_{\phi \in Sigma^+_{\mathfrak{a}}}(\dim \mathfrak{g}^\phi)\phi$, so $\mathfrak{a}$ acts on $\mathfrak{n}$ with trace $-2\rho_{\mathfrak{a}}$. Now UAN acts on the representation space V_μ of μ by

$$\gamma_{\mu,\sigma}(uan) = e^{\rho_a + i\sigma}(a)\mu(u).$$

That specifies the associated G–homogeneous complex vector bundle

$$p : \mathbb{E}_{\mu,\sigma} \to G/UAN = G(x) = Y. \tag{8.1.5}$$

LEMMA 8.1.6. *There is a unique assignment of complex structures to the parts $p^{-1}S_{[gx]}$ of $\mathbb{E}_{\mu,\sigma}$ over the holomorphic arc components of Y, such that each restriction $\mathbb{E}_{\mu,\sigma}|_{S_{[gx]}} \to S_{[gx]}$ is an $N_{[gx]}$–homogeneous holomorphic vector bundle. The assignment is a G–equivariant real analytic tangent space distribution on $\mathbb{E}_{\mu,\sigma}$.*

PROOF. Lemma 7.1.4 says that $p^{-1}S_{[gx]}$ has a unique complex structure for which $\mathbb{E}_{\mu,\sigma}|_{S_{[gx]}} \to S_{[gx]}$ is an $\mathrm{Ad}(g)M^\dagger$–homogeneous holomorphic vector bundle. Each $\mathrm{Ad}(g)(an)$ is trivial on $S_{[gx]} = gS_{[x]}$ and multiplies all fibers of $\mathbb{E}_{\mu,\sigma}|_{S_{[gx]}}$ by the same scalar $e^{\rho_{\mathfrak{a}} + i\sigma}(a)$. Now the complex structure on $p^{-1}S_{[gx]}$ is invariant by the action of $\mathrm{Ad}(g)N_{[x]} = N_{[gx]}$, so $\mathbb{E}_{\mu,\sigma}|_{S_{[gx]}} \to S_{[gx]}$ is an $N_{[gx]}$–homogeneous holomorphic vector bundle. Finally, the assignment of complex structures to the $p^{-1}S_{[gx]}$ is G–invariant by uniqueness, thus also real analytic. □

If $z \in Y = G(x)$ we have the holomorphic tangent space T_z to $S_{[z]}$ at z. Evidently $\{T_z\}_{z \in Y}$ is a G–invariant complex tangent space distribution on Y, so it is real analytic. Thus $\mathbb{T} := \bigcup_{z \in Y} T_z$ is a G–homogeneous real analytic sub–bundle of the complexified tangent bundle of Y. Given non-negative integers p and q, the space of *partially smooth (p,q)–forms* on Y with values in $\mathbb{E}_{\mu,\sigma}$ is

$$\begin{aligned} A^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) : &\text{ measurable sections } \alpha \text{ of } \mathbb{E}_{\mu,\sigma} \otimes \Lambda^p\mathbb{T}^* \otimes \Lambda^q\overline{\mathbb{T}^*} \\ &\text{where } \alpha \text{ is } C^\infty \text{ on each holomorphic arc component of } Y. \end{aligned} \tag{8.1.7}$$

If $\alpha \in A^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$ and $z \in Y$ then $\alpha|_{S_{[z]}}$ is a smooth (p,q)–form on $S_{[z]}$ with values in $\mathbb{E}_{\mu,\sigma}|_{S_{[z]}}$, in the ordinary sense. The $\overline{\partial}$ operator of X specifies operators $\overline{\partial} : A^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) \to A^{p,q+1}(Y;\mathbb{E}_{\mu,\sigma})$.

We need hermitian metrics for the harmonic theory. Let θ be a Cartan involution of G with $\theta(H) = H$ and denote $K = \{g \in G \mid \theta(g) = g\}$ as usual. Then $K \cap N_{[x]} = K \cap M^\dagger$ can be assumed to contain U, and we have an $M^\dagger$–invariant hermitian metric on the complex manifold $S_{[x]}$. Every holomorphic arc component of $G(x)$ is an $S_{[kx]}$, $k \in K$. Give $S_{[kx]}$ the hermitian metric such that the $k : S_{[x]} \to S_{[kx]}$ are hermitian isometries. In other words, we have a K–invariant hermitian metric on the fibers of the bundle $\mathbb{T} \to Y$. Similarly the unitary structure of E_μ specifies an $M^\dagger$–invariant hermitian metric on the fibers of $\mathbb{E}_{\mu,\sigma} \to Y$. Now we have K–invariant hermitian metrics on the fibers of the bundles $\mathbb{E}_{\mu,\sigma} \otimes \Lambda^p\mathbb{T}^* \otimes \Lambda^q\overline{\mathbb{T}^*} \to Y$. As in (7.1.5), that specifies Hodge–Kodaira operators

$$A^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) \xrightarrow{\#} A^{n-p,n-q}(Y;\mathbb{E}^*_{\mu,\sigma}) \xrightarrow{\widetilde{\#}} A^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) \tag{8.1.8}$$

where $n = \dim_{\mathbb{C}} S_{[x]}$. It also specifies a pre Hilbert space

$$A_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) = \left\{ \alpha \in A^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) \,\middle|\, \int_{K/Z} \left(\int_{S_{[kx]}} \alpha\bar{\wedge}\#\alpha \right) d(kZ) < \infty \right\} \tag{8.1.9}$$

whose inner product is $\langle \alpha, \beta \rangle = \int_{K/Z} \left(\int_{S_{[kx]}} \alpha\bar{\wedge}\#\beta \right) d(kZ)$.

We define *square integrable partially-(p,q)-form* on Y with values in $\mathbb{E}_{\mu,\sigma}$ to mean an element of

$$L_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) : \text{ Hilbert space completion of } A_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma}). \tag{8.1.10}$$

$\overline{\partial}$ is densely defined on $L_2^{p,q}(\mathbb{E}_{\mu,\sigma})$ with formal adjoint $\overline{\partial}^* = -\widetilde{\#}\overline{\partial}\#$; this follows from the corresponding standard fact (7.1.8) over each holomorphic arc component. The analogue of the Hodge–Kodaira–Laplacian is

$$\Box = (\overline{\partial} + \overline{\partial}^*)^2 = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}, \tag{8.1.11}$$

which is elliptic and essentially self adjoint over every holomorphic arc component. Now $\Box$ is essentially self adjoint on $L_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$ from the domain consisting of C^∞ forms with support compact modulo Z. We write $\Box$ for the closure, which is the unique self–adjoint extension on $L_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$. The kernel

$$H_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) = \{\omega \in L_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) \mid \Box\omega = 0\} \tag{8.1.12}$$

is the space of *square integrable partially harmonic (p,q)–forms* on Y with values in $\mathbb{E}_{\mu,\sigma}$. $H_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$ is the subspace of $A_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$ consisting of all elements ω such that $\omega|_{S_{[kx]}}$ is harmonic a.e. in K/Z. It is a closed subspace of $L_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$ and there is an orthogonal direct sum decomposition

$$L_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) = c\ell\,\overline{\partial}L_2^{p,q-1}(Y;\mathbb{E}_{\mu,\sigma}) \oplus \overline{\partial}^*L_2^{p,q+1}(Y;\mathbb{E}_{\mu,\sigma}) \oplus H_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) \tag{8.1.13}$$

obtained by applying (7.1.10) to each holomorphic arc component.

LEMMA 8.1.14. *The action $[\widetilde{\pi}^{p,q}_{\mu,\sigma}\alpha](z) = g(\alpha(g^{-1}z))$ of G on $L_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$ is a unitary representation.*

PROOF. $\mathbb{E}_{\mu,\sigma} \otimes \Lambda^p \mathbb{T}^* \otimes \Lambda^q \overline{\mathbb{T}}^*$ has fiber $E_\mu^{p,q} := E_\mu \otimes \Lambda^p T_x^* \otimes \Lambda^q \overline{T}_x^*$ over x. If $\mu^{p,q}$ denotes the representation of U on $E_\mu^{p,q}$ then UAN acts on $E_\mu^{p,q}$ by

$$\gamma_{\mu,\sigma}^{p,q}(uan) = e^{\rho_{\mathfrak{a}}+i\sigma}(a)\mu^{p,q}(u) = e^{\rho_{\mathfrak{a}}}(a) \cdot {}'\gamma_{\mu,\sigma}^{p,q}(uan)$$

where ${}'\gamma_{\mu,\sigma}^{p,q} = \mu^{p,q} \otimes e^{i\sigma}$ is unitary. Since $e^{\rho_{\mathfrak{a}}}(a)$ is the square root of the determinant of uan on the real tangent space $\mathfrak{g}/(\mathfrak{u}+\mathfrak{a}+\mathfrak{n})$ to Y at x, now $\widetilde{\pi}_{\mu,\sigma}^{p,q}$ is the unitarily induced representation $\operatorname{Ind}_{UAN}^G({}'\gamma_{\mu,\sigma}^{p,q})$. □

The representation $\widetilde{\pi}_{\mu,\sigma}^{p,q}$ commutes with $\overline{\partial}$, hence also with $\overline{\partial}^*$, so $H_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$ is a closed G–invariant subspace of $L_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$. Thus we have

(8.1.15) $\pi_{\mu,\sigma}^{p,q}$: unitary representation of G on $H_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$ and $\pi_{\mu,\sigma}^q = \pi_{\mu,\sigma}^{0,q}$

The program of Section 8 is to represent the various H–series of unitary representation classes of G by the various $\pi_{\mu,\sigma}^q$.

8.2. We set up $\pi_{\mu,\sigma}^q$ as an induced representation from $N_{[x]} = M^\dagger AN$. Write $\mathbb{E}_\mu = \mathbb{E}_{\mu,\sigma}|_{S_{[x]}} \to S_{[x]}$. It is the $M^\dagger$–homogeneous hermitian holomorphic vector bundle defined by $[\mu] \in \widehat{U}$ as in Lemma 7.1.4. That gives us the unitary representations η_μ^q of $M^\dagger$ on $H_2^{0,q}(S_{[x]};\mathbb{E}_\mu)$. The formula $\eta_{\mu,\sigma}^q(man) = e^{i\sigma}(a)\eta_\mu^q(m)$ defines a unitary representation of $N_{[x]} = M^\dagger AN$ on $H_2^{0,q}(S_{[x]};\mathbb{E}_\mu)$.

THEOREM 8.2.1. $[\pi_{\mu,\sigma}^q] = [\operatorname{Ind}_{N_{[x]}}^G(\eta_{\mu,\sigma}^q)]$.

PROOF. Let $\widetilde{\pi} = \widetilde{\pi}_{\mu,\sigma}^{0,q}$, the representation of G on $L_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$. Let ${}'\gamma$ denote the representation of UAN on $E_\mu \otimes \Lambda^q(\overline{T}_x^*)$; it is the ${}'\gamma_{\mu,\sigma}^{0,q} \otimes e^{i\sigma}$ of the proof of Lemma 8.1.14. That lemma was proved (if $p=0$) by showing $[\widetilde{\pi}] = [\operatorname{Ind}_{UAN}^G({}'\gamma)]$.

Let $\widetilde{\eta}$ denote the representation of $M^\dagger$ on $L_2^{0,q}(S_{[x]};\mathbb{E}_\mu)$, and ${}'\eta$ the representation of $M^\dagger AN$ by ${}'\eta(man) = e^{i\sigma}(a)\widetilde{\eta}(m)$. Then $[\widetilde{\eta}] = [\operatorname{Ind}_U^{M^\dagger}(\mu^{0,q})]$, and so $[{}'\eta] = [\operatorname{Ind}_{UAN}^{M^\dagger AN}(\gamma)]$.

Induction by stages now says that $\widetilde{\pi}$ is unitarily equivalent to $\operatorname{Ind}_{M^\dagger AN}^G({}'\eta)$. We need the equivalence. Let f be in the representation space of $\operatorname{Ind}_{M^\dagger AN}^G({}'\eta)$. In other words $f : G \to L_2^{0,q}(S_{[x]};\mathbb{E}_\mu)$ is Borel measurable, f transforms by $f(gman) = e^{-\rho_{\mathfrak{a}}}(a) \cdot {}'\eta(man)^{-1}f(g)$ for $g \in G$ and $man \in M^\dagger AN$, and we have global norms $\int_{K/Z} ||f(k)||^2 d(kZ) < \infty$. For almost all $g \in G$ we may view $f(g) \in L_2^{0,q}(S_{[x]};\mathbb{E}_\mu)$ as a Borel–measurable function $M^\dagger AN \to E_\mu^q = E_\mu \otimes \Lambda^q(\overline{T}_x^*)$ such that

$$f(g)(puan) = {}'\gamma(uan)^{-1}[f(g)(p)] \text{ for } p \in M^\dagger AN\,, uan \in UAN$$
$$\text{and } \int_{M^\dagger/U} ||f(g)(m)||^2 d(mU) < \infty.$$

Now define

(8.2.2) $$F = \Gamma(f) : G \to E_\mu^q = E_\mu \otimes \Lambda^q(\overline{T}_x^*) \text{ by } F(g) = f(g)(1).$$

Then F is Borel measurable. Use ${}'\eta = \operatorname{Ind}_{UAN}^{M^\dagger AN}({}'\gamma)$ to compute

$$\begin{aligned} F(guan) &= f(guan)(1) = [e^{-\rho_{\mathfrak{a}}}(a) \cdot {}'\eta(uan)^{-1}f(g)](1) \\ &= e^{-\rho_{\mathfrak{a}}}(a) \cdot {}'\gamma(uan)^{-1}[f(g)(1)] = e^{-\rho_{\mathfrak{a}}}(a) \cdot {}'\gamma(uan)^{-1}F(g) \end{aligned}$$

and

$$\int_{K/Z}\Big\{\int_{M^\dagger/U}||F(km)||^2\,d(mU)\Big\}d(kZ)=\int_{K/Z}\Big\{\int_{M^\dagger/U}||f(k)(m)||^2\,d(mU)\Big\}d(kZ)<\infty.$$

Thus $f\mapsto\Gamma(f)=F$ is the desired equivalence $\widetilde{\pi}\simeq\operatorname{Ind}_{M^\dagger AN}^{G}('\eta)$.

In the construction just above, f is in the representation space of $\operatorname{Ind}_{N[x]}^{g}(\eta^q_{\mu,\sigma})$ precisely when almost every $f(g)$ is annihilated by the Hodge–Kodaira–Laplace operator of $\mathbb{E}_\mu\to S_{[x]}$. That is equivalent to $\Box\,\Gamma(f)=0$. Thus the equivalence Γ of (8.2.2) restricts to the equivalence asserted in Theorem 8.2.1. $\Box$

8.3. We now come to the geometric realization of the various H–series of unitary representations of G. Note that these are the standard induced representations. They are unitary, in fact tempered.

We construct a particular positive $\mathfrak{h}_{\mathbb{C}}$–root system Σ^+. The choice $P=MAN$ us a choice $\Sigma^+_{\mathfrak{a}}$ of positive $\mathfrak{a}$–root system on $\mathfrak{g}$. Let $\overline{M}=M/Z_G(G^0)$, and choose a simple $\mathfrak{t}_{\mathbb{C}}$–root system $\Pi_{\mathfrak{t}}$ on $\mathfrak{m}_{\mathbb{C}}$ such that the parabolic subalgebra $\mathfrak{q}\cap\overline{\mathfrak{m}}_{\mathbb{C}}$ of $\overline{\mathfrak{m}}_{\mathbb{C}}$ is specified by $\Pi_{\mathfrak{t}}$ and a subset $\Phi_{\mathfrak{t}}$. Σ^+ will be the positive $\mathfrak{h}_{\mathbb{C}}$–root system on $\mathfrak{g}_{\mathbb{C}}$ determined by $\Sigma^+_{\mathfrak{a}}$ and $\Sigma^+_{\mathfrak{t}}$, and Π is the simple root system for Σ^+. From Proposition 6.5.5 the measurable orbit $G(x)$ is integrable exactly when $\mathfrak{q}=\mathfrak{q}_\Phi$ with $\Phi=\Phi_{\mathfrak{t}}\cup(\Pi\setminus\Pi_{\mathfrak{t}})$. As we had done before for $\mathfrak{g}$ let

$$\rho_{\mathfrak{t}}=\tfrac12\sum_{\phi\in\Sigma^+_{\mathfrak{t}}}\phi\,,\Delta_{M,T}=\prod_{\mathfrak{t}^+}(e^{\phi/2}-e^{-\phi/2})\,,\varpi_{\mathfrak{t}}(\nu)=\prod_{\mathfrak{t}^+}\langle\nu,\phi\rangle.$$

Replacing G be a $\mathbb{Z}_2$ extension if necessary, Lemma 4.4.3 ensures that $e^{\rho_{\mathfrak{t}}}$ and $\Delta_{M,T}$ are well defined on T. As we did before for (G,H) denote

(8.3.1) $L_{\mathfrak{t}}=\{\nu\in i\mathfrak{t}^*\mid e^\nu$ is well defined on $T^0\}$ and $L''_{\mathfrak{t}}=\{\nu\in L_{\mathfrak{t}}\mid\varpi_{\mathfrak{t}}(\nu)\neq0\}$.

Then $\rho_{\mathfrak{t}}\in L''_{\mathfrak{t}}$. If $\nu+\rho_{\mathfrak{t}}\in L''_{\mathfrak{t}}$ then $\Sigma^+_{\mathfrak{t}}$ specifies $q_M(\nu+\rho_{\mathfrak{t}})\in\mathbb{Z}$ as in (7.2.2).

Since $U=Z_M(M^0)U^0$ and $U\cap M^0=U^0$, $\widehat{U}$ consists of all $[\chi\otimes\mu^0]$ with $\chi\in\widehat{Z_M(M^0)}$ consistent with $[\mu^0]\in\widehat{U^0}$.

We review some aspects of our basic setup and then come to our main result.

G is a general real reductive Lie group as defined in §3.1, Q is a parabolic subgroup of $\overline{G}_{\mathbb{C}}$, and $Y=G(x)\subset X=\overline{G}_{\mathbb{C}}$ is a measurable integrable orbit partially complex orbit of flag type as described in §6.5. $H=T\times A\in\operatorname{Car}(G)$, $P=MAN$ is an associated cuspidal parabolic subgroup of G, and we suppose that $U=\{m\in M\mid m(x)=x\}$ is compact modulo $Z_G(G^0)$. $S_{[x]}$ is the holomorphic arc component of Y through x and its G–normalizer is $N_{[x]}=M^\dagger AN$ denotes the G–normalizer of the holomorphic arc component $S_{[x]}$.

Recall that $\pi^q_{\mu,\sigma}$ denotes the unitary representation of G on $H^{0,q}_2(Y;\mathbb{E}_{\mu,\sigma})$ and that ${}^H\pi^q_{\mu,\sigma}$ denotes the sum of its irreducible subrepresentations.

THEOREM 8.3.2. *Let $[\mu]\in\widehat{U}$, say $[\mu]=[\chi\otimes\mu^0]$ as above. Let ν be the highest weight of μ^0 in the $\mathfrak{t}_{\mathbb{C}}$–root system $\Sigma^+_{\mathfrak{t}}\cap\Phi^r_{\mathfrak{t}}$ of $\mathfrak{u}_{\mathbb{C}}$. Then $\nu\in L_{\mathfrak{t}}$ and $\mu\in\widehat{U}_\zeta$ where $\zeta\in\widehat{Z}$ agrees with e^ν on $Z\cap M^0$ and where $[\chi]\in\widehat{Z_M(M^0)}_\zeta$. Let $\sigma\in\mathfrak{a}^*$ and suppose that $\nu+\rho_{\mathfrak{t}}\in L''_{\mathfrak{t}}$.*

1. *If $q\neq q(\nu+\rho_{\mathfrak{t}})$ then $H^{0,q}_2(Y;E_{\mu,\sigma})=0$, so $\pi^q_{\mu,\sigma}$ does not occur*

2. *If $q=q(\nu+\rho_{\mathfrak{t}})$ then G acts on $H^{0,q}_2(Y;E_{\mu,\sigma})$ by the H–series representation $\pi_{\chi,\nu+\rho_{\mathfrak{t}},\sigma}$. Every H–series representation is obtained in this way.*

PROOF. Theorem 7.2.3 says that the representation η_μ^q of $M^\dagger$ on $H_2^{0,q}(S_{[x]};\mathbb{E}_\mu)$ is trivial if $q \neq q_M(\nu+\rho_{\mathfrak{k}})$, and if $q = q_M(\nu+\rho_{\mathfrak{k}})$ it is equivalent to the relative discrete series representation $\eta_{\chi,\nu+\rho_{\mathfrak{k}}}$ of $M^\dagger$. Then the representation of M on $H_2^{0,q}(S_{[x]};\mathbb{E}_\mu)$ is the relative discrete series representation (which we temporarily denote $\eta'_{\chi,\nu+\rho_{\mathfrak{k}}}$) $\operatorname{Ind}_{M^\dagger}^{M}(\eta_{\chi,\nu+\rho_{\mathfrak{k}}})$ of M. Finally, if $\sigma \in \mathfrak{a}^*$ then G acts on $H_2^{0,q}(Y;E_{\mu,\sigma})$ by the H–series representation $\operatorname{Ind}_{MAN}^{G}\big(\operatorname{Ind}_{M^\dagger AN}^{MAN}(\eta_{\chi,\nu+\rho_{\mathfrak{k}}}\otimes e^{i\sigma})\big) = \pi_{\chi,\nu+\rho_{\mathfrak{k}},\sigma}$ by Theorem 8.2.1. □

8.4. Theorem 8.3.2 gives explicit geometric realizations for the standard tempered representations, i.e., for the various H–series classes of unitary representations of G. Theorem 7.2.3 is the special case of the relative discrete series. In view of the Plancherel Theorem 5.1.1 we now have, for every $\zeta \in \widehat{Z}$, explicit geometric realizations for a subset of $\widehat{G}_\zeta$ that supports Plancherel measure there.

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