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Semisimple weakly symmetric pseudo-Riemannian manifolds

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Abstract

We develop the classification of weakly symmetric pseudo-Riemannian manifolds G/H where G is a semisimple Lie group and H is a reductive subgroup. We derive the classification from the cases where G is compact, and then we discuss the (isotropy) representation of H on the tangent space of G/H and the signature of the invariant pseudo-Riemannian metric. As a consequence we obtain the classification of semisimple weakly symmetric manifolds of Lorentz signature $(n-1, 1)$ and trans-Lorentzian signature $(n-2, 2)$.

Keywords Pseudo-Riemannian manifold · Weakly symmetric space · Real form family · Lorentz manifold · Trans-Lorentz manifold

Mathematics Subject Classification Primary 53C30 · 53C35 · 22E15 · Secondary 53C50 · 22E46

1 Introduction

There have been a number of important extensions of the theory of Riemannian symmetric spaces. Weakly symmetric spaces, introduced by Selberg [9], play key roles in number theory, Riemannian geometry and harmonic analysis. See [12]. pseudo-Riemannian symmetric spaces, including semisimple symmetric spaces, play central but complementary roles in number theory, differential geometry and relativity, Lie group representation theory and harmonic analysis. Much of the activity there has been on the Lorentz cases, which are of particular interest in physics. Here we work out the classification of weakly sym-

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metric pseudo-Riemannian manifolds G/H where G is a semisimple Lie group and H is a reductive subgroup. We do this in a way that allows us to derive the signatures of all invariant pseudo-Riemannian metrics. (All such metrics are necessarily weakly symmetric.) In particular we obtain explicit listings for invariant pseudo-Riemannian metrics of Riemannian (Table 3), Lorentzian (Table 4) and trans-Lorentzian (Table 5) signature. All our listings, Tables 1 through 5, are collected in Sect. 6 at the end of the paper.

This treatment of weakly symmetric pseudo-Riemannian manifolds is a major extension of the classical paper of Berger [1]. Even in the Riemannian case it adds new information: the signatures of invariant metrics that may be non-Riemannian. The Lorentzian case is of course of physical interest. And the trans-Lorentzian case is related to conformal and other parabolic structures as described in [3], with structural information from [11, Theorem 8.13.3].

Our analysis in the weakly symmetric setting uses the classifications of Krämer [7], Brion [2], Mikityuk [8] and Yakimova [15–18] for the weakly symmetric Riemannian manifolds. We pass from these weakly symmetric Riemannian cases to our weakly symmetric pseudo-Riemannian classification by a combination of semisimple Lie group methods and ideas extending those of Wolf and Gray [13,14]. The construction here is more group-theoretic than that of [4].

To start, we show how a weakly symmetric pseudo-Riemannian manifold (M, ds^2), $M = G/H$ with G semisimple and H reductive in G, belongs to a family of such spaces associated to a compact weakly symmetric Riemannian manifold $M_u = G_u/H_u$. There G_u and H_u are compact real forms of the complex Lie groups $G_{\mathbb{C}}$ and $H_{\mathbb{C}}$. More generally, whenever G_u is a compact connected semisimple Lie group and H_u is a closed connected subgroup, we have the complexification $(G_u)_{\mathbb{C}}/(H_u)_{\mathbb{C}}$ of G_u/H_u.

Definition 1.1 The **real form family** of G_u/H_u consists of $(G_u)_{\mathbb{C}}/(H_u)_{\mathbb{C}}$ and all G_0/H_0 with the same complexification $(G_u)_{\mathbb{C}}/(H_u)_{\mathbb{C}}$. $\diamondsuit$

If G_0/H_0 is in the real form family of G_u/H_u, we have a Cartan involution θ of G_0 that preserves H_0 and (G_u, H_u) is the corresponding compact real form of (G_0, H_0). But the point here is that this is reversible:

Lemma 1.2 *Let G_u be a compact connected semisimple Lie group and H_u a closed connected subgroup. Let σ be an involutive automorphism of G_u that preserves H_u. Then there is a unique G_0/H_0 in the real form family of G_u/H_u such that G_0 is simply connected, H_0 is connected, and $\sigma = \theta|_{G_u}$ where θ is the holomorphic extension to $(G_u)_{\mathbb{C}}$ of a Cartan involution of G_0 that preserves H_0. Up to covering, every space G_0/H_0 in the real form family of G_u/H_u is obtained in this way.*

In Sect. 2 we recall Krämer's classification [7] of the spaces $M_u = G_u/H_u$ for the cases where M_u is not symmetric but is weakly symmetric with G_u simple. See (2.1). Note that in all but two cases there is an "intermediate" subgroup K_u, where $H_u \subsetneqq K_u \subsetneqq G_u$ with both G_u/K_u and K_u/H_u symmetric. In the cases where an intermediate group K_u is present we work out the real form families in steps, from H_u to K_u to G_u, using commuting involution methods of Cartan, Berger, and Wolf and Grey. When no intermediate group K_u is available we manage the calculation with some basic information on G_2, $Spin(7)$ and $Spin(8)$.

In Sect. 3 we calculate the H-irreducible subspaces of the real tangent space of spaces $M = G/H$ found in Sect. 2, and in each of the twelve cases there we work out the possible signatures of the G-invariant pseudo-Riemannian metrics. The results are gathered in Table 1.

In Sect. 4 we recall the Brion–Mikityuk classification [2,8] as formulated by Yakimova [17,18]. See (4.1) below. The exposition is taken from [12]. Those are the cases where M_u is weakly symmetric and irreducible, G_u is semisimple but not simple, and G_u/H_u is principal. In this context, G_u semisimple, "principal" just means that the center $Z_{H_{\mathbb{C}}}$ of $H_{\mathbb{C}}$ is the product of its intersection with the complexifications of the centers of the simple factors of $G_{\mathbb{C}}$. For the first eight of the nine cases of (4.1) we work out the resulting spaces $M = G/H$ of the real form family, the H-irreducible subspaces of the real tangent space, and the resulting contributions to the signatures of the G-invariant pseudo-Riemannian metrics. The results are gathered in Table 2. The ninth case of (4.1) is a pattern rather than a formula; there we obtain the signature information by applying our notion of "Riemannian unfolding" to the information contained in Tables 1 and 2.

Finally, in Sect. 5 we extract some signature information from Berger's [1, §50, Table II on page 157], and combine it with certain cases from our Tables 1 and 2, to classify the semisimple pseudo-Riemannian weakly symmetric spaces of Riemannian signature $(n, 0)$, Lorentzian signature $(n-1, 1)$ and trans-Lorentzian signature $(n-2, 2)$. It is interesting to note the prevalence of Riemannian signature here. The examples of signature $(n-2, 2)$ are also quite interesting: they are related to conformal and other parabolic geometries ([3]). This data is collected in Tables 3, 4 and 5 found in Sect. 6 at the end of the paper.

Some of the methods here extend classifications of Wolf and Gray [13,14], concerning the isotropy representation of H_0 on $\mathfrak{g}_0/\mathfrak{h}_0$ where $\mathfrak{h}_0$ is the fixed point set of a semisimple automorphism of a semisimple algebra $\mathfrak{g}_0$. Those papers, however, are only peripherally concerned with signatures of invariant metrics. There also is a small overlap with the papers [5,6] of Knop, Krötz, Pecher and Schlichtkrull on reductive real spherical pairs, which are oriented toward algebraic geometry and not concerned with signatures of invariant metrics; we learned of those papers when most of this paper was completed.

2 Real form families for G_u simple

For the cases where M_u is a Riemannian symmetric space we have the classification of Élie Cartan and its extension by Berger [1], which we need not repeat here.

For the cases where M_u is not symmetric but is weakly symmetric with G_u simple, the Krämer classification is given by

Weakly Symmetric Coset Spaces of a Compact Connected Simple Lie Group				
$M_u = G_u/H_u$ weakly symmetric				G_u/K_u symmetric
	G_u	H_u	conditions	K_u with $H_u \subset K_u \subset G_u$
Riemannian symmetric spaces with symmetry s				$(H_u = K_u)$
circle bundles over hermitian symmetric spaces dual to a non-tube domain:				
(1)	$SU(m+n)$	$SU(m) \times SU(n)$	$m > n \geqq 1$	$S[U(m) \times U(n)]$
(2)	$SO(2n)$	$SU(n)$	n odd, $n \geqq 5$	$U(n)$
(3)	E_6	$Spin(10)$		$Spin(10) \times Spin(2)$
(4)	$SU(2n+1)$	$Sp(n)$	$n \geqq 2$	$U(2n) = S[U(2n) \times U(1)]$
(5)	$SU(2n+1)$	$Sp(n) \times U(1)$	$n \geqq 2$	$U(2n) = S[U(2n) \times U(1)]$
constant positive curvature spheres:				
(6)	$Spin(7)$	G_2		(there is none)
(7)	G_2	$SU(3)$		(there is none)
weakly symmetric spaces of Cayley type:				
(8)	$SO(10)$	$Spin(7) \times SO(2)$		$SO(8) \times SO(2)$
(9)	$SO(9)$	$Spin(7)$		$SO(8)$
(10)	$Spin(8)$	G_2		$Spin(7)$
(11)	$SO(2n+1)$	$U(n)$	$n \geqq 2$	$SO(2n)$
(12)	$Sp(n)$	$Sp(n-1) \times U(1)$	$n \geqq 3$	$Sp(n-1) \times Sp(1)$

(2.1)

In order to deal with entries other than (6) and (7) we rely on

Lemma 2.2 *Let $M_u = G_u/H_u$ be one of the entries in* (2.1) *excluding entries* (6) *and* (7)*, so we have the corresponding symmetric space G_u/K_u where $H_u \subset K_u \subset G_u$. Let σ be an automorphism of $\mathfrak{h}_u$ that extends to $\mathfrak{g}_u$. Then $\sigma(\mathfrak{k}_u) = \mathfrak{k}_u$. Further, in the Riemannian metric on M_u defined by the negative of the Killing form of $\mathfrak{g}_u$, K_u/H_u is a totally geodesic submanifold of M_u and itself is a Riemannian symmetric space.*

Proof For entries (1), (2) and (3) of (2.1), $\mathfrak{k}_u = \mathfrak{h}_u + \mathfrak{z}_{\mathfrak{g}_u}(\mathfrak{h}_u)$, so it is preserved by σ. For the other entries (4), (5), (8), (9), (10), (11) and (12), with $\mathfrak{g}_u$ acting as usual on a real vector space V, we proceed as follows: $\dim V = 4n+2, 4+2, 10, 9, 8, 2n+1$ or $4n$, respectively, for entries (4), (5), (8), (9), (10), (11) and (12). Let W be the subspace of V on which $[\mathfrak{h}_u, \mathfrak{h}_u]$ acts trivially. The action of H_u on $W^\perp$ is $(\mathbb{R}^2, \{1\})$, $(\mathbb{R}^2.U(1))$, $(\mathbb{R}^2, SO(2))$, $(\mathbb{R}, \{1\})$, $(\mathbb{R}, \{1\})$, $(\mathbb{R}, \{1\})$ or $(\mathbb{R}^4, T)$, respectively, where T is a circle subgroup of $Sp(1)$. $W^\perp$ is H_u-invariant and K_u is its G_u-stabilizer. Thus $\sigma(\mathfrak{k}_u) = \mathfrak{k}_u$.

For the last statement note that K_u/H_u is a circle S^1 for entries (1), (2) and (3); $S^1 \times SU(2n)/Sp(n)$ for entry (4); $SU(2n)/Sp(n)$ for entry (5); the sphere S^7 for entries (8), (9) and (10); $SO(2n)/U(n)$ for entry (11); and the sphere S^2 for entry (12). □

We'll run through the cases of (2.1). When there is an "intermediate" group K_u, we make use of Berger's work [1]. In the other two cases the situation is less complicated and we can work directly. Afterwards we will collect the classification of real form families as the first column in Table 1 below.

Case (1): $M_u = SU(m+n)/[SU(m) \times SU(n)]$, $m > n \geqq 1$. Then $\widetilde{M}_u = SU(m+n)/S[U(m) \times U(n)]$ is a Grassmann manifold. We start with Berger's classification [1, §50] (Table 2 on page 157). There we need only consider the cases $\widetilde{M} = G/K$ where either (1) $G = SL(m+n;\mathbb{C})$ and $K = S[GL(m;\mathbb{C}) \times GL(n;\mathbb{C})]$ or (2) G is a real form of $SL(m+n;\mathbb{C})$, K is a real form of $S[GL(m;\mathbb{C}) \times GL(n;\mathbb{C})]$, and $K \subset G$. In these cases K is not semisimple. The possibilities are

$$\begin{aligned}
&\text{(i) } \widetilde{M} = SL(m+n;\mathbb{C})/S[GL(m;\mathbb{C}) \times GL(n;\mathbb{C})] \text{ and } M = SL(m+n;\mathbb{C})/[SL(m;\mathbb{C}) \times SL(n;\mathbb{C})]\\
&\text{(ii) } \widetilde{M} = SL(m+n;\mathbb{R})/S[GL(m;\mathbb{R}) \times GL(n;\mathbb{R})] \text{ and } M = SL(m+n;\mathbb{R})/[SL(m;\mathbb{R}) \times SL(n;\mathbb{R})]\\
&\text{(iii) } \widetilde{M} = SL(m'+n';\mathbb{H})/S[GL(m';\mathbb{H}) \times GL(n';\mathbb{H})] \text{ where } m = 2m' \text{ and } n = 2n'; \text{ and}\\
&\qquad M = SL(m'+n';\mathbb{H})/[SL(m';\mathbb{H}) \times SL(n';\mathbb{H})]\\
&\text{(iv) } \widetilde{M} = SU(m-k+\ell, n-\ell+k)/S[U(m-k,k) \times U(n-\ell,\ell)] \text{ for } k \leqq m \text{ and } \ell \leqq n; \text{ and}\\
&\qquad M = SU(m-k+\ell, n-\ell+k)/[SU(m-k,k) \times SU(\ell, n-\ell)]
\end{aligned} \tag{2.3}$$

where $GL(k;\mathbb{H}) := SL(k;\mathbb{H}) \times \mathbb{R}^+$.

Case (2): $M_u = SO(2n)/SU(n)$, n odd, $n \geqq 5$. Then $\widetilde{M_u} = SO(2n)/U(n)$. In Berger's classification [1, §50] (Table 2 on page 157) we need only consider the cases $\widetilde{M} = G/K$ where either (1) $G = SO(2n;\mathbb{C})$ and $K = GL(n;\mathbb{C})$ or (2) G is a real form of $SO(2n;\mathbb{C})$, K is a real form of $GL(n;\mathbb{C})$, and $K \subset G$. As K is not semisimple the possibilities are

$$\begin{aligned}
&\text{(i) } \widetilde{M} = SO(2n;\mathbb{C})/GL(n;\mathbb{C}) \text{ and } M = SO(2n;\mathbb{C})/SL(n;\mathbb{C})\\
&\text{(ii) } \widetilde{M} = SO^*(2n)/U(k,\ell) \text{ where } k+\ell = n \text{ and } M = SO^*(2n)/SU(k,\ell)\\
&\text{(iii) } \widetilde{M} = SO(2k,2\ell)/U(k,\ell) \text{ where } k+\ell = n \text{ and } M = SO(2k,2\ell)/SU(k,\ell)\\
&\text{(iv) } \widetilde{M} = SO(n,n)/GL(n;\mathbb{R}) \text{ and } M = SO(n,n)/SL(n;\mathbb{R})
\end{aligned} \tag{2.4}$$

Case (3): $M_u = E_6/Spin(10)$. Then $\widetilde{M_u} = E_6/[Spin(10) \times Spin(2)]$. Again, in [1, §50] we need only consider the cases $\widetilde{M} = G/K$ where either (1) $G = E_{6,\mathbb{C}}$ and $K = Spin(10;\mathbb{C}) \times Spin(2;\mathbb{C})$ or (2) G is a real form of $E_{6,\mathbb{C}}$, K is a real form of $Spin(10;\mathbb{C}) \times Spin(2;\mathbb{C})$, and $K \subset G$. Berger writes E_6^1 for $E_{6,C_4} = E_{6(6)}$, E_6^2 for $E_{6,A_5A_1} = E_{6(2)}$, E_6^3 for $E_{6,D_5T_1} = E_{6(14)}$ and E_6^4 for $E_{6,F_4} = E_{6(-26)}$. The possibilities are

$$\begin{aligned}
&\text{(i) } \widetilde{M} = E_{6,\mathbb{C}}/[Spin(10;\mathbb{C}) \times Spin(2;\mathbb{C})] \text{ and } M = E_{6,\mathbb{C}}/Spin(10;\mathbb{C})\\
&\text{(ii) } \widetilde{M} = E_6/[Spin(10) \times Spin(2)] \text{ and } M = E_6/Spin(10)\\
&\text{(iii) } \widetilde{M} = E_{6,C_4}/[Spin(5,5) \times Spin(1,1)] \text{ and } M = E_{6,C_4}/Spin(5,5)\\
&\text{(iv) } \widetilde{M} = E_{6,A_5A_1}/[SO^*(10) \times SO(2)] \text{ and } M = E_{6,A_5A_1}/SO^*(10)\\
&\text{(v) } \widetilde{M} = E_{6,A_5A_1}/[Spin(4,6) \times Spin(2)] \text{ and } M = E_{6,A_5A_1}/Spin(4,6)\\
&\text{(vi) } \widetilde{M} = E_{6,D_5T_1}/[Spin(10) \times Spin(2)] \text{ and } M = E_{6,D_5T_1}/Spin(10)\\
&\text{(vii) } \widetilde{M} = E_{6,D_5T_1}/[Spin(2,8) \times Spin(2)] \text{ and } M = E_{6,D_5T_1}/Spin(2,8)\\
&\text{(viii) } \widetilde{M} = E_{6,D_5T_1}/[SO^*(10) \times SO(2)] \text{ and } M = E_{6,D_5T_1}/SO^*(10)\\
&\text{(ix) } \widetilde{M} = E_{6,F_4}/[Spin(1,9) \times Spin(1,1)] \text{ and } M = E_{6,F_4}/Spin(1,9)
\end{aligned} \tag{2.5}$$

Case (4): $M_u = SU(2n+1)/Sp(n)$. Then $\widetilde{M}_u$ is the complex projective space $SU(2n+1)/S[U(2n) \times U(1)]$, and $K_u/H_u = U(2n)/Sp(n)$. In [1, §50] we need only consider the cases $\widetilde{M} = G/K$ where either (1) $G = SL(2n+1;\mathbb{C})$ and $K = GL(2n;\mathbb{C})$, or (2) G is a real form of $SL(2n+1;\mathbb{C})$, K is a real form of $GL(2n;\mathbb{C})$, and $K \subset G$; and the cases (3) $K = GL(2n;\mathbb{C})$ and $H = Sp(n;\mathbb{C})$, or (4) K is a real form of

$GL(2n;\mathbb{C})$, H is a real form of $Sp(n;\mathbb{C})$, and $H \subset K$. The possibilities for $\widetilde{M}$ are $SL(2n+1;\mathbb{C})/S[GL(2n;\mathbb{C}) \times GL(1;\mathbb{C})]$, $SL(2n+1;\mathbb{R})/S[GL(2n;\mathbb{R}) \times GL(1;\mathbb{R})]$, and $SU(2n+1-k,k)/S[U(2n-k,k) \times U(1)]$. The possibilities for K/H are $GL(2n;\mathbb{C})/Sp(n;\mathbb{C}) = [SL(2n;\mathbb{C})/Sp(n;\mathbb{C})] \times \mathbb{C}^*$, $[SU^*(2n)/Sp(k,\ell)] \times U(1)$ $(k+\ell = n)$, $GL(2n;\mathbb{R})/Sp(n;\mathbb{R})$, $U(n,n)/Sp(n;\mathbb{R})$, and $U(2k,2\ell)/Sp(k,\ell)$ $(k+\ell=n)$. Fitting these together, the real form family of $M_u = SU(2n+1)/Sp(n)$ consists of

$$\begin{aligned} &\text{(i) } M = SL(2n+1;\mathbb{C})/Sp(n;\mathbb{C}) \\ &\text{(ii) } M = SL(2n+1;\mathbb{R})/Sp(n;\mathbb{R}) \\ &\text{(iii) } M = SU(n+1,n)/Sp(n;\mathbb{R}) \\ &\text{(iv) } M = SU(2n+1-2\ell,2\ell)/Sp(n-\ell,\ell) \end{aligned} \tag{2.6}$$

Case (5): $M_u = SU(2n+1)/[Sp(n) \times U(1)]$. Then $\widetilde{M}_u = SU(2n+1)/S[U(2n) \times U(1)]$, complex projective space, and $K_u/H_u = SU(2n)/Sp(n)$. As before the cases of $\widetilde{M}$ are

$$\begin{aligned} &SL(2n+1;\mathbb{C})/S[GL(2n;\mathbb{C}) \times GL(1;\mathbb{C})], \quad SL(2n+1;\mathbb{R})/S[GL(2n;\mathbb{R}) \times GL(1;\mathbb{R})] \\ &SU(2n+1-k,k)/S[U(2n-k,k) \times U(1)] \end{aligned}$$

The possibilities for K/H are

$$\begin{aligned} &GL(2n;\mathbb{C})/[Sp(n;\mathbb{C}) \times \mathbb{C}^*], \quad GL(2n+1;\mathbb{R})/[Sp(n;\mathbb{R}) \times \mathbb{R}^*], \\ &U(2n+1-2\ell,2\ell)/[Sp(n-\ell,\ell) \times U(1)] \end{aligned}$$

Fitting these together, the real form family of $M_u = SU(2n+1)/[Sp(n) \times U(1)]$ consists of

$$\begin{aligned} &\text{(i) } M = SL(2n+1;\mathbb{C})/[Sp(n;\mathbb{C}) \times \mathbb{C}^*] \\ &\text{(ii) } M = SL(2n+1;\mathbb{R})/[Sp(n;\mathbb{R}) \times \mathbb{R}^+] \\ &\text{(iii) } M = SU(n+1,n)/[Sp(n;\mathbb{R}) \times \mathbb{R}^+] \\ &\text{(iv) } M = SU(2n+1-2\ell,2\ell)/[Sp(n-\ell,\ell)] \times U(1)] \end{aligned} \tag{2.7}$$

Case (6): $M_u = Spin(7)/G_2$. Neither G_2 nor $Spin(7)$ has an outer automorphism. Further, G_2 is a non-symmetric maximal subgroup of $Spin(7)$, so any involutive automorphism of $Spin(7)$ that is the identity on G_2 is itself the identity. Thus the involutive automorphisms of $Spin(7)$ that preserve G_2 have form $\mathrm{Ad}(s)$ with $s \in G_2$. Now the real form family of $M_u = Spin(7)/G_2$ consists of

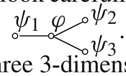
$$\begin{aligned} &\text{(i) } M = Spin(7;\mathbb{C})/G_{2,\mathbb{C}} \\ &\text{(ii) } M = Spin(7)/G_2 \\ &\text{(iii) } M = Spin(3,4)/G_{2,A_1A_1} \end{aligned} \tag{2.8}$$

Case (7): $M_u = G_2/SU(3)$. $SU(3)$ is a non-symmetric maximal subgroup of G_2, so any involutive automorphism of G_2 that is the identity on $SU(3)$ is itself the identity. Thus the involutive automorphisms of G_2 that preserve $SU(3)$ either have form $\mathrm{Ad}(s)$ with $s \in SU(3)$ or act by $z \mapsto z^{-1}$ on the center $Z_{SU(3)} (\cong \mathbb{Z}_3)$. Further, G_{2,A_1A_1} is the only noncompact real form of $G_{2,\mathbb{C}}$. Now the real form family of $M_u = G_2/SU(3)$ consists of

$$\begin{aligned} &\text{(i) } M = G_{2,\mathbb{C}}/SL(3;\mathbb{C}) \\ &\text{(ii) } M = G_2/SU(3) \\ &\text{(iii) } M = G_{2,A_1A_1}/SU(1,2) \\ &\text{(iv) } M = G_{2,A_1A_1}/SL(3;\mathbb{R}) \end{aligned} \tag{2.9}$$

Case (8): $M_u = SO(10)/[Spin(7) \times SO(2)]$. Here $\widetilde{M}_u$ is the Grassmann manifold $SO(10)/[SO(8) \times SO(2)]$, and $K_u/H_u = [SO(8) \times SO(2)]/[Spin(7) \times SO(2)]$. The possibilities for $\widetilde{M}$, as described by Berger [1, §50] (Table 2 on page 157) are

$$SO(10;\mathbb{C})/[SO(8;\mathbb{C}) \times SO(2;\mathbb{C})],$$
$$SO(9-a,a+1)/[SO(8-a,a) \times SO(1,1)],$$
$$SO(8-a,a+2)/[SO(8-a,a) \times SO(0,2)],$$
$$SO(10-a,a)/[SO(8-a,a) \times SO(2,0)], \quad \text{and } SO^*(10)/[SO^*(8) \times SO(2)].$$

To see the possibilities for K/H we must first look carefully at $SO(8)/Spin(7)$. Label the Dynkin diagram and simple roots of $Spin(8)$ by . Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{spin}(8)$ implicit in that diagram, and define three 3-dimensional subalgebras

$$\mathfrak{t}_1 : \psi_2 = \psi_3, \ \ \mathfrak{t}_2 : \psi_3 = \psi_1, \ \ \mathfrak{t}_3 : \psi_1 = \psi_2.$$

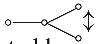
They are the respective Cartan subalgebras of three $\mathfrak{spin}(7)$ subalgebras

$$\mathfrak{s}_1 := \mathfrak{spin}(7)_1, \mathfrak{s}_2 := \mathfrak{spin}(7)_2 \text{ and } \mathfrak{s}_3 := \mathfrak{spin}(7)_3.$$

$Spin(8)$ has center $Z_{Spin(8)} = \{1, a_1, a_2, a_3\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, numbered so that the analytic subgroups S_i for the $\mathfrak{s}_i$ have centers $Z_{S_i} = \{1, a_i\} \cong \mathbb{Z}_2$. In terms of the Clifford algebra construction of the spin groups and an orthonormal basis $\{e_j\}$ of $\mathbb{R}^8$ we may take $a_1 = -1$, $a_2 = e_1e_2 \dots e_8$ and $a_3 = a_1a_2 = -e_1e_2 \dots e_8$. Thus Z_{S_1} is the kernel of the universal covering group projection $\pi : Spin(8) \to SO(8)$. Note that

$\pi(S_1) = SO(7)$ and

$\pi : S_i \to SO(8)$ is an isomorphism onto a $Spin(7)$-subgroup πS_i for $i = 2, 3$.

The outer automorphism group of $Spin(8)$ is given by the permutations of $\{\psi_1, \psi_2, \psi_3\}$. It is generated by the triality automorphism $\tau : \psi_1 \to \psi_2 \to \psi_3 \to \psi_1$, equivalently $\tau : S_1 \to S_2 \to S_3 \to S_1$, equivalently $\tau : a_1 \to a_2 \to a_3 \to a_1$. It follows that the outer automorphism group of $SO(8)$ is given by , and the $SO(8)$-conjugacy classes of $Spin(7)$-subgroups of $SO(8)$ are represented by πS_2 and πS_3. It follows that no $Spin(7)$-subgroup of $SO(8)$ can be invariant under an outer automorphism of $SO(8)$. See [10] for a detailed exposition.

Let σ be an involutive automorphism of $SO(8)$ that preserves the $Spin(7)$-subgroup πS_2. As noted just above, σ is inner on $SO(8)$. σ is nontrivial on πS_2 because πS_2 is a non-symmetric maximal connected subgroup. As πS_2 is simply connected it follows that the fixed point set of $\sigma|_{\pi S_2}$ is connected. Express $\sigma = \mathrm{Ad}(s)$. Then $s^2 = \pm I$, and $s \in \pi S_2$ because πS_2 is its own normalizer in $SO(8)$.

We may assume $s \in T$ where T is the maximal torus of $SO(8)$ with Lie algebra $\mathfrak{t}$. Let $t \in T$ with $\det t = -1$. Then $\mathrm{Ad}(t)$ is an outer automorphism of $SO(8)$ so $\pi S_3' := \mathrm{Ad}(t)(\pi S_2)$ is conjugate of πS_3. Compute $\sigma(\pi S_3') = \mathrm{Ad}(st)(\pi S_2) = \mathrm{Ad}(ts)(\pi S_2) = \mathrm{Ad}(t)(\pi S_2) = \pi S_3'$, so $s \in \pi S_3'$ as above. According to [10, Theorem 4] $(\pi S_2 \cap \pi S_3') = \{\pm I\}G_2$, so now $s \in \{\pm I\}G_2$. As $-I \notin G_2$ we conclude $s^2 = I$.

We can replace s by $-s$ if necessary and assume that $s \in G_2$. The group G_2 has only one conjugacy class of nontrivial automorphisms. If σ_{G_2} is the identity then $\sigma_{\pi S_2}$ is the identity, because G_2 is a non-symmetric maximal connected subgroup of πS_2. But then σ is the identity because πS_2 is a non-symmetric maximal connected subgroup of $SO(8)$.

Now suppose that $\sigma|_{G_2}$ is not the identity. Then σ leads to real forms G_{2,A_1A_1} of $G_{2,\mathbb{C}}$ and $Spin(3,4)$ of $Spin(7;\mathbb{C})$. Thus we may assume that $s = \left(\begin{smallmatrix} +I_4 & 0 \\ 0 & -I_4 \end{smallmatrix}\right) \in T$. In Clifford algebra terms, a unit vector e acts on $\mathbb{R}^8$ by reflection in the hyperplane $e^{\perp}$. Thus the π^{-1}-image of s is $\{\pm e_5e_6e_7e_8\}$, and σ leads to the real form $SO(4,4)$ of $SO(8;\mathbb{C})$.

Now we look at the possibilities for K/H. Recall $K_u/H_u = [SO(8)\times SO(2)]/[Spin(7)\times SO(2)]$. So K/H must be one of

$$\begin{aligned}
&[SO(8;\mathbb{C})\times SO(2;\mathbb{C})]/[Spin(7;\mathbb{C})\times SO(2;\mathbb{C})],\\
&[SO(8)\times SO(2)]/[Spin(7)\times SO(2)], \quad [SO(8)\times SO(1,1)]/[Spin(7)\times SO(1,1)],\\
&[SO(4,4)\times SO(2)]/[Spin(3,4)\times SO(2)],\\
&[SO(4,4)\times SO(1,1)]/[Spin(3,4)\times SO(1,1)].
\end{aligned}$$

We conclude that the real form family of $SO(10)/[Spin(7)\times SO(2)]$ consists of

$$\begin{aligned}
&\text{(i) } M = SO(10;\mathbb{C})/[Spin(7;\mathbb{C})\times SO(2;\mathbb{C})]\\
&\text{(ii) } M = SO(10)/[Spin(7)\times SO(2)]\\
&\text{(iii) } M = SO(9,1)/[Spin(7,0)\times SO(1,1)]\\
&\text{(iv) } M = SO(8,2)/[Spin(7,0)\times SO(0,2)]\\
&\text{(v) } M = SO(6,4)/[Spin(4,3)\times SO(2,0)]\\
&\text{(vi) } M = SO(5,5)/[Spin(3,4)\times SO(1,1)]
\end{aligned} \tag{2.10}$$

Case (9): $M_u = SO(9)/Spin(7)$. Then $\widetilde{M}_u$ is the sphere $SO(9)/SO(8)$ and $K_u/H_u = SO(8)/Spin(7)$. From the considerations of the case $M_u = SO(10)/[Spin(7)\times SO(2)]$ we see that here, $\widetilde{M}$ must be one of

$$SO(9;\mathbb{C})/SO(8;\mathbb{C}), \quad SO(8-a,a+1)/SO(8-a,a), \quad \text{or } SO(9-a,a)/SO(8-a,a)$$

while K/H must be one of

$$SO(8;\mathbb{C})/Spin(7;\mathbb{C}), \quad SO(8)/Spin(7), \quad \text{or} \quad SO(4,4)/Spin(3,4).$$

Thus the real form family of $M_u = SO(9)/Spin(7)$ consists of

$$\begin{aligned}
&\text{(i) } M = SO(9;\mathbb{C})/Spin(7;\mathbb{C})\\
&\text{(ii) } M = SO(9)/Spin(7)\\
&\text{(iii) } M = SO(8,1)/Spin(7)\\
&\text{(iv) } M = SO(5,4)/Spin(3,4)
\end{aligned} \tag{2.11}$$

Case (10): $M_u = Spin(8)/G_2$. Topologically, $M_u = S^7\times S^7$, and $\widetilde{M}_u = Spin(8)/Spin(7) = SO(8)/SO(7) = S^7$ and $K_u/H_u = SO(7)/G_2 = S^7$. The possibilities for $\widetilde{M}$ are $Spin(8;\mathbb{C})/Spin(7;\mathbb{C})$, $Spin(8-a,a)/Spin(7-a,a)$ for $0 \leqq a \leqq 7$ and $Spin(8-a,a)/Spin(8-a,a-1)$ for $1 \leqq a \leqq 8$, and for K/H are $Spin(7;\mathbb{C})/G_{2,\mathbb{C}}$, $SO(7)/G_2$ and $Spin(3,4)/G_{2,A_1A_1}$. Now the real form family of M_u consists of

$$\begin{aligned}
&\text{(i) } M = Spin(8;\mathbb{C})/G_{2,\mathbb{C}}\\
&\text{(ii) } M = Spin(8)/G_2\\
&\text{(iii) } M = Spin(7,1)/G_2\\
&\text{(iv) } M = Spin(4,4)/G_{2,A_1A_1}\\
&\text{(v) } M = Spin(3,5)/G_{2,A_1A_1}
\end{aligned} \tag{2.12}$$

Case (11): $M_u = SO(2n+1)/U(n)$. Then $\widetilde{M}_u = SO(2n+1)/SO(2n)$ and $K_u/H_u = SO(2n)/U(n)$. The possibilities for $\widetilde{M}$ are

$$\begin{aligned}
&SO(2n+1;\mathbb{C})/SO(2n;\mathbb{C}), \quad SO(n,n+1)/SO^*(2n)\\
&SO(2n+1;\mathbb{C})/SO(2n-k,k) \quad \text{for} \quad 0 \leqq k \leqq 2n,\\
&SO(2n+1-k,k)/SO(2n-k,k) \quad \text{for} \quad 0 \leqq k \leqq 2n\\
&SO(2n-k,k+1)/SO(2n-k,k) \quad \text{for} \quad 0 \leqq k \leqq 2n
\end{aligned}$$

The possibilities for K/H are

$$\begin{aligned}
&SO(2n;\mathbb{C})/GL(n;\mathbb{C}), \quad SO(2n-2k,2k)/U(n-k,k) \quad \text{for} \quad 0 \leqq k \leqq n\\
&SO^*(2n)/U(n), \quad SO^*(2n)/GL(n/2;\mathbb{H}) \ \text{for } n \text{ even}, \ SO(n,n)/GL(n;\mathbb{R})
\end{aligned}$$

Putting these together, the real form family of M_u consists of

$$\begin{aligned}
&\text{(i)} \ M = SO(2n+1;\mathbb{C})/GL(n;\mathbb{C})\\
&\text{(ii)} \ M = SO(2n+1-2k,2k)/U(n-k,k) \text{ for } 0 \leqq k \leqq n\\
&\text{(iii)} \ M = SO(2n-2k,2k+1)/U(n-k,k) \text{ for } 0 \leqq k \leqq n\\
&\text{(iv)} \ M = SO(n,n+1)/GL(n;\mathbb{R})
\end{aligned} \tag{2.13}$$

Case (12): $M_u = Sp(n)/[Sp(n-1) \times U(1)]$. Here $\widetilde{M}_u$ is the quaternionic projective space $Sp(n)/[Sp(n-1) \times Sp(1)]$ and K_u/H_u is $[Sp(n-1) \times Sp(1)]/[Sp(n-1) \times U(1)] = S^2$. The possibilities for $\widetilde{M}$ are

$$\begin{aligned}
&Sp(n;\mathbb{C})/[Sp(n-1;\mathbb{C}) \times Sp(1;\mathbb{C})], \quad Sp(n;\mathbb{R})/[Sp(n-1;\mathbb{R}) \times Sp(1;\mathbb{R})]\\
&Sp(n-k,k)/[Sp(n-1-k,k) \times Sp(1,0)] \quad \text{for } 0 \leqq k \leqq n-1\\
&Sp(n-k,k)/[Sp(n-k,k-1) \times Sp(0,1)] \quad \text{for } 1 \leqq k \leqq n
\end{aligned}$$

The possibilities for K/H are

$$\begin{aligned}
&[Sp(n-1;\mathbb{C}) \times Sp(1;\mathbb{C})]/[Sp(n-1;\mathbb{C}) \times GL(1;\mathbb{C})]\\
&[Sp(n-1-k,k) \times Sp(1,0)]/[Sp(n-1-k,k) \times U(1,0)] \text{ for } 0 \leqq k \leqq n-1\\
&[Sp(n-k,k-1) \times Sp(0,1)]/[Sp(n-k,k-1) \times U(0,1)] \text{ for } 1 \leqq k \leqq n\\
&[Sp(n-1;\mathbb{R}) \times Sp(1;\mathbb{R})]/[Sp(n-1;\mathbb{R}) \times GL(1;\mathbb{R})]\\
&[Sp(n-1;\mathbb{R}) \times Sp(1;\mathbb{R})]/[Sp(n-1;\mathbb{R}) \times U(1)]
\end{aligned}$$

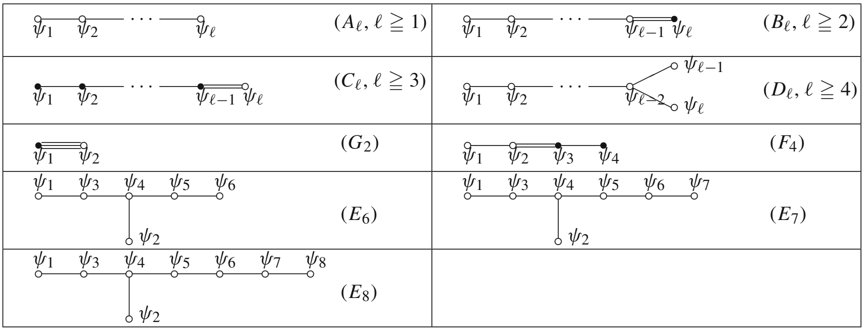
Now the real form family of M_u consists of

$$\begin{aligned}
&\text{(i)} \ M = Sp(n;\mathbb{C})/[Sp(n-1;\mathbb{C}) \times GL(1;\mathbb{C})]\\
&\text{(ii)} \ M = Sp(n-k,k)/[Sp(n-1-k,k) \times U(1,0)] \text{ for } 0 \leqq k \leqq n-1\\
&\text{(iii)} \ M = Sp(n-k,k)/[Sp(n-k,k-1) \times U(0,1)] \text{ for } 1 \leqq k \leqq n\\
&\text{(iv)} \ M = Sp(n;\mathbb{R})/[Sp(n-1;\mathbb{R}) \times GL(1;\mathbb{R})]\\
&\text{(v)} \ M = Sp(n;\mathbb{R})/[Sp(n-1;\mathbb{R}) \times U(1)]
\end{aligned} \tag{2.14}$$

As mentioned earlier, all the real form family classification results of Sect. 2 are tabulated as the first column in Table 1, located in Sect. 6 below.

3 Isotropy representations and signature

We will describe the isotropy representations for the weakly symmetric spaces $M = G/H$ of Sect. 2 using the Bourbaki order for the simple root system $\Psi = \Psi_G = \{\psi_1, \ldots, \psi_\ell\}$ of G. The result will appear in the twelve sub-headers on Table 1, the consequence for the decomposition of the tangent space will appear in the second column of Table 1, and the resulting possible signatures of G-invariant Riemannian metric will be in the third column. The Bourbaki order of the simple roots is

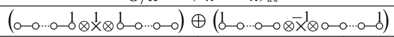
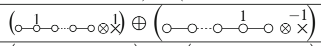
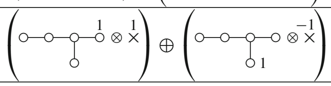


(3.1)

where, if there are two root lengths, the black dots indicate the short roots. We will use the notation

$$\begin{aligned}
&\xi_i : \text{ fundamental highest weight, } \frac{2\langle \xi_i, \psi_j\rangle}{\langle \psi_j, \psi_j\rangle} = \delta_{i,j}\\
&\pi_\lambda : \text{ irreducible representation of } \mathfrak{g} \text{ of highest weight } \lambda\\
&\nu_\lambda : \text{ irreducible representation of } \mathfrak{k} \text{ of highest weight } \lambda\\
&\tau_\lambda : \text{ irreducible representation of } \mathfrak{h} \text{ of highest weight } \lambda\\
&\pi_{\lambda,\mathbb{R}}, \nu_{\lambda,\mathbb{R}}, \tau_{\lambda,\mathbb{R}} : \text{ corresponding real representations}
\end{aligned} \tag{3.2}$$

Here, if $\pi_\lambda(\mathfrak{g})$ preserves a real form of the representation space of π_λ then $\pi_{\lambda,\mathbb{R}}$ is the representation on that real form. Otherwise, $(\pi_\lambda \oplus \pi_{\lambda^*})_{\mathbb{R}}$ is the representation on the invariant real form of the representation space of $\pi_\lambda \oplus \pi_{\lambda^*}$, where π_{λ^*} is the complex conjugate of π_λ ; $\nu_{\lambda,\mathbb{R}}$ and $\tau_{\lambda,\mathbb{R}}$, etc., are defined similarly. Thus, for example, the isotropy (tangent space) representations $\nu_{G/K}$ of the compact irreducible symmetric spaces G/K that correspond to non-tube bounded symmetric domains are

G/K	conditions	weights	$\nu_{G/K} = (\nu_\lambda \oplus \overline{\nu_\lambda})_{\mathbb{R}}$
$\frac{SU(m+n)}{S(U(m)\times U(n))}$	$m > n \geqq 1$	$\lambda = \xi_{m-1} + \xi_m + \xi_{m+1}$	
$\frac{SO(2n)}{U(n)}$	n odd, $n \geqq 3$	$\lambda = \xi_2 + \xi_n$	
$\frac{E_6}{Spin(10)\times Spin(2)}$		$\lambda = \xi_5 + \xi_6$	

(3.3)

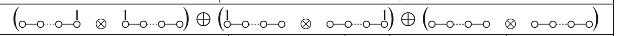
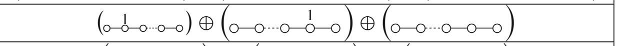
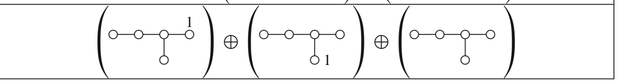
Here the $\times$ corresponds to the (1-dimensional) center of $\mathfrak{k}$, and with a over the $\times$ we have the unitary character ζ^a which is the a^{th} power of a basic character ζ on that center. We note that

Lemma 3.4 *Let G_u/H_u be a circle bundle over an irreducible hermitian symmetric space G_u/K_u dual to a non-tube domain, in other words one of the spaces* (1), (2) *or* (3) *of* (2.1).

Let $\nu_{G/K}$ denote the representation of K_u on the real tangent space $\mathfrak{g}_u/\mathfrak{k}_u$, from (3.3). Then $\nu_{G/K}|_{H_u}$ is irreducible.

Proof In view of the conditions from (3.3), $\nu_\lambda|_{H_u} \neq \overline{\nu_\lambda|_{H_u}}$ with the one exception of $SU(3)/SU(2)$. It follows, with that exception, that $\nu_{G/K}|_{H_u}$ is irreducible and $\tau_{G/H} = \nu_{G/K}|_{H_u} \oplus \tau_{0,\mathbb{R}}$. In the case of $SU(3)/SU(2)$, $\dim \mathfrak{g}/\mathfrak{k} = 4$ while τ_λ cannot have a trivial summand in $\mathfrak{g}/\mathfrak{k}$. If τ_λ reduces on $\mathfrak{g}/\mathfrak{k}$ it is the sum of two 2-dimensional real representations. But $SU(2)$ does not have a nontrivial 2-dimensional real representation: the 2-dimensional complex representation of $SU(2)$ is quaternionic, not real. Thus, in the case of $SU(3)/SU(2)$, again $\nu_{G/K}|_{H_u}$ is irreducible and $\tau_{G/H} = \nu_{G/K}|_{H_u} \oplus \tau_{0,\mathbb{R}}$. □

Now, in the cases of (3.3) and Lemma 3.4, the isotropy representations of the corresponding weakly symmetric spaces involve suppressing the $\times$ and adding a trivial representation, as follows.

G/H	weights	$\tau_{G/H} = (\tau_\lambda \oplus \overline{\tau_\lambda})_{\mathbb{R}} \oplus \tau_{0,\mathbb{R}}$
$\frac{SU(m+n)}{SU(m)\times SU(n))}$	$\lambda = \xi_{m-1} + \xi_{m+1}$	
$\frac{SO(2n)}{SU(n)}$	$\lambda = \xi_2$	
$\frac{E_6}{Spin(10)}$	$\lambda = \xi_5$	

(3.5)

Now we run through the cases of Sect. 2.

Case (1): $M_u = SU(m+n)/[SU(m) \times SU(n)]$, $m > n \geqq 1$. Consider the spaces listed in (2.3). The first three have form $SL(m+n;\mathbb{F})/[SL(m;\mathbb{F}) \times SL(n;\mathbb{F})]$. In block form matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the real tangent space of G/H is given by b, c and the (real, complex, real) scalar matrices in the places of a and d. This says that the irreducible summands of the real isotropy representation have dimensions mn, mn and 1 for $\mathbb{F} = \mathbb{R}$; $2mn$, $2mn$, 1 and 1 for $\mathbb{F} = \mathbb{C}$; and $4mn$, $4mn$ and 1 for $\mathbb{F} = \mathbb{H}$. Each $\mathrm{Ad}(H)$-invariant space $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$ is null for the Killing form of $\mathfrak{g}$, but they are paired, so together they contribute signature (mnd, mnd) to any invariant pseudo-Riemannian metric on G/H. Thus the possibilities for signature of invariant pseudo-Riemannian metrics here are

$$
\begin{aligned}
&SL(m+n;\mathbb{R})/[SL(m;\mathbb{R}) \times SL(n;\mathbb{R})] : \ (mn+1, mn), (mn, mn+1) \\
&SL(m+n;\mathbb{C})/[SL(m;\mathbb{C})\times SL(n;\mathbb{C})] : \ (2mn+1, 2mn+1), (2mn, 2mn+2), (2mn+2, 2mn) \\
&SL(m+n;\mathbb{H})/[SL(m;\mathbb{H}) \times SL(n;\mathbb{H})] : \ (4mn+1, 4mn), (4mn, 4mn+1).
\end{aligned}
$$

Now consider the fourth space, $G/H = SU(m-k+\ell, n-\ell+k)/[SU(m-k,k) \times SU(\ell, n-\ell)]$. In the notation of (3.2) and (3.5), the complex tangent space of G/K is the sum of ad $(\mathfrak{h})$-invariant subspaces $\mathfrak{s}_+$ and $\mathfrak{s}_-$, the holomorphic and antiholomorphic tangent spaces of G/K, where $\mathfrak{h}$ acts irreducibly on $\mathfrak{s}_+$ by $\tau_{\xi_1} \otimes \tau_{\xi_{m+n-1}}$ and on $\mathfrak{s}_-$ by $\tau_{\xi_{m-1}} \otimes \tau_{\xi_{m+1}}$. The Killing form $\kappa_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ is null on $\mathfrak{s}_+$ and on $\mathfrak{s}_-$ but pairs them, and the Killing form κ of $\mathfrak{g}$ is the real part of $\kappa_{\mathbb{C}}$. Now the irreducible summands of the isotropy representation of H on the real tangent space of G/H have dimensions $2mn$ and 1. Note the signs of certain inner products: $\mathbb{C}^{x,y} \otimes \mathbb{C}^{z,w} = \mathbb{C}^{xz+yw,xw+yz}$. Thus summand of dimension $2mn$ contributes $(2(m-k)(n-\ell)+2k\ell, 2(m-k)\ell+2(n-\ell)k)$ or $(2(m-k)\ell+2(n-\ell)k, 2(m-k)(n-\ell)+2k\ell)$ to the signature of any invariant pseudo-Riemannian metric on G/H. Now the possibilities for the signature of invariant pseudo-Riemannian metrics here are

$$
\begin{aligned}
&(2(mn - m\ell - nk + 2k\ell) + 1, 2(m\ell + nk - 2k\ell)),\\
&\quad(2(mn - m\ell - nk + 2k\ell), 2(m\ell + nk - 2k\ell) + 1),\\
&(2(m\ell + nk - 2k\ell) + 1, 2(mn - m\ell - nk + 2k\ell)),\\
&\quad(2(m\ell + nk - 2k\ell), 2(mn - m\ell - nk + 2k\ell) + 1).
\end{aligned}
$$

Case (2): $M_u = SO(2n)/SU(n)$. We now consider the spaces listed in (2.4). In the first and fourth cases H has form $SL(n;\mathbb{F})$. For (i), $\mathfrak{g}$ consists of all $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ with $a + a^t = 0 = d + d^t$ and $c = b^t$. The symmetry of $\mathfrak{g}$ over $\mathfrak{k}$ is $\mathrm{Ad}(J)$ where $J = \left(\begin{smallmatrix} 0 & I\\ -I & 0\end{smallmatrix}\right)$ as above, and the (-1)-eigenspace $\mathfrak{s}$ of $\mathrm{Ad}(J)$ on $\mathfrak{g}$ consists of all $\left(\begin{smallmatrix} a & b\\ b & -a\end{smallmatrix}\right)$ with a antisymmetric and b symmetric. Thus the contribution of $\mathfrak{s}$ to the Killing form of $\mathfrak{g}$ has signature $(n(n-1), n(n-1))$ from real a and pure imaginary a. For (iv), the matrices are real, so the contribution of $\mathfrak{s}$ to the Killing form of $\mathfrak{g}$ has signature $\left(\frac{n(n-1)}{2}, \frac{n(n-1)}{2}\right)$. Thus the possibilities for the signature of invariant pseudo-Riemannian metrics here are

$$
\begin{aligned}
SO(2n;\mathbb{C})/SL(n;\mathbb{C}):\ &(n(n-1)+2, n(n-1)),\ (n(n-1)+1, n(n-1)+1),\\
&(n(n-1), n(n-1)+2);\\
SO(n,n)/SL(n;\mathbb{R}):\ &\left(\tfrac{1}{2}n(n-1)+1,\ \tfrac{1}{2}n(n-1)\right),\ \left(\tfrac{1}{2}n(n-1),\ \tfrac{1}{2}n(n-1)+1\right).
\end{aligned}
$$

In cases (ii) and (iii) of (2.4) we argue as above for the last case of (2.3). Note the signs of the inner products: $\Lambda^2(\mathbb{C}^{k,\ell}) = \mathbb{C}^{a,b}$ where $a = \frac{1}{2}(k(k-1) + \ell(\ell-1))$ and $b = k\ell$. Thus the possibilities for the signature of invariant pseudo-Riemannian metrics here are

$$
\begin{aligned}
SO^*(2n)/SU(k,\ell):\ &(k(k-1)+\ell(\ell-1)+1, 2k\ell), (k(k-1)+\ell(\ell-1), 2k\ell+1),\\
&(2k\ell+1, k(k-1)+\ell(\ell-1)), (2k\ell, k(k-1)+\ell(\ell-1)+1);\\
SO(2k,2\ell)/SU(k,\ell):\ &(k(k-1)+\ell(\ell-1))+1, 2k\ell), (k(k-1)+\ell(\ell-1)), 2k\ell+1),\\
&(2k\ell+1, k(k-1)+\ell(\ell-1)), (2k\ell, k(k-1)+\ell(\ell-1)+1).
\end{aligned}
$$

Case (3): $M_u = E_6/Spin(10)$. We now consider the spaces $M = G/H$ listed in (2.5). The representation of $\mathfrak{h}$ on the complexified tangent space of M is the sum of its two half spin representations, whose spaces $\mathfrak{s}_\pm$ are null under the Killing form but are paired. In case (i) this means that the contribution of $\mathfrak{s} = \mathfrak{s}_+ + \mathfrak{s}_-$ to the (real) Killing form is (32, 32). In the Riemannian cases where $H = Spin(10)$ the contribution is (32, 0) or (0, 32). In the other four cases for which H has form $Spin(a,b)$ the contribution is (16, 16), because those half spin representations of H are real. Finally, in the two cases where $H = SO^*(10)$, each half spin representation restricts on the maximal compact subgroup $G_u \cap H \cong U(5)$ of H to the sum of three irreducible representations, $(\circ\!-\!\overset{1}{\circ}\!-\!\circ\!-\!\circ\ \overset{a}{\times}) \oplus (\circ\!-\!\circ\!-\!\circ\!-\!\overset{1}{\circ}\ \overset{b}{\times}) \oplus (\circ\!-\!\circ\!-\!\circ\!-\!\circ\ \overset{c}{\times})$ with $a = 3$, $b = 1$ and $c = 5$, or its dual. The signature of the Killing form of $\mathfrak{g}$ on the real tangent space of G/K is (20, 12) both for case (iv) and for case (viii). Thus the possibilities for the signature of invariant pseudo-Riemannian metrics here are

$$
\begin{aligned}
&E_{6,\mathbb{C}}/Spin(10;\mathbb{C}):\ (34,32), (33,33), (32,34);\\
&\text{cases } H = Spin(10):\ (17,16),\ (16,17);\\
&\text{cases } H = SO^*(10):\ (21,12),\ (20,13),\ (13,20),\ (12,21).
\end{aligned}
$$

Case (4): $M_u = SU(2n+1)/Sp(n)$. Consider the four spaces listed in (2.6). The isotropy representation of $\mathfrak{h}$ on the real tangent space of G/H is the sum of the isotropy representation

$\tau_{K/H}$ on the real tangent space of K/H and the restriction $\nu_{G/K}|_H$ of the isotropy representation of K on the real tangent space of G/K. Thus the signatures of the Killing form on the minimal nondegenerate summands in the real isotropy representation are as the second column of Table 1, and the possible signatures of invariant pseudo-Riemannian metric on G/H, are given by the third column there.

Case (5): $M_u = SU(2n+1)/[Sp(n) \times U(1)]$. The four spaces listed in (2.7) are minor variations on those of (2.6). The commutative factor of H is central in K, where it delivers a trivial factor in the representation of $\mathfrak{h}$ on $\mathfrak{k}/\mathfrak{h}$ and the identity character χ, a nontrivial rotation ρ, or a dilation δ plus $1/\delta$, in the representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{k}$. In this notation, the representation of $\mathfrak{h}$ on the real tangent space of G/H, and the signature of the restriction of the Killing form of $\mathfrak{g}$ there, are listed in Table 1.

Case (6): $M_u = Spin(7)/G_2$. In the three cases of (2.8), the representation τ_{ξ_1} of $\mathfrak{h}$ on the real tangent space of G/H, the signature of the Killing form of $\mathfrak{g}$ on that tangent space, and the possible signatures of invariant pseudo-Riemannian metric, are as listed in Table 1.

Case (7): $M_u = G_2/SU(3)$. In the four cases of (2.9), the representation of $\mathfrak{h}$ on the real tangent space of G/H, the signature of the Killing form of $\mathfrak{g}$ on that tangent space, and the possible signatures of invariant pseudo-Riemannian metric, are given by the sum of the vector representation τ_{ξ_1} and its dual τ_{ξ_2}, and listed in Table 1.

Case (8): $M_u = SO(10)/[Spin(7) \times SO(2)]$. We run through the cases of (2.10). The representation of K_u on the real tangent space of the Grassmannian $SO(10)/[SO(8)\times SO(2)]$ remains irreducible on $Spin(7) \times SO(2)$, and the representation of H_u on the tangent space of $K_u/H_u = [SO(8) \times SO(2)]/[Spin(7) \times SO(2)] = S^7$ is just the vector representation. In the cases of (2.10), there is no further decomposition when the identity component of the center of H is a compact. But it splits when that component is noncompact. Thus the isotropy representation of $\mathfrak{h}$ on the real tangent space of G/H, the signature of the Killing form on the minimal nondegenerate summands in the real isotropy representation, and the possible signatures of invariant pseudo-Riemannian metrics on G/H, are as listed in Table 1.

Case (9): $M_u = SO(9)/Spin(7)$. The cases of (2.11) are essentially the same as those of (2.10), but with the central subgroup of H removed and with τ_{ξ_3} no longer tensored with a 2-dimensional representation. The isotropy representation of $\mathfrak{h}$ on the real tangent space of G/H, the signature of the Killing form on the minimal nondegenerate summands in the real isotropy representation, and the possible signatures of invariant pseudo-Riemannian metric on G/H, follow immediately as listed in Table 1.

Case (10): $M_u = Spin(8)/G_2$. In the cases of (2.12), the representation of H on the tangent space of G/H is the sum of two copies of the 7-dimensional representation τ_{ξ_1} of G_2. The isotropy representation of $\mathfrak{h}$ on the real tangent space of G/H, the signature of the Killing form on the minimal nondegenerate summands in the real isotropy representation, and the possible signatures of invariant pseudo-Riemannian metric on G/H, are listed in Table 1.

Case (11): $M_u = SO(2n+1)/U(n)$. The representation of H on the real tangent space of $G/K = SO(2n+1)/SO(2n)$ is the restriction $(\tau_{\xi_2} \otimes (\zeta \oplus \overline{\zeta})_{\mathbb{R}})$ of the vector representation of $SO(2n)$, and on the real tangent space of $K/H = SO(2n)/U(n)$ is $((\tau_{\xi_2} \otimes \chi) \oplus (\tau_{\xi_{n-2}} \otimes \overline{\chi}))_{\mathbb{R}}$ as indicated in the second line of (3.3). Now the isotropy representation of $\mathfrak{h}$ on the real tangent space of G/H, the signature of the Killing form on the minimal nondegenerate summands in the real isotropy representation, and the possible signatures of invariant pseudo-Riemannian metric on G/H, are given as stated below.

Case (12): $M_u = Sp(n)/[Sp(n-1) \times U(1)]$. The representation of H on the real tangent space of $G/K = Sp(n)/[Sp(n-1) \times Sp(1)]$ is the restriction $\tau_{\xi_1,\mathbb{R}} \otimes ((\zeta_{+1} \oplus \zeta_{-1})_{\mathbb{R}} \oplus (\zeta_{-1} \oplus \zeta_{+1})_{\mathbb{R}})$ of the representation $\tau_{\xi_1,\mathbb{R}} \otimes (\tau_{\xi_n} \oplus \tau_{\xi_n})_{\mathbb{R}}$ of K. The representation of H on the real tangent space of K/H is trivial on $Sp(n-1)$ and is $(\zeta_{+1} \oplus \zeta_{-1})_{\mathbb{R}}$ on $U(1)$. The results are listed below in Table 1. There "metric-irreducible" means minimal subspace nondegenerate for the Killing form of $\mathfrak{g}$.

4 Real form families for G_u semisimple but not simple

Table (4.1) just below is Yakimova's formulation ([17,18]) of the principal case diagram of Mikityuk [8], with some indices shifted to facilitate descriptions of the real form families. See [12, Section 12.8] for the details. It gives the irreducible compact spherical pairs and nonsymmetric compact weakly symmetric spaces. There, $\mathfrak{sp}(n)$ corresponds to the compact group $Sp(n)$. In each of the nine entries of Table (4.1), $\mathfrak{g}$ is the sum of the algebras on the top row and $\mathfrak{h}$ is the sum of the algebras on the bottom row. We continue the numbering from (2.1).

The spaces of (2.1), and entries (1) through (8) in (4.1), all are principal. Entry (9) of (4.1) is a little more complicated; see [12, Section 12.8]. There the $\mathfrak{g}_i$ are semisimple but not necessarily simple.

Compact Irred Nonsymmetric Weakly Symmetric $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{g}$ is Semisimple but not Simple		
(13) $\mathfrak{su}(n)$ $\mathfrak{su}(n+1)$ $\mathfrak{su}(n)$ $\mathfrak{u}(1)$	(16) $\mathfrak{su}(n+2)$ $\mathfrak{sp}(m+1)$ $\mathfrak{u}(n)$ $\mathfrak{su}(2)=\mathfrak{sp}(1)$ $\mathfrak{sp}(m)$	(19) $\mathfrak{sp}(n+1)$ $\mathfrak{sp}(\ell+1)$ $\mathfrak{sp}(m+1)$ $\mathfrak{sp}(n)$ $\mathfrak{sp}(1)$ $\mathfrak{sp}(\ell)$ $\mathfrak{sp}(m)$
(14) $\mathfrak{sp}(n+2)$ $\mathfrak{sp}(2)$ $\mathfrak{sp}(n)$ $\mathfrak{sp}(2)$	(17) $\mathfrak{su}(n+2)$ $\mathfrak{sp}(m+1)$ $\mathfrak{su}(n)$ $\mathfrak{su}(2)=\mathfrak{sp}(1)$ $\mathfrak{sp}(m)$	(20) $\mathfrak{sp}(n+1)$ $\mathfrak{sp}(2)$ $\mathfrak{sp}(m+1)$ $\mathfrak{sp}(n)$ $\mathfrak{sp}(1)$ $\mathfrak{sp}(1)$ $\mathfrak{sp}(m)$
(15) $\mathfrak{so}(n)$ $\mathfrak{so}(n+1)$ $\mathfrak{so}(n)$	(18) $\mathfrak{sp}(n+1)$ $\mathfrak{sp}(m+1)$ $\mathfrak{sp}(n)$ $\mathfrak{sp}(1)$ $\mathfrak{sp}(m)$	(21) $\mathfrak{g}_1$ $\dots$ $\mathfrak{g}_n$ $\dots$ $\mathfrak{z}_{\mathfrak{h}}$ $\mathfrak{h}'_1$ $\dots$ $\mathfrak{h}'_n$

(4.1)

Definition 4.2 Let $M_u = G_u/H_u$ be one of the entries in (4.1) excluding entry (21). For entries (13), (14), (15), (16), (17) and (18) express $\mathfrak{g}_u = \mathfrak{g}_{1,u} \oplus \mathfrak{g}_{2,u}$ with $\mathfrak{g}_{u,i}$ nonzero and simple. For entries (19) and (20) express $\mathfrak{g}_u = \mathfrak{g}_{1,u} \oplus \mathfrak{g}_{2,u} \oplus \mathfrak{g}_{3,u}$ with $\mathfrak{g}_{u,i}$ nonzero and simple. Let $\mathfrak{h}_{u,i}$ denote the image of $\mathfrak{h}_u$ under the projection $\mathfrak{g}_u \to \mathfrak{g}_{u,i}$. Then each $(\mathfrak{g}_{u,i}, \mathfrak{h}_{u,i})$ corresponds to a compact simply connected irreducible Riemannian symmetric (or at least weakly symmetric) space $M_{u,i} = G_{u,i}/H_{u,i}$, and $\widetilde{M}_u = \prod M_{u,i}$ is the **Riemannian unfolding** of M_u.

Now we run through the cases of (4.1). The results will be summarized in Table 2 below.

(13)	$\mathfrak{su}(n)$ $\mathfrak{su}(n+1)$ $\mathfrak{su}(n)$ $\mathfrak{u}(1)$	$[SU(n)SU(n+1)]/S[U(n) \times U(1)]$

The real form family of $SU(n)/SU(n)$ consists of the G_1/H_1 given by

$$SL(n;\mathbb{C})/SL(n;\mathbb{C}),\ SL(n;\mathbb{R})/SL(n;\mathbb{R}),\ SL(n';\mathbb{H})/SL(n';\mathbb{H}) \text{ if } n = 2n',$$
$$\text{and one of the } SU(k,\ell)/SU(k,\ell) \text{ where } k+\ell = n.$$

The real form family of $SU(n+1)/S[U(n)\times U(1)]$ consists of the $\widetilde{M}_2 = G_2/H_2$ in (2.3) with (m,n) replaced by $(n,1)$. That family is

$$SL(n+1;\mathbb{C})/S[GL(n;\mathbb{C})\times GL(1;\mathbb{C})],\ SL(n+1;\mathbb{R})/S[GL(n;\mathbb{R})\times GL(1;\mathbb{R})],$$
$$SU(n-a+1,a)/S[U(n-a,a)\times U(1,0)],\ SU(n-a,1+a)/S[U(n-a,a)\times U(0,1)].$$

We can fold these together exactly in the cases where H_1 is the semisimple part of H_2, so the possibilities for G/H are

$$\begin{aligned}
&\text{(i) } [SL(n;\mathbb{C})\times SL(n+1;\mathbb{C})]/[SL(n;\mathbb{C})\times GL(1;\mathbb{C})]\\
&\text{(ii) } [SL(n;\mathbb{R})\times SL(n+1;\mathbb{R})]/[SL(n;\mathbb{R})\times GL(1;\mathbb{R})]\\
&\text{(iii) } [SU(k,\ell)\times SU(k+1,\ell)]/[SU(k,\ell)\times U(1)]\\
&\text{(iv) } [SU(k,\ell)\times SU(k,\ell+1)]/[SU(k,\ell)\times U(1)]
\end{aligned} \tag{4.3}$$

In case (i), the metric irreducible summands of the tangent space have signatures (n^2-1, n^2-1) and $(2n,2n)$. In case (ii), those signatures are $\left(\frac{n(n+1)}{2}-1, \frac{n(n-1)}{2}\right)$ and (n,n). In case (iii), those signatures are $(2k\ell, k^2+\ell^2-1)$ and $(2\ell,2k)$. In case (iv), those signatures are $(2k\ell, k^2+\ell^2-1)$ and $(2k,2\ell)$.

(14)	$\begin{tikzcd}[column sep=small] \mathfrak{sp}(n+2) \arrow[d, no head] \arrow[dr, no head] & \mathfrak{sp}(2) \arrow[d, no head] \\ \mathfrak{sp}(n) & \mathfrak{sp}(2) \end{tikzcd}$	$[Sp(n+2)\times Sp(2)]/[Sp(n)\times Sp(2)]$

For $Sp(n+2)/[Sp(n)\times Sp(2)$, we have the following possibilities:

$$Sp(n+2;\mathbb{C})/[Sp(n;\mathbb{C})\times Sp(2;\mathbb{C})],\quad Sp(n+2;\mathbb{R})/[Sp(n;\mathbb{C})\times Sp(2;\mathbb{R})]$$
$$Sp(n-a+b,2-b+a)/[Sp(n-a,a)\times Sp(b,2-b)] \text{ for } 0 \leqq a \leqq n \text{ and } 0 \leqq b \leqq 2$$

Thus we have the following possibilities for this case:

$$\begin{aligned}
&\text{(i) } [Sp(n+2;\mathbb{C})\times Sp(2;\mathbb{C})]/[Sp(n;\mathbb{C})\times Sp(2;\mathbb{C})]\\
&\text{(ii) } [Sp(n+2;\mathbb{R})\times Sp(2;\mathbb{R})]/[Sp(n;\mathbb{R})\times Sp(2;\mathbb{R})]\\
&\text{(iii) } [Sp(n-a+b,2-b+a)\times Sp(b,2-b)]/[Sp(n-a,a)\times Sp(b,2-b)]\\
&\qquad \text{for } 0 \leqq a \leqq n \text{ and } b \in \{0,1,2\}
\end{aligned} \tag{4.4}$$

In case (i), the metric irreducible summands of the tangent space have signatures $(10,10)$ and $(8n,8n)$. In case (ii), those signatures are $(6,4)$ and $(4n,4n)$. In case (iii), those signatures are $(8b-4b^2, 4b^2-8b+10)$ and $(8n+4(a-n)b+4a(b-2), 4(n-a)b+4a(2-b))$.

(15)	$\begin{tikzcd}[column sep=small] \mathfrak{so}(n) \arrow[dr, no head] & & \mathfrak{so}(n+1) \arrow[dl, no head] \\ & \mathfrak{so}(n) & \end{tikzcd}$	$[SO(n)\times SO(n+1)]/SO(n)$

For $SO(n+1)/SO(n)$, the possibilities are

$$SO(n+1;\mathbb{C})/SO(n,\mathbb{C}),\ \text{and } SO(n-a+b,1-b+a)/SO(n-a,a)$$
$$\text{for } 0 \leqq a \leqq n \text{ and } 0 \leqq b \leqq 1.$$

Thus the possibilities for M in this case are:

$$\begin{aligned} &\text{(i)}\ [SO(n,\mathbb{C}) \times SO(n+1;\mathbb{C})]/SO(n,\mathbb{C}) \\ &\text{(ii)}\ [SO(n-a,a) \times SO(n-a,a+1)]/SO(n-a,a) \text{ for } 0 \leqq a \leqq n \\ &\text{(iii)}\ [SO(n-a,a) \times SO(n-a+1,a)]/SO(n-a,a) \text{ for } 0 \leqq a \leqq n \end{aligned} \tag{4.5}$$

In case (i), the metric irreducible subspaces of the real tangent space have signatures $\left(\frac{n(n-1)}{2}, \frac{n(n-1)}{2}\right)$ and (n,n). In case (ii), those signatures are $\left((n-a)a, \frac{n(n-1)}{2} - (n-a)a\right)$ and $(n-a,a)$. In case (iii), those signatures are $\left((n-a)a, \frac{n(n-1)}{2} - (n-a)a\right)$ and $(a, n-a)$.

(16)

$$\begin{tikzcd}[column sep=small] \mathfrak{su}(n+2) \arrow[d, dash] \arrow[dr, dash] & & \mathfrak{sp}(m+1) \arrow[dl, dash] \arrow[d, dash] \\ \mathfrak{u}(n) & \mathfrak{su}(2) = \mathfrak{sp}(1) & \mathfrak{sp}(m) \end{tikzcd}$$

$[SU(n+2) \times Sp(m+1)]/[U(n) \times SU(2) \times Sp(m)]$

Let $G_u/H_u = [SU(n+2) \times Sp(m+1)]/[U(n) \times SU(2) \times Sp(m)]$ as in entry (14) on Table 4.1. The real form family of $M_1 = SU(n+2)/S[U(n) \times U(2)]$ consists of the G_1/H_1 in (2.3) with (m,n) replaced by $(n,2)$. That family is

$$\begin{aligned} &SL(n+2;\mathbb{C})/S[GL(n;\mathbb{C}) \times GL(2;\mathbb{C})], \quad SL(n+2;\mathbb{R})/S[GL(n;\mathbb{R}) \times GL(2;\mathbb{R})] \\ &SL(n'+1;\mathbb{H})/S[GL(n';\mathbb{H}) \times GL(1;\mathbb{H})] \text{ where } n = 2n' \\ &SU(n-a+b, 2-b+a)/S[U(n-a,a) \times U(b,2-b)] \text{ for } a \leqq n \text{ and } b \leqq 2. \\ &SL(4;\mathbb{R})/GL'(2;\mathbb{C}), \quad SU^*(4)/[GL'(2;\mathbb{C})], \quad SU(2,2)/[SL(2;\mathbb{C}) \times \mathbb{R}]. \end{aligned}$$

where $GL'(m;\mathbb{C}) := \{g \in GL(m;\mathbb{C}) \mid |\det(g)| = 1\}$ and $GL(k;\mathbb{H}) := SL(k;\mathbb{H}) \times \mathbb{R}^+$. The real form family of $M_2 = Sp(m+1)/[Sp(m) \times Sp(1)]$ consists of the

$$\begin{aligned} &Sp(m+1;\mathbb{C})/[Sp(m;\mathbb{C}) \times Sp(1;\mathbb{C})], \quad Sp(m+1;\mathbb{R})/[Sp(m;\mathbb{R}) \times Sp(1;\mathbb{R})], \text{ and} \\ &Sp(m-a+b, 1-b+a)/[Sp(m-a,a) \times Sp(b,1-b)] \text{ for } 0 \leqq a \leqq m \text{ and } 0 \leqq b \leqq 1. \end{aligned}$$

Fitting these together, the real form family of $M_u = [SU(n+2) \times Sp(m+1)]/[U(n) \times SU(2) \times Sp(m)]$ consists of the

$$\begin{aligned} &[SL(n+2;\mathbb{C}) \times Sp(m+1;\mathbb{C})]/[GL(n;\mathbb{C}) \times SL(2;\mathbb{C}) \times Sp(m;\mathbb{C})] \\ &[SL(n+2;\mathbb{R}) \times Sp(m+1;\mathbb{R})]/[GL(n;\mathbb{R}) \times SL(2;\mathbb{R}) \times Sp(m;\mathbb{R})] \\ &[SU(n-a_1+b_1, 2-b_1+a_1) \times Sp(m-a_2+b_2, 1-b_2+a_2)] \\ &\quad /[U(n-a_1,a_1) \times SU(2) \times Sp(m-a_2,a_2)] \\ &\qquad \text{where } 0 \leqq a_1 \leqq n, 0 \leqq a_2 \leqq m, b_1 = 0,2, b_2 = 0,1 \\ &[SU(n+1-a, a+1) \times Sp(m+1;\mathbb{R})]/[U(n-1,a) \times SU(1,1) \\ &\quad \times Sp(m;\mathbb{R})] \text{ for } 0 \leqq a \leqq n \\ &[SL(4;\mathbb{R}) \times Sp(m+1;\mathbb{C})]/[GL'(2;\mathbb{C}) \times Sp(m;\mathbb{C})], \\ &\quad [SU^*(4) \times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C}) \times Sp(m;\mathbb{C}) \times T] \\ &[SU(2,2) \times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C}) \times Sp(m;\mathbb{C}) \times \mathbb{R}] \end{aligned}$$

This list is not convenient for analysis of the metric irreducible subspaces of the tangent space, so we refine it as follows.

$$
\begin{aligned}
&\text{(i) } [SL(n+2;\mathbb{C}) \times Sp(m+1;\mathbb{C})]/[GL(n;\mathbb{C}) \times SL(2;\mathbb{C}) \times Sp(m;\mathbb{C})]\\
&\text{(ii) } [SL(n+2;\mathbb{R}) \times Sp(m+1;\mathbb{R})]/[GL(n;\mathbb{R}) \times SL(2;\mathbb{R}) \times Sp(m;\mathbb{R})]\\
&\text{(iii) } [SL(n'+1;\mathbb{H}) \times Sp(m-a,1+a)]/[GL(n';\mathbb{H})\\
&\qquad \times SU(2) \times Sp(m-a,a)] \text{ for } 0 \leqq a \leqq n\\
&\text{(iv) } [SL(n'+1;\mathbb{H}) \times Sp(m-a+1,a)]/[GL(n';\mathbb{H})\\
&\qquad \times SU(2) \times Sp(m-a,a)] \text{ for } 0 \leqq a \leqq n\\
&\text{(v) } [SU(n-a_1+b_1, 2-b_1+a_1) \times Sp(m-a_2, 1+a_2)]\\
&\quad /[U(n-a_1,a_1) \times SU(2) \times Sp(m-a_2,a_2)],\\
&\qquad \text{where } 0 \leqq a_1 \leqq n, 0 \leqq a_2 \leqq m, b_1 \in \{0,2\} \qquad (4.6)\\
&\text{(vi) } [SU(n-a_1+b_1, 2-b_1+a_1) \times Sp(m-a_2+1, a_2)]\\
&\quad /[U(n-a_1,a_1) \times SU(2) \times Sp(m-a_2,a_2)],\\
&\qquad \text{where } 0 \leqq a_1 \leqq n, 0 \leqq a_2 \leqq m, b_1 \in \{0,2\}\\
&\text{(vii) } [SU(n+1-a,a+1) \times Sp(m+1;\mathbb{R})]/[U(n-a,a)\\
&\qquad \times SU(1,1) \times Sp(m;\mathbb{R})] \text{ for } 0 \leqq a \leqq n\\
&\text{(viii) } [SL(4;\mathbb{R}) \times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C}) \times Sp(m;\mathbb{C}) \times T]\\
&\text{(ix) } [SU^*(4) \times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C}) \times Sp(m;\mathbb{C}) \times T]\\
&\text{(x) } [SU(2,2) \times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C}) \times Sp(m;\mathbb{C}) \times \mathbb{R}]
\end{aligned}
$$

In case (i), the metric irreducible subspaces of the real tangent space have signatures $(3,3)$, $(4n,4n)$ and $(4m,4m)$. In case (ii), those signatures are $(2,1)$, $(2n,2n)$ and $(2m,2m)$. In case (iii), those signatures are $(0,3)$, $(4n,4n)$ and $(4m-4a,4a)$. In case (iv), those signatures are $(0,3)$, $(4n,4n)$ and $(4a,4m-4a)$. In case (v), those signatures are $(0,3)$, $(4(n-a_1)-2b_1(n-2a_1), 2b_1(n-2a_1)+4a_1)$ and $(4m-4a_2, 4a_2)$. In case (vi), those signatures are $(0,3)$, $(4(n-a_1)-2b_1(n-2a_1), 2b_1(n-2a_1)+4a_1)$ and $(4a_2, 4m-4a_2)$. In case (vii), those signatures are $(2,1)$, $(2n,2n)$ and $(2m,2m)$. In case (viii), those signatures are $(3,3)$, $(6,2)$ and $(4m,4m)$. In case (ix), those signatures are $(3,3)$, $(2,6)$ and $(4m,4m)$. In case (x), those signatures are $(3,3)$, $(4,4)$ and $(4m,4m)$.

(17)
$$
\begin{tikzcd}[arrows=dash]
\mathfrak{su}(n+2) \arrow[d] \arrow[dr] & & \mathfrak{sp}(m+1) \arrow[dl] \arrow[d] \\
\mathfrak{su}(n) & \mathfrak{su}(2)=\mathfrak{sp}(1) & \mathfrak{sp}(m)
\end{tikzcd}
$$
$[SU(n+2) \times Sp(m+1)]/[SU(n) \times SU(2) \times Sp(m)]$

Using the calculations in Sect. 2, the real form family for $SU(n+2)/[SU(n) \times SU(2)]$ is

$$
\begin{aligned}
&SL(n+2;\mathbb{C})/[SL(n;\mathbb{C}) \times SL(2;\mathbb{C})], \quad SL(n+2;\mathbb{R})/[SL(n;\mathbb{R}) \times SL(2;\mathbb{R})],\\
&SL(4;\mathbb{R})/SL(2;\mathbb{C}),\ SU^*(4)/SL(2;\mathbb{C}), \quad SU(2,2)/SL(2;\mathbb{C}),\\
&SL(n'+1;\mathbb{H})/[SL(n';\mathbb{H}) \times SL(1;\mathbb{H})] \text{ where } n = 2n', \text{ and}\\
&SU(n-a+b, 2-b+a)/[SU(n-a,a) \times SU(b,2-b)] \text{ for } a \leqq n \text{ and } b \leqq 2.
\end{aligned}
$$

Combining that with the possibilities for $Sp(n+1)/[Sp(n) \times Sp(1)]$, and refining the result as appropriate for computation of the metric irreducible subspaces of the real tangent space, this case gives us

$$\begin{aligned}
&\text{(i)}\ [SL(n+2;\mathbb{C})\times Sp(m+1;\mathbb{C})]/[SL(n;\mathbb{C})\times SL(2;\mathbb{C})\times Sp(m;\mathbb{C})]\\
&\text{(ii)}\ [SL(n+2;\mathbb{R})\times Sp(m+1;\mathbb{R})]/[SL(n;\mathbb{R})\times SL(2;\mathbb{R})\times Sp(m;\mathbb{R})]\\
&\text{(iii)}\ [SL(n'+1;\mathbb{H})\times Sp(m-a,1+a)]\\
&\qquad /[SL(n';\mathbb{H})\times SU(2)\times Sp(m-a,a)] \text{ for } 0\leqq a\leqq n\\
&\text{(iv)}\ [SL(n'+1;\mathbb{H})\times Sp(m-a+1,a)]\\
&\qquad /[GL(n';\mathbb{H})\times SU(2)Sp(m-a,a)] \text{ for } 0\leqq a\leqq n\\
&\text{(v)}\ [SU(n-a_1+b_1,2-b_1+a_1)\times Sp(m-a_2,1+a_2)]\\
&\quad /[SU(n-a_1,a_1)\times SU(2)\times Sp(m-a_2,a_2)],\\
&\qquad \text{where } 0\leqq a_1\leqq n,\ 0\leqq a_2\leqq m,\ b_1\in\{0,2\} \qquad\qquad (4.7)\\
&\text{(vi)}\ [SU(n-a_1+b_1,2-b_1+a_1)\times Sp(m-a_2+1,a_2)]\\
&\quad /[SU(n-a_1,a_1)\times SU(2)\times Sp(m-a_2,a_2)],\\
&\qquad \text{where } 0\leqq a_1\leqq n,\ 0\leqq a_2\leqq m,\ b_1\in\{0,2\}\\
&\text{(vii)}\ [SU(n+1-a,a+1)\times Sp(m+1;\mathbb{R})]\\
&\qquad /[SU(n-a,a)\times SU(1,1)\times Sp(m;\mathbb{R})] \text{ for } 0\leqq a\leqq n\\
&\text{(viii)}\ [SL(4;\mathbb{R})\times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C})\times Sp(m;\mathbb{C})]\\
&\text{(ix)}\ [SU^*(4)\times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C})\times Sp(m;\mathbb{C})]\\
&\text{(x)}\ [SU(2,2)\times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C})\times Sp(m;\mathbb{C})]
\end{aligned}$$

In case (i), the metric irreducible subspaces of the real tangent space have signatures $(3,3)$, $(1,0)$, $(0,1)$, $(4n,4n)$ and $(4m,4m)$. In case (ii), those signatures are $(2,1)$, $(1,0)$, $(2n,2n)$ and $(2m,2m)$. In case (iii), those signatures are $(0,3)$, $(1,0)$, $(4n,4n)$ and $(4m-4a,4a)$. In case (iv), those signatures are $(0,3)$, $(1,0)$, $(4n,4n)$ and $(4a,4m-4a)$. In case (v), those signatures are $(0,3)$, $(0,1)$, $(4(n-a_1)-2b_1(n-2a_1), 2b_1(n-2a_1)+4a_1)$ and $(4m-4a_2,4a_2)$. In case (vi), those signatures are $(0,3)$, $(0,1)$, $(4(n-a_1)-2b_1(n-2a_1), 2b_1(n-2a_1)+4a_1)$ and $(4a_2,4m-4a_2)$. In case (vii), those signatures are $(2,1)$, $(0,1)$, $(2n,2n)$ and $(2m,2m)$. In case (viii), those signatures are $(3,3)$, $(0,1)$, $(6,2)$ and $(4m,4m)$. In case (ix), those signatures are $(3,3)$, $(0,1)$, $(2,6)$ and $(4m,4m)$. In case (x), those signatures are $(3,3)$, $(1,0)$, $(4,4)$ and $(4m,4m)$.

(18)
$$\begin{tikzcd}[arrows=dash]
\mathfrak{sp}(n+1) \arrow[d] \arrow[dr] & & \mathfrak{sp}(m+1) \arrow[dl] \arrow[d] \\
\mathfrak{sp}(n) & \mathfrak{sp}(1) & \mathfrak{sp}(m)
\end{tikzcd}
\qquad \frac{Sp(n+1)\times Sp(m+1)}{Sp(n)\times Sp(m)\times Sp(1)}$$

Here are the possibilities for real forms:

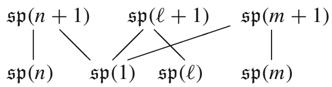
$$\begin{aligned}
&\text{(i)}\ [Sp(n+1;\mathbb{C})\times Sp(m+1;\mathbb{C})]/[Sp(n;\mathbb{C})\times Sp(1;\mathbb{C})\times Sp(m;\mathbb{C})]\\
&\text{(ii)}\ [Sp(n+1;\mathbb{R})\times Sp(m+1;\mathbb{R})]/[Sp(n;\mathbb{R})\times Sp(1;\mathbb{R})\times Sp(m;\mathbb{R})]\\
&\text{(iii)}\ [Sp(n-a_1+b_1,1-b_1+a_1)\times Sp(m-a_2+b_2,1-b_2+a_2)]\\
&\qquad /[Sp(n-a_1,a_1)\times Sp(1)\times Sp(m-a_2,a_2)] \text{ where } 0\leqq a_1\leqq n, \qquad\qquad (4.8)\\
&\qquad 0\leqq a_2\leqq m,\ b_1,b_2\in\{0,1\}\\
&\text{(iv)}\ Sp(n+1;\mathbb{C})/[Sp(n;\mathbb{C})\times Sp(1)] \text{ where } m=n\\
&\text{(v)}\ Sp(n+1;\mathbb{C})/[Sp(n;\mathbb{C})\times Sp(1;\mathbb{R})] \text{ where } m=n
\end{aligned}$$

The first three correspond to inner automorphisms of G_u, preserving each simple factor, and the last two to an involutive automorphism α that interchanges the two simple factors. Then α is given by the interchange $(x_1, x_2) \mapsto (x_2, x_1)$ and is the identity on the common $Sp(1)$ factor of H_u, so the corresponding real form G/H is given by $G = Sp(n+1; \mathbb{C})$ of $G_{\mathbb{C}}$ and $H = Sp(n; \mathbb{C}) \times Sp(1)$ of $H_{\mathbb{C}}$. In detail we are using

Lemma 4.9 *Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be Lie algebras, $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, and α an involutive automorphism of $\mathfrak{m}$ that exchanges the $\mathfrak{m}_i$ (so $\mathfrak{m}_1 \cong \mathfrak{m}_2$). Write $\mathfrak{m} = \mathfrak{m}_+ + \mathfrak{m}_-$, sum the (± 1)-eigenspaces of α. Then the corresponding Lie algebra $\mathfrak{m}_+ + \sqrt{-1}\,\mathfrak{m}_- \cong (\mathfrak{m}_1)_{\mathbb{C}}$ as a real Lie algebra.*

Proof Identify the $\mathfrak{m}_i$ by means of α, so $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_1$ with α given by $\alpha(x, y) = (y, x)$. Then $\mathfrak{m}_+ = \{(x, x) \mid x \in \mathfrak{m}_1\}$ and $\mathfrak{m}_- = \{(y, -y) \mid y \in \mathfrak{m}_1\}$, and $\mathfrak{m}_+ + \sqrt{-1}\,\mathfrak{m}_- = \{(x, x) + \sqrt{-1}\,(y, -y)\} = \{(z, \overline{z}) \mid z \in (\mathfrak{m}_1)_{\mathbb{C}}\}$, which is isomorphic to $(\mathfrak{m}_1)_{\mathbb{C}}$ as a real Lie algebra. □

In case (i), the metric irreducible subspaces of the real tangent space are of signatures $(3, 3)$, $(4n, 4n)$ and $(4m, 4m)$. In case (ii), those signatures are $(2, 1)$, $(2n, 2n)$ and $(2m, 2m)$. In case (iii), those signatures are $(0, 3)$, $(4(n - a_1) - 4b_1(n - 2a_1), 4a_1 + 4b_1(n - 2a_1))$ and $(4(m - a_2) - 4b_2(m - 2a_2), 4a_2 + 4b_2(m - 2a_2))$. In case (iv), those signatures are $(3, 0)$ and $(4n, 4n)$. In case (v), those signatures are $(1, 2)$ and $(4n, 4n)$.

(19) 

$\frac{Sp(n+1)\times Sp(\ell+1)\times Sp(m+1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)}$

Let $M_u = G_u/H_u = [Sp(n+1) \times Sp(\ell+1) \times Sp(m+1)]/[Sp(n) \times Sp(\ell) \times Sp(m) \times Sp(1)]$, $n \leqq \ell \leqq m$. Then $M_{1,u} = Sp(n+1)/[Sp(n) \times Sp(1)]$, $M_{2,u} = Sp(\ell+1)/[Sp(\ell) \times Sp(1)]$ and $M_{3,u} = Sp(m+1)/[Sp(n) \times Sp(1)]$. Let α be an involutive automorphism of G_u. It induces a permutation $\overline{\alpha}$ of $\{M_{1,u}, M_{2,u}, M_{3,u}\}$. Up to conjugacy, and using $\alpha^2 = 1$, the possibilities are (a) α is inner and $\overline{\alpha} = 1$, and (b) α is outer, $n = \ell$, $\overline{\alpha}$ exchanges $M_{1,u}$ and $M_{2,u}$, and $\overline{\alpha}(M_{3,u}) = M_{3,u}$. In case (b) we argue as in (4.8). Now the possibilities for $M = G/H$ are

$$
\begin{aligned}
&\text{(i) } [Sp(n+1; \mathbb{C}) \times Sp(\ell+1; \mathbb{C}) \times Sp(m+1; \mathbb{C})]/[Sp(n; \mathbb{C}) \\
&\qquad \times Sp(\ell; \mathbb{C}) \times Sp(m; \mathbb{C}) \times Sp(1; \mathbb{C})] \\
&\text{(ii) } [Sp(n+1; \mathbb{R}) \times Sp(\ell+1; \mathbb{R}) \times Sp(m+1; \mathbb{R})]/[Sp(n; \mathbb{R}) \\
&\qquad \times Sp(\ell; \mathbb{R}) \times Sp(m; \mathbb{R}) \times Sp(1; \mathbb{R})] \\
&\text{(iii) } [Sp(n - a_1 + b_1, 1 - b_1 + a_1) \times Sp(\ell - a_2 + b_2, 1 - b_2 + a_2) \\
&\qquad \times Sp(m - a_3 + b_3, 1 - b_3 + a_3)] \\
&\qquad /[Sp(n - a_1, a_1) \times Sp(\ell - a_2, a_2) \times Sp(m - a_3, a_3) \times Sp(1)] \qquad (4.10)\\
&\qquad\quad \text{where } 0 \leqq a_1 \leqq n, 0 \leqq a_2 \leqq \ell, 0 \leqq a_3 \leqq m, b_1, b_2, b_3 \in \{0, 1\} \\
&\text{(iv) } [Sp(n+1; \mathbb{C}) \times Sp(m+1; \mathbb{R})]/[Sp(n; \mathbb{C}) \times Sp(1; \mathbb{R}) \times Sp(m; \mathbb{R})] \text{ if } n = \ell \\
&\text{(v) } [Sp(n+1; \mathbb{C}) \times Sp(m+1-a, a)]/[Sp(n; \mathbb{C}) \times Sp(1) \times Sp(m-a, a)], \\
&\qquad 0 \leqq a \leqq m, \quad \text{if } n = \ell \\
&\text{(vi) } [Sp(n+1; \mathbb{C}) \times Sp(m-a, a+1)]/[Sp(n; \mathbb{C}) \times Sp(1) \times Sp(m-a, a)], \\
&\qquad 0 \leqq a \leqq m, \quad \text{if } n = \ell
\end{aligned}
$$

Here the first three cases correspond to inner automorphisms, case (a), and the remaining three correspond to outer automorphisms α, case (b). There we apply Lemma 4.9 to the interchange $G_{1,u} \leftrightarrow G_{2,u}$ defined by α, $\alpha|_{G_{3,u}}$ is any involutive automorphism.

In case (i), the signatures of the metric irreducible subspaces of the real tangent space of $M = G/H$ are $(3, 3)$, $(3, 3)$, $(4n, 4n)$, $(4l, 4l)$ and $(4m, 4m)$. In case (ii) those signatures are $(2, 1)$, $(2, 1)$, $(2n, 2n)$, $(2\ell, 2\ell)$ and $(2m, 2m)$. In case (iii) those signatures are $(0, 3)$, $(0, 3)$, $(4(n-a_1) - 4b_1(n-2a_1), 4a_1 + 4b_1(n-2a_1))$, $(4(\ell - a_2) - 4b_2(\ell - 2a_2), 4a_2 + 4b_2(\ell - 2a_2))$ and $(4(m-a_3) - 4b_3(m-2a_3), 4a_3 + 4b_3(m-2a_3))$. In case (iv) those signatures are $(1, 2)$, $(2, 1)$, $(4n, 4n)$ and $(2m, 2m)$. In case (v) those signatures are $(3, 0)$, $(0, 3)$, $(4n, 4n)$ and $(4a, 4m-4a)$. In case (vi) those signatures are $(3, 0)$, $(0, 3)$, $(4n, 4n)$ and $(4m-4a, 4a)$.

$$(20)\qquad \begin{array}{ccccccc} \mathfrak{sp}(n+1) & & \mathfrak{sp}(2) & & \mathfrak{sp}(m+1) \\ | & \diagdown & \diagup \quad \diagdown & \diagup & | \\ \mathfrak{sp}(n) & \mathfrak{sp}(1) & & \mathfrak{sp}(1) & \mathfrak{sp}(m) \end{array} \qquad \frac{Sp(n+1)\times Sp(2)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$$

The real form family members defined by involutive inner automorphisms of G_u are straightforward now. If $m = n$ we also have the automorphism α that is the interchange $Sp(n+1) \leftrightarrow Sp(m+1)$ and preserves $Sp(2)$. Then $Sp(2)$ goes to a real form of $Sp(2;\mathbb{C})$ that contains $Sp(1;\mathbb{C})$ as a symmetric subgroup. Again making use of Lemma 4.9, the result is

$$\begin{aligned}
&\text{(i)}\ [Sp(n+1;\mathbb{C}) \times Sp(2;\mathbb{C}) \times Sp(m+1;\mathbb{C})]/\\
&\qquad [Sp(n;\mathbb{C}) \times Sp(1;\mathbb{C}) \times Sp(1;\mathbb{C}) \times Sp(m;\mathbb{C})]\\
&\text{(ii)}\ [Sp(n+1;\mathbb{R}) \times Sp(2;\mathbb{R}) \times Sp(m+1;\mathbb{R})]/\\
&\qquad [Sp(n;\mathbb{R}) \times Sp(1;\mathbb{R}) \times Sp(1;\mathbb{R}) \times Sp(m;\mathbb{R})]\\
&\text{(iii)}\ [Sp(n-a_1+b_1, 1-b_1+a_1) \times Sp(1,1) \times Sp(m-a_2+b_2, 1-b_2+a_2)]\\
&\qquad /[Sp(n-a_1, a_1) \times Sp(1) \times Sp(1) \times Sp(m-a_2, a_2)] \qquad\qquad (4.11)\\
&\qquad \text{where } 0 \leqq a_1 \leqq n,\ 0 \leqq a_2 \leqq m, \text{ and } b_1,\ b_2 \in \{0, 1\}\\
&\text{(iv)}\ [Sp(n-a_1+b_1, 1-b_1+a_1) \times Sp(2) \times Sp(m-a_2+b_2, 1-b_2+a_2)]\\
&\qquad /[Sp(n-a_1, a_1) \times Sp(1) \times Sp(1) \times Sp(m-a_2, a_2)]\\
&\qquad \text{where } 0 \leqq a_1 \leqq n,\ 0 \leqq a_2 \leqq m, \text{ and } b_1,\ b_2 \in \{0, 1\}\\
&\text{(v)}\ [Sp(n+1;\mathbb{C}) \times Sp(2;\mathbb{R})]/[Sp(n;\mathbb{C}) \times Sp(1;\mathbb{C})] \text{ where } m = n\\
&\text{(vi)}\ [Sp(n+1;\mathbb{C}) \times Sp(1,1)]/[Sp(n;\mathbb{C}) \times Sp(1;\mathbb{C})] \text{ where } m = n
\end{aligned}$$

In case (i), the metric irreducible subspaces of the real tangent space have signatures $(3, 3)$, $(3, 3)$, $(4n, 4n)$, $(4, 4)$ and $(4m, 4m)$. In case (ii), those signatures are $(2, 1)$, $(2, 1)$, $(2n, 2n)$, $(2, 2)$ and $(2m, 2m)$. In case (iii), those signatures are $(0, 3)$, $(0, 3)$, $(4(n-a_1)-4b_1(n-2a_1), 4a_1 + 4b_1(n-2a_1))$, $(4, 0)$, and $(4(m-a_2) - 4b_2(m-2a_2), 4a_2 + 4b_2(m-2a_2))$. In case (iv), those signatures are $(0, 3)$, $(0, 3)$, $(4(n-a_1) - 4b_1(n-2a_1), 4a_1 + 4b_1(n-2a_1))$, $(0, 4)$, and $(4(m-a_2) - 4b_2(m-2a_2), 4a_2 + 4b_2(m-2a_2))$. In case (v), those signatures are $(3, 3)$, $(4n, 4n)$ and $(3, 1)$. In case (vi), those signatures are $(3, 3)$, $(4n, 4n)$ and $(1, 3)$.

We summarize the computations for G semisimple but not simple, except for item (21), in Table 2, located in Sect. 6 below.

(21)	$\begin{array}{cccc} & \mathfrak{g}_1 & \cdots & \mathfrak{g}_n \\ \diagup & \vert & \cdots & \vert \\ \mathfrak{z}_{\mathfrak{h}} & \mathfrak{h}'_1 & \cdots & \mathfrak{h}'_n \end{array}$	$\{(G_{u,1}/H'_{u,1}) \times \cdots \times (G_{u,n}/H'_{u,n})\}/diag(Z_{H_u})$

Case (21) requires some discussion. To pass to the group level we assume that the semisimple groups $G_{u,i}$ are compact and simply connected, so $G_u = \prod G_{u,i}$ also is compact and simply connected, that the central subgroup $Z_{\mathfrak{h}}$ of H is connected, and that H_u is connected. Thus $M_u = G_u/H_u$ is simply connected Let $H_{u,i}$ be the projection of H_u to $G_{u,i}$, say $H_{u,i} = H'_{u,i} Z_{u,i}$ where $Z_{u,i} = p_i(Z_{\mathfrak{h}})$ is the projection of $Z_{\mathfrak{h}}$ to H_u. Then $M_{u,i} = G_{u,i}/H_{u,i}$ is weakly symmetric with non-semisimple isotropy $H_{u,i}$. Further, each $M_{u,i}$ is symmetric, or is the complexification of $M_{u,i}$, or is one of the spaces of Cases (1) through (20) of Tables 1 and 2. Combining these requirements, each $M_{u,i} = G_{u,i}/H_{u,i}$ is one of the following:

- a compact irreducible hermitian symmetric space, or
- one of the spaces of cases (5), (8), (11) or (12) in Table 1, or
- one of the spaces of cases (13) or (16) in Table 2.

Thus either $M_i = G_i/H_i$ is on Berger's list of pseudo-Riemannian symmetric spaces, or it is listed under Case (5), (8), (11) or (12) in Table 1, or it is listed under Case (13) or (16) in Table 2.

Let $M = G/H$ be a pseudo-Riemannian weakly symmetric space with the same complexification as $M_u = G_u/H_u$. Then M corresponds to an involutive automorphism σ of $\mathfrak{g}_u$ that preserves $\mathfrak{h}_u$. It necessarily preserves $\mathfrak{z}_{\mathfrak{h}}$ as well. Now permute the simple factors $\mathfrak{g}_{u,i}$ of $\mathfrak{g}_u$ so that σ exchanges $\mathfrak{g}_{u,2i-1}$ and $\mathfrak{g}_{u,2i}$ for $2i \leqq s$ and preserves each $\mathfrak{g}_{u,i}$ for $s < i \leqq s+t$. For $j = 2i \leqq s$ we then have $(\mathfrak{g}_{j,\mathbb{C}}, \mathfrak{h}_{j,\mathbb{C}})$ corresponding to indices $(2i-1, i)$, and for $i > s$ we have $(\mathfrak{g}_i, \mathfrak{h}_i)$ where $\mathfrak{g}_i$ (resp. $\mathfrak{h}_i$, resp. $\mathfrak{z}_i$) is a real form of $\mathfrak{g}_{u,i,\mathbb{C}}$ (resp $\mathfrak{h}_{u,i,\mathbb{C}}$, resp. $\mathfrak{z}_{u,i,\mathbb{C}}$). It is implicit here that σ preserves the center Z_{H_u} of H_u so that Z_{H_u} is a subgroup of the center $\widetilde{Z_{H_u}} = \prod Z_{H_{u,i}}$ of $\widetilde{H_u} = \prod H_{u,i}$. Thus $H_u \subset \widetilde{H_u}$ and we have

$$\varphi_u : M_u = G_u/H_u \to G_u/\widetilde{H}_u = \widetilde{M}_u \text{ by } gH_u \mapsto g\widetilde{H}_u$$
$$\text{where } \widetilde{M}_u = \prod M_{u,i} \text{ and } \widetilde{H}_u = \prod H_{u,i}. \tag{4.12}$$

Since everything is σ-stable here, $H \subset \widetilde{H}$ where $\widetilde{H} = \prod H_i$ and we have a well defined projection

$$\varphi : M = G/H \to G/\widetilde{H} = \widetilde{M} \text{ by } gH \mapsto g\widetilde{H} \text{ where } \widetilde{M} = \prod M_i \text{ and } \widetilde{H} = \prod H_i. \tag{4.13}$$

Conversely, let $M_i = G_i/H_i$ be irreducible weakly symmetric pseudo-Riemannian manifolds, not all symmetric, where each G_i is semisimple but each H_i has center Z_i of dimension 1. Thus each Z_i^0 is a circle group or the multiplicative group of positive reals or (if G_i is complex) the multiplicative group $\mathbb{C}^*$, and each $H_i = H'_i Z_i^0$ with H'_i semisimple. Suppose that $G = G_1 \times \cdots \times G_\ell$ has a Cartan involution θ that preserves each G_i, each H_i and thus each Z_i^0. Then we have the compact real forms

$$G_u = \prod G_{u,i},\ \widetilde{H}_u = \prod H_{u,i},\ \widetilde{Z_u^0} = \prod Z_{u,i}^0, \text{ and } \widetilde{M}_u = \prod M_{u,i}$$

where $M_{u,i} = G_{u,i}/H_{u,i}$ and $Z_{u,i}$ is the center of $H_{u,i}$. Consider the set $\mathcal{S}$ of all closed connected θ-invariant subgroups $S_u \subset \widetilde{Z_u}$ such that the projections $S_u \to Z_{u,i}^0$ all are surjective. The set $\mathcal{S}$ is nonempty—it contains $\widetilde{Z_u^0}$—so it has elements of minimal dimension.

Let Z_u denote one of them and define $H = (\prod H_i')Z_u$. Then $M = G/H$ belongs to the real form family of Case (21). This constructs every element in that real form family.

Following [12, Proposition 12.8.4] and Tables 1 and 2, the metric signature of the weakly symmetric pseudo-Riemannian manifold G/H in the real form family of Case (21) is given as follows. First, we have the metric irreducible subspaces $S_{i,j}$ of the real tangent space of G_i/H_i, and their signatures $(a_{i,j}, b_{i,j})$. That gives us the metric irreducible subspaces, with their signatures, for $G/\widetilde{H}$. To this collection we add the metric irreducible subspaces of the fiber $\widetilde{\mathfrak{h}}/\mathfrak{h}$ of $\widetilde{\mathfrak{h}} \to \mathfrak{h}$ implicit in (4.13).

5 Special signatures: Riemannian, Lorentz, and trans-Lorentz

We go through Berger's classification [1] and our Tables 1 and 2 to pick out the cases where $M = G/K$ can have an invariant weakly symmetric pseudo-Riemannian metric of signature $(n, 0)$, $(n-1, 1)$ or $(n-2, 2)$. Of course this gives the classification of the weakly symmetric pseudo-Riemannian manifolds of those signatures with G semisimple and H reductive in G; they are certain products $G/H = \prod G_i/H_i$ from Berger [1] for the pseudo-Riemannian symmetric cases and from Tables 1 and 2 for the nonsymmetric pseudo-Riemannian weakly symmetric cases.

We will refer to $(n, 0)$, $(n-1, 1)$ and $(n-2, 2)$ as *special signatures*. Now we run through the cases of Table 1, then the cases of Table 2, and finally the symmetric cases from [1].

From Table 1.

Case (1): Since $m > n \geqq 1$ we know $mn \geqq 2$. Then, of the first three cases, only $SL(3;\mathbb{R})/SL(2;\mathbb{R})$ can have special signature; it is $(3, 2)$.

For the fourth case of Case (1), $\frac{SU(m-k+\ell,n-\ell+k)}{SU(m-k,k)\times SU(\ell,n-\ell)}$, both $2m\ell+2nk-4k\ell = 2(m-k)\ell + 2(n-\ell)k$ and $2mn-2m\ell-2nk+4k\ell = 2(m-k)(n-\ell)+2k\ell$ are even, so it is enough to see when one of them is 0 or 2. If $2(m-k)\ell + 2(n-\ell)k = 0$, then $2(m-k)\ell = 0$ and $2(n-\ell)k = 0$, so $\ell = 0$ or $k = m$, and $k = 0$ or $\ell = n$. If $k = \ell = 0$, or if $k = m$ and $\ell = n$, then the metric irreducibles have signatures $(2mn, 0)$ and $(0, 1)$; the other two cases of (k, ℓ) trivialize M. That leaves us with $\frac{SU(m,n)}{SU(m)\times SU(n)}$, which has invariant metrics of signatures $(2mn+1, 0)$ and $(2mn, 1)$.

If $2(m-k)\ell+2(n-\ell)k = 2$, then $2(m-k)\ell = 0$ and $2(n-\ell)k = 2$, or $2(m-k)\ell = 2$ and $2(n-\ell)k = 0$. If $2(m-k)\ell = 2$ then $(m-k)\ell = 1$, and either $n = \ell$ or $k = 0$; if $k = 0$ then $m = \ell = 1$ and we have $\frac{SU(2,n-1)}{SU(1)\times SU(1,n-1)}$ if $n = \ell$ then $(m-k) = \ell = 1$ and we have $\frac{SU(2,k)}{SU(1,k)\times SU(1)}$.

Since $m \geqq 2$, we then have $k = n = 1$ and $\ell = 0$, or $k = m-1$ and $\ell = n = 1$; then $SU(m-1, 2)/SU(m-1, 1)$ has invariant metric of signature $(2m-1, 2)$. If $2(m-k)(n-\ell) + 2k\ell = 0$, then $k = 0$ and $\ell = n$, or $\ell = 0$ and $k = m$. As expected this shows that $SU(m+n)/SU(m)\times SU(n)$ has metrics of signatures $(2mn+1, 0)$ and $(2mn, 1)$. If $2(m-k)(n-\ell) + 2k\ell = 2$, then $k = \ell = n = 1$, or $\ell = 0,\ k = m-1,\ n = 1$. Then $SU(m, 1)/SU(m-1, 1)$ has a metric of signature $(2m-1, 2)$. Summarizing,

$$
\begin{aligned}
&SL(3;\mathbb{R})/SL(2;\mathbb{R}):\ (3,2)\\
&SU(m+n)/[SU(m)\times SU(n)]:\ (2mn+1,0),\ (2mn,1)\\
&SU(m,n)/[SU(m)\times SU(n)]:\ (2mn+1,0),\ (2mn,1)\\
&SU(n-1,2)/SU(n-1,1):\ (2n-1,2)\\
&SU(n,1)/SU(n-1,1):\ (2n-1,2)
\end{aligned}
$$

Case (2): Here n is odd and $\geqq 5$ by (2.1). The first and fourth cases are excluded because $\frac{1}{2}n(n-1) \leqq 2$ would give $n < 3$, so we only need to discuss the second and third cases. There $(k(k-1)+\ell(\ell-1), 2k\ell)$ is the signature. Since $k(k-1)+\ell(\ell-1) \geqq \frac{1}{2}(k+\ell)^2 - n = \frac{n^2}{2} - n > 2$ we are reduced to considering $2k\ell \leqq 2$. if $k = 0$ then $\ell = n$, G/H is $SO^*(2n)/SU(n)$ or $SO(2n)/SU(n)$, n odd, and the possible signatures are $(n(n-1)+1, 0)$ and $(n(n-1), 1)$. It is the same for $\ell = 0$. Now we may suppose $k\ell > 0$; so $k = \ell = 1$ because $2k\ell \leqq 2$. So $n = k + \ell = 2$. But n is odd. Summarizing, we have

$$SO^*(2n)/SU(n),\ \ SO(2n)/SU(n):\ (n(n-1)+1, 0),\ \ (n(n-1), 1)$$

Case (3): From Table 1, the spaces $E_6/Spin(10)$ and $E_{6,D_5T_1}/Spin(10)$ have invariant metrics of special signatures only for signatures $(33, 0)$ and $(32, 1)$.

Case (4): We may assume $n \geqq 2$, so the first three cases of Case (4) are excluded. For the fourth, $\frac{SU(2n+1-2\ell,2\ell)}{Sp(n-\ell,\ell)}$, we need $\ell = n$ or $\ell = 0$, leading to signatures $(2n^2 + 3n - 1, 1)$ and $(2n^2 + 3n, 0)$. Summarizing,

$$\frac{SU(2n+1)}{Sp(n)},\ \ \frac{SU(2n,1)}{Sp(n)}:\ (2n^2+3n-1, 1),\ \ (2n^2+3n, 0).$$

Case (5): As above, $n \geqq 2$, and that excludes the first three cases of Case (5). For the fourth, $\frac{SU(2n+1-2\ell,2\ell)}{Sp(n-\ell,\ell)\times U(1)}$, we need $\ell = n$ or $\ell = 0$, leading to special signature $(2n^2 + 3n - 1, 0)$. Summarizing,

$$SU(2n+1)/[Sp(n)\times U(1)],\ \ SU(2n,1)/[Sp(n)\times U(1)]:\ (2n^2+3n-1, 0).$$

Case (6): The space $Spin(7)/G_2$ has an invariant metric of special signature $(7, 0)$.

Case (7): The space $G_2/SU(3)$ has an invariant metric of special signature $(6, 0)$, and the space $G_{2,A_1A_1}/SU(1, 2)$ has an invariant metric of special signature $(4, 2)$.

Case (8): The spaces $SO(10)/[Spin(7) \times SO(2)]$ and $SO(8, 2)/[Spin(7) \times SO(2)]$ each has an invariant metric of special signature $(23, 0)$.

Case (9): The spaces $SO(9)/Spin(7)$ and $SO(8, 1)/Spin(7)$ have invariant metrics of signatures $(15, 0)$.

Case (10): The spaces $Spin(8)/G_2$ and $Spin(7, 1)/G_2$ have invariant metrics of signatures $(14, 0)$.

Case (11): Here $n \geqq 2$. That excludes the first and fourth cases of Case (11). For the second case, if $k = 0$ or $k = n$, the signatures of the real tangent space irreducibles are $(2n, 0)$ and $(n^2 - n, 0)$, so the spaces $SO(2n+1)/U(n)$ and $SO(2n, 1)/U(n)$ have invariant metrics of special signature $(n^2 + n, 0)$; and $SO(5)/U(2)$ and $SO(4, 1)/U(2)$ have invariant metrics of special signature $(4, 2)$. If $k = 1$ or $k = n - 1$, the signatures of the irreducibles are $(2, 2n - 2)$ and $(2n - 2, (n - 1)(n - 2))$, leading to $n = 2$ where $SO(3, 2)/U(1, 1)$ has metrics of special signature $(4, 2)$. If $1 < k < n - 1$ there is no invariant metric of special signature. Summarizing,

$$SO(2n+1)/U(n),\ \ SO(2n,1)/U(n):\ (n^2+n, 0)$$
$$SO(5)/U(2),\ \ SO(4,1)/U(2),\ \ SO(3,2)/U(1,1):\ (4, 2)$$

Case (12): Here $n \geqq 3$. That excludes the first and fourth cases of Case (12). In the second and third cases we exclude the range $0 < k < n-1$ where $4k, 4n-4k-4 \geqq 4$. There, for $k = 0$ and $k = n-1$ we note that $Sp(n)/[Sp(n-1)\times U(1)]$ and $Sp(n-1, 1)/[Sp(n-1)\times U(1)]$ have metrics of special signatures $(4n - 2, 0)$ and $(4n - 4, 2)$.

From Table 2.

Case (13): Here $n \geqq 2$. That excludes the first and second cases of Case (13). It also excludes the possibility $k\ell \neq 0$ in the third and fourth cases. That leaves $G/H = \frac{SU(n)\times SU(n+1)}{SU(n)\times U(1)}$ and $G/H = \frac{SU(n)\times SU(n,1)}{SU(n)\times U(1)}$, which have invariant metrics of special signature $(n^2+2n-1, 0)$.

Case (14): Here $n \geqq 1$. That excludes the first two cases of Case (14). In the third case, $8b - 4b^2$ implies $b \leqq 2$, and $b = 1$ is excluded because of a metric irreducible $(4, 6)$. For $b = 0$ and $b = 2$, the signatures of the metric irreducibles are $(0, 10)$ and $(8n - 8a, 8a)$, so $a = 0$ or $a = n$, leading to

$$[Sp(n, 2) \times Sp(2)]/[Sp(n) \times Sp(2)] \text{ and } [Sp(n+2) \times Sp(2)]/[Sp(n) \times Sp(2)]$$

with invariant metric of special signature $(8n + 10, 0)$.

Case (15): Here $n \geqq 3$ since G is semisimple. That excludes the first case of Case (15). In the second and third cases $a = 0$ and $a = n$ lead to $[SO(n) \times SO(n, 1)]/SO(n)$ and $[SO(n) \times SO(n+1)]/SO(n)$ with invariant metric of special signature $(\frac{n(n+1)}{2}, 0)$, and the cases $a = 1$ and $a = n-1$ lead only to $[SO(2, 1) \times SO(2, 2)]/SO(2, 1)$ and $[SO(2, 1) \times SO(3, 1)]/SO(2, 1)$ with invariant metric of special signature $(4, 2)$. The cases $1 < a < n-1$ do not lead to special signature.

Case (16): Here $n + m \geqq 1$. The first, second, seventh, eighth, ninth and tenth cases of Case (16) are excluded at a glance, reducing the discussion to the third, fourth, fifth and sixth cases. The third and fourth require $n = 0$ and then further require $a = 0$ or $a = m$, leading to $[Sp(1) \times Sp(m, 1)]/[Sp(m) \times Sp(m)]$ and $[Sp(1) \times Sp(m+1)]/[Sp(m) \times Sp(m)]$ with invariant metrics of special signature $(3 + 4m, 0)$. These are in fact included in the fifth and sixth cases. For the fifth and sixth cases, we must have $a_1 = 0$ or $a_1 = n$, and $a_2 = 0$ or $a_2 = m$. Then we arrive at the spaces

$$\frac{SU(n+2)\times Sp(m+1)}{U(n)\times SU(2)\times Sp(m)}, \quad \frac{SU(n,2)\times Sp(m+1)}{U(n)\times SU(2)\times Sp(m)}, \quad \frac{SU(n+2)\times Sp(m,1)}{U(n)\times SU(2)\times Sp(m)}, \quad \frac{SU(n,2)\times Sp(m,1)}{U(n)\times SU(2)\times Sp(m)},$$

which have invariant metrics of special signature $(4n + 4m + 3, 0)$.

Case (17): This essentially is a simplification of Case (16). By the considerations there, we have that the spaces

$$\frac{SU(n+2)\times Sp(m+1)}{SU(n)\times SU(2)\times Sp(m)}, \quad \frac{SU(n,2)\times Sp(m+1)}{SU(n)\times SU(2)\times Sp(m)}, \quad \frac{SU(n+2)\times Sp(m,1)}{SU(n)\times SU(2)\times Sp(m)}, \quad \frac{SU(n,2)\times Sp(m,1)}{SU(n)\times SU(2)\times Sp(m)}$$

have invariant metrics of special signatures $(4n + 4m + 4, 0)$ and $(4n + 4m + 3, 1)$.

Case (18): Here $n+m \geq 1$. The first, second, fourth and fifth cases of Case (18) are excluded at a glance, so we only need to consider the third case. There, the signatures of the irreducibles are $(0, 3)$, $(4(n - a_1) - 4b_1(n - 2a_1), 4a_1 + 4b_1(n - 2a_1))$ and $(4(m - a_2) - 4b_2(m - 2a_2), 4a_2 + 4b_2(m - 2a_2))$, where $b_1, b_2 = \{0, 1\}$. Thus we must have $a_1 = 0$ or $a_1 = n$, and $a_2 = 0$ or $a_2 = m$. That brings us to the spaces

$$\frac{Sp(n+1)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(m)}, \quad \frac{Sp(n,1)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(m)}, \quad \frac{Sp(n,1)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(m)}, \quad \frac{Sp(n+1)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(m)},$$

which have invariant metrics of special signature $(4n + 4m + 3, 0)$.

Case (19): The first case of Case (19) is excluded at a glance. Visibly, the second and fourth cases require $\ell = m = n = 0$, where G/H is $[Sp(1; \mathbb{R}) \times Sp(1; \mathbb{R}) \times Sp(1; \mathbb{R})]/Sp(1; \mathbb{R})$

or $[Sp(1;\mathbb{C}) \times Sp(1;\mathbb{R})]/Sp(1;\mathbb{R})$; they have invariant metrics of special signature $(4, 2)$. The fifth and sixth cases require $n = \ell = 0$ and $a = 0$ or $a = m$, leading to

$$[Sp(1;\mathbb{C}) \times Sp(m+1)]/[Sp(1) \times Sp(m)] \text{ and } [Sp(1;\mathbb{C}) \times Sp(m,1)]/[Sp(1) \times Sp(m)]$$

which have invariant metrics of special signature $(4m + 6, 0)$.

For the third case, we must have $a_1 = 0$ or $a_1 = n$, $a_2 = 0$ or $a_2 = \ell$, and $a_3 = 0$ or $a_3 = m$. In other words, G/H must be one of

$$\frac{Sp(n+1)\times Sp(\ell+1)\times Sp(m+1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)},\ \frac{Sp(n+1)\times Sp(\ell+1)\times Sp(m,1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)},$$
$$\frac{Sp(n+1)\times Sp(\ell,1)\times Sp(m+1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)},\ \frac{Sp(n+1)\times Sp(\ell,1)\times Sp(m,1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)},$$
$$\frac{Sp(n,1)\times Sp(\ell+1)\times Sp(m+1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)},\ \frac{Sp(n,1)\times Sp(\ell+1)\times Sp(m,1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)},$$
$$\frac{Sp(n,1)\times Sp(\ell,1)\times Sp(m+1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)},\ \frac{Sp(n,1)\times Sp(\ell,1)\times Sp(m,1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)},$$

These all have invariant metrics of special signature $(4n + 4\ell + 4m + 6, 0)$.

Case (20): The first, second, fifth and sixth cases are excluded at a glance. For the third and fourth cases, we must have $a_1 = 0$ or $a_1 = n$, and $a_2 = 0$ or $a_2 = m$. Then the spaces

$$\frac{Sp(n+1)\times Sp(1,1)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)},\ \frac{Sp(n,1)\times Sp(1,1)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)},$$
$$\frac{Sp(n,1)\times Sp(1,1)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)},\ \frac{Sp(n+1)\times Sp(1,1)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$$
$$\frac{Sp(n+1)\times Sp(2)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)},\ \frac{Sp(n,1)\times Sp(2)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)},$$
$$\frac{Sp(n,1)\times Sp(2)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)},\ \frac{Sp(n+1)\times Sp(2)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$$

have invariant metrics of special signature $(4n + 4m + 10, 0)$.

From Berger's [1, Table II].

The irreducible pseudo-Riemannian symmetric spaces G/H of [1] fall into two classes: the real form families for which G_u is simple and those for the compact group manifolds $G_u = L_u \times L_u$ where H_u is the diagonal $\delta L_u = \{(x, x) \mid x \in L_u\}$. First consider the group manifolds. There the real tangent space of G/H is $\mathfrak{m} = \{(\xi, -\xi) \mid \xi \in \mathfrak{l}\}$ and the invariant pseudo-Riemannian metrics come from multiples of the Killing form of $\mathfrak{l}$. Thus $G/H = (L \times L)/diag(L)$ has an invariant pseudo-Riemannian metric of special signature if and only if (i) the Killing form of $\mathfrak{l}$ is definite, or (ii) the Killing form of $\mathfrak{l}$ has signature $\pm(\dim \mathfrak{l} - 1, 1)$, or (iii) the Killing form of $\mathfrak{l}$ has signature $\pm(\dim \mathfrak{l} - 2, 2)$.

The case (i) is the case where G/H is a compact simple group manifold with bi-invariant metric. The cases (ii) and (iii) occurs only for the group manifold $SL(2;\mathbb{R})$ (up to covering); that group manifold has bi-invariant metrics of signatures $(2, 1)$ and $(1, 2)$.

For the moment we put the group manifold cases aside and consider the cases where G_u is simple. Start with the compact simple classical groups: $SU(n)$ for $n \geqq 2$, $Sp(n)$ for $n \geqq 2$, and $SO(n)$ for $n \geqq 7$.

For $G_u = SU(n)$, $n \geqq 2$, we have the following cases:

(1) $SL(n;\mathbb{R})/SO(n)$ and $SU(n)/SO(n)$ with signature $(\frac{n^2+n}{2} - 1, 0)$.
(2) $SL(2;\mathbb{C})/SO(2;\mathbb{C})$ with signature $(2, 2)$.
(3) $SL(2;\mathbb{R})/\mathbb{R}$ with signature $(1, 1)$.
(4) $SL(2;\mathbb{C})/SL(2;\mathbb{R})$(or $SL(2;\mathbb{C})/SU(1, 1)$) with signature $(2, 1)$.
(5) $SU^*(2n)/Sp(n)$ and $SU(2n)/Sp(n)$ with signature $(2n^2 - n - 1, 0)$.

(6) $SL(2;\mathbb{C})/SU^*(2)$ with signature $(3,0)$.
(7) $SU(m,n)/S(U(m)\times U(n))$ and $SU(m+n)/S(U(m)\times U(n))$ with signature $(2mn,0)$.
(8) $SL(n;\mathbb{C})/SU(n)$ with signature $(n^2-1,0)$.
(9) $SL(3;\mathbb{R})/[SL(2;\mathbb{R})\times R]$ with signature $(2,2)$.
(10) $SL(4;\mathbb{R})/Sp(2;\mathbb{R})$ with signature $(3,2)$.
(11) $SL(4;\mathbb{R})/GL'(2;\mathbb{C})$ with signature $(6,2)$.
(12) $SU^*(4)/Sp(1,1)$ with signature $(4,1)$.
(13) $SU^*(4)/GL'(2;\mathbb{C})$ with signature $(6,2)$.
(14) $SU(2,1)/SO(2,1)$ with signature $(3,2)$.
(15) $SU(2,2)/Sp(2;\mathbb{R})$ with signature $(3,2)$.
(16) $SU(2,2)/Sp(1,1)$ with signature $(4,1)$.
(17) We discuss the case $SU(m-a+b,n-b+a)/S(U(m-a,a)\times U(n-b,b))$ with signature $(2mn-2(m-a)b-2(n-b)a, 2(m-a)b+2(n-b)a) = 2(m-a)(n-b)+2ab, 2(m-a)b+2(n-b)a)$. If $(m-a)(n-b)+ab=0$, or $(m-a)b+(n-b)a=0$, we are in the Riemannian case (7) just above. If $(m-a)(n-b)+ab=1$, or $(m-a)b+(n-b)a=1$, we have $SU(n-1,2)/U(n-1,1)$ and $SU(n,1)/U(n-1,1)$ with signature $(2n-2,2)$.

For $G_u = SO(n)$, $n \geqq 7$, we have the following cases:

(1) $SO^*(2n)/U(n)$ and $SO(2n)/U(n)$ with signature $(n^2-n,0)$.
(2) $SO(m,n)/[SO(m)\times SO(n)]$ and $SO(m+n)/[SO(m)\times SO(n)]$ with signature $(mn,0)$.
(3) $SO(n;\mathbb{C})/SO(n)$ with signature $(\frac{n^2-n}{2},0)$.
(4) $SO(n-3,3)/[SO(n-3,1)\times SO(2)]$ and $SO(n-1,1)/[SO(n-3,1)\times SO(2)]$ with signature $(2n-6,2)$.
(5) Finally (for $SO(n)$) we discuss the case $SO(m-a+b,n-b+a)/[SO(m-a,a)\times SO(n-b,b)]$ with signature $(mn-(m-a)b-(n-b)a = (m-a)(n-b)+ab, (m-a)b+(n-b)a)$. We need to see when one of the above two numbers in the signature is 0, 1, or 2.

If $(m-a)(n-b)+ab=0$, or $(m-a)b+(n-b)a=0$, we are in case (2) of $SO(n)$ just above. If $(m-a)(n-b)+ab=1$ or $(m-a)b+(n-b)a=1$, we have $SO(n-1,2)/SO(n-1,1)$ and $SO(n,1)/SO(n-1,1)$ with invariant metric of signature $(n-1,1)$. The discussion for these cases is similar to case (17) for $SU(n)$ because the equations are the same.

Now we consider the cases where $(m-a)(n-b)+ab=2$ or $(m-a)b+(n-b)a=2$.

First let $(m-a)b+(n-b)a=2$. If $(m-a)b=1$ then $a=b=1$ and $m=n=2$, contradicting our assumption $n \geqq 7$. Thus either $(m-a)b=0$ and $(n-b)a=2$, or $(m-a)b=2$ and $(n-b)a=0$. Then we have the following solutions:

(1) $m=a=1, n=b+2$	(2) $m=a=2, n=b+1$
(3) $b=0, n=1, a=2$	(4) $b=0, n=2, a=1$
(5) $a=0, m=1, b=2$	(6) $a=0, m=2, b=1$
(7) $n=b=1, m-a=2$	(8) $n=b=2, m-a=1$

We may assume $m \leqq n$. As $m+n \geqq 7$ the solutions are (1), (2), (5) and 6. That leads us to

$$SO(n-1,3)/[SO(n-1,1)\times SO(2)] : (2n-2,2) \text{ and}$$
$$SO(n-2,3)/SO(n-2,2) : (n-2,2).$$

A similar discussion of the case $(m-a)(n-b)+ab=2$ leads to

$$SO(n+1,1)/[SO(n-1,1)\times SO(2)]:\ (2n-2,2) \text{ and}$$
$$SO(n-1,2)/SO(n-2,2):\ (n-2,2).$$

For $G_u = Sp(n)$, $n \geqq 2$, we have the following cases:

(1) $Sp(n;\mathbb{R})/U(n)$ and $Sp(n)/U(n)$ with signature $(n^2+n,0)$.
(2) $Sp(m,n)/[Sp(m)\times Sp(n)]$ and $Sp(m+n)/[Sp(m)\times Sp(n)]$ with signature $(4mn,0)$.
(3) $Sp(n;\mathbb{C})/Sp(n)$ with signature $(2n^2+n,0)$.
(4) $Sp(2;\mathbb{R})/[Sp(1;\mathbb{R})\times Sp(1;\mathbb{R})]$ with signature $(2,2)$.
(5) $Sp(2;\mathbb{R})/U(1,1)$ and $Sp(1,1)/U(1,1)$ with signature $(4,2)$.
(6) $Sp(2;\mathbb{R})/Sp(1;\mathbb{C})$ and $Sp(1,1)/Sp(1;\mathbb{C})$ with signature $(3,1)$.
(7) We discuss the case $Sp(m-a+b,n-b+a)/[Sp(m-a,a)\times Sp(n-b,b)]$ with signature $(4mn-4(m-a)b-4(n-b)a, 4(m-a)b+4(n-b)a)$. It is enough to discuss $4mn-4(m-a)b-4(n-b)a = 4(m-a)(n-b)+4ab = 0$ or $4(m-a)b+4(n-b)a = 0$. It gives case (2) just above.

Finally we look for special signature in real form families where G_u is a compact simple exceptional group.

(1) $G_2^*/[SU(2)\times SU(2)]$ and $G_2/[SU(2)\times SU(2)]$ with signature $(8,0)$.
(2) $F_{4,C_3A_1}/[Sp(3)\times SU(2)]$ and $F_4/[Sp(3)\times SU(2)]$ with signature $(28,0)$.
(3) $F_{4,B_4}/SO(9)$ and $F_4/SO(9)$ with signature $(16,0)$.
(4) $E_{6,C_4}/Sp(4)$ and $E_6/Sp(4)$ with signature $(42,0)$.
(5) $E_{6,A_5A_1}/[SU(6)\times SU(2)]$ and $E_6/[SU(6)\times SU(2)]$ with signature $(40,0)$.
(6) $E_{6,D_5T_1}/[SO(10)\times T_1]$ and $E_6/[SO(10)\times T_1]$ with signature $(32,0)$.
(7) $E_{6,F_4}/F_4$ and E_6/F_4 with signature $(26,0)$.
(8) $E_{7,A_7}/SU(8)$ and $E_7/SU(8)$ with signature $(70,0)$.
(9) $E_{7,D_6A_1}/[SO(12)\times SU(2)]$ and $E_7/[SO(12)\times SU(2)]$ with signature $(64,0)$.
(10) $E_{7,E_6T_1}/[E_6\times T_1]$ and $E_7/[E_6\times T_1]$ with signature $(54,0)$.
(11) $E_{8,D_8}/SO(16)$ and $E_8/SO(16)$ with signature $(128,0)$.
(12) $E_{8,E_7A_1}/[E_7\times SU(2)]$ and $E_8/[E_7\times SU(2)]$ with signature $(112,0)$.

6 Tables of Results

The basic special signature results are in Tables 3, 4 and 5 below. As indicated earlier, the semisimple Riemannian symmetric spaces are (up to local isometry) the products of spaces from Table 3, the semisimple Lorentzian spaces are (up to local isometry) the products of spaces from Table 3 and one space from Table 4, and the semisimple trans-Lorentzian spaces are (up to local isometry) the products of spaces from Table 3 and either one space from Table 5 or two spaces from Table 4.

Table 1 Weakly symmetric pseudo-Riemannian G/H, G simple, H reductive

G/H	Metric-irreducibles	Metric signatures
(1) Real form family of $SU(m+n)/[SU(m)\times SU(n)]$, $m>n \geqq 1$; $\tau = ((\tau_{\xi_1}\otimes\tau_{\xi_{m+n-1}})\oplus(\tau_{\xi_{m-1}}\otimes\tau_{\xi_{m+1}}))_{\mathbb{R}}\oplus\tau_{0,\mathbb{R}}$		
$\frac{SL(m+n;\mathbb{C})}{SL(m;\mathbb{C})\times SL(n;\mathbb{C})}$	$(2mn, 2mn)$, $(1,0)$, $(0,1)$	$(2mn+2, 2mn)$, $(2mn+1, 2mn+1)$, $(2mn, 2mn+2)$
$\frac{SL(m+n;\mathbb{R})}{SL(m;\mathbb{R})\times SL(n;\mathbb{R})}$	(mn, mn), $(1,0)$	$(mn+1, mn)$, $(mn, mn+1)$
$\frac{SL(m+n;\mathbb{H})}{SL(m;\mathbb{H})\times SL(n;\mathbb{H})}$	$(4mn, 4mn)$, $(1,0)$	$(4mn+1, 4mn)$, $(4mn, 4mn+1)$
$\frac{SU(m-k+\ell,n-\ell+k)}{SU(m-k,k)\times SU(\ell,n-\ell)}$	$(2m\ell+2nk-4k\ell, 2mn-2m\ell-2nk+4k\ell)$ $(0,1)$	$(2m\ell+2nk-4k\ell+1, 2mn-2m\ell-2nk+4k\ell)$ $(2m\ell+2nk-4k\ell, 2mn-2m\ell-2nk+4k\ell+1)$
(2) Real form family of $SO(2n)/SU(n)$; $\tau=(\tau_{\xi_2}\oplus\tau_{\xi_{n-2}})_{\mathbb{R}}\oplus\tau_{0,\mathbb{R}}$		
$SO(2n;\mathbb{C})/SL(n;\mathbb{C})$	$(n(n-1), n(n-1))$, $(1,0)$, $(0,1)$	$(n(n-1)+2, n(n-1))$ $(n(n-1)+1, n(n-1)+1)$ $(n(n-1), n(n-1)+2)$
$SO^*(2n)/SU(k,\ell)$	$(k(k-1)+\ell(\ell-1), 2k\ell)$, $(0,1)$	$(k(k-1)+\ell(\ell-1)+1, 2k\ell)$ $(k(k-1)+\ell(\ell-1), 2k\ell+1)$ $(2k\ell+1, k(k-1)+\ell(\ell-1))$ $(2k\ell, k(k-1)+\ell(\ell-1)+1)$
$SO(2k,2\ell)/SU(k,\ell)$	$((k(k-1)+\ell(\ell-1), 2k\ell)$, $(0,1)$	$(k(k-1)+\ell(\ell-1)+1, 2k\ell)$ $(k(k-1)+\ell(\ell-1), 2k\ell+1)$ $(2k\ell+1, k(k-1)+\ell(\ell-1))$ $(2k\ell, k(k-1)+\ell(\ell-1)+1)$
$SO(n,n)/SL(n;\mathbb{R})$	$(\frac{1}{2}n(n-1), \frac{1}{2}n(n-1))$, $(1,0)$	$(\frac{1}{2}n(n-1)+1, \frac{1}{2}n(n-1))$ $(\frac{1}{2}n(n-1), \frac{1}{2}n(n-1)+1)$
(3) Real form family of $E_6/Spin(10)$; $\tau=(\tau_{\xi_4}\oplus\tau_{\xi_5})_{\mathbb{R}}\oplus\tau_{0,\mathbb{R}}$		
$E_{6,\mathbb{C}}/Spin(10;\mathbb{C})$	$(32,32)$, $(1,0)$, $(0,1)$	$(34,32)$, $(33,33)$, $(32,34)$
$E_6/Spin(10)$	$(0,32)$, $(0,1)$	$(33,0)$, $(32,1)$, $(1,32)$, $(0,33)$
$E_{6,C_4}/Spin(5,5)$	$(16,16)$, $(1,0)$	$(17,16)$, $(16,17)$
$E_{6,A_5A_1}/SO^*(10)$	$(20,12)$, $(0,1)$	$(21,12)$, $(20,13)$, $(13,20)$, $(12,21)$
$E_{6,A_5A_1}/Spin(4,6)$	$(16,16)$, $(0,1)$	$(17,16)$, $(16,17)$

Table 1 continued

G/H	Metric-irreducibles	Metric signatures
$E_{6,D_5T_1}/Spin(10)$	$(32,0),(0,1)$	$(33,0),(32,1),(1,32),(0,33)$
$E_{6,D_5T_1}/Spin(2,8)$	$(16,16),(0,1)$	$(17,16),(16,17)$
$E_{6,D_5T_1}/SO^*(10)$	$(12,20),(0,1)$	$(21,12),(20,13),(13,20),(12,21)$
$E_{6,F_4}/Spin(1,9)$	$(16,16),(1,0)$	$(17,16),(16,17)$
(4) Real form family of $SU(2n+1)/Sp(n)$; $\tau=(\tau_1\oplus\tau_1)_{\mathbb{R}}\oplus(\tau_2\oplus\tau_2)_{\mathbb{R}}\oplus\tau_{0,\mathbb{R}}$		
$SL(2n+1;\mathbb{C})/Sp(n;\mathbb{C})$	$(1,0),(0,1)$ $(4n,4n)$ $(2n^2-n-1,2n^2-n-1)$	$(2n^2+3n+1,2n^2+3n-1)$ $(2n^2+3n,2n^2+3n)$ $(2n^2+3n-1,2n^2+3n+1)$
$SL(2n+1;\mathbb{R})/Sp(n;\mathbb{R})$	$(1,0)$ $(2n,2n)$ (n^2-1,n^2-n)	(n^2+n+1,n^2+2n-1) (n^2+2n,n^2+n) (n^2+n,n^2+2n) (n^2+2n-1,n^2+n+1)
$SU(n+1,n)/Sp(n;\mathbb{R})$	$(0,1)$ $(2n,2n)$ (n^2-n,n^2-1)	(n^2+n+1,n^2+2n-1) (n^2+2n,n^2+n) (n^2+n,n^2+2n) (n^2+2n-1,n^2+n+1)
$\frac{SU(2n+1-2\ell,2\ell)}{Sp(n-\ell,\ell)}$	$(0,1)$ $(4\ell,4n-4\ell)$ $(4n\ell-4\ell^2,2n^2-4n\ell+4\ell^2-n-1)$	$(4n\ell-4\ell^2+4\ell+1,2n^2-4n\ell+4\ell^2+3n-4\ell-1)$ $(4n\ell-4\ell^2+4n-4\ell+1,2n^2-4n\ell+4\ell^2+4\ell-n-1)$ $(2n^2-4n\ell+4\ell^2+3n-4\ell,4n\ell-4\ell^2+4\ell)$ $(2n^2-4n\ell+4\ell^2+4\ell-n,4n\ell-4\ell^2+4n-4\ell)$ $(4n\ell-4\ell^2+4\ell,2n^2-4n\ell+4\ell^2+3n-4\ell)$ $(4n\ell-4\ell^2+4n-4\ell,2n^2-4n\ell+4\ell^2+4\ell-n)$ $(2n^2-4n\ell+4\ell^2+3n-4\ell-1,4n\ell-4\ell^2+4\ell+1)$ $(2n^2-4n\ell+4\ell^2+4\ell-n-1,4n\ell-4\ell^2+4n-4\ell+1)$
(5) Real form family of $SU(2n+1)/[Sp(n)\times U(1)]$; $\tau=((\tau_1\otimes\zeta_1)\oplus(\tau_1\otimes\zeta_{-1}))_{\mathbb{R}}\oplus((\tau_2\otimes\zeta_2)\oplus(\tau_2\otimes\zeta_{-2}))_{\mathbb{R}}$		
$\frac{SL(2n+1;\mathbb{C})}{Sp(n;\mathbb{C})\times\mathbb{C}^*}$	$(4n,4n),(2n^2-n-1,2n^2-n-1)$	$(2n^2+3n-1,2n^2+3n-1)$
$\frac{SL(2n+1;\mathbb{R})}{Sp(n;\mathbb{R})\times\mathbb{R}^*}$	$(n^2-1,n^2-n),(2n,2n)$	$(n^2+2n-1,n^2+n),(n^2+n,n^2+2n-1)$

Table 1 continued

G/H	Metric-irreducibles	Metric signatures
$\frac{SU(n+1,n)}{Sp(n;\mathbb{R})\times U(1)}$	(n^2-n, n^2-1), $(2n, 2n)$	(n^2+n, n^2+2n-1), (n^2+2n-1, n^2+n)
$\frac{SU(2n+1-2\ell,2\ell)}{Sp(n-\ell,\ell)\times U(1)}$	$(4\ell, 4n-4\ell)$ $(4n\ell-4\ell^2, 2n^2-4n\ell+4\ell^2-n-1)$	$(4n\ell-4\ell^2+4\ell, 2n^2-4n\ell+4\ell^2+3n-4\ell-1)$ $(4n\ell-4\ell^2+4n-4\ell, 2n^2-4n\ell+4\ell^2+4\ell-n-1)$ $(2n^2-4n\ell+4\ell^2+3n-4\ell-1, 4n\ell-4\ell^2+4\ell)$ $(2n^2-4n\ell+4\ell^2+4\ell-n-1, 4n\ell-4\ell^2+4n-4\ell)$
(6) Real form family of $Spin(7)/G_2$; $\tau=\tau_{\xi_1,\mathbb{R}}$		
$Spin(7;\mathbb{C})/G_{2,\mathbb{C}}$	$(7, 7)$	$(7, 7)$
$Spin(7)/G_2$	$(0, 7)$	$(7, 0) and (0, 7)$
$Spin(3,4)/G_{2,A_1A_1}$	$(4, 3)$	$(4, 3)$ and $(3, 4)$
(7) Real form family of $G_2/SU(3)$; $\tau=(\tau_{\xi_1}\oplus\tau_{\xi_2})_{\mathbb{R}}$		
$G_{2,\mathbb{C}}/SL(3;\mathbb{C})$	$(6, 6)$	$(6, 6)$
$G_2/SU(3)$	$(0, 6)$	$(6, 0)$ and $(0, 6)$
$G_{2,A_1A_1}/SU(1,2)$	$(4, 2)$	$(4, 2)$ and $(2, 4)$
$G_{2,A_1A_1}/SL(3;\mathbb{R})$	$(3, 3)$	$(3, 3)$
(8) Real form family of $SO(10)/[Spin(7)\times SO(2)]$; $\tau=\tau_{\xi_1,\mathbb{R}}\oplus(\tau_{\xi_3,\mathbb{R}}\otimes(\zeta_1\oplus\zeta_{-1}))_{\mathbb{R}}$		
$\frac{SO(10;\mathbb{C})}{Spin(7;\mathbb{C})\times SO(2;\mathbb{C})}$	$(16, 16)$, $(7, 7)$	$(23, 23)$
$\frac{SO(10)}{Spin(7)\times SO(2)}$	$(0, 16)$, $(0, 7)$	$(23, 0)$, $(16, 7)$, $(7, 16)$, $(0, 23)$
$\frac{SO(9,1)}{Spin(7,0)\times SO(1,1)}$	$(8, 8)$, $(0, 7)$	$(15, 8)$, $(8, 15)$
$\frac{SO(8,2)}{Spin(7,0)\times SO(0,2)}$	$(16, 0)$, $(0, 7)$	$(23, 0)$, $(16, 7)$, $(7, 16)$, $(0, 23)$
$\frac{SO(6,4)}{Spin(4,3)\times SO(2,0)}$	$(8, 8)$, $(4, 3)$	$(12, 11)$, $(11, 12)$
$\frac{SO(5,5)}{Spin(3,4)\times SO(1,1)}$	$(8, 8)$, $(4, 3)$	$(12, 11)$, $(11, 12)$

Table 1 continued

G/H	Metric-irreducibles	Metric signatures
(9) Real form family of $SO(9)/Spin(7)$; $\tau = \tau_{\xi_3,\mathbb{R}} \oplus \tau_{\xi_1,\mathbb{R}}$		
$SO(9;\mathbb{C})/Spin(7;\mathbb{C})$	(7, 7) and (8, 8)	(15, 15)
$SO(9)/Spin(7)$	(0, 7) and (0, 8)	(15, 0), (8, 7), (7, 8), (0, 15)
$SO(8,1)/Spin(7)$	(0, 7) and (8, 0)	(15, 0), (8, 7), (7, 8), (0, 15)
$SO(5,4)/Spin(3,4)$	(4, 3) and (4, 4)	(8, 7), (7, 8)
(10) Real form family of $Spin(8)/G_2$; $\tau = \tau_{\xi_1,\mathbb{R}} \oplus \tau_{\xi_1,\mathbb{R}}$		
$Spin(8;\mathbb{C})/G_{2,\mathbb{C}}$	(7, 7) and (7, 7)	(14, 14)
$Spin(8)/G_2$	(0, 7) and (0, 7)	(14, 0), (7, 7), (0, 14)
$Spin(7,1)/G_2$	(7, 0) and (0, 7)	(14, 0), (7, 7), (0, 14)
$Spin(4,4)/G_{2,A_1A_1}$	(4, 3) and (4, 3)	(8, 6), (7, 7), (6, 8)
$Spin(3,5)/G_{2,A_1A_1}$	(4, 3) and (3, 4)	(8, 6), (7, 7), (6, 8)
(11) Real form family of $SO(2n+1)/U(n)$; $\tau = (\tau_{\xi_2,\mathbb{R}} \otimes (\zeta \oplus \zeta_{-1}))_{\mathbb{R}}$		
$\frac{SO(2n+1;\mathbb{C})}{GL(n;\mathbb{C})}$	$(2n,2n)$, $\big(n(n-1), n(n-1)\big)$	$\big(n(n+1), n(n+1)\big)$
$\frac{SO(2n+1-2k,2k)}{U(n-k,k)}$	$(2k, 2n-2k)$ $(2kn-2k^2, n^2-2kn+2k^2-n)$	$(n^2-2nk+2k^2+n-2k, 2nk-2k^2+2k)$ $(n^2-2nk+2k^2-n+2k, 2nk-2k^2+2n-2k)$ $(2nk-2k^2+2k, n^2-2nk+2k^2+n-2k)$ $(2nk-2k^2+2n-2k, n^2-2nk+2k^2-n+2k)$
$\frac{SO(2n-2k,2k+1)}{U(n-k,k)}$	$(2n-2k, 2k)$ $(2kn-2k^2, n^2-2kn+2k^2-n)$	$(n^2-2nk+2k^2+n-2k, 2nk-2k^2+2k)$ $(n^2-2nk+2k^2-n+2k, 2nk-2k^2+2n-2k)$ $(2nk-2k^2+2k, n^2-2nk+2k^2+n-2k)$ $(2nk-2k^2+2n-2k, n^2-2nk+2k^2-n+2k)$
$SO(n,n+1)/GL(n;\mathbb{R})$	(n,n) $\big(\frac{1}{2}n(n-1), \frac{1}{2}n(n-1)\big)$	$\big(\frac{1}{2}n(n+1), \frac{1}{2}n(n+1)\big)$

Table 1 continued

G/H	Metric-irreducibles	Metric signatures
(12) Real form family of $Sp(n)/[Sp(n-1) \times U(1)]$; $\tau = (\tau_{\xi_1} \otimes (\zeta_{+1} \oplus \zeta_{-1})_{\mathbb{R}}) \oplus (\zeta_{+1} \oplus \zeta_{-1})_{\mathbb{R}}$		
$\frac{Sp(n;\mathbb{C})}{Sp(n-1;\mathbb{C}) \times GL(1;\mathbb{C})}$	$(2, 2)$ $(4n-4, 4n-4)$	$(4n-2, 4n-2)$
$\frac{Sp(n-k,k)}{Sp(n-1-k,k) \times U(1)}$	$(0, 2)$ $(4k, 4n-4k-4)$	$(4n-4k-2, 4k)$ $(4n-4k-4, 4k+2)$ $(4k+2, 4n-4k-4)$ $(4k, 4n-4k-2)$
$\frac{Sp(n-k,k)}{Sp(n-k,k-1) \times U(1)}$	$(0, 2)$ $(4n-4k, 4k-4)$	$(4n-4k, 4k-2)$ $(4n-4k+2, 4k-4)$ $(4k-4, 4n-4k+2)$ $(4k-2, 4n-4k)$
$\frac{Sp(n;\mathbb{R})}{Sp(n-1;\mathbb{R}) \times GL(1;\mathbb{R})}$	$(1, 1)$, $(2n-2, 2n-2)$	$(2n-1, 2n-1)$
$\frac{Sp(n;\mathbb{R})}{Sp(n-1;\mathbb{R}) \times U(1)}$	$(2, 0)$, $(2n-2, 2n-2)$	$(2n, 2n-2)$, $(2n-2, 2n)$

Table 2 Weakly Symmetric pseudo-Riemannian homogeneous spaces G/H, G/H not symmetric, G semisimple but not simple, H reductive

G/H	Metric-irreducibles
(13) Real form family of $[SU(n) \times SU(n+1)]/S[U(n) \times U(1)]$	
$[SL(n;\mathbb{C}) \times SL(n+1;\mathbb{C})]/[SL(n;\mathbb{C}) \times GL(1;\mathbb{C})]$	$(n^2-1, n^2-1), (2n, 2n)$
$[SL(n;\mathbb{R}) \times SL(n+1;\mathbb{R})]/[SL(n;\mathbb{R}) \times GL(1;\mathbb{R})]$	$\left(\frac{n(n+1)}{2}-1, \frac{n(n-1)}{2}\right), (n, n)$
$[SU(k,\ell) \times SU(k+1,\ell)]/[SU(k,\ell) \times U(1)]$	$(2kl, k^2+l^2-1), (2l, 2k)$
$[SU(k,\ell) \times SU(k,\ell+1)]/[SU(k,\ell) \times U(1)]$	$(2kl, k^2+l^2-1), (2k, 2l)$
(14) Real form family of $[Sp(n+2) \times Sp(2)]/[Sp(n) \times Sp(2)]$	
$[Sp(n+2;\mathbb{C}) \times Sp(2;\mathbb{C})]/[Sp(n;\mathbb{C}) \times Sp(2;\mathbb{C})]$	$(10, 10), (8n, 8n)$
$[Sp(n+2;\mathbb{R}) \times Sp(2;\mathbb{R})]/[Sp(n;\mathbb{R}) \times Sp(2;\mathbb{R})]$	$(6, 4), (4n, 4n)$
$[Sp(n-a+b, 2-b+a) \times Sp(b, 2-b)]/[Sp(n-a, a) \times Sp(b, 2-b)]$	$(8b-4b^2, 4b^2-8b+10)$ $(8n+4(a-n)b+4a(b-2), 4(n-a)b+4a(2-b))$
(15) Real form family of $[SO(n) \times SO(n+1)]/SO(n)$	
$[SO(n;\mathbb{C}) \times SO(n+1;\mathbb{C})]/SO(n,\mathbb{C})$	$\left(\frac{n(n-1)}{2}, \frac{n(n-1)}{2}\right), (n, n)$
$[SO(n-a,a) \times SO(n-a,a+1)]/SO(n-a,a)$	$\left((n-a)a, \frac{n(n-1)}{2}-(n-a)a\right), (n-a, a)$
$[SO(n-a,a) \times SO(n-a+1,a)]/SO(n-a,a)$	$\left((n-a)a, \frac{n(n-1)}{2}-(n-a)a\right), (a, n-a)$
(16) Real form family of $[SU(n+2) \times Sp(m+1)]/[U(n) \times SU(2) \times Sp(m)]$	
$[SL(n+2;\mathbb{C}) \times Sp(m+1;\mathbb{C})]/[GL(n;\mathbb{C}) \times SL(2;\mathbb{C}) \times Sp(m;\mathbb{C})]$	$(3, 3), (4n, 4n), (4m, 4m)$
$[SL(n+2;\mathbb{R}) \times Sp(m+1;\mathbb{R})]/[GL(n;\mathbb{R}) \times SL(2;\mathbb{R}) \times Sp(m;\mathbb{R})]$	$(2, 1), (2n, 2n), (2m, 2m)$
$[SL(n'+1;\mathbb{H}) \times Sp(m-a, 1+a)]/[GL(n';\mathbb{H}) \times SU(2) \times Sp(m-a, a)]$	$(0, 3), (2n, 2n), (4m-4a, 4a)$
$[SL(n'+1;\mathbb{H}) \times Sp(m-a+1, a)]/[GL(n';\mathbb{H}) \times SU(2) \times Sp(m-a, a)]$	$(0, 3), (2n, 2n), (4a, 4m-4a)$
$\frac{SU(n-a_1+b_1, 2-b_1+a_1) \times Sp(m-a_2, 1+a_2)}{U(n-a_1, a_1) \times SU(2) \times Sp(m-a_2, a_2)}$	$(0, 3)$ $(4(n-a_1)-2b_1(n-2a_1), 2b_1(n-2a_1)+4a_1)$ $(4m-4a_2, 4a_2)$
$\frac{SU(n-a_1+b_1, 2-b_1+a_1) \times Sp(m-a_2+1, a_2)}{U(n-a_1, a_1) \times SU(2) \times Sp(m-a_2, a_2)}$	$(0, 3)$ $(4(n-a_1)-2b_1(n-2a_1), 2b_1(n-2a_1)+4a_1)$ $(4a_2, 4m-4a_2)$

Table 2 continued

G/H	Metric-irreducibles
$[SU(n+1-a,a+1)\times Sp(m+1;\mathbb{R})]/[U(n-a,a)\times SU(1,1)\times Sp(m;\mathbb{R})]$	$(2,1),(2n,2n),(2m,2m)$
$[SL(4;\mathbb{R})\times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C})\times Sp(m;\mathbb{C})\times T]$	$(3,3),(6,2),(4m,4m)$
$[SU^*(4)\times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C})\times Sp(m;\mathbb{C})\times T]$	$(3,3),(2,6),(4m,4m)$
$[SU(2,2)\times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C})\times Sp(m;\mathbb{C})\times \mathbb{R}]$	$(3,3),(4,4),(4m,4m)$
(17) Real form family of $[SU(n+2)\times Sp(m+1)]/[SU(n)\times SU(2)\times Sp(m)]$	
$[SL(n+2;\mathbb{C})\times Sp(m+1;\mathbb{C})]/[SL(n;\mathbb{C})\times SL(2;\mathbb{C})\times Sp(m;\mathbb{C})]$	$(3,3),(1,0),(0,1),(4n,4n),(4m,4m)$
$[SL(n+2;\mathbb{R})\times Sp(m+1;\mathbb{R})]/[SL(n;\mathbb{R})\times SL(2;\mathbb{R})\times Sp(m;\mathbb{R})]$	$(2,1),(1,0),(2n,2n),(2m,2m)$
$[SL(n'+1;\mathbb{H})\times Sp(m-a,1+a)]/[SL(n';\mathbb{H})\times SU(2)\times Sp(m-a,a)]$	$(0,3),(1,0),(2n,2n),(4m-4a,4a)$
$[SL(n'+1;\mathbb{H})\times Sp(m-a+1,a)]/[SL(n';\mathbb{H})\times SU(2)\times Sp(m-a,a)]$	$(0,3),(1,0),(2n,2n),(4a,4m-4a)$
$\frac{SU(n-a_1+b_1,2-b_1+a_1)\times Sp(m-a_2,1+a_2)}{SU(n-a_1,a_1)\times SU(2)\times Sp(m-a_2,a_2)}$	$(0,3),(0,1)$ $(4(n-a_1)-2b_1(n-2a_1),2b_1(n-2a_1)+4a_1)$ $(4m-4a_2,4a_2)$
$\frac{SU(n-a_1+b_1,2-b_1+a_1)\times Sp(m-a_2+1,a_2)}{SU(n-a_1,a_1)\times SU(2)\times Sp(m-a_2,a_2)}$	$(0,3),(0,1)$ $(4(n-a_1)-2b_1(n-2a_1),2b_1(n-2a_1)+4a_1)$ $(4a_2,4m-4a_2)$
$[SU(n+1-a,a+1)\times Sp(m+1;\mathbb{R})]/[SU(n-a,a)\times SU(1,1)\times Sp(m;\mathbb{R})]$	$(2,1),(0,1),(2n,2n),(2m,2m)$
$[SL(4;\mathbb{R})\times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C})\times Sp(m;\mathbb{C})]$	$(3,3),(0,1),(6,2),(4m,4m)$
$[SU^*(4)\times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C})\times Sp(m;\mathbb{C})]$	$(3,3),(0,1)(2,6),(4m,4m)$
$[SU(2,2)\times Sp(m+1;\mathbb{C})]/[SL(2;\mathbb{C})\times Sp(m;\mathbb{C})]$	$(3,3),(1,0),(4,4),(4m,4m)$
(18) Real form family of $[Sp(n+1)\times Sp(m+1)]/[Sp(n)\times Sp(m)\times Sp(1)]$	
$[Sp(n+1;\mathbb{C})\times Sp(m+1;\mathbb{C})]/[Sp(n;\mathbb{C})\times Sp(m;\mathbb{C})\times Sp(1;\mathbb{C})]$	$(3,3),(4n,4n),(4m,4m)$
$[Sp(n+1;\mathbb{R})\times Sp(m+1;\mathbb{R})]/[Sp(n;\mathbb{R})\times Sp(m;\mathbb{R})\times Sp(1;\mathbb{R})]$	$(2,1),(2n,2n),(2m,2m)$
$\frac{Sp(n-a_1+b_1,1-b_1+a_1)\times Sp(m-a_2+b_2,1-b_2+a_2)}{\{Sp(n-a_1,a_1)\times Sp(m-a_2,a_2)\times Sp(1)}$	$(0,3)$ $(4(n-a_1)-4b_1(n-2a_1),4a_1+4b_1(n-2a_1))$ $(4(m-a_2)-4b_2(m-2a_2),4a_2+4b_2(m-2a_2))$
$Sp(n+1;\mathbb{C})/[Sp(n;\mathbb{C})\times Sp(1)]$ where $m=n$	$(3,0),(4n,4n)$
$Sp(n+1;\mathbb{C})/[Sp(n;\mathbb{C})\times Sp(1;\mathbb{R})]$ where $m=n$	$(1,2),(4n,4n)$

Table 2 continued

G/H	Metric-irreducibles
(19) Real form family of $[Sp(n+1)\times Sp(\ell+1)\times Sp(m+1)]/[Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)]$	
$\frac{Sp(n+1;\mathbb{C})\times Sp(l+1;\mathbb{C})\times Sp(m+1;\mathbb{C})}{Sp(n;\mathbb{C})\times Sp(l;\mathbb{C})\times Sp(m;\mathbb{C})\times Sp(1;\mathbb{C})}$	$(3,3), (3,3), (4n,4n), (4l,4l), (4m,4m)$
$\frac{Sp(n+1;\mathbb{R})\times Sp(l+1;\mathbb{R})\times Sp(m+1;\mathbb{R})}{Sp(n;\mathbb{R})\times Sp(l;\mathbb{R})\times Sp(m;\mathbb{R})\times Sp(1;\mathbb{R})}$	$(2,1), (2,1), (2n,2n), (2l,2l), (2m,2m)$
$\frac{Sp(n-a_1+b_1,1-b_1+a_1)\times Sp(l-a_2+b_2,1-b_2+a_2)\times Sp(m-a_3+b_3,1-b_3+a_3)}{Sp(n-a_1,a_1)\times Sp(l-a_2,a_2)\times Sp(m-a_3,a_3)\times Sp(1)}$	$(0,3), (0,3),$ $(4(n-a_1)-4b_1(n-2a_1), 4a_1+4b_1(n-2a_1))$ $(4(l-a_2)-4b_2(l-2a_2), 4a_2+4b_2(l-2a_2))$ $(4(m-a_3)-4b_3(m-2a_3), 4a_3+4b_3(m-2a_3))$
$[Sp(n+1;\mathbb{C})\times Sp(m+1;\mathbb{R})]/[Sp(n;\mathbb{C})\times Sp(1;\mathbb{R})\times Sp(m;\mathbb{R})]$ where $n=\ell$	$(1,2), (2,1), (4n,4n), (2m,2m)$
$[Sp(n+1;\mathbb{C})\times Sp(m+1-a,a)]/[Sp(n;\mathbb{C})\times Sp(1)\times Sp(m-a,a)]$, $n=\ell$	$(3,0), (0,3), (4n,4n), (4a,4m-4a)$
$[Sp(n+1;\mathbb{C})\times Sp(m-a,a+1)]/[Sp(n;\mathbb{C})\times Sp(1)\times Sp(m-a,a)]$, $n=\ell$	$(3,0), (0,3), (4n,4n), (4m-4a,4a)$
(20) Real form family of $[Sp(n+1)\times Sp(2)\times Sp(m+1)]/[Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)]$	
$\frac{Sp(n+1;\mathbb{C})\times Sp(2;\mathbb{C})\times Sp(m+1;\mathbb{C})}{Sp(n;\mathbb{C})\times Sp(l;\mathbb{C})\times Sp(1;\mathbb{C})\times Sp(m;\mathbb{C})}$	$(3,3), (3,3), (4n,4n), (4,4), (4m,4m)$
$\frac{Sp(n+1;\mathbb{R})\times Sp(2;\mathbb{R})\times Sp(m+1;\mathbb{R})}{Sp(n;\mathbb{R})\times Sp(l;\mathbb{R})\times Sp(1;\mathbb{R})\times Sp(m;\mathbb{R})}$	$(2,1), (2,1), (2n,2n), (2,2), (2m,2m)$
$\frac{Sp(n-a_1+b_1,1-b_1+a_1)\times Sp(1,1)\times Sp(m-a_2+b_2,1-b_2+a_2)}{Sp(n-a_1,a_1)\times Sp(1)\times Sp(1)\times Sp(m-a_2,a_2)}$	$(0,3), (0,3), (4,0)$ $(4(n-a_1)-4b_1(n-2a_1), 4a_1+4b_1(n-2a_1))$ $(4(m-a_2)-4b_2(m-2a_2), 4a_2+4b_2(m-2a_2))$
$\frac{Sp(n-a_1+b_1,1-b_1+a_1)\times Sp(2)\times Sp(m-a_2+b_2,1-b_2+a_2)}{Sp(n-a_1,a_1)\times Sp(1)\times Sp(1)\times Sp(m-a_2,a_2)}$	$(0,3), (0,3), (0,4)$ $(4(n-a_1)-4b_1(n-2a_1), 4a_1+4b_1(n-2a_1))$ $(4(m-a_2)-4b_2(m-2a_2), 4a_2+4b_a2(m-2a_2))$
$[Sp(n+1;\mathbb{C})\times Sp(2;\mathbb{R})]/[Sp(n;\mathbb{C})\times Sp(1;\mathbb{C})]$ where $m=n$	$(3,3), (4n,4n), (3,1)$
$[Sp(n+1;\mathbb{C})\times Sp(1,1)]/[Sp(n;\mathbb{C})\times Sp(1;\mathbb{C})]$ where $m=n$	$(3,3), (4n,4n), (1,3)$

Table 3 Weakly symmetric pseudo-Riemannian G/H, G semisimple and H reductive, of Riemannian signature

Type of $\mathfrak{g}$	G/H: irreducible cases of Riemannian signature	Metric signature
A	$SU(m+n)/[SU(m)\times SU(n)]$ and $SU(m,n)/[SU(m)\times SU(n)]$	$(2mn+1,0)$
D	$SO(2n)/SU(n)$ and $SO^*(2n)/SU(n)$	$(n(n-1)+1,0)$
E	$E_6/Spin(10)$ and $E_{6,D_5T_1}/Spin(10)$	$(33,0)$
A	$SU(2n+1)/Sp(n)$ and $SU(2n,1)/Sp(n)$	$(2n^2+3n,0)$
A	$SU(2n+1)/[Sp(n)\times U(1)]$ and $SU(2n,1)/[Sp(n)\times U(1)]$	$(2n^2+3n-1,0)$
B	$Spin(7)/G_2$	$(7,0)$
G	$G_2/SU(3)$	$(6,0)$
D	$SO(10)/[Spin(7)\times SO(2)]$ and $SO(8,2)[Spin(7)\times SO(2)]$	$(23,0)$
B	$SO(9)/Spin(7)$ and $SO(8,1)/Spin(7)$	$(15,0)$
D	$Spin(8)/G_2$ and $Spin(7,1)/G_2$	$(14,0)$
B	$SO(2n+1)/U(n)$ and $SO(2n,1)/U(n)$	$(n^2+n,0)$
C	$Sp(n)/[Sp(n-1)\times U(n)]$ and $Sp(n-1,1)/[Sp(n-1)\times U(n)]$	$(4n-2,0)$
A+A	$[SU(n)\times SU(n+1)]/[SU(n)\times U(1)]$ and $[SU(n)\times SU(n,1)]/[SU(n)\times U(1)]$	$(n^2+2n-1,0)$
C+C	$[Sp(n,2)\times Sp(2)]/[Sp(n)\times Sp(2)]$ and $[Sp(n+2)\times Sp(2)]/[Sp(n)\times Sp(2)]$	$(8n+10,0)$
B+D	$[SO(n)\times SO(n,1)]/SO(n)$ and $[SO(n)\times SO(n+1)]/SO(n)$	$\left(\frac{n(n+1)}{2},0\right)$
A+C	$\frac{SU(n+2)\times Sp(m+1)}{U(n)\times SU(2)\times Sp(m)}$, $\frac{SU(n,2)\times Sp(m+1)}{U(n)\times SU(2)\times Sp(m)}$, $\frac{SU(n+2)\times Sp(m,1)}{U(n)\times SU(2)\times Sp(m)}$, $\frac{SU(n,2)\times Sp(m,1)}{U(n)\times SU(2)\times Sp(m)}$	$(4n+4m+3,0)$
A+C	$\frac{SU(n+2)\times Sp(m+1)}{SU(n)\times SU(2)\times Sp(m)}$, $\frac{SU(n,2)\times Sp(m+1)}{SU(n)\times SU(2)\times Sp(m)}$, $\frac{SU(n+2)\times Sp(m,1)}{SU(n)\times SU(2)\times Sp(m)}$, $\frac{SU(n,2)\times Sp(m,1)}{SU(n)\times SU(2)\times Sp(m)}$	$(4n+4m+4,0)$
C+C	$\frac{Sp(n+1)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(m)}$, $\frac{Sp(n,1)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(m)}$, $\frac{Sp(n,1)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(m)}$, $\frac{Sp(n+1)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(m)}$	$(4n+4m+3,0)$
C+C	$[Sp(1;\mathbb{C})\times Sp(m+1)]/[Sp(1)\times Sp(m)]$ and $[Sp(1;\mathbb{C})\times Sp(m,1)]/[Sp(1)\times Sp(m)]$	$(4m+6,0)$
C+C+C	$\frac{Sp(n+1)\times Sp(\ell+1)\times Sp(m+1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)}$, $\frac{Sp(n+1)\times Sp(\ell+1)\times Sp(m,1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)}$, $\frac{Sp(n+1)\times Sp(\ell,1)\times Sp(m+1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)}$	
	$\frac{Sp(n+1)\times Sp(\ell,1)\times Sp(m,1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)}$, $\frac{Sp(n,1)\times Sp(\ell+1)\times Sp(m+1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)}$, $\frac{Sp(n,1)\times Sp(\ell+1)\times Sp(m,1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)}$	
	$\frac{Sp(n,1)\times Sp(\ell,1)\times Sp(m+1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)}$, $\frac{Sp(n,1)\times Sp(\ell,1)\times Sp(m,1)}{Sp(n)\times Sp(\ell)\times Sp(m)\times Sp(1)}$	$(4n+4\ell+4m+6,0)$

Table 3 continued

Type of $\mathfrak{g}$	G/H: irreducible cases of Riemannian signature	Metric signature
C+C+C	$\frac{Sp(n+1)\times Sp(1,1)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$, $\frac{Sp(n,1)\times Sp(1,1)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$, $\frac{Sp(n,1)\times Sp(1,1)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$ $\frac{Sp(n+1)\times Sp(1,1)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$, $\frac{Sp(n+1)\times Sp(2)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$, $\frac{Sp(n,1)\times Sp(2)\times Sp(m+1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$ $\frac{Sp(n,1)\times Sp(2)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$, $\frac{Sp(n+1)\times Sp(2)\times Sp(m,1)}{Sp(n)\times Sp(1)\times Sp(1)\times Sp(m)}$	$(4n+4m+10,0)$
Various	See discussion of Case (21)	see Case (21)
A+A	$[SU(n)\times SU(n)]/diag(SU(n))$ and $SL(n;\mathbb{C})/SU(n)$	$(n^2-1,0)$
BD+BD	$[SO(n)\times SO(n)]/diag(SO(n))$ and $SO(n;\mathbb{C})/SO(n)$	$(n(n-1)/2,0)$
C+C	$[Sp(n)\times Sp(n)]/diag(Sp(n))$ and $Sp(n;\mathbb{C})/Sp(n)$	$(2n^2+n,0)$
G+G	$[G_2\times G_2]/diag(G_2)$ and $G_{2,\mathbb{C}}/G_2$	$(14,0)$
F+F	$[F_4\times F_4]/diag(F_4)$ and $F_{4,\mathbb{C}}/F_4$	$(52,0)$
E+E	$[E_6\times E_6]/diag(E_6)$ and $E_{6,\mathbb{C}}/E_6$	$(78,0)$
E+E	$[E_7\times E_7]/diag(E_7)$ and $E_{7,\mathbb{C}}/E_7$	$(133,0)$
E+E	$[E_8\times E_8]/diag(E_8)$ and $E_{8,\mathbb{C}}/E_8$	$(248,0)$
A	$SL(n;\mathbb{R})/SO(n)$ and $SU(n)/SO(n)$	$(\frac{n^2+n}{2}-1,0)$
A	$SU(2n)/Sp(n)$ and $SU(2n)^*/Sp(n)$	$(2n^2-n-1,0)$
A	$SU(m+n)/S(U(m)\times U(n))$ and $SU(m,n)/S(U(m)\times U(n))$	$(2mn,0)$
D	$SO(2n)/U(n)$ and $SO^*(2n)/U(n)$	$(n^2-n,0)$
BD	$SO(m+n)/[SO(m)\times SO(n)]$ and $SO(m,n)/[SO(m)\times SO(n)]$	$(mn,0)$
C	$Sp(n)/U(n)$ and $Sp(n;\mathbb{R})/U(n)$	$(n^2+n,0)$
C	$Sp(m+n)/[Sp(m)\times Sp(n)]$ and $Sp(m,n)/[Sp(m)\times Sp(n)]$	$(4mn,0)$
G	$G_2/[SU(2)\times SU(2)]$ and $G_{A_1A_1}/[SU(2)\times SU(2)]$	$(8,0)$
F	$F_4/Spin(9)$ and $F_{4,B_4}/Spin(9)$	$(16,0)$
F	$F_4/[Sp(3)\times SU(2)]$ and $F_{4,C_3A_1}/[Sp(3)\times SU(2)]$	$(28,0)$
E	$E_6/Sp(4)$ and $E_{6,C_4}/Sp(4)$	$(42,0)$
E	$E_6/[SU(6)\times SU(2)]$ and $E_{6,A_5A_1}/[SU(6)\times SU(2)]$	$(40,0)$
E	$E_6/[SO(10)\times SO(2)]$ and $E_{6,D_5T_1}/[SO(10)\times SO(2)]$	$(32,0)$
E	E_6/F_4 and $E_{6,F_4}/F_4$	$(26,0)$
E	$E_7/SU(8)$ and $E_{7,A_7}/SU(8)$	$(70,0)$
E	$E_7/[SO(12)\times SU(2)]$ and $E_{7,D_6A_1}/[SO(12)\times SU(2)]$	$(64,0)$
E	$E_7/[E_6\times T_1]$ and $E_{7,E_6T_1}/[E_6\times T_1]$	$(54,0)$
E	$E_8/SO(16)$ and $E_{8,D_8}/SO(16)$	$(128,0)$
E	$E_8/[E_7\times SU(2)]$ and $E_{8,E_7A_1}/[E_7\times SU(2)]$	$(112,0)$

Table 4 Weakly symmetric pseudo-Riemannian G/H, G semisimple and H reductive, of Lorentz signature

Type of $\mathfrak{g}$	G/H: irreducible cases of Lorentz signature	Metric signature
A	$SU(m+n)/[SU(m) \times SU(n)]$ and $SU(m,n)/[SU(m) \times SU(n)]$	$(2mn, 1)$
D	$SO(2n)/SU(n)$ and $SO^*(2n)/SU(n)$	$(n(n-1), 1)$
E	$E_6/Spin(10)$ and $E_{6,D_5T_1}/Spin(10)$	$(32, 1)$
A	$SU(2n+1)/Sp(n)$ and $SU(2n,1)/Sp(n)$	$(2n^2+3n-1, 1)$
A+C	$\frac{SU(n+2)\times Sp(m+1)}{SU(n)\times SU(2)\times Sp(m)}$, $\frac{SU(n,2)\times Sp(m+1)}{SU(n)\times SU(2)\times Sp(m)}$, $\frac{SU(n+2)\times Sp(m,1)}{SU(n)\times SU(2)\times Sp(m)}$, $\frac{SU(n,2)\times Sp(m,1)}{SU(n)\times SU(2)\times Sp(m)}$	$(4n+4m+3, 1)$
A+A	$SL(2;\mathbb{R}) = [SL(2;\mathbb{R}) \times SL(2;\mathbb{R})]/diag(SL(2;\mathbb{R}))$ group manifold viewed as Lorentz manifold	$(2, 1)$
A+A	$SL(2;\mathbb{C})/SL(2;\mathbb{R})$ viewed as Lorentz manifold	$(2, 1)$
A	$SL(2;\mathbb{R})/\mathbb{R}$	$(1, 1)$
A	$SU^*(4)/Sp(1,1)$	$(4, 1)$
A	$SU(2,2)/Sp(1,1)$	$(4, 1)$
BD	$SO(n,1)/SO(n-1,1)$ and $SO(n-1,2)/SO(n-1,1)$	$(n-1, 1)$
C	$Sp(2;\mathbb{R})/Sp(1;\mathbb{C})$ and $Sp(1,1)/Sp(1;\mathbb{C})$	$(3, 1)$

Table 5 Weakly symmetric pseudo-Riemannian G/H, G semisimple and H reductive, of trans-Lorentz signature

Type of $\mathfrak{g}$	G/H: irreducible cases of trans-Lorentz signature	Metric signature
A	$SL(3;\mathbb{R})/SL(2;\mathbb{R})$	$(3, 2)$
A	$SU(n-1,2)/SU(n-1,1)$ and $SU(n,1)/SU(n-1,1)$	$(2n-1, 2)$
G	$G_{2,A_1A_1}/SU(1,2)$	$(4, 2)$
B	$SO(4,1)/U(2)$, $SO(5)/U(2)$ and $SO(3,2)/U(1,1)$	$(4, 2)$
C	$Sp(n)/[Sp(n-1) \times U(1)]$ and $Sp(n-1,1)/[Sp(n-1) \times U(1)]$	$(4n-4, 2)$
BD	$[SO(2,1) \times SO(2,2)]/SO(2,1)$ and $[SO(2,1) \times SO(3,1)]/SO(2,1)$	$(4, 2)$
A+A	group manifold $SL(2;\mathbb{R}) = [SL(2;\mathbb{R}) \times SL(2;\mathbb{R})]/diag(SL(2;\mathbb{R}))$ viewed as trans-Lorentz manifold	$(1, 2)$
A	$SL(2;\mathbb{C})/SO(2;\mathbb{C})$	$(2, 2)$
A	$SL(2;\mathbb{C})/SL(2;\mathbb{R}) = SL(2;\mathbb{C})/SU(1,1))$	$(1, 2)$
A	$SL(3;\mathbb{R})/[SL(2;\mathbb{R}) \times R]$	$(2, 2)$
A	$SL(4;\mathbb{R})/Sp(2;\mathbb{R})$	$(3, 2)$
A	$SL(4;\mathbb{R})/GL'(2;\mathbb{C})$ and $SU^*(4)/GL'(2;\mathbb{C})$	$(6, 2)$
A	$SU(2,1)/SO(2,1)$	$(3, 2)$
A	$SU(2,2)/Sp(2;\mathbb{R})$	$(3, 2)$
A	$SU(n-1,2)/U(n-1,1)$ and $SU(n,1)/U(n-1,1)$	$(2n-2, 2)$
BD	$SO(n-3,3)/[SO(n-3,1) \times SO(2)]$ and $SO(n-1,1)/[SO(n-3,1) \times SO(2)]$	$(2n-6, 2)$
BD	$SO(n-2,3)/SO(n-2,2)$ and $SO(n-1,2)/SO(n-2,2)$	$(n-2, 2)$
C	$Sp(2;\mathbb{R})/[Sp(1;\mathbb{R}) \times Sp(1;\mathbb{R})]$	$(2, 2)$
C	$Sp(2;\mathbb{R})/U(1,1)$ and $Sp(1,1)/U(1,1)$	$(4, 2)$

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