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Solvability, Structure, and Analysis for Minimal Parabolic Subgroups

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Abstract We examine the structure of the Levi component MA in a minimal parabolic subgroup $P = MAN$ of a real reductive Lie group G and work out the cases where M is metabelian, equivalently where $\mathfrak{p}$ is solvable. When G is a linear group we verify that $\mathfrak{p}$ is solvable if and only if M is commutative. In the general case M is abelian modulo the center Z_G, we indicate the exact structure of M and P, and we work out the precise Plancherel Theorem and Fourier Inversion Formulae. This lays the groundwork for comparing tempered representations of G with those induced from generic representations of P.

Keywords Parabolic subgroup · Plancherel formula · Fourier inversion formula

1 Introduction

Let G be a real reductive Lie group and $P = MAN$ a minimal parabolic subgroup. Later we will be more precise about conditions on the structure of G, but first we recall the *unitary principal series* representations of G. They are the induced representations $\pi_{\chi,\nu,\sigma} = \mathrm{Ind}_P^G(\eta_{\chi,\nu,\sigma})$ defined as follows. First, ν is the highest weight of an irreducible representation η_ν of the identity component M^0 of M. Second, χ is an irreducible representation of the M-centralizer $Z_M(M^0)$ that agrees with η_ν on the center $Z_{M^0} = Z_M(M^0) \cap M^0$ of M^0. Third, σ is a real linear functional on the Lie algebra $\mathfrak{a}$ of A, in other words $e^{i\sigma}$ is a unitary character on A. Write $\eta_{\chi,\nu}$ for the representation $\chi \otimes \eta_\nu$ of M, and let $\eta_{\chi,\nu,\sigma}$ denote the representation $man \mapsto e^{i\sigma}(a)\eta_{\chi,\nu}(m)$ of P. These data define the principal series representation $\pi_{\chi,\nu,\sigma} = \mathrm{Ind}_P^G(\eta_{\chi,\nu,\sigma})$ of G.

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Now consider a variation in which an irreducible unitary representation of N is incorporated. A few years ago, we described Plancherel almost all of the unitary dual $\widehat{N}$ in terms of strongly orthogonal roots ([16,17]). Using those "stepwise square integrable" representations π_λ of N we arrive at representations $\eta_{\chi,\nu,\sigma,\lambda}$ of P and $\pi_{\chi,\nu,\sigma,\lambda} = \mathrm{Ind}_P^G(\eta_{\chi,\nu,\sigma,\lambda})$ of G. The representations $\pi_{\chi,\nu,\sigma,\lambda}$ have not yet been studied, at least in terms of their relation to tempered representations and harmonic analysis on G. In this paper, we lay some of the groundwork for that study.

Clearly this is much simpler when M is commutative modulo the center Z_G. Then there is a better chance of finding a clear relation between the $\pi_{\chi,\nu,\sigma,\lambda}$ and the tempered representation theory of G. In this paper, we see just when M is commutative mod Z_G. That turns out to be equivalent to solvability of P, and leads to a straightforward construction both of the Plancherel Formula and the Fourier Inversion Formula for P and of the principal series representations of G. It would also be interesting to see whether solvability of P simplifies the operator-theoretic formulation [1] of stepwise square integrability.

It will be obvious to the reader that if any parabolic subgroup of a real Lie group has commutative Levi component, then that parabolic is a minimal parabolic. For this reason, we only deal with minimal parabolics.

The concept of "stepwise square integrable" representation is basic to this note and to many of the references after 2012. It came out of conversations with Maria Laura Barberis concerning the application of square integrability [10] to her work with Isabel Dotti on abelian complex structures. The first developments were [16] and [17], and we follow the notation in those papers.

2 Lie Algebra Structure

Let $\mathfrak{g}$ be a real reductive Lie algebra. In other words $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is semisimple and $\mathfrak{z}$ is the center of $\mathfrak{g}$. As usual, $\mathfrak{g}_{\mathbb{C}}$ denotes the complexification of $\mathfrak{g}$, so $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{z}_{\mathbb{C}}$, direct sum of the respective complexifications of $\mathfrak{g}'$ and $\mathfrak{z}$. Choose a Cartan involution θ of $\mathfrak{g}$ and decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ into (± 1)-eigenspaces of θ. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and let $\mathfrak{m}$ denote the $\mathfrak{k}$-centralizer of $\mathfrak{a}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}$, so $\mathfrak{h} := \mathfrak{t} + \mathfrak{a}$ is a "maximally split" Cartan subalgebra of $\mathfrak{g}$.

We denote root systems by $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, $\Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and $\Delta(\mathfrak{g}, \mathfrak{a})$. Choose consistent positive root subsystems $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, $\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and $\Delta^+(\mathfrak{g}, \mathfrak{a})$. In other words, if $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ then $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ if and only if either (i) $\alpha|_{\mathfrak{a}} \neq 0$ and $\alpha|_{\mathfrak{a}} \in \Delta^+(\mathfrak{g}, \mathfrak{a})$, or (ii) $\alpha|_{\mathfrak{a}} = 0$ and $\alpha|_{\mathfrak{t}_{\mathbb{C}}} \in \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. We write $\Psi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, $\Psi(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and $\Psi(\mathfrak{g}, \mathfrak{a})$ for the corresponding simple root systems. Note that

$$\begin{aligned} &\Psi(\mathfrak{g}, \mathfrak{a}) = \{\psi|_{\mathfrak{a}} \mid \psi \in \Psi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ and } \psi|_{\mathfrak{a}} \neq 0\} \text{ and} \\ &\Psi(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\psi|_{\mathfrak{t}_{\mathbb{C}}} \mid \psi \in \Psi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ and } \psi|_{\mathfrak{a}} = 0\}. \end{aligned} \tag{2.1}$$

If $\mathfrak{g}$ is the underlying structure of a complex simple Lie algebra $\mathfrak{l}$ then $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{l} \oplus \bar{\mathfrak{l}}$ and $\Psi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is the union of the simple root systems of the two summands. In that case $\Psi(\mathfrak{g}, \mathfrak{a})$ looks like the simple root system of $\mathfrak{l}$, but with every root of multiplicity 2.

On the group level, the centralizer $Z_G(\mathfrak{a}) = Z_G(A) = M \times A$ where $A = \exp(\mathfrak{a})$ and $M = Z_K(A)$.

Lemma 2.1 *The following conditions are equivalent:* (1) *if* $\alpha \in \Psi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ *then* $\alpha|_{\mathfrak{a}} \neq 0$, (2) *the Lie algebra* $\mathfrak{m}$ *of* M *is abelian, and* (3) $\mathfrak{m}$ *is solvable.*

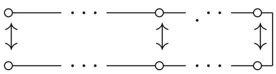
Proof If (1) fails we have $\psi \in \Psi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ such that $\psi|_{\mathfrak{a}} = 0$, so $0 \neq \psi|_{\mathfrak{t}_{\mathbb{C}}} \in \Psi(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then $\mathfrak{m}_{\mathbb{C}}$ contains the simple Lie algebra with simple root $\psi|_{\mathfrak{t}_{\mathbb{C}}}$, and (3) fails. If (3) fails then (2) fails. Finally suppose that (2) fails. Since $\mathfrak{m}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} + \sum_{\alpha \in \Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} \mathfrak{m}_\alpha$, the root system $\Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is not empty. In particular $\Psi(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \neq \emptyset$, and (2.1) provides $\psi \in \Psi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ with $\psi|_{\mathfrak{a}} = 0$, so (1) fails. □

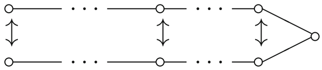
Decompose the derived algebra as a direct sum of simple ideals, $\mathfrak{g}' = \bigoplus \mathfrak{g}_i$. Then the minimal parabolic subalgebras $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ of $\mathfrak{g}$ decompose as direct sums $\mathfrak{p} = \mathfrak{z} \oplus \bigoplus \mathfrak{p}_i$ where $\mathfrak{p}_i = \mathfrak{m}_i + \mathfrak{a}_i + \mathfrak{n}_i$ is a minimal parabolic subalgebra of $\mathfrak{g}_i$. Thus $\mathfrak{m}$ is abelian (resp. solvable) if and only if each of the $\mathfrak{m}_i$ is abelian (resp. solvable). The classification of real reductive Lie algebras $\mathfrak{g}$ with $\mathfrak{m}$ abelian (resp. solvable) is thus reduced to the case where $\mathfrak{g}$ is simple. This includes the case where $\mathfrak{g}$ is the underlying real structure of a complex simple Lie algebra.

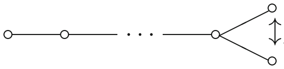
The Satake diagram for $\Psi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is the Dynkin diagram, using the arrow convention rather than the black dot convention, with the following modifications. If ψ' and ψ'' have the same non-zero restriction to $\mathfrak{a}$ then the corresponding nodes on the diagram are joined by a two-headed arrow. In the case where $\mathfrak{g}$ is complex this joins two roots that are complex conjugates of each other. If $\psi|_{\mathfrak{a}} = 0$ then the corresponding node on the diagram is changed from a circle to a black dot. Condition (1) of Lemma 2.1 says that the Satake diagram for $\Psi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ has no black dots. Combining the classification with Lemma 2.1 we arrive at

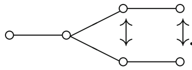
Theorem 2.2 *Let* $\mathfrak{g}$ *be a simple real Lie algebra and let* $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ *be a minimal parabolic subalgebra. Let* $\mathfrak{t}$ *be a Cartan subalgebra of* $\mathfrak{m}$, *so* $\mathfrak{h} := \mathfrak{t} + \mathfrak{a}$ *is a maximally split Cartan subalgebra of* $\mathfrak{g}$. *Then the following conditions are equivalent*: (i) $\mathfrak{m}$ *is abelian,* (ii) $\mathfrak{m}$ *is solvable,* (iii) $\mathfrak{a}$ *contains a regular element of* $\mathfrak{g}$, (iv) $\mathfrak{g}_{\mathbb{C}}$ *has a Borel subalgebra stable under complex conjugation of* $\mathfrak{g}_{\mathbb{C}}$ *over* $\mathfrak{g}$, *and* (v) $\mathfrak{g}$ *appears on the following list.*

1. *Cases* $\mathfrak{h} = \mathfrak{a}$ *and* $\mathfrak{m} = 0$ (called split real forms or Cartan normal forms): $\mathfrak{g}$ *is one of* $\mathfrak{sl}(\ell + 1; \mathbb{R})$, $\mathfrak{so}(\ell, \ell + 1)$, $\mathfrak{sp}(\ell; \mathbb{R})$, $\mathfrak{so}(\ell, \ell)$, $\mathfrak{g}_{2,A_1A_1}$, $\mathfrak{f}_{4,A_1C_3}$, $\mathfrak{e}_{6,C_4}$, $\mathfrak{e}_{7,A_7}$, *or* $\mathfrak{e}_{8,D_8}$. *In this case, since* $\mathfrak{h} = \mathfrak{a}$, *the restricted roots all have multiplicity* 1.
2. *Cases where* $\mathfrak{g}$ *is the underlying real structure of a complex simple Lie algebra*: $\mathfrak{g}$ *is one of* $\mathfrak{sl}(\ell + 1; \mathbb{C})$, $\mathfrak{so}(2\ell + 1; \mathbb{C})$, $\mathfrak{sp}(\ell; \mathbb{C})$, $\mathfrak{so}(2\ell; \mathbb{C})$, $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, *or* $\mathfrak{e}_8$. *Here* $\mathfrak{h} = i\mathfrak{a} + \mathfrak{a}$ *and* $\mathfrak{m} = i\mathfrak{a}$, *and the restricted roots all have multiplicity* 2.
3. *Four remaining cases*:

(3a) $\mathfrak{g} = \mathfrak{su}(\ell, \ell + 1)$ *with Satake diagram* . *In this case* $\Delta(\mathfrak{g}, \mathfrak{a})$ *is of type* B_ℓ, *the long indivisible roots have multiplicity* 2, *the short indivisible roots have multiplicity* 1, *and the divisible roots also have multiplicity* 1.

(3b) $\mathfrak{g} = \mathfrak{su}(\ell, \ell)$ *with Satake diagram* . *In this case* $\Delta(\mathfrak{g}, \mathfrak{a})$ *is of type* C_ℓ, *the long restricted roots have multiplicity* 1, *and the short ones have multiplicity* 2.

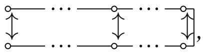
(3c) $\mathfrak{g} = \mathfrak{so}(\ell - 1, \ell + 1)$ *with Satake diagram* . *In this case* $\Delta(\mathfrak{g}, \mathfrak{a})$ *is of type* $B_{\ell-1}$, *the long restricted roots have multiplicity* 1, *and the short ones have multiplicity* 2.

(3d) $\mathfrak{g} = \mathfrak{e}_{6,A_1A_5}$ *with Satake diagram* . *In this case* $\Delta(\mathfrak{g}, \mathfrak{a})$ *is of type* F_4, *the long restricted roots have multiplicity* 1, *and the short ones have multiplicity* 2.

(*These real Lie algebras are often called the* Steinberg normal forms *of their complexifications.*)

Corollary 2.3 *Let* $\mathfrak{g}$ *be a simple real Lie algebra and let* $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ *be a minimal parabolic subalgebra with* $\mathfrak{m}$ *abelian. Let* $\mathfrak{t}$ *be a Cartan subalgebra of* $\mathfrak{m}$, *so* $\mathfrak{h} := \mathfrak{t} + \mathfrak{a}$ *is a maximally split Cartan subalgebra of* $\mathfrak{g}$.

If $\mathfrak{g} \neq \mathfrak{su}(\ell, \ell + 1)$ *then* $\Delta(\mathfrak{g}, \mathfrak{a})$ *is non-multipliable, in other words if* $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ *then* $2\alpha \notin \Delta(\mathfrak{g}, \mathfrak{a})$.

If $\mathfrak{g} = \mathfrak{su}(\ell, \ell + 1)$ , *let* $\alpha_1, \dots, \alpha_{2\ell}$ *be the simple roots of* $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ *in the usual order and* $\psi_i = \alpha_i|_\mathfrak{a}$. *Then the multipliable roots in* $\Delta^+(\mathfrak{g}, \mathfrak{a})$ *are just the* $\frac{1}{2}\beta_u = (\psi_u + \dots \psi_\ell) = (\alpha_u + \cdots + \alpha_\ell)|_\mathfrak{a}$ *for* $1 \leqq u \leqq \ell$; *there* $\beta_u = 2(\psi_u + \cdots + \psi_\ell) = (\alpha_u + \cdots + \alpha_{2\ell-u+1})|_\mathfrak{a}$.

3 Structure of the Minimal Parabolic Subgroup

As in Section 2, $\mathfrak{g}$ is a real simple Lie algebra, $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is a minimal parabolic subalgebra, and we assume that $\mathfrak{m}$ is abelian. G is a connected Lie group with Lie algebra $\mathfrak{g}$ and $P = MAN$ is the minimal parabolic subgroup with Lie algebra $\mathfrak{p}$. In this section, we work out the detailed structure of M and P, essentially by adapting the results of K. D. Johnson ([5,6]). We move the commutativity criteria of Theorem 2.2 from $\mathfrak{m}$ to M when G is linear and describe the metabelian (in fact abelian mod Z_G) structure of M for G in general.

We use the following notation. $\widetilde{G}$ is the connected simply connected Lie group with Lie algebra $\mathfrak{g}$; $G_\mathbb{C}$ is the connected simply connected complex Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$; G' is the analytic subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{g}$; and $\overline{G}$ and $\overline{G_\mathbb{C}}$ are the adjoint groups of G and $G_\mathbb{C}$. G' is called the *algebraically simply connected* group for $\mathfrak{g}$. We write $\widetilde{P} = \widetilde{M}\widetilde{A}\widetilde{N} \subset \widetilde{G}$, $P' = M'A'N' \subset G'$, and $\overline{P} = \overline{MAN} \subset \overline{G}$ for the corresponding minimal parabolic subgroups, aligned so that $\widetilde{G} \to G'$ maps $\widetilde{M} \to M'$, $\widetilde{A} \cong A'$, and $\widetilde{N} \cong N'$; $G' \to \overline{G}$ maps $M' \to \overline{M}$, $A' \cong \overline{A}$, and $N' \cong \overline{N}$; and $G \to \overline{G}$ maps $M \to \overline{M}$, $A \cong \overline{A}$, and $N \cong \overline{N}$.

3.1 The Linear Case

In order to discuss M/M^0, we need the following concept from [5].

$$r = r(\mathfrak{g}) \text{ is the number of white dots in the Satake diagram of } \mathfrak{g} \quad (3.1)$$
$$\text{not adjacent to a black dot and not attached to another dot by an arrow.}$$

However, we only need it for the case where $\mathfrak{m}$ is abelian, so there are no black dots. In fact, running through the cases of Theorem 2.2, we have the following.

Proposition 3.1 *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, let $P = MAN$ be a minimal parabolic subgroup of G, and assume that $\mathfrak{m}$ is abelian. Then $r(\mathfrak{g})$ is given by*

1. *Cases $\mathfrak{h} = \mathfrak{a}$ and $\mathfrak{m} = 0$: then $r(\mathfrak{g}) = \dim \mathfrak{h}$, the rank of $\mathfrak{g}$.*
2. *Cases where $\mathfrak{g}$ is the underlying real structure of a complex simple Lie algebra: then $r(\mathfrak{g}) = 0$.*
3. *Four remaining cases:*
   (3a) *Case $\mathfrak{g} = \mathfrak{su}(\ell, \ell + 1)$: then $r(\mathfrak{g}) = 0$.*
   (3b) *Case $\mathfrak{g} = \mathfrak{su}(\ell, \ell)$: then $r(\mathfrak{g}) = 1$.*
   (3c) *Case $\mathfrak{g} = \mathfrak{so}(\ell - 1, \ell + 1)$: then $r(\mathfrak{g}) = \ell - 2$.*
   (3d) *Case $\mathfrak{g} = \mathfrak{e}_{6,A_1A_5}$: then $r(\mathfrak{g}) = 2$.*

The first and second assertions in the following Proposition are mathematical folklore; we include their proofs for continuity of exposition. The third part is from [5]. Recall that $\mathfrak{g}$ satisfies the conditions of Theorem 2.2.

Proposition 3.2 *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, let $P = MAN$ be a minimal parabolic subgroup of G, and assume that $\mathfrak{m}$ is abelian. Let G' be the algebraically simply connected group for $\mathfrak{g}$.*

(1) *G is linear if and only if $G' \to \overline{G}$ factors into $G' \to G \to \overline{G}$,*
(2) *if G is linear with minimal parabolic $P = MAN$, then $M = F \times M^0$ where $F \subset (\exp(i\mathfrak{a}) \cap K)$ is an elementary abelian 2-group, and*
(3) *$M' = F' \times M'^0$ where $F' \cong \mathbb{Z}_2^\ell$, and if G is linear then F is a quotient of F'.*

Proof If G is linear it is contained in its complexification, which is covered by $G_{\mathbb{C}}$; (1) follows.

Let G be linear. The Cartan involution θ of G extends to is complexification G_c and defines the compact real forms $\mathfrak{g}_u := \mathfrak{k} + i\mathfrak{s}$ and G_u of $\mathfrak{g}_{\mathbb{C}}$ and G_c. Here the complexification $M_c A_c$ of MA is the centralizer of $\mathfrak{a}$ in G_c. Its maximal compact subgroup is the centralizer of the torus $\exp(i\mathfrak{a})$ in the compact connected group G_u, so it is connected. Thus $M^0 \exp(i\mathfrak{a})$ is the centralizer of $\exp(i\mathfrak{a})$ in G_u. Now the centralizer M of $\mathfrak{a}$ in K is $(M^0 \exp(i\mathfrak{a})) \cap K = M^0(\exp(i\mathfrak{a}) \cap K)$. Thus $M = (\exp(i\mathfrak{a}) \cap K) \cdot M^0$.

By construction, θ preserves $\exp(i\mathfrak{a}) \cap K$. If $x \in (\exp(i\mathfrak{a}) \cap K)$ then $\theta(x) = x$ because $x \in K$ and $\theta(x) = x^{-1}$ because $x \in \exp(\mathfrak{a}_{\mathbb{C}})$, so $x = x^{-1}$. Now $\exp(i\mathfrak{a}) \cap K$ is an elementary abelian 2-group, so $\exp(i\mathfrak{a}) \cap K = F \times (\exp(i\mathfrak{a}) \cap M^0)$ for an elementary abelian 2-subgroup F, and (2) follows.

Statement (3) is Theorem 3.5 in Johnson's paper [5]. □

Now we have a characterization of the linear case.

Theorem 3.3 *Let G be a connected real reductive Lie group and $P = MAN$ a minimal parabolic subgroup. If G is linear, then M is abelian if and only if $\mathfrak{m}$ is abelian, and the following conditions are equivalent:* (1) *M is abelian,* (2) *M is solvable, and* (3) *P is solvable.*

Proof Since G is linear it has form G'/Z where G' is its algebraically simply connected covering group, the analytic subgroup of $G_{\mathbb{C}}$ for $\mathfrak{g}$. If M' is abelian then its Lie algebra $\mathfrak{m}$ is abelian. Conversely suppose that $\mathfrak{m}$ is abelian. Then M^0 is abelian and Proposition 3.2(2) ensures that M is abelian. We have proved that if G is linear then M is abelian if and only if $\mathfrak{m}$ is abelian.

Statements (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate so we need to only prove (3) $\Rightarrow$ (1). If P is solvable then M is solvable so $\mathfrak{m}$ is abelian by Theorem 2.2, and M is abelian as proved just above. $\square$

3.2 The Finite Groups D_n and $D_{\mathfrak{g}}$

In order to deal with the non-linear cases, we need certain finite groups that enter into the description of the component groups of minimal parabolics. Those are the D_n for $\mathfrak{g}$ classical or of type $\mathfrak{g}_2$, and $D_{\mathfrak{g}}$ for the other exceptional cases.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Consider the multiplicative subgroup

$$D_n = \{\pm e_{i_1} \ldots e_{i_{2\ell}} \mid 1 \leqq i_1 < \cdots < i_{2\ell} \leqq n\} \tag{3.2}$$

of decomposable even invertible elements in the Clifford algebra of $\mathbb{R}^n$. It is contained in $Spin(n)$ and has order 2^n, and we need it for $\mathfrak{g}$ classical. Denote $\overline{D_n} = \{\mathrm{diag}(\pm 1, \ldots, \pm 1) \mid \det \mathrm{diag}(\pm 1, \ldots, \pm 1) = 1\} \cong \mathbb{Z}_2^{n-1}$. Then $\overline{D_n}$ is the image of D_n in $SO(n)$ under the usual map (vector representation) $v : Spin(n) \to SO(n)$. Here $\{\pm 1\}$ is the center and also the derived group of D_n, and $\overline{D_n} \cong D_n/\{\pm 1\}$. Also, D_n is related to the 2-tori of Borel and Serre [3] and to 2-torsion in integral cohomology [2]. Note that D_3 is isomorphic to the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$.

As usual we write $\widehat{D_n}$ for the unitary dual of D_n. It contains the 2^{n-1} characters that factor through $\overline{D_n}$. Those are the 1-dimensional representations

$$\varepsilon_{i_1,\ldots,i_{2\ell}} : \mathrm{diag}(a_1, \ldots, a_n) \mapsto a_{i_1} a_{i_2} \ldots a_{i_{2\ell}}. \tag{3.3}$$

There are also representations of degree > 1:

$$\begin{aligned} &\text{If } n = 2k \text{ even let } \sigma_\pm \text{ denote the restriction of the half-spin representations from} \\ &\quad Spin(n) \text{ to } D_n. \\ &\text{If } n = 2k+1 \text{ odd let } \sigma \text{ denote the restriction of the spin representation from} \\ &\quad Spin(n) \text{ to } D_n. \end{aligned} \tag{3.4}$$

Note that $\deg \sigma_\pm = 2^{k-1}$ for $n = 2k$ and $\deg \sigma = 2^k$ for $n = 2k+1$. These representations enumerate $\widehat{D_n}$, as follows. We will need this for the Plancherel formula for P.

Proposition 3.4 ([6, Section 3]) *The representations $\sigma_\pm$ of D_{2k} and σ of D_{2k+1} are irreducible, and*

$$\begin{aligned} &\textit{If } n = 2k \textit{ even then } \widehat{D_n} = \{\sigma_+ , \sigma_- , \varepsilon_{i_1,\dots,i_{2\ell}} \mid 1 \leqq i_1 < \cdots < i_{2\ell} \leqq n \textit{ and} \\ &\quad 0 \leqq \ell < k\}. \\ &\textit{If } n = 2k+1 \textit{ odd then } \widehat{D_n} = \{\sigma , \varepsilon_{i_1,\dots,i_{2\ell}} \mid 1 \leqq i_1 < \cdots < i_{2\ell} \leqq n \textit{ and} \\ &\quad 0 \leqq \ell \leqq k\}. \end{aligned} \tag{3.5}$$

Now we describe the analogs of the D_n for the split exceptional Lie algebras of types $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, or $\mathfrak{e}_8$. Following [6, Section 8] the natural inclusions, $\mathfrak{f}_4 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8$ exponentiate to inclusions

$$\widetilde{Z}_{E_{8,D_8}} \subset \widetilde{F}_{4,C_1C_3} \subset \widetilde{E}_{6,C_4} \subset \widetilde{K}_{E_{7,A_7}} \subset \widetilde{E}_{8,D_8},$$

where $\widetilde{Z}_{E_{8,D_8}}$ is the center of $\widetilde{E}_{8,D_8}$ and the others are split real simply connected exceptional Lie groups. For the first inclusion, note that the maximal compact subgroups satisfy

$$\begin{aligned} &\widetilde{K}_{F_{4,C_1C_3}} \subset \widetilde{K}_{E_{6,C_4}} \subset \widetilde{K}_{E_{7,A_7}} \subset \widetilde{K}_{E_{8,D_8}} \text{ given by} \\ &\quad (Sp(1) \times Sp(3)) \subset Sp(4) \subset SU(8) \subset Spin(16) \end{aligned}$$

and $\widetilde{Z}_{E_{8,D_8}} = \{1, e_1 \cdot e_2 \cdot \cdots \cdot e_{16}\} \cong \mathbb{Z}_2$ in the spin group using Clifford multiplication.

Let U_7 denote the group of permutations of $\{1, 2, \dots, 8\}$ generated by products of 4 commuting transpositions, e.g., by $\tau := (12)(34)(56)(78)$ and its conjugates in the permutation group, viewed as a subgroup $\cong \mathbb{Z}_2^3$ of $SU(8)$. Let V_7 be the group generated by $\omega_1 := iI$, $\omega_2 := \mathrm{diag}(-1, -1, 1, 1, 1, 1, -1, -1)$, $\omega_3 := \mathrm{diag}(-1, -1, 1, 1, -1, -1, 1, 1)$, and $\omega_4 := \mathrm{diag}(-1, 1, -1, 1, -1, 1, -1, 1)$, viewed as a subgroup $\cong \mathbb{Z}_4 \times \mathbb{Z}_2^3$ of $SU(8)$. Now let U_6 denote the subgroup $\{1, (13)(24)(57)(68), (15)(26)(37)(48), (17)(28)(37)(48)\} \subset U_7$, so $U_6 \cong Z_2^2$ and let V_6 denote the subgroup of V_7 generated by $\omega_1\omega_4$, ω_2, and ω_3. Finally, let W_4 denote the group generated by $\{\tau\omega_1 , \omega_2 , \omega_3 , \tau\omega_4\}$.

Proposition 3.5 ([6, Section 9] *Define $W_6 = U_6V_6 \cup \tau\omega_1U_6V_6$, $W_7 = U_7V_7$, and $W_8 = W_7 \cup \tau W_7$. Then*

1. *W_4 is a group of order 2^5 with $[W_4, W_4] = \{\pm 1\} \cong \mathbb{Z}_2$, $W_4/[W_4, W_4] \cong \mathbb{Z}_2^4$, and W_4 has center $Z_{W_4} = \{\pm 1, \pm\omega_2 , \pm\omega_3 , \pm\omega_2\omega_3\}$. The action of W_4 on $\mathbb{C}^8$, as a subgroup of $Sp(4)$, breaks into the sum of 4 irreducible inequivalent 2-dimensional subspaces $\mathbb{C}_j^2$, distinguished by the action of Z_{W_4}. Thus the unitary dual $\widehat{W_4} = \{w_{4,1} , w_{4,2} , w_{4,3} , w_{4,4} , \epsilon_1, \dots , \epsilon_{16}\}$, where the $w_{4,j}$ are the representations on the $\mathbb{C}_j^2$ and the ϵ_j are the (1-dimensional) representations that annihilate $[W_4, W_4]$.*

2. *W_6 is a group of order 2^7 with $[W_6, W_6] = \{\pm 1\} \cong \mathbb{Z}_2$ and $W_6/[W_6, W_6] \cong \mathbb{Z}_2^6$. Let $w_6 = w|_{W_6}$, where w is the (vector) representation of $Sp(4)$ on $\mathbb{C}^8$. Then w_6 is irreducible and $\widehat{W_6} = \{w_6\,, \varepsilon_1\,, \ldots, \varepsilon_{64}\}$ where the ε_j are the (1-dimensional) representations that annihilate $[W_6, W_6]$.*
3. *W_7 is a group of order 2^8 with derived group $[W_7, W_7] = \widetilde{Z}_{E_{8,D_8}}$, and $W_7/[W_7, W_7] \cong \mathbb{Z}_2^7$. Let $w_7 = w|_{W_7}$, where w is the (vector) representation of $SU(8)$ on $\mathbb{C}^8$. Then w_7 and w_7^* are inequivalent irreducible representations of W_7, and the unitary dual $\widehat{W_7} = \{w_7\,, w_7^*\,, \varepsilon'_1\,, \ldots, \varepsilon'_{128}\}$ where the ε'_j are the (1-dimensional) representations that annihilate $[W_7, W_7]$.*
4. *W_8 is a group of order 2^9 with $[W_8, W_8] = \widetilde{Z}_{E_{8,D_8}}$ and $W_8/[W_8, W_8] \cong \mathbb{Z}_2^8$. Let $w_8 = w|_{W_8}$, where w is the (vector) representation of $Spin(16)$ on $\mathbb{C}^{16}$. Then w_8 is irreducible and $\widehat{W_8} = \{w_8\,, \varepsilon''_1\,, \ldots, \varepsilon''_{256}\}$ where the ε''_j are the (1-dimensional) representations that annihilate $[W_8, W_8]$.*

3.3 The General Case

As before, G is a connected real simple Lie group with minimal parabolic $P = MAN$ such that $\mathfrak{m}$ is abelian. Recall that $\widetilde{G}$ is the universal covering group of G; G' is the algebraically simply connected Lie group with Lie algebra $\mathfrak{g}$; and $\overline{G}$ is the adjoint group. Also, $\widetilde{P}$, P', P, and $\overline{P}$ are the respective minimal parabolics. We specialize the summary section of [6] to our setting.

(a) $\widetilde{M} = \widetilde{F} \times \widetilde{M}^0$, where $\widetilde{F}$ is discrete, and if $\mathfrak{g}$ is the split real form of $\mathfrak{g}_{\mathbb{C}}$ then $\widetilde{M} = \widetilde{F}$ discrete,
(b) $\widetilde{F}$ is infinite if and only if G/K is a tube domain (hermitian symmetric space of tube type),
(c) $r(\mathfrak{g}) = 0 \Leftrightarrow \widetilde{M}$ is connected $\Leftrightarrow M'$ is connected,
(*d*) if $r(\mathfrak{g}) = 1$ then G/K is a tube domain and $\widetilde{F} \cong \mathbb{Z}$,
(e) if $r(\mathfrak{g}) > 1$ and G/K is a tube domain then $G' = Sp(n; \mathbb{R})$ and $\widetilde{M} = \widetilde{F} \cong \mathbb{Z}_2^{r(\mathfrak{g})-1} \times \mathbb{Z}$, and
(*f*) if $r(\mathfrak{g}) > 1$ and G/K is not a tube domain then $\widetilde{F}$ is a non-abelian group of order $2^{r(\mathfrak{g})+1}$.

Now we combine this information with Theorems 2.2 and 3.1, as follows. We use $\mathfrak{su}(1,1) = \mathfrak{sp}(1;\mathbb{R}) = \mathfrak{sl}(2;\mathbb{R})$, and $\mathfrak{so}(2,3) = \mathfrak{sp}(2;\mathbb{R})$.

Proposition 3.6 *Let G be a connected simple Lie group with Lie algebra $\mathfrak{g}$, let $P = MAN$ be a minimal parabolic subgroup of G, and assume that $\mathfrak{m}$ is abelian. Retain the notation $\widetilde{G}$, $\widetilde{P}$, and $\widetilde{M} = \widetilde{F} \times \widetilde{M}^0$ as above.*

1. *Cases $\mathfrak{h} = \mathfrak{a}$, where G is a split real Lie group and $r(\mathfrak{g}) =$* rank $\mathfrak{g}$. *Then $\widetilde{M} = \widetilde{F}$.*
   (1*a*) *If $\widetilde{M}$ is infinite then $\mathfrak{g} = \mathfrak{sp}(n; \mathbb{R})$, where $n \geqq 1$ and $r(\mathfrak{g}) = n$. In that case $\widetilde{M} \cong \mathbb{Z}_2^{n-1} \times Z$.*
   (1*b*) *If $\widetilde{M}$ is finite then $\widetilde{F}$ is a non-abelian* (but metabelian) *group of order $2^{r(\mathfrak{g})+1}$. In those cases*
   - $\mathfrak{sl}(n;\mathbb{R})$, *$n = 3$ or $n > 4$: $\widetilde{M} \cong D_n$ from* [6, Proposition 17.1].
   - $\mathfrak{so}(\ell, \ell+1)$ *and* $\mathfrak{so}(\ell, \ell)$, *$\ell \geqq 3$: $\widetilde{M} \cong D_\ell$ from* [6, Proposition 17.5].

$*\ \mathfrak{g}_{2,A_1A_1}$: $\widetilde{K} = Sp(1) \times Sp(1)$ *and* $\widetilde{M} \cong D_3$ *from* [6, Proposition 10.4].
$*\ \mathfrak{f}_{4,A_1C_3}$: $\widetilde{K} = Sp(1) \times Sp(3)$ *and* $\widetilde{M} \cong W_4$ (Proposition 3.5 above) *from* [6, Proposition 9.6].
$*\ \mathfrak{e}_{6,C_4}$: $\widetilde{K} = Sp(4)$ *and* $\widetilde{M} \cong W_6$ (Proposition 3.5 above) *from* [6, Proposition 9.5].
$*\ \mathfrak{e}_{7,A_7}$: $\widetilde{K} = SU(8)$ *and* $\widetilde{M} \cong W_7$ (Proposition 3.5 above) *from* [6, Proposition 9.3].
$*\ \mathfrak{e}_{8,D_8}$: $\widetilde{K} = Spin(16)$ *and* $\widetilde{M} \cong W_8$ (Proposition 3.5 above) *from* [6, Proposition 9.4].

2. *Cases where* $\mathfrak{g}$ *is the underlying real structure of a complex simple Lie algebra. Then* $r(\mathfrak{g}) = 0$, $\widetilde{F} = \{1\}$, $\widetilde{G} = G'$ (so G is linear)*, and* $\widetilde{M} = M'$ *is a torus group.*
3. *The four remaining cases*:
   (3*a*) $\mathfrak{g} = \mathfrak{su}(\ell, \ell+1)$*:* $r(\mathfrak{g}) = 0$, $\widetilde{F} = \{1\}$ *and* $\widetilde{M} \cong U(1)^{\ell-1} \times \mathbb{R}$ *from* [6, Proposition 17.3].
   (3*b*) $\mathfrak{g} = \mathfrak{su}(\ell, \ell)$*:* $r(\mathfrak{g}) = 1$, $\widetilde{F} \cong \mathbb{Z}$ *and* $\widetilde{M} \cong U(1)^{\ell-1} \times \mathbb{Z}$ *from* [6, Proposition 17.4].
   (3*c*) $\mathfrak{g} = \mathfrak{so}(\ell-1, \ell+1)$, $\ell \neq 1$*:* $r(\mathfrak{g}) = \ell - 2$*, and*
   $*$ *if* $\ell \neq 3$ *then* $\widetilde{F} \cong D_{\ell-1}$ *and* $\widetilde{M} \cong D_{\ell-1} \times \mathbb{R}$ *from* [6, Proposition 17.5],
   $*$ *if* $\ell = 3$ *and then* $\widetilde{F} \cong \mathbb{Z}$ *and* $\widetilde{M} \cong \mathbb{Z} \times \mathbb{R}$ *from* [6, Proposition 17.6].
   (3*d*) $\mathfrak{g} = \mathfrak{e}_{6,A_1A_5}$*:* $r(\mathfrak{g}) = 2$, $\widetilde{F} \cong D_3$ *and* $\widetilde{M} \cong D_3 \times Spin(8)$ *from* [6, §16] *and* [5, Theorem 3.5]

4 Stepwise Square Integrable Representations of the Nilradical

In this section, we recall the background for Fourier Inversion on connected simply connected nilpotent Lie groups, and its application to nilradicals of minimal parabolics. In Section 5, we use it to describe "generic" representations and the Plancherel Formula on the minimal parabolic. Then in Section 6 we come to the Fourier Inversion Formula on the parabolic. In the last section (the Appendix), we will run through the cases of Theorem 2.2 making explicit the Plancherel Formula and Fourier Inversion Formula for those nilradicals.
The basic decomposition is

$$\begin{aligned}
&N = L_1 L_2 \dots L_m, \text{ where} \\
&\quad \text{(a) each factor } L_r \text{ has unitary representations that are square integrable} \\
&\quad \text{modulo its center } Z_r, \\
&\quad \text{(b) each } L_r = L_1 L_2 \dots L_r \text{ is a normal subgroup of } N \text{ with} \\
&\quad N_r = N_{r-1} \rtimes L_r \text{ semidirect}, \\
&\quad \text{(c) decompose } \mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r \text{ and } \mathfrak{n} = \mathfrak{s} + \mathfrak{v} \text{ as vector direct sums where} \\
&\quad \mathfrak{s} = \oplus\, \mathfrak{z}_r \text{ and } \mathfrak{v} = \oplus\, \mathfrak{v}_r; \text{ then } [\mathfrak{l}_r, \mathfrak{z}_s] = 0 \text{ and } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v} \text{ for } r > s.
\end{aligned} \tag{4.1}$$

We will need the notation

$$
\begin{aligned}
&\text{(a) } d_r = \tfrac{1}{2}\dim(\mathfrak{l}_r/\mathfrak{z}_r) \text{ so } \tfrac{1}{2}\dim(\mathfrak{n}/\mathfrak{s}) = d_1 + \cdots + d_m\,, \text{ and} \\
&\quad c = 2^{d_1+\cdots+d_m} d_1! d_2! \ldots d_m! \\
&\text{(b) } b_{\lambda_r} : (x, y) \mapsto \lambda([x, y]) \text{ viewed as a bilinear form on } \mathfrak{l}_r/\mathfrak{z}_r \\
&\text{(c) } S = Z_1 Z_2 \ldots Z_m = Z_1 \times \cdots \times Z_m \text{ where } Z_r \text{ is the center of } L_r \\
&\text{(d) } P : \text{ polynomial } P(\lambda) = \mathrm{Pf}(b_{\lambda_1})\mathrm{Pf}(b_{\lambda_2}) \ldots \mathrm{Pf}(b_{\lambda_m}) \text{ on } \mathfrak{s}^* \\
&\text{(e) } \mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid P(\lambda) \neq 0\} \\
&\text{(f) } \pi_\lambda \in \widehat{N} \text{ where } \lambda \in \mathfrak{t}^* : \text{ irreducible unitary rep. of } N = L_1 L_2 \ldots L_m\,.
\end{aligned}
\tag{4.2}
$$

As $\exp : \mathfrak{n} \to N$ is a polynomial diffeomorphism, the Schwartz space $\mathcal{C}(N)$ consists of all C^∞ functions f on N such that $f \cdot \exp \in \mathcal{C}(\mathfrak{n})$, the classical Schwartz space of all rapidly decreasing C^∞ functions on the real vector space $\mathfrak{n}$. The general result, which we will specialize, is [16]

Theorem 4.1 *Let N be a connected simply connected nilpotent Lie group that satisfies* (4.1)*. Then Plancherel measure for N is concentrated on $\{[\pi_\lambda] \mid \lambda \in \mathfrak{t}^*\}$ where $[\pi_\lambda]$ denotes the unitary equivalence class of π_λ. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ, then the coefficient $f_{u,v}(x) = \langle u, \pi_\nu(x)v\rangle$ satisfies*

$$||f_{u,v}||^2_{\mathcal{L}^2(N/S)} = \frac{||u||^2||v||^2}{|P(\lambda)|}\,. \tag{4.3}$$

The distribution character Θ_{π_λ} of π_λ satisfies

$$\Theta_{\pi_\lambda}(f) = c^{-1}|P(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi) d\nu_\lambda(\xi) \textit{ for } f \in \mathcal{C}(N), \tag{4.4}$$

where c is given by (4.2)(a)*; $\mathcal{C}(N)$ is the Schwartz space; f_1 is the lift $f_1(\xi) = f(\exp(\xi))$; $\widehat{f_1}$ is its classical Fourier transform; $\mathcal{O}(\lambda)$ is the coadjoint orbit* $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$*; and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to* $\mathrm{Ad}^*(N)\lambda$*. The Plancherel Formula on N is*

$$\mathcal{L}^2(N) = \int_{\mathfrak{t}^*} \mathcal{H}_{\pi_\lambda} \widehat{\otimes} \mathcal{H}^*_{\pi_\lambda} |P(\lambda)| d\lambda \textit{ where } \mathcal{H}_{\pi_\lambda} \textit{ is the representation space of } \pi_\lambda \tag{4.5}$$

and the Fourier Inversion Formula is

$$f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|P(\lambda)| d\lambda \textit{ for } f \in \mathcal{C}(N) \textit{ with } c \textit{ as in } (4.2)(\mathrm{a}). \tag{4.6}$$

Definition 4.2 The representations π_λ of (4.2(f)) are the *stepwise square integrable* representations of N relative to (4.1). □

Nilradicals of minimal parabolics fit this pattern as follows. Start with the Iwasawa decomposition $G = KAN$. Here we use the root order of (2.1) so $\mathfrak{n}$ is the sum of the positive $\mathfrak{a}$-root spaces in $\mathfrak{g}$. Since $\Delta(\mathfrak{g}, \mathfrak{a})$ is a root system, if $\gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and $\gamma|_{\mathfrak{a}} \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ then $\gamma \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Define β_1 to be the maximal root, β_{r+1} a maximum among the positive roots orthogonal to $\{\beta_1, \dots, \beta_r\}$, etc. This constructs a maximal set $\{\beta_1, \dots, \beta_m\}$ of strongly orthogonal positive restricted roots. For $1 \leqq r \leqq m$ define

$$\begin{aligned} &\Delta_1^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \beta_1 - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\} \text{ and} \\ &\Delta_{r+1}^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \backslash (\Delta_1^+ \cup \dots \cup \Delta_r^+) \mid \beta_{r+1} - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\}. \end{aligned} \tag{4.7}$$

Then

$$\Delta_r^+ \cup \{\beta_r\} = \{\alpha \in \Delta^+ \mid \alpha \perp \beta_i \text{ for } i < r \text{ and } \langle \alpha, \beta_r \rangle > 0\}. \tag{4.8}$$

Note: if β_r is divisible then $\frac{1}{2}\beta_r \in \Delta_r^+$; See Corollary 2.3. Now define

$$\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum\nolimits_{\Delta_r^+} \mathfrak{g}_\alpha \text{ for } 1 \leqq r \leqq m. \tag{4.9}$$

Thus $\mathfrak{n}$ has an increasing foliation by ideals

$$\mathfrak{n}_r = \mathfrak{l}_1 + \mathfrak{l}_2 + \dots + \mathfrak{l}_r \text{ for } 1 \leqq r \leqq m. \tag{4.10}$$

The corresponding group level decomposition $N = L_1 L_2 \dots L_m$ and the semidirect product decompositions $N_r = N_{r-1} \rtimes L_r$ satisfy all the requirements of (4.1).

5 Generic Representations of the Parabolic

In this section, G is a connected real reductive Lie group, not necessarily linear, and the minimal parabolic subgroup $P = MAN$ is solvable. In other words, we are in the setting of Theorems 3.3 and 4.1. Recall $\mathfrak{t}^* = \{\lambda = (\lambda_1 + \dots + \lambda_m) \in \mathfrak{s}^* \mid \text{ each } \lambda_r \in \mathfrak{g}_{\beta_r} \text{ with } \mathrm{Pf}_{\mathfrak{l}_r}(\lambda_r) \neq 0\}$. For each $\lambda \in \mathfrak{t}^*$, we have the stepwise square integrable representation π_λ of N. Now we look at the corresponding representations of MAN. As before, the superscript ^{0} denotes identity component.

Theorem 3.3 shows that $\mathrm{Ad}(MA)$ is commutative. Also, $\mathrm{Ad}(M^0 A)$ acts $\mathbb{C}$ irreducibly on each complexified restricted root space $(\mathfrak{g}_\alpha)_{\mathbb{C}}$ by [14, Theorem 8.13.3]. But $\mathrm{Ad}(M^0 A)$ preserves each $\mathfrak{g}_\alpha$, so it is irreducible there. Thus, for each $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$, either $\mathrm{Ad}(M^0)$ is trivial on $\mathfrak{g}_\alpha$ and $\dim_{\mathbb{R}} \mathfrak{g}_\alpha = 1$, or $\mathrm{Ad}(M^0)$ is non-trivial on $\mathfrak{g}_\alpha$ and $\dim_{\mathbb{R}} \mathfrak{g}_\alpha = 2$. In the latter case $\mathrm{Ad}(M^0)|_{\mathfrak{g}_\alpha}$ must be the circle group of all proper rotations of $\mathfrak{g}_\alpha$. A glance at Theorem 2.2 makes this more explicit on the $\mathfrak{g}_{\beta_r}$.

Lemma 5.1 *In cases* (1) *and* (3) *of* Theorem 2.2*, each* $\dim_{\mathbb{R}} \mathfrak{g}_{\beta_r} = 1$*, so* $\mathrm{Ad}^*(M^0)$ *is trivial on each* $\mathfrak{g}_{\beta_r}$*. In case* (2) *each* $\dim_{\mathbb{R}} \mathfrak{g}_{\beta_r} = 2$*, so* $\mathrm{Ad}^*(M^0)$ *acts on* $\mathfrak{g}_{\beta_r}$ *as a circle group* $SO(2)$.

Recall the notation $\widetilde{M} = \widetilde{F} \times \widetilde{M}^0$ from Section 3.1 where $p : \widetilde{G} \to G$ is the universal cover and $\widetilde{P} = \widetilde{M}\widetilde{A}\widetilde{N}$ is $p^{-1}(P)$. Note $M = F \cdot M^0$ where $F = p(\widetilde{F})$. Combining [17, Lemma 3.4] with [17, Proposition 3.6] and specializing to the case of $\mathfrak{m}$ abelian, we have

Lemma 5.2 *When $\mathfrak{m}$ is abelian, $\mathrm{Ad}^*(\widetilde{F})$ acts trivially on $\mathfrak{s}^*$. Then in particular, $\mathrm{Ad}^*(F)|_{\mathfrak{s}^*}$ is trivial and $\mathrm{Ad}^*(M)|_{\mathfrak{s}^*} = \mathrm{Ad}^*(M^0)|_{\mathfrak{s}^*}$.*

Now combine Lemmas 5.1 and 5.2:

Proposition 5.3 *In cases* (1) *and* (3) *of* Theorem 2.2, *$\mathrm{Ad}^*(M)$ is trivial on $\mathfrak{s}^*$. In case* (2) *of* Theorem 2.2, *$\mathrm{Ad}^*(M)$ acts non-trivially as a circle group on each $\mathfrak{g}^*_{\beta_r}$, thus acts almost effectively as a torus group on $\mathfrak{s}^*$.*

Fix $\lambda \in \mathfrak{t}^*$. By Proposition 5.3 its $\mathrm{Ad}^*(M)$-stabilizer is all of M in cases (1) and (3) of Theorem 2.2, and in case (2) it has form

$$M_\diamond = FM^0_\diamond \text{ where } M^0_\diamond = \{x \in M^0 \mid \mathrm{Ad}(x)|_{\mathfrak{s}} = 1\}. \tag{5.1}$$

This is independent of the choice of $\lambda \in \mathfrak{t}^*$. Thus the kernel of the action of $\mathrm{Ad}(M^0)$ on $\mathfrak{s}^*$ is the codimension m subtorus of M^0 with Lie algebra $\mathfrak{m}_\diamond = \sqrt{-1}\{\xi \in \mathfrak{a} \mid$ every $\beta_r(\xi) = 0\}$.

Since $\mathrm{Ad}^*(A)$ acts on $\mathfrak{g}_{\beta_r}$ by positive real scalars, given by the real character e^{β_r}, we have a similar result for A: the $\mathrm{Ad}^*(A)$-stabilizer of any $\lambda \in \mathfrak{t}^*$ is

$$A_\diamond = \{\exp(\xi) \mid \xi \in \mathfrak{a} \text{ and every } \beta_r(\xi) = 0\}. \tag{5.2}$$

Its Lie algebra is $\mathfrak{a}_\diamond = \{\xi \in \mathfrak{a} \mid \text{ every } \beta_r(\xi) = 0\}$. Combining (5.1) and (5.2) we arrive at

Lemma 5.4 *The stepwise square integrable representations π_λ of N all have the same MA-stabilizer $M_\diamond A_\diamond$ on the unitary dual $\widehat{N}$.*

Specialize [17, Lemma 3.8] and [17, Lemma 5.4] to π_λ and $M_\diamond A_\diamond N$. The Mackey obstruction ([7], [9], or see [8]) vanishes as in [11] and [13]. Now π_λ extends to an irreducible unitary representation $\widetilde{\pi_\lambda}$ of $M_\diamond A_\diamond N$ on the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ. Compare [4]. Consider the unitarily induced representations

$$\pi_{\chi,\alpha,\lambda} := \mathrm{Ind}_{M_\diamond A_\diamond N}^{MAN} (\chi \otimes e^{i\alpha} \otimes \widetilde{\pi_\lambda}) \text{ for } \chi \in \widehat{M_\diamond},\ \alpha \in \mathfrak{a}^*_\diamond, \text{ and } \lambda \in \mathfrak{t}^*. \tag{5.3}$$

Note that χ is a (finite-dimensional) unitary representation of the metabelian group $M_\diamond$ and that the representation space of $\chi \otimes e^{i\alpha} \otimes \widetilde{\pi_\lambda}$ is $\mathcal{H}_\chi \widehat{\otimes}\, \mathbb{C} \widehat{\otimes} \mathcal{H}_{\pi_\lambda}$. So the representation space of $\pi_{\chi,\alpha,\lambda}$ is

$$\begin{aligned}\mathcal{H}_{\pi_{\chi,\alpha,\lambda}} =& \{L^2 \text{ functions } f : MAN \to \mathcal{H}_\chi \widehat{\otimes} \mathcal{H}_{\pi_\lambda} \mid \\ & f(xman) = \delta(a)^{-1/2} e^{-i\alpha(\log a)} (\chi(m)^{-1} \otimes \widetilde{\pi_\lambda}(man)^{-1})(f(x)), \\ & x \in MAN, man \in M_\diamond A_\diamond N\}.\end{aligned} \tag{5.4}$$

Unitarity of $\pi_{\chi,\alpha,\lambda}$ requires the $\delta(a)^{-1/2}$ term, as we will see when we discuss Dixmier–Pukánszky operators.

Definition 5.5 The $\pi_{\chi,\alpha,\lambda}$ of (5.3) are the *generic* irreducible unitary representations of *MAN*.

Now, specializing [17, Theorem 5.12],

Theorem 5.6 *Let G be a connected real reductive Lie group and $P = MAN$ a minimal parabolic subgroup. Suppose that P is solvable. Then the Plancherel measure for MAN is concentrated on the set of all generic unitary representation classes $[\pi_{\chi,\alpha,\lambda}] \in \widehat{MAN}$ and*

$$\mathcal{L}^2(MAN) = \int_{\chi \in \widehat{M_\diamond}} \left(\int_{\alpha \in \mathfrak{a}_\diamond^*} \left(\int_{\lambda \in \mathfrak{t}^*} \mathcal{H}_{\pi_{\chi,\alpha,\lambda}} \widehat{\otimes} \mathcal{H}^*_{\pi_{\chi,\alpha,\lambda}} |P(\lambda)| d\lambda \right) d\alpha \right) \deg(\chi) d\chi .$$

6 Fourier Inversion on the Parabolic

In this section, as before, G is a connected[1] real reductive Lie group whose minimal parabolic subgroup $P = MAN$ is solvable. We work out an explicit Fourier Inversion Formula for MAN. It uses the generic representations of Definition 5.5 and an operator to compensate non-unimodularity. That operator is the Dixmier–Pukánszky Operator on MAN and its domain is the Schwartz space $\mathcal{C}(MAN)$ of rapidly decreasing C^∞ functions.

The kernel of the modular function δ of MAN contains MN and is given on A as follows.

Lemma 6.1 [17, Lemmas 4.2 & 4.3] *Let* $\xi \in \mathfrak{a}$. *Then* $\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r) \in \mathbb{Z}$ *for* $1 \leqq r \leqq m$ *and*

(i) *the trace of* $\mathrm{ad}\,(\xi)$ *on* $\mathfrak{l}_r$ *is* $\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)\beta_r(\xi)$,

(ii) *the trace of* $\mathrm{ad}\,(\xi)$ *on* $\mathfrak{n}$ *and on* $\mathfrak{p}$ *is* $\frac{1}{2}\sum_r(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)\beta_r(\xi)$, *and*

(iii) *the determinant of* $\mathrm{Ad}(\exp(\xi))$ *on* $\mathfrak{n}$ *and on* $\mathfrak{p}$ *is* $\prod_r \exp(\beta_r(\xi))^{\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)}$.

The modular function $\delta = \mathrm{Det} \cdot \mathrm{Ad} : \mathrm{man} \mapsto \prod_r \exp(\beta_r(\log a))^{\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)}$.

Recall the *quasi-center determinant* $\mathrm{Det}_{\mathfrak{s}^*}(\lambda) := \prod_r(\beta_r(\lambda))^{\dim \mathfrak{g}_{\beta_r}}$. It is a polynomial function on $\mathfrak{s}^*$, and ([17, Proposition 4.7]) the product $\mathrm{Pf} \cdot \mathrm{Det}_{\mathfrak{s}^*}$ is an $\mathrm{Ad}(MAN)$-semi-invariant polynomial on $\mathfrak{s}^*$ of degree $\frac{1}{2}(\dim \mathfrak{n} + \dim \mathfrak{s})$ and of weight equal to that of the modular function δ.

[1] These results extend *mutatis mutandis* to all real reductive Lie groups G such that (a) the minimal parabolic subgroup of G^0 is solvable, (b) if $g \in G$ then $\mathrm{Ad}(g)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$, and (c) G has a closed normal abelian subgroup U such that (c1) U centralizes the identity component G^0; (c2) UG^0 has finite index in G; and (c3) $U \cap G^0$ is co-compact in the center of G^0. The extension is relatively straightforward using the methods of [12] as described in [12, Introduction] and [12, Section 1]. For continuity of exposition, we the leave details on that to the interested reader.

Our fixed decomposition $\mathfrak{n} = \mathfrak{v} + \mathfrak{s}$ gives $N = VS$ where $V = \exp(\mathfrak{v})$ and $S = \exp(\mathfrak{s})$. Now define

$$D : \text{ Fourier transform of Pf} \cdot \mathrm{Det}_{\mathfrak{s}^*}, \text{ acting on } MAN = MAVS \text{ by acting on } S. \tag{6.1}$$

See [17, Section 1] for a discussion of the Schwartz space $\mathcal{C}(MAN)$.

Theorem 6.2 ([17, Theorem 4.9]) *D is an invertible self-adjoint differential operator of degree $\frac{1}{2}(\dim \mathfrak{n} + \dim \mathfrak{s})$ on $\mathcal{L}^2(MAN)$ with dense domain $\mathcal{C}(MAN)$, and it is* $\mathrm{Ad}(MAN)$*-semi-invariant of weight equal to the modular function δ. In other words, $|D|$ is a Dixmier–Pukánszky Operator on MAN with domain $\mathcal{C}(MAN)$.*

For the Fourier Inversion Formula we also need to know the $\mathrm{Ad}^*(MA)$-orbits on $\mathfrak{t}^*$.

Proposition 6.3 *The* $\mathrm{Ad}^*(MA)$*-orbits on* $\mathfrak{t}^*$ *are the following.*

In cases (1) *and* (3) *of* Theorem 2.2*, the number of* $\mathrm{Ad}(MA)^*$*-orbits on* $\mathfrak{t}^*$ *is* 2^m. *Fix non-zero $\nu_r \in \mathfrak{g}_{\beta_r}$. Then the orbits are the*

$$\mathcal{O}_{(\varepsilon_1,\dots,\varepsilon_m)} = \{\lambda = \lambda_1 + \cdots + \lambda_m \mid \lambda_r \in \mathbb{R}^+ \varepsilon_r \nu_r \textit{ for } 1 \leqq r \leqq m \textit{ where each } \varepsilon_r = \pm 1\}.$$

In case (2) *of* Theorem 2.2*, there is just one* $\mathrm{Ad}(MA)^*$*-orbit on* $\mathfrak{t}^*$*, i.e.,* $\mathrm{Ad}(MA)^*$ *is transitive on* $\mathfrak{t}^*$.

Proof The assertions follow from Proposition 5.3, as follows. In case (2) of Theorem 2.2, where $\dim \mathfrak{g}_{\beta_r} = 2$, $\mathrm{Ad}^*(M)$ acts on $\mathfrak{s}^*$ by independent circle groups on the $\mathfrak{g}^*_{\beta_r}$ while $\mathrm{Ad}^*(A)$ acts on $\mathfrak{s}^*$ by independent positive scalar multiplication on the $\mathfrak{g}^*_{\beta_r}$. In cases (1) and (3) of Theorem 2.2, where $\dim \mathfrak{g}_{\beta_r} = 1$, $\mathrm{Ad}^*(M)$ acts trivially on $\mathfrak{s}^*$ while $\mathrm{Ad}^*(A)$ acts on $\mathfrak{s}^*$ by independent positive scalar multiplication on the $\mathfrak{g}^*_{\beta_r}$. □

We combine Lemma 5.4, Theorem 5.6, Theorem 6.2, and Proposition 6.3 for the Fourier Inversion Formula. We need notation from Proposition 6.3. For cases (1) and (3) of Theorem 2.2 we fix non-zero $\nu_r \in \mathfrak{g}_{\beta_r}$; then for each $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ we have the orbit $\mathcal{O}_\varepsilon$. As usual we write ℓ_x for left translate, $(\ell_x h)(y) = h(x^{-1}y)$ and r_y for right translate $(r_y h)(x) = h(xy)$. Using the structure of M as a quotient of $\widetilde{M}$ by a subgroup of the center of $\widetilde{G}$, from Proposition 3.6, [17, Theorem 6.1] specializes as follows.

Theorem 6.4 *Let G be a real reductive Lie group whose minimal parabolic subgroup $P = MAN$ is solvable. Given a generic representation $\pi_{\chi,\alpha,\lambda} \in \widehat{MAN}$, its distribution character is tempered and is given by*

$$\begin{aligned}\Theta_{\pi_{\chi,\alpha,\lambda}}(f) &= \mathrm{trace}\, \pi_{\chi,\alpha,\lambda}(f)\\ &= \int_M \mathrm{trace}\, \chi(m) \int_A e^{i\alpha(\log a)} \Theta_{\pi_\lambda}(\ell_{(ma)^{-1}} f) da\, dm \textit{ for } f \in \mathcal{C}(MAN).\end{aligned}$$

where Θ_{π_λ} *is given by* Theorem 4.1. *In Cases* (1) *and* (3) *of* Theorem 2.2 *the Fourier Inversion Formula is*

$$f(x) = c\int_{\chi\in\widehat{M}}\left(\int_{\alpha\in\mathfrak{a}^*_\diamondsuit}\sum_{\varepsilon}\left(\int_{\lambda\in\mathcal{O}_\varepsilon}\Theta_{\pi_{\chi,\alpha,\lambda}}(D(r(x)f))|\mathrm{Pf}(\lambda)|d\lambda\right)d\alpha\right)\deg(\chi)\,d\chi$$

where $c > 0$ *depends on* (4.2)(a) *and normalizations of Haar measures. In Case* (2) *of* Theorem 2.2*, the Fourier Inversion Formula is*

$$f(x) = c\sum\nolimits_{\chi\in\widehat{M_\diamondsuit}}\left(\int_{\alpha\in\mathfrak{a}^*_\diamondsuit}\left(\int_{\lambda\in\mathfrak{t}^*}\Theta_{\pi_{\chi,\alpha,\lambda}}(D(r(x)f))|\mathrm{Pf}(\lambda)|d\lambda\right)d\alpha\right)$$

where again, $c > 0$ *depends on* (4.2)(a) *and normalization of Haar measures.*

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7 Appendix: Explicit Decompositions

One can see the decompositions of Section 2 explicitly. This is closely related to the computations in [15, Section 8]. As noted in Corollary 2.3, the only case of Theorem 2.2 where there is any divisibility is $\mathfrak{g} = \mathfrak{su}(\ell, \ell+1)$, and in that case the only divisibility is given by the $\left\{\frac{1}{2}\beta_r\,, \beta_r\right\} \subset \Delta^+(\mathfrak{g}, \mathfrak{a})$. For ease of terminology we say that $\Delta(\mathfrak{g}, \mathfrak{a})$ is *non-multipliable* if $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ implies $2\alpha \notin \Delta(\mathfrak{g}, \mathfrak{a})$, *multipliable* otherwise.

We first consider the multipliable case $\mathfrak{g} = \mathfrak{su}(\ell, \ell+1)$. Here $\psi_i = \alpha_i|_{\mathfrak{a}}$, where $\{\alpha_1, \dots, \alpha_{2\ell}\}$ are the simple roots of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ in the usual order. The multipliable roots in $\Delta^+(\mathfrak{g}, \mathfrak{a})$ are just the $\frac{1}{2}\beta_r = (\alpha_r + \cdots + \alpha_\ell)|_{\mathfrak{a}}\,, 1 \leqq r \leqq \ell$, where $\beta_r = 2(\psi_r + \cdots + \psi_\ell) = (\alpha_r + \cdots + \alpha_{2\ell-r+1})|_{\mathfrak{a}}$. If $\alpha + \alpha' = \beta_r$ then, either $\alpha = \alpha' = \frac{1}{2}\beta_r$, or one of α, α' has form $\gamma_{r,u} = \psi_r + \psi_{r+1} + \cdots + \psi_u$, while the other is $\gamma'_{r,u} = \psi_r + \psi_{r+1} + \cdots + \psi_u + 2(\psi_{u+1} + \cdots + \psi_\ell)$. Now $\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \mathfrak{g}_{\frac{1}{2}\beta_r} + \sum_{r\leqq u\leqq n}(\mathfrak{g}_{\gamma_{r,u}} + \mathfrak{g}_{\gamma'_{r,u}})$. The conditions of (4.1) follow by inspection.

For the rest of this section, we assume that $\mathfrak{g} \neq \mathfrak{su}(\ell, \ell+1)$, in other words, following Corollary 2.3, that $\Delta(\mathfrak{g}, \mathfrak{a})$ is non-multipliable.

First suppose that $\Delta(\mathfrak{g}, \mathfrak{a})$ is of type $A_{\ell-1}$: ψ_1 — ψ_2 — $\psi_{\ell-2}$ … — $\psi_{\ell-1}$. Then $m = [\ell/2]$ and $\beta_r = \psi_r + \psi_{r+1} + \cdots + \psi_{\ell-r}$. If $\alpha\,, \alpha' \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ with $\alpha + \alpha' = \beta_r$ then one of $\alpha\,, \alpha'$ must have form $\gamma_{r,s} := \psi_r + \cdots + \psi_s$ and the other must be $\gamma'_{r,s} := \psi_{s+1} + \cdots + \psi_{\ell-r}$. Thus $\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum_{r\leqq s<\ell-r}(\mathfrak{g}_{\gamma_{r,s}} + \mathfrak{g}_{\gamma'_{r,s}})$. The conditions of (4.1) follow by inspection.

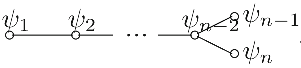
Next suppose that $\Delta(\mathfrak{g}, \mathfrak{a})$ is of type B_n: ψ_1 — ψ_2 … — ψ_{n-1} ψ_n. Then $\beta_1 = \psi_1 + 2(\psi_2 + \cdots + \psi_n)$, $\beta_2 = \psi_1$, $\beta_3 = \psi_3 + 2(\psi_4 + \cdots + \psi_n)$, $\beta_4 = \psi_3$, etc. If r is even, $\beta_r = \psi_{r-1}$ and $\mathfrak{l}_r = \mathfrak{g}_{\beta_r}$.

Now let r be odd, $\beta_r = \psi_r + 2(\psi_{r+1} + \cdots + \psi_n)$. If $\alpha, \alpha' \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ with $\alpha + \alpha' = \beta_r$ then one possibility is that one of $\alpha\,, \alpha'$ has form $\gamma_{r,u} := \psi_r + \psi_{r+1} + \cdots + \psi_u$ and the other is $\gamma'_{r,u} := \psi_{r+1} + \cdots + \psi_u + 2(\psi_{u+1} + \cdots + \psi_n)$ with $r < u < n$. Another possibility is that one of $\alpha\,, \alpha'$ is $\gamma_{r,u} - \psi_r$, while the other is $\gamma'_{r,u} + \psi_r$.

A third is that one of α, α' is $\gamma_{r,n} := \psi_r + \psi_{r+1} + \cdots + \psi_n$, while the other is $\gamma'_{r,n} := \psi_{r+1} + \cdots + \psi_n$. Then $\mathfrak{l}_r$ is the sum of $\mathfrak{g}_{\beta_r}$ with the sum of all these possible $\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha'}$, and the conditions of (4.1) follow by inspection.

Let $\Delta(\mathfrak{g}, \mathfrak{a})$ be of type C_n: 

. Then $\beta_r = 2(\psi_r + \ldots \psi_{n-1}) + \psi_n$ for $1 \leqq r < n$, and $\beta_n = \psi_n$. If $\alpha + \alpha' = \beta_r$ with $r < n$ then one of α, α' has form $\gamma_{r,u} = \psi_r + \psi_{r+1} + \cdots + \psi_u$, while the other is $\gamma'_{r,u} = \psi_r + \psi_{r+1} + \cdots + \psi_u + 2(\psi_{u+1} + \cdots + \psi_{n-1}) + \psi_n$, $r \leqq u < n$. Note that $\gamma_{r,n-1} = \psi_r + \psi_{r+1} + \cdots + \psi_{n-1}$ and $\gamma'_{r,n-1} = \psi_r + \psi_{r+1} + \cdots + \psi_n$. Now $\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum_{r \leqq u < n} (\mathfrak{g}_{\gamma_{r,u}} + \mathfrak{g}_{\gamma'_{r,u}})$. The conditions of (4.1) follow by inspection.

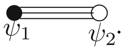
Let $\Delta(\mathfrak{g}, \mathfrak{a})$ be of type D_n: 

. Then $\beta_1 = \psi_1 + 2(\psi_2 + \cdots + \psi_{n-2}) + \psi_{n-1} + \psi_n$, $\beta_2 = \psi_1$, $\beta_3 = \psi_3 + 2(\psi_4 + \cdots + \psi_{n-2}) + \psi_{n-1} + \psi_n$, $\beta_4 = \psi_3$, etc., until $r = n - 3$.

If n is even then $m = n$, $\beta_{n-3} = \psi_{n-3} + 2\psi_{n-2} + \psi_{n-1} + \psi_n$, $\beta_{n-2} = \psi_{n-3}$, $\beta_{n-1} = \psi_{n-1}$, and $\beta_n = \psi_n$. Then, if $r \leqq n-2$ is even we have $\beta_r = \psi_{r-1}$. Thus $\mathfrak{l}_r = \mathfrak{g}_{\beta_r}$ for n even and either r even or $r = n - 1$.

If n is odd then $m = n - 1$, $\beta_{n-2} = \psi_{n-2} + \psi_{n-1} + \psi_n$, and $\beta_{n-1} = \psi_{n-2}$. Thus $\beta_r = \psi_{r-1}$ and $\mathfrak{l}_r = \mathfrak{g}_{\beta_r}$ for n odd and r even.

That leaves the cases where r is odd and $r \neq n - 1$, so $\beta_r = \psi_r + 2(\psi_{r+1} + \cdots + \psi_{n-2}) + \psi_{n-1} + \psi_n$. If $\alpha + \alpha' = \beta_r$, one possibility is that one of α, α' is of the form $\gamma_{r,u} := \psi_r + (\psi_{r+1} + \cdots + \psi_u)$ with $r + 1 \leqq u \leqq n - 2$, while the other is $\gamma'_{r,u} := (\psi_{r+1} + \cdots + \psi_u) + 2(\psi_{u+1} + \cdots + \psi_{n-2}) + \psi_{n-1} + \psi_n$, or that one of α, α' is of the form $\gamma_{r,u} - \psi_r$, while the other is $\gamma'_{r,u} + \psi_r$. A third possibility is that one of α, α' is $\gamma_{r,n-1} := \psi_r + \psi_{r+1} + \cdots + \psi_{n-1}$, while the other is $\gamma'_{r,n-1} := \psi_{r+1} + \cdots + \psi_{n-2} + \psi_n$. The fourth possibility is that one of α, α' is $\gamma_{r,n-1} - \psi_{n-1} + \psi_n$, while the other is $\gamma'_{r,n-1} + \psi_{n-1} - \psi_n$. Then $\mathfrak{l}_r$ is the sum of $\mathfrak{g}_{\beta_r}$ with the sum of all these possible $\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha'}$, and the conditions of (4.1) follow by inspection.

Let $\Delta(\mathfrak{g}, \mathfrak{a})$ be of type G_2 

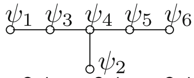
. Then $\beta_1 = 3\psi_1 + 2\psi_2$ and $\beta_2 = \psi_1$. If $\alpha + \alpha' = \beta_1$ then either one of α, α' is $3\psi_1 + \psi_2$ and the other is ψ_2, or one of α, α' is $2\psi_1 + \psi_2$ and the other is $\psi_1 + \psi_2$. Thus $\mathfrak{l}_1 = \mathfrak{g}_{\beta_1} + (\mathfrak{g}_{3\psi_1+\psi_2} + \mathfrak{g}_{\psi_2}) + (\mathfrak{g}_{2\psi_1+\psi_2} + \mathfrak{g}_{\psi_1+\Psi_2})$ and $\mathfrak{l}_2 = \mathfrak{g}_{\beta_2}$. The conditions of (4.1) follow.

Suppose that $\Delta(\mathfrak{g}, \mathfrak{a})$ is of type F_4 

. Then $\beta_1 = 2\psi_1 + 3\psi_2 + 4\psi_4 + 2\psi_4$, $\beta_2 = \psi_2 + 2\psi_3 + 2\psi_4$, $\beta_3 = \psi_2 + 2\psi_3$, and $\beta_4 = \psi_2$. Thus $\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum_{(\gamma,\gamma') \in S_r} (\mathfrak{g}_\gamma + \mathfrak{g}_{\gamma'})$, where

$$\begin{aligned}
S_1 = \{&(\psi_1,\ \psi_1 + 3\psi_2 + 4\psi_3 + 2\psi_4), (\psi_1 + \psi_2,\ \psi_1 + 2\psi_2 + 4\psi_3 + 2\psi_4),\\
&(\psi_1 + \psi_2 + \psi_3,\ \psi_1 + 2\psi_2 + 3\psi_3 + 2\psi_4), (\psi_1 + \psi_2 + 2\psi_3,\ \psi_1 + 2\psi_2 + 2\psi_3 + 2\psi_4),\\
&(\psi_1 + \psi_2 + \psi_3 + \psi_4,\ \psi_1 + 2\psi_2 + 3\psi_3 + \psi_4), (\psi_1 + 2\psi_2 + 2\psi_3,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4),\\
&(\psi_1 + \psi_2 + 2\psi_3 + \psi_4,\ \psi_1 + 2\psi_2 + 2\psi_3 + \psi_4);\\
S_2 = \{&\{\psi_4,\ \psi_2 + 2\psi_3 + \psi_4\}, \{\psi_3 + \psi_4,\ \psi_2 + \psi_3 + \psi_4\}\};\ S_3 = \{\psi_3,\ \psi_2 + \psi_3\};\ \text{ and } S_4 = \emptyset.
\end{aligned}$$

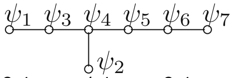
The conditions of (4.1) follow.

Suppose that $\Delta(\mathfrak{g}, \mathfrak{a})$ is of type E_6 

. Then the strongly orthogonal roots β_i are given by $\beta_1 = \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$, $\beta_2 = \psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6$, , $\beta_3 = \psi_3 + \psi_4 + \psi_5$ and $\beta_4 = \psi_4$. Now $\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum_{(\gamma,\gamma')\in S_r}(\mathfrak{g}_\gamma + \mathfrak{g}_{\gamma'})$, where

$$\begin{aligned}
S_1 = \{&(\psi_2, \psi_1 + \psi_2 + 2\psi_2 + 3\psi_4 + 2\psi_5 + \psi_6), (\psi_2 + \psi_4, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6),\\
&(\psi_2 + \psi_3 + \psi_4, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6), (\psi_2 + \psi_4 + \psi_5, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4\\
&+\psi_5 + \psi_6), (\psi_1 + \psi_2 + \psi_3 + \psi_4, \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6), (\psi_2 + \psi_3 + \psi_4 + \psi_5, \psi_1\\
&+ \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6), (\psi_2 + \psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5),\\
&(\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5, \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6), (\psi_2 + \psi_3 + 2\psi_4 + \psi_5, \psi_1 + \psi_2\\
&+ \psi_3 + \psi_4 + \psi_5 + \psi_6), (\psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6, \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5)\};\\
S_2 = \{&(\psi_1, \psi_3 + \psi_4 + \psi_5 + \psi_6, \psi_6, \psi_1 + \psi_3 + \psi_4 + \psi_5), (\psi_1 + \psi_3, \psi_4 + \psi_5 + \psi_6,\\
&\psi_5 + \psi_6, \psi_1 + \psi_3 + \psi_4)\};\\
S_3 = \{&(\psi_3, \psi_4 + \psi_5), (\psi_5, \psi_3 + \psi_4)\};\ \text{and } S_4 = \emptyset.
\end{aligned}$$

The conditions of (4.1) follow.

Next, suppose that $\Delta(\mathfrak{g}, \mathfrak{a})$ is of type E_7 

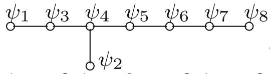
. Then the strongly orthogonal roots are $\beta_1 = 2\psi_1 + 2\psi_2 + 3\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7$, $\beta_2 = \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7$, $\beta_3 = \psi_7$, $\beta_4 = \psi_2 + \psi_3 + 2\psi_4 + \psi_5$, $\beta_5 = \psi_2$, $\beta_6 = \psi_3$, and $\beta_7 = \psi_5$. Now $\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum_{(\gamma,\gamma')\in S_r}(\mathfrak{g}_\gamma + \mathfrak{g}_{\gamma'})$, where

$$\begin{aligned}
S_1 = \{&(\psi_1,\ \psi_1 + 2\psi_2 + 3\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7),\\
&(\psi_1 + \psi_3,\ \psi_1 + 2\psi_2 + 2\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7),\\
&(\psi_1 + \psi_3 + \psi_4,\ \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7),\\
&(\psi_1 + \psi_2 + \psi_3 + \psi_4,\ \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7),\\
&(\psi_1 + \psi_3 + \psi_4 + \psi_5,\ \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7),\\
&(\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5,\ \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7),\\
&(\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6,\ \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6 + \psi_7),\\
&(\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7),\\
&(\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6 + \psi_7),\\
&(\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7,\ \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6),\\
&(\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7),\\
&(\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6 + \psi_7),\\
&(\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7,\ \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6),\\
&(\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6 + \psi_7),\\
&(\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6 + \psi_7),\\
&(\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6 + \psi_7,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6)\},
\end{aligned}$$

while

$$\begin{aligned}
S_2 = \{&(\psi_6, \psi_2+\psi_3+2\psi_4+2\psi_5+\psi_6+\psi_7), (\psi_5+\psi_6, \psi_2+\psi_3+2\psi_4+\psi_5+\psi_6+\psi_7),\\
&(\psi_6+\psi_7, \psi_2+\psi_3+2\psi_4+2\psi_5+\psi_6), (\psi_4+\psi_5+\psi_6, \psi_2+\psi_3+\psi_4+\psi_5+\psi_6+\psi_7),\\
&(\psi_5+\psi_6+\psi_7, \psi_2+\psi_3+2\psi_4+\psi_5+\psi_6), (\psi_2+\psi_4+\psi_5+\psi_6, \psi_3+\psi_4+\psi_5+\psi_6+\psi_7),\\
&(\psi_3+\psi_4+\psi_5+\psi_6, \psi_2+\psi_4+\psi_5+\psi_6+\psi_7), (\psi_4+\psi_5+\psi_6+\psi_7, \psi_2+\psi_3\\
&+\psi_4+\psi_5+\psi_6)\};\\
S_4 = \{&(\psi_4,\ \psi_2+\psi_3+\psi_4+\psi_5), (\psi_2+\psi_4,\ \psi_3+\psi_4+\psi_5), (\psi_3+\psi_4,\ \psi_2+\psi_4+\psi_5),\\
&(\psi_4+\psi_5,\ \psi_2+\psi_3+\psi_4)\};
\end{aligned}$$

and $S_3 = S_5 = S_6 = S_7 = \emptyset$. The conditions of (4.1) follow.

Finally, suppose that $\Delta(\mathfrak{g}, \mathfrak{a})$ is of type E_8 

. Then $\beta_1 = 2\psi_1 + 3\psi_2 + 4\psi_3 + 6\psi_4 + 5\psi_5 + 4\psi_6 + 3\psi_7 + 2\psi_8$, $\beta_2 = 2\psi_1 + 2\psi_2 + 3\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7$, $\beta_3 = \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6 + \psi_7$, $\beta_4 = \psi_7$, $\beta_5 = \psi_2 + \psi_3 + 2\psi_4 + \psi_5$, $\beta_6 = \psi_2$, $\beta_7 = \psi_3$, and $\beta_8 = \psi_5$. Now $\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum_{(\gamma,\gamma')\in S_r}(\mathfrak{g}_\gamma + \mathfrak{g}_{\gamma'})$, where

$$\begin{aligned}
S_4 &= S_6 = S_7 = S_8 = \emptyset;\\
S_5 = \{&(\psi_4,\ \psi_2+\psi_3+\psi_4+\psi_5), (\psi_2+\psi_4,\ \psi_3+\psi_4+\psi_5),\\
&(\psi_3+\psi_4,\ \psi_2+\psi_4+\psi_5), (\psi_4+\psi_5,\ \psi_2+\psi_3+\psi_4)\};\\
S_3 = \{&(\psi_6,\ \psi_2+\psi_3+2\psi_4+2\psi_5+\psi_6+\psi_7), (\psi_5+\psi_6,\ \psi_2+\psi_3+2\psi_4+\psi_5+\psi_6+\psi_7),\\
&(\psi_6+\psi_7,\ \psi_2+\psi_3+2\psi_4+2\psi_5+\psi_6), (\psi_4+\psi_5+\psi_6,\ \psi_2+\psi_3+\psi_4+\psi_5+\psi_6+\psi_7),\\
&(\psi_5+\psi_6+\psi_7,\ \psi_2+\psi_3+2\psi_4+\psi_5+\psi_6), (\psi_2+\psi_4+\psi_5+\psi_6,\ \psi_3+\psi_4\\
&+\psi_5+\psi_6+\psi_7),\\
&(\psi_3+\psi_4+\psi_5+\psi_6,\ \psi_2+\psi_4+\psi_5+\psi_6+\psi_7), (\psi_4+\psi_5+\psi_6+\psi_7,\ \psi_2+\psi_3\\
&+\psi_4+\psi_5+\psi_6)\};
\end{aligned}$$

$$\begin{aligned}
S_1 = \{&(\psi_8,\ \ 2\psi_1+3\psi_2+4\psi_3+6\psi_4+5\psi_5+4\psi_6+3\psi_7+\psi_8),\\
&(\psi_7+\psi_8,\ \ 2\psi_1+3\psi_2+4\psi_3+6\psi_4+5\psi_5+4\psi_6+2\psi_7+\psi_8),\\
&(\psi_6+\psi_7+\psi_8,\ \ 2\psi_1+3\psi_2+4\psi_3+6\psi_4+5\psi_5+3\psi_6+2\psi_7+\psi_8),\\
&(\psi_5+\psi_6+\psi_7+\psi_8,\ \ 2\psi_1+3\psi_2+4\psi_3+6\psi_4+4\psi_5+3\psi_6+2\psi_7+\psi_8),\\
&(\psi_4+\psi_5+\psi_6+\psi_7+\psi_8,\ \ 2\psi_1+3\psi_2+4\psi_3+5\psi_4+4\psi_5+3\psi_6+2\psi_7+\psi_8),\\
&(\psi_2+\psi_4+\psi_5+\psi_6+\psi_7+\psi_8,\ \ 2\psi_1+2\psi_2+4\psi_3+5\psi_4+4\psi_5+3\psi_6+2\psi_7+\psi_8),\\
&(\psi_3+\psi_4+\psi_5+\psi_6+\psi_7+\psi_8,\ \ 2\psi_1+3\psi_2+3\psi_3+5\psi_4+4\psi_5+3\psi_6+2\psi_7+\psi_8),\\
&(\psi_1+\psi_3+\psi_4+\psi_5+\psi_6+\psi_7+\psi_8,\ \ \psi_1+3\psi_2+3\psi_3+5\psi_4+4\psi_5+3\psi_6+2\psi_7+\psi_8),\\
&(\psi_2+\psi_3+\psi_4+\psi_5+\psi_6+\psi_7+\psi_8,\ \ 2\psi_1+2\psi_2+3\psi_3+5\psi_4+4\psi_5+3\psi_6+2\psi_7+\psi_8),\\
&(\psi_1+\psi_2+\psi_3+\psi_4+\psi_5+\psi_6+\psi_7+\psi_8,\ \ \psi_1+2\psi_2+3\psi_3+5\psi_4+4\psi_5+3\psi_6\\
&+2\psi_7+\psi_8), (\psi_2+\psi_3+2\psi_4+\psi_5+\psi_6+\psi_7+\psi_8,\ \ 2\psi_1+2\psi_2+3\psi_3+4\psi_4+4\psi_5\\
&+3\psi_6+2\psi_7+\psi_8), (\psi_1+\psi_2+\psi_3+2\psi_4+\psi_5+\psi_6+\psi_7+\psi_8,\ \ \psi_1+2\psi_2+3\psi_3\\
&+4\psi_4+4\psi_5+3\psi_6+2\psi_7+\psi_8), (\psi_2+\psi_3+2\psi_4+2\psi_5+\psi_6+\psi_7+\psi_8,\ \ 2\psi_1+2\psi_2\\
&+3\psi_3+4\psi_4+3\psi_5+3\psi_6+2\psi_7+\psi_8), (\psi_1+\psi_2+2\psi_3+2\psi_4+\psi_5+\psi_6+\psi_7+\psi_8,\\
&\psi_1+2\psi_2+2\psi_3+4\psi_4+4\psi_5+3\psi_6+2\psi_7+\psi_8), (\psi_1+\psi_2+\psi_3+2\psi_4+2\psi_5+\psi_6
\end{aligned}$$

$$
\begin{aligned}
&+\psi_7+\psi_8, \quad \psi_1+2\psi_2+3\psi_3+4\psi_4+3\psi_5+3\psi_6+2\psi_7+\psi_8), (\psi_2+\psi_3+2\psi_4+2\psi_5\\
&+2\psi_6+\psi_7+\psi_8, \quad 2\psi_1+2\psi_2+3\psi_3+4\psi_4+3\psi_5+2\psi_6+2\psi_7+\psi_8),\\
&(\psi_1+\psi_2+2\psi_3+2\psi_4+2\psi_5+\psi_6+\psi_7+\psi_8, \quad \psi_1+2\psi_2+2\psi_3+4\psi_4+3\psi_5+3\psi_6\\
&+2\psi_7+\psi_8), (\psi_1+\psi_2+\psi_3+2\psi_4+2\psi_5+2\psi_6+\psi_7+\psi_8, \quad \psi_1+2\psi_2+3\psi_3+4\psi_4\\
&+3\psi_5+2\psi_6+2\psi_7+\psi_8), (\psi_2+\psi_3+2\psi_4+2\psi_5+2\psi_6+2\psi_7+\psi_8, \quad 2\psi_1+2\psi_2+3\psi_3\\
&+4\psi_4+3\psi_5+2\psi_6+\psi_7+\psi_8), (\psi_1+\psi_2+2\psi_3+3\psi_4+2\psi_5+\psi_6+\psi_7+\psi_8,\\
&\psi_1+2\psi_2+2\psi_3+3\psi_4+3\psi_5+3\psi_6+2\psi_7+\psi_8), (\psi_1+\psi_2+2\psi_3+2\psi_4+2\psi_5+2\psi_6+\psi_7\\
&+\psi_8, \quad \psi_1+2\psi_2+2\psi_3+4\psi_4+3\psi_5+2\psi_6+2\psi_7+\psi_8), (\psi_1+\psi_2+\psi_3+2\psi_4+2\psi_5\\
&+2\psi_6+2\psi_7+\psi_8, \quad \psi_1+2\psi_2+3\psi_3+4\psi_4+3\psi_5+2\psi_6+\psi_7+\psi_8), (\psi_1+2\psi_2+2\psi_3\\
&+3\psi_4+2\psi_5+\psi_6+\psi_7+\psi_8, \quad \psi_1+\psi_2+2\psi_3+3\psi_4+3\psi_5+3\psi_6+2\psi_7+\psi_8),\\
&(\psi_1+\psi_2+2\psi_3+3\psi_4+2\psi_5+2\psi_6+\psi_7+\psi_8, \quad \psi_1+2\psi_2+2\psi_3+3\psi_4+3\psi_5+2\psi_6\\
&+2\psi_7+\psi_8), (\psi_1+\psi_2+2\psi_3+2\psi_4+2\psi_5+2\psi_6+2\psi_7+\psi_8, \quad \psi_1+2\psi_2+2\psi_3+4\psi_4\\
&+3\psi_5+2\psi_6+\psi_7+\psi_8), (\psi_1+2\psi_2+2\psi_3+3\psi_4+2\psi_5+2\psi_6+\psi_7+\psi_8, \quad \psi_1+\psi_2\\
&+2\psi_3+3\psi_4+3\psi_5+2\psi_6+2\psi_7+\psi_8), (\psi_1+\psi_2+2\psi_3+3\psi_4+3\psi_5+2\psi_6+\psi_7+\psi_8,\\
&\psi_1+2\psi_2+2\psi_3+3\psi_4+2\psi_5+2\psi_6+2\psi_7+\psi_8), (\psi_1+\psi_2+2\psi_3+3\psi_4+2\psi_5+2\psi_6\\
&+2\psi_7+\psi_8, \quad \psi_1+2\psi_2+2\psi_3+3\psi_4+3\psi_5+2\psi_6+\psi_7+\psi_8)\};
\end{aligned}
$$

while

$$
\begin{aligned}
S_2 = \{&(\psi_1, \quad \psi_1+2\psi_2+3\psi_3+4\psi_4+3\psi_5+2\psi_6+\psi_7),\\
&(\psi_1+\psi_3, \quad \psi_1+2\psi_2+2\psi_3+4\psi_4+3\psi_5+2\psi_6+\psi_7),\\
&(\psi_1+\psi_3+\psi_4, \quad \psi_1+2\psi_2+2\psi_3+3\psi_4+3\psi_5+2\psi_6+\psi_7),\\
&(\psi_1+\psi_2+\psi_3+\psi_4, \quad \psi_1+\psi_2+2\psi_3+3\psi_4+3\psi_5+2\psi_6+\psi_7),\\
&(\psi_1+\psi_3+\psi_4+\psi_5, \quad \psi_1+2\psi_2+2\psi_3+3\psi_4+2\psi_5+2\psi_6+\psi_7),\\
&(\psi_1+\psi_2+\psi_3+\psi_4+\psi_5, \quad \psi_1+\psi_2+2\psi_3+3\psi_4+2\psi_5+2\psi_6+\psi_7),\\
&(\psi_1+\psi_3+\psi_4+\psi_5+\psi_6, \quad \psi_1+2\psi_2+2\psi_3+3\psi_4+2\psi_5+\psi_6+\psi_7),\\
&(\psi_1+\psi_2+\psi_3+2\psi_4+\psi_5, \quad \psi_1+\psi_2+2\psi_3+2\psi_4+2\psi_5+2\psi_6+\psi_7),\\
&(\psi_1+\psi_2+\psi_3+\psi_4+\psi_5+\psi_6, \quad \psi_1+\psi_2+2\psi_3+3\psi_4+2\psi_5+\psi_6+\psi_7),\\
&(\psi_1+\psi_3+\psi_4+\psi_5+\psi_6+\psi_7, \quad \psi_1+2\psi_2+2\psi_3+3\psi_4+2\psi_5+\psi_6),\\
&(\psi_1+\psi_2+2\psi_3+2\psi_4+\psi_5, \quad \psi_1+\psi_2+\psi_3+2\psi_4+2\psi_5+2\psi_6+\psi_7),\\
&(\psi_1+\psi_2+\psi_3+2\psi_4+\psi_5+\psi_6, \quad \psi_1+\psi_2+2\psi_3+2\psi_4+2\psi_5+\psi_6+\psi_7),\\
&(\psi_1+\psi_2+\psi_3+\psi_4+\psi_5+\psi_6+\psi_7, \quad \psi_1+\psi_2+2\psi_3+3\psi_4+2\psi_5+\psi_6),\\
&(\psi_1+\psi_2+2\psi_3+2\psi_4+\psi_5+\psi_6, \quad \psi_1+\psi_2+\psi_3+2\psi_4+2\psi_5+\psi_6+\psi_7),\\
&(\psi_1+\psi_2+\psi_3+2\psi_4+2\psi_5+\psi_6, \quad \psi_1+\psi_2+2\psi_3+2\psi_4+\psi_5+\psi_6+\psi_7),\\
&(\psi_1+\psi_2+\psi_3+2\psi_4+\psi_5+\psi_6+\psi_7, \quad \psi_1+\psi_2+2\psi_3+2\psi_4+2\psi_5+\psi_6)\}.
\end{aligned}
$$

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