Differential Geometry and its Applications 56 (2018) 355–372

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Homogeneity for a class of Riemannian quotient manifolds

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ARTICLE INFO

Article history:
Received 14 December 2016
Available online 6 November 2017
Communicated by M.G. Eastwood

ABSTRACT

We study Riemannian coverings $\varphi : \widetilde{M} \to \Gamma\backslash\widetilde{M}$ where $\widetilde{M}$ is a normal homogeneous space G/K_1 fibered over another normal homogeneous space $M = G/K$ and K is locally isomorphic to a nontrivial product $K_1 \times K_2$. The most familiar such fibrations $\pi : \widetilde{M} \to M$ are the natural fibrations of Stiefel manifolds $SO(n_1+n_2)/SO(n_1)$ over Grassmann manifolds $SO(n_1+n_2)/[SO(n_1)\times SO(n_2)]$ and the twistor space bundles over quaternionic symmetric spaces (= quaternion-Kaehler symmetric spaces = Wolf spaces). The most familiar of these coverings $\varphi : \widetilde{M} \to \Gamma\backslash\widetilde{M}$ are the universal Riemannian coverings of spherical space forms. When $M = G/K$ is reasonably well understood, in particular when G/K is a Riemannian symmetric space or when K is a connected subgroup of maximal rank in G, we show that the Homogeneity Conjecture holds for $\widetilde{M}$. In other words we show that $\Gamma\backslash\widetilde{M}$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement. In order to find all the isometries of constant displacement on $\widetilde{M}$ we work out the full isometry group of $\widetilde{M}$, extending Élie Cartan's determination of the full group of isometries of a Riemannian symmetric space. We also discuss some pseudo-Riemannian extensions of our results.

1. Introduction

Some years ago I studied Riemannian covering spaces $S \to \Gamma\backslash S$ where S is homogeneous. I conjectured that $\Gamma\backslash S$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement (now usually called *Clifford translations* or *Clifford–Wolf isometries*) on S. I'll call that the *Homogeneity Conjecture*. This paper proves the conjecture for a class of normal Riemannian homogeneous spaces $\widetilde{M} = G/K_1$ that fiber over homogeneous spaces $M = G/K$ where K_1 is a local direct factor of K. The principal examples are those for which K is the fixed point set of an automorphism of G and the Lie algebra $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ with $\dim \mathfrak{k}_1 \neq 0 \neq \mathfrak{k}_2$. Those include the cases where G/K is an hermitian symmetric space, or a Grassmann manifold of rank > 1, or a quaternion-Kaehler symmetric space (Wolf space), one of the irreducible nearly-Kaehler manifolds of F_4, E_6, E_7 or E_8, or everybody's favorite 5-symmetric space E_8/A_4A_4. See [35] and [36] for a complete list.

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[1] Research partially supported by Collaboration Grant 419528 from the Simons Foundation.

https://doi.org/10.1016/j.difgeo.2017.10.006
0926-2245/

For lack of a better term I'll refer to such spaces $\widetilde{M} = G/K_1$ as *isotropy–split* homogeneous spaces and to the fibration $\pi : \widetilde{M} \to M = G/K$ as an *isotropy–splitting* fibration.

Here we use isotropy–splitting fibrations $\pi : \widetilde{M} \to M$ as a bootstrap device to study Riemannian coverings $\varphi : \widetilde{M} \to \Gamma\backslash\widetilde{M}$. Specifically, π is the projection given by $G/K_1 \to G/K$ with $K = K_1K_2$ where M and $\widetilde{M}$ are normal Riemannian homogeneous spaces of the of the same group G and each $\dim K_i > 0$. In particular, $\pi : \widetilde{M} \to M$ is a principal K_2-bundle. The point is to choose the splitting of K so that M is reasonably well understood. The most familiar example is the case where $\widetilde{M}$ is a Stiefel manifold and M is the corresponding Grassmann manifold. More generally we study the situation where

G is a compact connected simply connected Lie group,

$K = K_1K_2$ where the K_i are closed connected subgroups of G such that

$$\text{(i) } K = (K_1 \times K_2)/(K_1 \cap K_2), \text{ (ii) } \mathfrak{k}_2 \perp \mathfrak{k}_1 \text{ and } \text{ (iii) } \dim \mathfrak{k}_1 \neq 0 \neq \dim \mathfrak{k}_2 \,, \tag{1.1}$$

the centralizers $Z_G(K_1) = Z_{K_1}\widetilde{K_2}$ and $Z_G(K_2) = Z_{K_2}\widetilde{K_1}$ with $K_1 = \widetilde{K_1}^0$ and $K_2 = \widetilde{K_2}^0$, and

$M = G/K$ and $\widetilde{M} = G/K_1$ are normal Riemannian homogeneous spaces of G.

Thus we may assume that the metrics on M and $\widetilde{M}$ are the normal Riemannian metrics defined by the negative of the Killing form of G. Note that $\widetilde{M}$ and M are simply connected, because G is simply connected and K_1 and K are connected.

Lemma 1.2. *There is no nonzero G-invariant vector field on M. In other words, if $\mathfrak{m} = \mathfrak{k}^\perp$ then $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})|_{\mathfrak{m}}$ has no nonzero fixed vector.*

Proof. The centralizer $Z_G(K)$ is finite by (1.1). □

Lemma 1.2 is of course obvious whenever $\operatorname{rank} K = \operatorname{rank} G$, in other words when the Euler characteristic $\chi(M) \neq 0$. The point here is that it holds as well when $\operatorname{rank} K < \operatorname{rank} G$.

Important examples of M include the irreducible Riemannian symmetric spaces G/K with K not simple, the irreducible nearly-Kaehler manifolds of F_4, E_6, E_7 or E_8, and the very interesting 5-symmetric space E_8/A_4A_4. We will list these examples in detail and work out the precise structure of the group $I(\widetilde{M})$ of all isometries of $\widetilde{M}$. That is Theorem 3.12, and Corollary 3.5 identifies all the Killing vector fields on $\widetilde{M}$ of constant length. Killing vector fields of constant length are the infinitesimal version of isometries of constant displacement. After that we come to the main result, Theorem 5.6, which identifies all the isometries of constant displacement on $\widetilde{M}$. Applying it to a Riemannian covering $\widetilde{M} \to \Gamma\backslash\widetilde{M}$ we prove the Homogeneity Conjecture for isotropy–split manifolds. Then we sketch the mathematical background and current state for the Homogeneity Conjecture.

In Section 2 we view (compact) isotropy–splitting fibrations from the viewpoint of the Borel–de Siebenthal classification ([4], or see [32]) of pairs (G, K) where G is a compact connected simply connected simple Lie group and K is a maximal subgroup of equal rank in G. This yields an explicit list. We then run through the cases where G/K is a compact irreducible Riemannian symmetric space with $\operatorname{rank} K < \operatorname{rank} G$; the only ones that yield isotropy–splitting fibrations are the fibrations of real Stiefel manifolds over odd dimensional oriented real Grassmann manifolds. These are examples with which one can calculate explicitly, and to which our principal results apply.

In Section 3 we work out the full group of isometries of $\widetilde{M}$. The method combines ideas from Élie Cartan's description of the full isometry group of a Riemannian symmetric space, Carolyn Gordon's work on isometry groups of noncompact homogeneous spaces, and a theorem of Silvio Reggiani. The result is Theorem 3.12. One consequence, Corollary 3.5, is a complete description of the Killing vector fields of constant length on $\widetilde{M}$.

Section 4 is a digression in which we show that an appropriate form of Theorem 3.12 holds in the equal rank case without the need for an isotropy–splitting fibration.

In Section 5 we study isometries of constant displacement on $\widetilde{M}$ in the equal rank case, in other words when the Euler–Poincaré characteristic $\chi(M) \neq 0$. In that setting we give a classification of homogeneous Riemannian coverings $\widetilde{M} \to \Gamma\backslash\widetilde{M}$. The arguments are modeled in part on those of the group manifold case of Riemannian coverings $S \to \Gamma\backslash S$ in [30]. The result is Theorem 5.6, which is the principal result of this paper. The main application is Corollary 5.7, which applies Theorem 5.6 to Riemannian coverings $\widetilde{M} \to \Gamma\backslash\widetilde{M}$.

In Section 6 we study isometries of constant displacement on $\widetilde{M}$ when $\chi(M) = 0$. We work out a modification of the proof of Theorem 5.6, proving Theorem 6.1, which characterizes the isometries of constant displacement on $\widetilde{M}$ for $\chi(M) = 0$.

In Section 7 we specialize Theorem 5.6 to the case where M is a compact irreducible Riemannian symmetric space. From the classification and the isotropy–splitting requirement, the only cases are the natural fibrations of Stiefel manifolds over odd dimensional oriented real Grassmann manifolds. There we characterize the isometries of constant on $\widetilde{M}$ by a matrix calculation.

In Section 8 we apply our results on constant displacement isometries to the Homogeneity Conjecture. The main result of this paper, Theorem 8.1, proves the conjecture for $\widetilde{M}$ when $\operatorname{rank} K = \operatorname{rank} G$, and also when M is a Riemannian symmetric space. In particular it proves the conjecture for $\widetilde{M}$ when $\widetilde{M} \to M$ is one of the fibrations described in Section 2. We then describe the current state of the art for the Homogeneity Conjecture, its infinitesimal variation, and its extension to Finsler manifolds. Earlier work had proved it in many special cases, for example for Riemannian symmetric spaces, and its validity for isotropy–split manifolds extends our understanding of the area.

In Section 9 we show how our results on compact isotropy–split manifolds carry over (or, rather, often do not carry over) to the noncompact case. There we see that $\widetilde{M}$ is pseudo-Riemannian and we can't talk about isometries of constant displacement. Thus, in that setting, we concentrate on the isometry group and on Killing vector fields of constant length. Of special interest here is the case where the base M of the isotropy–splitting fibration $\widetilde{M} \to M$ is a Riemannian symmetric space of noncompact type, but other cases of special interest are those for which the "compact dual" isotropy–splitting fibration $\widetilde{M}^u \to M^u$ has 3-symmetric or 5-symmetric base.

2. Some special classes of isotropy–splitting fibrations

In this section we describe a number of interesting examples of isotropy–splitting fibrations $\widetilde{M} \to M$. Those are examples with which one can calculate explicitly, and to which our principal results apply.

Fix a compact connected simply connected Lie group G and a maximal connected subgroup K with $\operatorname{rank} K = \operatorname{rank} G$. The Borel–de Siebenthal classification of all such pairs (G, K) is in [4], or see [32].

We recall that classification. We may assume that G is simple. Fix a maximal torus $T \subset K$ of G and a positive root system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Express the maximal root $\beta = \sum_{\psi\in\Psi_G} n_\psi \psi$ where Ψ_G is the simple root system for $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. The coefficients n_ψ are positive integers, and the possibilities for $\mathfrak{k}$ correspond to the simple roots ψ_0 for which either $n_{\psi_0} = 1$ or $n_{\psi_0} > 1$ with n_{ψ_0} prime. Fix one such, ψ_0, and write n_0 for n_{ψ_0}.

If $n_0 = 1$ the simple root system $\Psi_K = \Psi_G \setminus \{\psi_0\}$. This is the case where G/K is an hermitian symmetric space. If $n_0 > 1$ then $\Psi_K = (\Psi_G \setminus \{\psi_0\}) \cup \{-\beta\}$. In this case either $n_0 = 2$ and G/K is a non-hermitian symmetric space, or $n_0 = 3$ and G/K is a nearly-Kaehler manifold, or $n_0 = 5$ and $G/K = E_8/A_4A_4$.

If $\mathcal{D}_G$ is the Dynkin diagram of $\mathfrak{g}$ then the diagram $\mathcal{D}_K$ of $\mathfrak{k}$ is obtained as follows. If $n_0 = 1$ then delete the vertex ψ_0 from $\mathcal{D}_G$. If $n_0 > 1$ the delete the vertex ψ_0 and adjoin the vertex $-\beta$. The simple root(s) not orthogonal to β, in other words the attachment points for $-\beta$ to $\mathcal{D}_G$, may or may not disconnect $\mathcal{D}_G$. If there is disconnection then K splits into the form K_1K_2 of interest to us.

2.1. Hermitian symmetric space base

If $n_0 = 1$ then $K = SK'$ where S is a circle group and $K' = [K, K]$ is semisimple. The corresponding fibrations are

$$G/K' \to G/K \text{ circle bundle over a compact hermitian symmetric space and}$$
$$G/S \to G/K \text{ principal } K'\text{-bundle over a compact hermitian symmetric space.}$$

In addition, if $\mathfrak{g} = \mathfrak{su}(s+t)$ we can have $\mathfrak{k}' = \mathfrak{su}(s) \oplus \mathfrak{su}(t)$, leading to fibrations

$$SU(s+t)/SU(s) \to SU(s+t)/S(U(s)U(t)) \text{ and } SU(s+t)/U(s) \to SU(s+t)/S(U(s)U(t)),$$
$$SU(s+t)/SU(t) \to SU(s+t)/S(U(s)U(t)) \text{ and } SU(s+t)/U(t) \to SU(s+t)/S(U(s)U(t)).$$

2.2. Quaternion-Kaehler symmetric space base

If $n_0 = 2$ then K is simple except in the cases

$$G/K = SO(s+t)/SO(s)SO(t) \text{ with } 2 < s \leqq t \text{ and } st \text{ even},$$
$$G/K = Sp(s+t)/Sp(s)Sp(t) \text{ with } 1 \leqq s \leqq t,$$
$$G/K = G_2/A_1A_1\,, F_4/A_1C_3\,, E_6/A_1A_5\,, E_7/A_1D_6 \text{ or } E_8/A_1E_7\,.$$

In the SO cases, G/K is a quaternion-Kaehler symmetric space for $s = 3$ and for $s = 4$. In the Sp cases G/K is a quaternion-Kaehler symmetric space for $s = 1$. In the exceptional group cases G/K always is a quaternion-Kaehler symmetric space.

2.3. Nearly-Kaehler 3-symmetric space base

If $n_0 = 3$ then either K is simple and $G/K = G_2/A_2$ or E_8/A_8, or K is not simple and G/K is one of the nearly-Kaehler manifolds F_4/A_2A_2, $E_6/A_2A_2A_2$, E_7/A_2A_5 or E_8/A_2E_6. In the F_4 case one of the A_2 is given by long roots and the other is given by short roots. In each case we have $\mathfrak{k} = \mathfrak{a}_2 \oplus \mathfrak{k}''$ where the $\mathfrak{a}_2$ is given by long roots. The 3-symmetry on G/K is given by one of the central elements of $\exp(\mathfrak{a}_2) = SU(3)$. It defines the almost-complex structure on G/K, which satisfies the nearly-Kaehler condition. The corresponding fibrations are

$$G/K'' \to G/K \text{ principal } SU(3)\text{-bundle and } G/SU(3) \to G/K \text{ principal } K''\text{-bundle.}$$

2.4. 5-symmetric space base

If $n_0 = 5$ then $G/K = E_8/A_4A_4$, where the first A_4 acts on the complexified tangent space by a sum of 5 dimensional representations and the second A_4 acts by a sum of 10 dimensional representations. This leads to two different principal $SU(5)$-bundles $E_8/SU(5) \to E_8/SU(5)SU(5)$.

2.5. Odd real Grassmann manifold base

The Borel–de Sibenthal classification, just described, gives the classification of irreducible compact Riemannian symmetric spaces S with Euler characteristic $\chi(S) \neq 0$. There are other symmetric spaces to which our results will apply, corresponding to the isotropy–split fibrations $\pi : \widetilde{M} \to M$ where the base M

is an irreducible compact Riemannian symmetric space G/K such that $\operatorname{rank} G > \operatorname{rank} K$. According to the classification of symmetric spaces, the only such G/K are

$$SU(n)/SO(n), SU(2n)/Sp(n), SO(2s+2t+2)/[SO(2s+1)\times SO(2t+1)], E_6/F_4, E_6/Sp(4), (K\times K)/diag(K).$$

Note that $SU(4)/SO(4) = SO(6)/[SO(3) \times SO(3)]$. Thus the only such symmetric spaces G/K that satisfy (1.1) are the oriented real Grassmann manifolds $SO(2s + 2t + 2)/[SO(2s + 1) \times SO(2t + 1)]$. Thus the corresponding fibrations are

$$\pi : \widetilde{M} \to M \text{ given by } G/K_1 \to G/K_1K_2 \text{ where}$$
$$G = SO(2s + 2t + 2)/SO(2s + 1), K_1 = SO(2s + 1) \text{ and } K_2 = SO(2t + 1).$$

The odd spheres are completely understood ([28] and [32]), and in any case they do not lead to isotropy–split fibrations, so we put those cases aside and assume $s, t > 0$.

3. The isometry group of $\widetilde{M}$

We look at an isotropy–splitting fibration $\pi : \widetilde{M} \to M$, given by $G/K_1 \to G/K$ in (1.1). As noted there we assume that the metrics on M and $\widetilde{M}$ are the normal Riemannian metrics defined by the negative of the Killing form of G. Now we work out the isometry groups $I(\widetilde{M})$.

Lemma 3.1. *The right action of $\widetilde{K_2}$ on $\widetilde{M}$, given by $r(k_2)(gK_1) = gK_1k_2^{-1} = gk_2^{-1}K_1$, is a well defined action by isometries. The fiber of $\pi : \widetilde{M} \to M$ through gK_1 is $r(K_2)(gK_1)$.*

$$\text{Let } F \text{ denote the fiber } r(K_2)(1K_1) \text{ of } \pi : \widetilde{M} \to M, \text{ so } gF \text{ is the fiber } \pi^{-1}(gK). \tag{3.2}$$

We have larger (than G) transitive groups of isometries of $\widetilde{M}$ given by

$$\widetilde{G} = G \times r(\widetilde{K_2}) \text{ and } \widetilde{G}^0 = G \times r(K_2) \text{ acting by } (g, r(k_2)) : xK_1 \mapsto g(xK_1)k_2^{-1} = gxk_2^{-1}K_1\,. \tag{3.3}$$

Every $\widetilde{g} = (g, r(k_2)) \in \widetilde{G}$ sends fiber to fiber in $\widetilde{M} \to M$ and induces the isometry $g : M \to M$ of M.

Specializing a theorem of Reggiani [25, Corollary 1.3] we have

Theorem 3.4. *Suppose that the Riemannian manifold $\widetilde{M} = G/K_1$ is irreducible. Then $\widetilde{G}^0$ is the identity component $I^0(\widetilde{M})$ of its isometry group.*

Corollary 3.5. *Suppose that the Riemannian manifold $\widetilde{M} = G/K_1$ is irreducible.*

(1) *The algebra of all Killing vector fields on $\widetilde{M}$ is $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus dr(\mathfrak{k}_2)$.*
(2) *The set of all constant length Killing vector fields on $\widetilde{M}$ is $\mathfrak{l} \oplus dr(\mathfrak{k}_2)$ where $\mathfrak{l} = \{\xi \in \mathfrak{g} \mid \xi$ defines a constant length Killing vector field on $\widetilde{M}\}$.*
(3) *$\mathfrak{l} = \{\xi \in \mathfrak{g} \mid \xi$ defines a constant length Killing vector field on $M\}$.*
(4) *If $\operatorname{rank} K = \operatorname{rank} G$ then $\mathfrak{l} = 0$, so the set of all constant length Killing vector fields on $\widetilde{M}$ is $dr(\mathfrak{k}_2)$. That applies in particular to the special classes of Sections 2.1 through 2.4.*

Proof. The first assertion is immediate from Theorem 3.4. For the second assertion, $dr(\mathfrak{k}_2)$ consists of Killing vector fields of constant length on $\widetilde{M}$ because every $\xi \in dr(\mathfrak{k}_2)$ is centralized by the transitive isometry group G.

For the third assertion, let ξ be a Killing vector field of constant length on $\widetilde{M}$. Using Theorem 3.4 express $\xi = \xi' + \xi''$ where $\xi' \in \mathfrak{g}$ and $\xi'' \in dr(\mathfrak{k}_2)$. The fibers of $\pi : \widetilde{M} \to M$ are just the orbits of $r(K_2)$ and are group manifolds, so ξ'' is a Killing vector field of constant length on $\widetilde{M}$. Further, $\xi' \perp \xi''$ at every point of $\widetilde{M}$. Now ξ' is a Killing vector field of constant length on $\widetilde{M}$. It follows that ξ' is a Killing vector field of constant length on M as well.

For the fourth assertion, let $\operatorname{rank} K = \operatorname{rank} G$, so the Euler–Poincaré characteristic $\chi(M) > 0$. Then the vector field ξ' (of the argument for (3) just above) must have a zero on M. Thus $\xi' = 0$ and $\xi = \xi'' \in dr(\mathfrak{k}_2)$. $\square$

Corollary 3.6. *Every isometry of $\widetilde{M}$ normalizes $r(K_2)$ and thus sends fiber to fiber in $\pi : \widetilde{M} \to M$.*

Now we start to extend this to a structure theorem for the full isometry group $I(\widetilde{M})$ under the constraint of (1.1).

The normalizer of K_1 in G also normalizes the centralizer of K_1, thus normalizes K_2 and thus normalizes K. That shows

Lemma 3.7. *The normalizer of K_1 in G is contained in the normalizer of K in G.*

Now we follow the basic idea of É. Cartan's determination of the holonomy group and then the isometry group of a Riemannian symmetric space ([6], [7]; or see [32]). Write $\operatorname{Out}(G)$ for the quotient $\operatorname{Aut}(G)/\operatorname{Int}(G)$ of the automorphism group by the normal subgroup of inner automorphism, and similarly $\operatorname{Out}(K_1) = \operatorname{Aut}(K_1)/\operatorname{Int}(K_1)$. We also need the relative group

$$\operatorname{Out}(G, K_1) = \{\alpha \in \operatorname{Aut}(G) \mid \alpha(K_1) = K_1\}/\{\alpha \in \operatorname{Int}(G) \mid \alpha(K_1) = K_1\} \subset \operatorname{Out}(G, K). \tag{3.8}$$

The inclusion in (3.8) follows because K_1 is a local direct factor of K. In many cases $\operatorname{Out}(G, K_1) = \operatorname{Out}(G, K)$ because $\mathfrak{k}_2$ is the $\mathfrak{g}$-centralizer of $\mathfrak{k}_1$ and $\mathfrak{k}_1 \not\cong \mathfrak{k}_2$. But there are exceptions, such as orthocomplementation (which exchanges the two factors of K) in the cases of Stiefel manifold fibrations

$$SO(2k)/SO(k) \to SO(2k)/[SO(k) \times SO(k)],$$
$$SU(2k)/U(k) \to SU(2k)/S(U(k) \times U(k)) \text{ and}$$
$$Sp(2k)/Sp(k) \to Sp(2k)/[Sp(k) \times Sp(k)].$$

There are other exceptions, including $E_6/[A_2A_2A_2]$, but neither F_4/A_2A_2 nor E_8/A_4A_4 is an exception.

Lemma 3.9. *Suppose that* $\operatorname{rank} K = \operatorname{rank} G$. *Let $\alpha \in \operatorname{Aut}(G)$ preserve K_1 (and thus also K_2 so $\alpha(K) = K$). Then the following conditions are equivalent:* (i) *$\alpha|_K$ is an inner automorphism of K,* (ii) *as an isometry, $\alpha \in I^0(M)$, and* (iii) *as an isometry, $\alpha \in I^0(\widetilde{M})$.*

Proof. Suppose that $\alpha|_K$ is an inner automorphism. Then we have $k_0 \in K$ such that $\alpha(k) = k_0 k k_0^{-1}$ for every $k \in K$. Thus $\alpha' := \operatorname{Ad}(k_0^{-1}) \cdot \alpha$ is an isometry of M that belongs to the same component of $I(M)$ as α. Let T be a maximal torus of K that contains k_0. Then $\alpha' := \operatorname{Ad}(k_0^{-1}) \cdot \alpha$ is an isometry of M that belongs to the same component $I^0(M)\alpha$ as α. Now $\alpha'(t) = t$ for every $t \in T$ so there is an element $t_0 \in T$ such that $\alpha'(g) = t_0 g t_0^{-1}$ for every $g \in G$. Consequently $\alpha'' := \operatorname{Ad}(t_0^{-1}) \cdot \alpha'$ is the identity in $I(M)$ and belongs to the same component of $I(M)$ as α. It follows that α'' is the identity in $I(\widetilde{M})$ and belongs to the same component of $I(\widetilde{M})$ as α. Thus, as an isometry, $\alpha \in I^0(M)$ and $\alpha \in I^0(\widetilde{M})$. We have shown that (i) implies (ii) and (iii).

Suppose that $\alpha|_K$ is an outer automorphism. Then $\alpha \in I(M)$ represents a non-identity component of the isotropy subgroup at $1K$, i.e. $\alpha \notin I^0(M)$. In view of Corollary 3.6 we have a natural continuous homomorphism of $I(\widetilde{M})$ to $I(M)$ that maps $I^0(\widetilde{M})$ onto $I^0(M)$. Thus $\alpha \notin I^0(\widetilde{M})$. We have shown that if (i) fails then (ii) and (iii) fail. Thus (ii) implies (i) and (iii) implies (i). That completes the proof. □

We reformulate Lemma 3.9 as

Lemma 3.10. *Let* $\operatorname{rank} K = \operatorname{rank} G$. *Let* $\widetilde{H}$ *denote the isotropy subgroup of* $I(\widetilde{M})$ *at the base point* $\widetilde{x_0} = 1K_1$. *Then the identity component* $\widetilde{H}^0$ *is* $K_1 \cdot \{(k_2, r(k_2)) \in \widetilde{G} \mid k_2 \in K_2\}$ *and*

$$\widetilde{H} = \bigcup_{\alpha \in \mathrm{Out}(G,K_1)\ ,\beta \in \mathrm{Out}(G,K_2)} K_1\alpha \cdot \left\{(k_2, r(k_2)) \in \widetilde{G} \mid k_2 \in \widetilde{K_2}\beta\right\}$$

Given $\alpha, \alpha' \in \mathrm{Out}(G, K_1)$ *and* $\beta, \beta' \in \mathrm{Out}(G, K_2)$, *the components* $K_1\alpha \cdot \left\{(k_2, r(k_2)) \in \widetilde{G} \mid k_2 \in \widetilde{K_2}\beta\right\} = K_1\alpha \cdot \left\{(k_2, r(k_2)) \in \widetilde{G} \mid k_2 \in \widetilde{K_2}\beta\right\}$ *if and only if both* $\alpha = \alpha'$ *and* $\beta = \beta'$ *modulo inner automorphisms.*

Proof. The fiber $F = r(K_2)\widetilde{x_0}$ is the group manifold K/K_1, and $\widetilde{H}$ preserves F by Corollary 3.5. The isotropy subgroup of $I(F)$ at $\widetilde{x_0}$ has identity component $diag(K_2) = \{(k_2, r(k_2)) \in \widetilde{G} \mid k_2 \in K_2\}$. The group $diag(K_2)$ is connected and is contained in $\widetilde{H}$ because it leaves $\widetilde{x_0}$ fixed, so $diag(K_2) \subset \widetilde{H}^0$. Also $K_1 = G \cap \widetilde{H} \subset \widetilde{H}^0$. It follows that $\widetilde{H}^0 = K_1 \cdot diag(K_2)$, as asserted.

The inclusion $\bigcup_{\alpha \in \mathrm{Out}(G,K_1)\ ,\beta \in \mathrm{Out}(G,K_2)} K_1\alpha \cdot \left\{(k_2, r(k_2)) \in \widetilde{G} \mid k_2 \in \widetilde{K_2}\beta\right\} \subset \widetilde{H}$ is clear.

Now let $h \in \widetilde{H}$. Then $h(F) = F$ by Corollary 3.6, so conjugation by h gives an automorphism β of $diag(K_2) = \{(k_2, r(k_2)) \in \widetilde{G} \mid k_2 \in K_2\}$, and we view β as an element of $\mathrm{Out}(G, K_2)$. Furthermore, conjugation by h gives an automorphism α of G and we view α as an element of $\mathrm{Out}(G, K_1)$. Thus $\widetilde{H}$ is contained in $\bigcup_{\alpha \in \mathrm{Out}(G,K_1)\ ,\beta \in \mathrm{Out}(G,K_2)} K_1\alpha \cdot \left\{(k_2, r(k_2)) \in \widetilde{G} \mid k_2 \in \widetilde{K_2}\beta\right\}$, and they are equal, as asserted.

Finally, the last statement is immediate from Lemma 3.9. □

We now define two subgroups of isometry groups by

$$G^\dagger = \bigcup_{\alpha \in \mathrm{Out}(G,K_1)} G\alpha \subset I(M) \text{ and } \widetilde{G}^\dagger = \bigcup_{\alpha \in \mathrm{Out}(G,K_1)\ ,\beta \in \mathrm{Out}(G,K_2)} G\alpha \cdot r(K_2)\beta \subset I(\widetilde{M}). \tag{3.11}$$

Here $g\alpha$ acts on M by $xK \mapsto g\alpha(x)K$ and on $\widetilde{M}$ by $xK_1 \mapsto g\alpha(x)K_1$, and $r(k_2)\beta$ acts on $\widetilde{M}$ by $xK_1 \mapsto x\beta(k_2)^{-1}K_1$.

Theorem 3.12. *Let* $\pi : \widetilde{M} \to M$ *be an isotropy–split fibration as in* (1.1). *Suppose that* $\operatorname{rank} K = \operatorname{rank} G$. *Then the identity component* $I^0(\widetilde{M}) = \widetilde{G}^0$ *and the full isometry groups* $I(\widetilde{M}) = \widetilde{G}^\dagger$.

Proof. The first statement repeats Theorem 3.4. As G is transitive on $\widetilde{M}$ one has $I(\widetilde{M}) = G\widetilde{H}$, and the assertion follows from Lemma 3.10. □

4. Digression: the isometry group without a splitting fibration

Theorem 3.12 holds without the splitting fibration. That result is useful and we indicate it here.

A "degenerate" form of Lemma 3.9 holds as follows: Let A be a compact connected semisimple Lie group and B a closed connected subgroup of maximal rank. Let $N = A/B$, coset space with the normal Riemannian metric from the negative of the Killing form of A. Let α be an automorphism of A that preserves B. Then $\alpha|_B$ is an inner automorphism of B if and only if, as an isometry, $\alpha \in I^0(N)$. The proof is immediate from the proof of Lemma 3.9.

Next, a "degenerate" form of Lemma 3.10 holds as follows: Let H denote the isotropy subgroup of $I(N)$ at the base point $1K$. Then $H = \bigcup_{\alpha \in \mathrm{Out}(A,B)} H^0\alpha$. Given $\alpha, \alpha' \in \mathrm{Out}(A, B)$ the components $H^0\alpha = H^0\alpha'$

if and only if $\alpha = \alpha'$ modulo inner automorphisms. The argument follows by specializing the proof of Lemma 3.10

Finally, a "degenerate" form of Theorem 3.12 holds as follows: Let A be a compact connected semisimple Lie group and B a closed connected subgroup of maximal rank. Let $N = A/B$, coset space with the normal Riemannian metric from the negative of the Killing form of A. Then $I^0(N)$ is given by Theorem 3.4 and, in view of the remarks just above, $I(N) = \bigcup_{\alpha \in \mathrm{Out}(A,B)} A\alpha$.

We summarize these comments as

Theorem 4.1. *Let A be a compact connected semisimple Lie group and B a closed connected subgroup of maximal rank. Let $N = A/B$, coset space with the normal Riemannian metric from the negative of the Killing form of A. Then $I^0(N)$ is given by Theorem 3.4 and $I(N) = \bigcup_{\alpha \in \mathrm{Out}(A,B)} A\alpha$.*

5. Isometries of constant displacement: case $\chi(M) \neq 0$

Fix an isotropy–splitting fibration $\pi : \widetilde{M} \to M$ as in (1.1). In this section we look at isometries of constant displacement on $\widetilde{M} = G/K_1$ where the Euler–Poincaré characteristic $\chi(M) \neq 0$, in other words where $\operatorname{rank} K = \operatorname{rank} G$. Then $\chi(M) = |W_G|/|W_K| > 0$ where W denotes the Weyl group. Some important examples are the isotropy–splitting fibrations described in Sections 2.1, 2.2, 2.3 and 2.4.

In Section 6 we will look at cases where $\chi(M) = 0$, and in Section 7 we will consider the remaining cases where M is an irreducible Riemannian symmetric space.

Lemma 5.1. *If $\operatorname{rank} K = \operatorname{rank} G$ and $\widetilde{g} = (g, r(k_2)) \in \widetilde{G}$ then there is a fiber $xF = \pi^{-1}(xK)$ of $\widetilde{M} \to M$ that is invariant under the action of $\widetilde{g}$ on $\widetilde{M}$.*

Proof. Every element of the compact connected Lie group $\widetilde{G}$ belongs to a maximal torus, thus is conjugate to an element of $K \times r(K_2)$, and consequently has a fixed point on M. □

We need an observation concerning the geodesics in $\widetilde{M}$ and F.

Lemma 5.2. *The isotropy–split manifold $\widetilde{M}$ is a geodesic orbit space, i.e. every geodesic is the orbit of a one-parameter subgroup of G. The fiber F of $\widetilde{M} \to M$ is totally geodesic in $\widetilde{M}$ and also is a geodesic orbit space. Every geodesic of $\widetilde{M}$ tangent to F is of the form $t \mapsto \exp(t\xi)x$ with $x \in F$ and $\xi \in \mathfrak{k}_2$.*

Proof. Recall that $\widetilde{M}$ is a normal homogeneous space relative to the group G and the Riemannian metric given by the negative of the Killing form κ of G. Write $\mathfrak{g} = \mathfrak{k}_1 + \mathfrak{m}_1$ where $\mathfrak{m}_1 = \mathfrak{k}_1^\perp$ relative to κ. Write $\langle \cdot\,, \cdot \rangle$ for $-\kappa$. It is positive definite on $\mathfrak{g}$. If $\xi, \eta, \zeta \in \mathfrak{m}_1$ then $\mathrm{ad}\,(\xi)$ is antisymmetric relative to $\langle \cdot\,, \cdot \rangle$ so $0 = \langle [\xi, \eta], \zeta \rangle + \langle \eta, [\xi, \zeta] \rangle = \langle [\xi, \eta]_{\mathfrak{m}_1}, \zeta \rangle + \langle \eta, [\xi, \zeta]_{\mathfrak{m}_1} \rangle$. In other words (see [18, Definition 1.3]),

$$\text{the } G\text{-homogeneous space } \widetilde{M} \text{ is naturally reductive relative to } G \text{ and } \mathfrak{g} = \mathfrak{k}_1 + \mathfrak{m}_1\,. \tag{5.3}$$

If $\xi \in \mathfrak{m}_1$ now ([19] or see [18]) $t \mapsto \exp(t\xi) \cdot 1K_1$ is a geodesic in $\widetilde{M}$. In particular $\widetilde{M}$ is a geodesic orbit space and F is totally geodesic in $\widetilde{M}$. But F is a Riemannian symmetric space under $K_2 \times r(K_2)$ with the metric obtained by restriction of $\langle \cdot\,, \cdot \rangle$. Thus every geodesic of $\widetilde{M}$ tangent to F at $1K_1$ has form $t \mapsto \exp(t\xi)(1K_1)$ with $\xi \in \mathfrak{k}_2$. As K_2 acts transitively on F with finite kernel every geodesic in F has form $t \mapsto \exp(t\xi)x$ with $x \in F$ and $\xi \in \mathfrak{k}_2$. □

Our principal results, starting with Proposition 5.4 just below, will depend on a certain flat rectangle argument. The idea is that we have two commuting Killing vector fields ξ_1 and ξ_2, typically $\xi_2 \in dr(\mathfrak{k}_2)$ and $\xi_1 \in \mathfrak{g}$, such that $\xi_1 \perp \mathfrak{k}_1$ and both $g \times r(k_2) = \exp(\xi_1 + \xi_2)$ and $r(k_2) = \exp(\xi_2)$ have the same constant displacement. Then the $\exp(t_1\xi_1 + t_2\xi_2)(1K_1)$, for $0 \leqq t_i \leqq 1$, form a flat rectangle. There $r(k_2)$ is displacement along one side while $g \times r(k_2)$ is displacement along the diagonal. Since these displacements are the same we argue that $\xi_1 = 0$.

Proposition 5.4. *Suppose that* $\operatorname{rank} K = \operatorname{rank} G$. *Let* Γ *be a subgroup of* $\widetilde{G}$ *such that every* $\gamma \in \Gamma$ *is an isometry of constant displacement on* $\widetilde{M}$. *Then* $\Gamma \subset (Z_G \times r(K_2))$ *where* Z_G *denotes the center of* G.

Note: Proposition 5.4 applies in particular to the isotropy–split fibrations $\widetilde{M} \to M$ described in Sections 2.1 through 2.4.

Proof. Let $\gamma = (g, r(k_2)) \in \Gamma$. By Lemma 5.1 and conjugacy of maximal tori in G, we have $h \in G$ such that $\gamma(hF) = hF$. Since both $Z_G \times r(K_2)$ and "constant displacement" are fixed under $\mathrm{Ad}(G)$ we may replace γ by its $\mathrm{Ad}(G)$-conjugate $(h^{-1}, 1)(g, r(k_2))(h, 1)$, which preserves F and still consists of isometries of constant displacement. That done, $\gamma \in (K_1K_2 \times r(K_2))$.

The group K_1 fixes the base point $\widetilde{x_0} = 1K_1 \in \widetilde{M}$. If $k \in K_2$ then $K_1k\widetilde{x} = kK_1\widetilde{x} = k\widetilde{x}$. Now K_1 fixes every point of F, so $\gamma|_F \in (K_2 \times r(K_2))$. As F is totally geodesic in $\widetilde{M}$, $\gamma|_F$ is an isometry of constant displacement on F. Now [29, Theorem 4.5.1] says that either $\gamma|_F \in (K_2 \times r(\{1\}))$ or $\gamma|_F \in r(K_2)$.

Suppose that $\gamma|_F = zk_2 \in Z_GK_2$. Then $\gamma = zk_1k_2$ and also has constant displacement on F, hence on $\widetilde{M}$. Let $T_i \subset K_i$ be a maximal torus, so $T := T_1T_2$ is a maximal torus of K, and thus of G. Replace γ by a conjugate and assume $\gamma = zk_2 \in Z_GT_2$. Lemma 5.2 gives us $\xi \in \mathfrak{t}_2$ such that $\exp(t\xi) \cdot 1K_1$, $0 \leqq t \leqq 1$, is the minimizing geodesic in $\widetilde{M}$ from $1K_1$ to zk_2K_1. In particular the (constant) displacement of $\gamma = zk_2$ is $||\xi||$. Let w belong to the Weyl group $W(G,T)$, say $w = \mathrm{Ad}(s)|_{\mathfrak{t}}$ where s normalizes T. Then $w(\gamma) = szk_2s^{-1}$ has the same constant displacement $||\xi||$ as does γ. Note that $w(\gamma) \cdot 1K_1 = \exp(w(\xi)) \cdot 1K_1$. Decompose $w(\xi) = w(\xi)' + w(\xi)''$ where $w(\xi)' \in \mathfrak{t}_1$ and $w(\xi)'' \in \mathfrak{t}_2$. Then $\exp(w(\xi)) \cdot 1K_1 = \exp(w(\xi)'') \cdot 1K_1$ so $||w(\xi)''|| = ||w(\xi)||$. This says $w(\xi) \in \mathfrak{t}_2$ for every $w \in W(G,T)$. But $W(G,T)$ acts irreducibly on $\mathfrak{t}$, so an orbit $\neq 0$ cannot be confined to a proper subspace. This contradicts $\gamma = k_2 \in K_2$. We conclude $\gamma \in (Z_G \times r(K_2))$.

We have just shown that every $\gamma \in \Gamma$ is $\mathrm{Ad}(G)$-conjugate to an element of $Z_G \times r(K_2)$. As G centralizes both Z_G and $r(K_2)$ it follows that $\Gamma \subset (Z_G \times r(K_2))$. $\square$

Proposition 5.4 holds whether or not the maximal rank subgroup K of G is a maximal subgroup, describing the groups of isometries of constant displacement on $\widetilde{M}$ that are contained in the identity component $I^0(\widetilde{M})$. Next, we look in the other components of $I(\widetilde{M})$. That will require an understanding of the full isometry group $I(M)$.

Lemma 5.5. *Suppose that* $\operatorname{rank} K = \operatorname{rank} G$. *Let* $\alpha \in \mathrm{Out}(G, K_1)$ *and* $\gamma \in \widetilde{G}\alpha$ *such that both* γ *and* γ^2 *are isometries of constant displacement on* $\widetilde{M}$. *Then* $\alpha|_{K_1}$ *is an inner automorphism of* K_1 *and* $\gamma \in (Z_G \times r(K_2))$.

Proof. Let $\gamma = (\alpha \cdot g) \times (r(k_2) \cdot \beta)$ as in (3.11) and Theorem 3.12, using $\alpha \cdot g = \alpha(g) \cdot \alpha$. Exactly as in the proof of Proposition 5.4 we may assume that $(g \times r(k_2) \cdot \beta)F = F$. Now $gF = F$ and $\gamma(F) = (\alpha \cdot g)(F) = \alpha(F)$. But $\alpha K_1 = K_1$ and (1.1) together imply $\alpha(K_2) = K_2$. Thus $\alpha(K) = K$, in other words $1K$ is a fixed point for α on G/K; equivalently, $\alpha(F) = F$. Now $\gamma(F) = F$. As F is totally geodesic in $\widetilde{M}$, $\gamma|_F$ has constant displacement on F, so $\gamma|_F \in r(K_2)$. In particular $\beta = 1$ and $\gamma \in ((\alpha \cdot Z_GK_1) \times r(K_2))$.

Now we argue along the lines of the proof of Proposition 5.4. Both γ and $r(k_2)$ have the same constant displacement (call it c) on F, thus on $\widetilde{M}$. Following de Siebenthal [26] we have an α-invariant maximal torus $T_1 \subset K_1$ such that (after a K_1-conjugation) $\alpha k_1 \in \alpha T_1^\alpha$ where T_1^α is the fixed point set of α on T_1. Express $\alpha k_1 = \alpha \cdot \exp(\xi_1)$ where $\xi_1 \in \mathfrak{t}_1^\alpha$. Let T_2 be a maximal torus of K_2 such that $k_2 = \exp(\xi_2)$ for some $\xi_2 \in \mathfrak{t}_2$. Let $\xi = \xi_1 + \xi_2$. We may assume the ξ_i chosen so that $\alpha \cdot \exp(t\xi) \cdot 1K_1$, $0 \leqq t \leqq 1$, is a minimizing geodesic from $1K_1$ to $\gamma(1K_1)$. Then $\exp(t\xi_2)\cdot 1K_1$, $0 \leqq t \leqq 1$, also is a minimizing geodesic from $1K_1$ to $\gamma(1K_1)$. Now the corresponding vector fields $d\alpha(\eta)$ and η_2 on $\widetilde{M}$ satisfy $||d\alpha(\eta)|| = c = ||\eta_2||$ at every point of $\widetilde{M}$.

If $\xi_1 \neq 0$ then, as we move a little bit away from $1K_1$ in some direction orthogonal to F, $||d\alpha(\eta_1)||$ increases from 0. That increase in $||d\alpha(\eta_1)||$ would cause an increase in $||d\alpha(\eta)||$ because $d\alpha(\eta_1)$ and $d\alpha(\eta_2)$ would remain close to orthogonal. We conclude $\xi_1 = 0$. Now $\gamma = \alpha \times r(k_2)$. Again, if $\alpha \neq 1$ then, as we move away from $1K_1$ in some direction, the displacement of α would increase from 0, and that would cause an increase in the displacement of γ. We conclude that α is inner and $\gamma \in Z_G \times r(K_2)$. □

Finally we come to the main result of this section.

Theorem 5.6. *Suppose that* $\operatorname{rank} K = \operatorname{rank} G$. *If* Γ *is a group of isometries of constant displacement on* $\widetilde{M}$ *then* $\Gamma \subset (Z_G \times r(K_2))$ *where* Z_G *denotes the center of* G. *Conversely, if* $\Gamma \subset (Z_G \times r(K_2))$ *then every* $\gamma \in \Gamma$ *is an isometry of constant displacement on* $\widetilde{M}$.

Proof. $\Gamma \subset I(\widetilde{M})$, so Theorem 3.12 says $\Gamma \subset \widetilde{G}^\dagger = \bigcup_{\alpha \in \mathrm{Out}(G,K_1,K_2)} \widetilde{G}\alpha$. Lemma 5.5 implies $\Gamma \subset \widetilde{G}$, and from Proposition 5.4 we conclude that $\Gamma \subset (Z_G \times r(K_2))$. Conversely, if $\Gamma \subset (Z_G \times r(K_2))$ then G centralizes Γ so every $\gamma \in \Gamma$ is of constant displacement. □

Corollary 5.7. *Let* $\widetilde{M} \to \Gamma\backslash\widetilde{M}$ *be a Riemannian covering whose deck transformation group* Γ *consists of isometries of constant displacement. Then* $\Gamma \subset (Z_G \times r(K_2))$ *and* $\Gamma\backslash\widetilde{M}$ *is homogeneous.*

6. Isometries of constant displacement: case $\chi(M) = 0$

In this section we study the cases where $\chi(M) = 0$, in other words where $\operatorname{rank} K < \operatorname{rank} G$. We know that the identity component $I^0(\widetilde{M}) = G \times r(K_2)$ by Theorem 3.4. We will prove the following analog of Proposition 5.4 for $\operatorname{rank} K < \operatorname{rank} G$. This uses an argument of Cámpoli [5].

Theorem 6.1. *Let* $\pi : \widetilde{M} \to M$ *as in* (1.1) *with* $\chi(M) = 0$. *If* Γ *is a group of isometries of constant displacement on* $\widetilde{M}$, *and if* $\Gamma \subset I^0(\widetilde{M})$, *then* $\Gamma \subset (Z_G \times r(K_2))$. *Conversely if* $\Gamma \subset (Z_G \times r(K_2))$ *then every* $\gamma \in \Gamma$ *is an isometry of constant displacement on* $\widetilde{M}$.

Proof. Let $\gamma = (g, r(k_2) \in \Gamma$. It descends to an isometry g of M. If g has a fixed point on M, in other words if it preserves a fiber of $\pi : \widetilde{M} \to M$, then the argument of Proposition 5.4 proves $\gamma \in Z_G \times r(K_2)$. Now suppose that g does not have a fixed point on M.

Let $t \to \sigma(t)$ denote the minimizing geodesic in $\widetilde{M}$ from $1K_1$ to $\gamma(1K_1)$. Then $\sigma(t) = \exp(t\xi)(1K_1)$ where $\xi = \xi_1 + dr(\xi_2)$ with $\xi_1 \in \mathfrak{g}$, $\xi_1 \perp \mathfrak{k}_1$, and $\xi_2 \in \mathfrak{k}_2$. Here ξ_1 belongs to the Lie algebra $\mathfrak{t}_1$ of a maximal torus T_1 of G such that $\mathfrak{t}_1 = \mathfrak{t}_1' + \mathfrak{t}_1''$ where $\xi \in \mathfrak{t}_1'$, $\mathfrak{t}_1' \perp \mathfrak{k}_1$, and $\mathfrak{t}_1''$ is the Lie algebra of a maximal torus of K_1. The isometry γ has constant displacement equal to $||\xi_1 + dr(\xi_2)||$. Note that $\xi_1 \neq 0$ because g does not have a fixed point on M.

Every conjugate of γ has the same constant displacement. In particular if w belongs to the Weyl group $W(G, T_1)$ then $||\xi_1 + dr(\xi_2)|| = ||p(w(\xi_1)) + dr(\xi_2)||$ where $p : \mathfrak{t}_1 \to \mathfrak{t}_1'$ is orthogonal projection. As $\xi_1 \perp dr(\xi_2) \perp p(w(\xi_1))||$ it follows that $||\xi_1|| = ||p(w(\xi_1))||$. From that, $w(\xi_1) \in \mathfrak{t}_1'$ for every $w \in W(G, T_1)$. But $W(G, T_1)$ acts irreducibly on $\mathfrak{t}_1$ so it cannot preserve the subspace $\mathfrak{t}_1'$. This implies $\xi_1 = 0$. That contradicts the assumption that g has no fixed point on M. In other words, $\gamma \in (Z_G \times r(K_2))$, as asserted. The theorem follows. □

In the next section we look at the special case of Stiefel manifold fibrations over odd dimensional Grassmann manifolds.

7. Isotropy–split bundles over odd real Grassmannians

In this section we extend the theory described in Sections 3, 5 and 6 to include isotropy–split bundles $\pi : \widetilde{M} \to M$ where the base M is an irreducible compact Riemannian symmetric space G/K such that $\operatorname{rank} G > \operatorname{rank} K$. According to the classification, the only such G/K are

$$SU(n)/SO(n), SU(2n)/Sp(n), SO(2s+2+2t)/[SO(2s+1)\times SO(1+2t)], E_6/F_4, E_6/Sp(4), (K\times K)/diag(K).$$

Note that $SU(4)/SO(4) = SO(6)/[SO(3) \times SO(3)]$. Thus the only such symmetric spaces G/K that satisfy (1.1) are the oriented real Grassmann manifolds $SO(2s + 2t + 2)/[SO(2s + 1) \times SO(2t + 1)]$ of odd real dimension. Thus we look at

$$\begin{aligned} &\pi : \widetilde{M} \to M \text{ given by } G/K_1 \to G/K_1K_2 \text{ where} \\ &\qquad G = SO(2s + 2 + 2t), K_1 = SO(2s + 1) \text{ and } K_2 = SO(1 + 2t). \end{aligned} \tag{7.1}$$

The odd spheres are completely understood ([28] and [32]), and in any case they do not lead to isotropy–split fibrations, so we put those cases aside and assume $s, t > 0$.

Theorem 3.4 and the first three statements of Corollary 3.5 are valid here. We still have the relative groups (3.8) and the isometry groups (3.11), but neither K_1 nor K_2 has an outer automorphism. However, following Cartan, the symmetry s at the base point $1K$ of G/K gives another component of $I(\widetilde{M})$. In fact it is clear that $s = \left(\begin{smallmatrix} I_{2s+1} & 0 \\ 0 & -I_{1+2t} \end{smallmatrix}\right)$. Thus

Proposition 7.2. *The full isometry group $I(\widetilde{M})$ is the 2-component group $O(2s + 2 + 2t) \times r(SO(1 + 2t))$. The set of all constant length Killing vector fields on $\widetilde{M}$ is $dr(\mathfrak{so}(1 + 2t))$.*

Lemma 7.3. *Every element of $sI^0(\widetilde{M})$ has a fixed point on M.*

Proof. If $g \in I^0(\widetilde{M})$ then the matrix sg has determinant -1. Let $R(\theta)$ denote the rotation matrix $\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$. Then $\mathbb{R}^{2s+2+2t}$ has an orthonormal basis $\{e_i\}$ in which sg has matrix

$$\operatorname{diag}\{R(\theta_1), \ldots, R(\theta_s), 1, -1, R(\theta'_1), \ldots, R(\theta'_t)\}.$$

Now $\operatorname{Span}\{e_1, \ldots, e_{2s+1}\}$ is the fixed point. □

Combining Theorem 6.1 with Proposition 7.2 and Lemma 7.3 we have

Theorem 7.4. *Let $\pi : \widetilde{M} \to M$ as in (7.1). If Γ is a group of isometries of constant displacement on the Stiefel manifold $\widetilde{M}$ then $\Gamma \subset (\{\pm I\} \times r(SO(1 + 2t)))$. Conversely if $\Gamma \subset (\{\pm I\} \times r(SO(1 + 2t)))$ then every $\gamma \in \Gamma$ is an isometry of constant displacement on $\widetilde{M}$.*

Now, as in Corollary 5.7, we have

Corollary 7.5. *Let $\pi : \widetilde{M} \to M$ be the isotropy–split fibration (7.1). Let $\widetilde{M} \to \Gamma\backslash\widetilde{M}$ be a Riemannian covering whose deck transformations $\gamma \in \Gamma$ have constant displacement. Then $\Gamma \subset (\{\pm I\} \times r(SO(1 + 2t)))$ and $\Gamma\backslash\widetilde{M}$ is a Riemannian homogeneous space.*

8. Applications to the homogeneity conjecture

The background to the Homogeneity Conjecture consists of three papers from the 1960's concerning Riemannian coverings $S \to \Gamma\backslash S$ where S is a Riemannian homogeneous space. The first one, [27], studies the case where S has constant sectional curvature and classifies the quotients $\Gamma\backslash S$ that are Riemannian homogeneous. There it is shown that if $\Gamma\backslash S$ is homogeneous then every $\gamma \in \Gamma$ is of constant displacement. The second one, [28], also for the case of constant sectional curvature, shows that if every $\gamma \in \Gamma$ is of constant displacement then $\Gamma\backslash S$ is homogeneous. The third one, [29], extends these results to symmetric spaces: Let S be a connected simply connected Riemannian symmetric space and $S \to \Gamma\backslash S$ a Riemannian covering; then $\Gamma\backslash S$ is homogeneous if and only if every $\gamma \in \Gamma$ is of constant displacement. Thus

Homogeneity Conjecture. *Let S be a connected simply connected Riemannian homogeneous space and $S \to \Gamma\backslash S$ a Riemannian covering. Then $\Gamma\backslash S$ is homogeneous if and only if every $\gamma \in \Gamma$ is of constant displacement.*

We note that one direction of the Homogeneity Conjecture is easy: If $\Gamma\backslash S$ is homogeneous then $I^0(\Gamma\backslash S)$ lifts to a subgroup G of $I(S)$, and G normalizes Γ by construction. Then G centralizes Γ because G is connected and Γ is discrete. Also, G is transitive on S because $\Gamma\backslash S$ is homogeneous. If $x, y \in S$ and $\gamma \in \Gamma$ choose $g \in G$ with $y = gx$. Then the displacement $\delta_\gamma(x) = \mathrm{dist}(x, \gamma x) = \mathrm{dist}(gx, g\gamma x) = \mathrm{dist}(gx, \gamma gx) = \mathrm{dist}(y, \gamma y) = \delta_\gamma(y)$. Thus if $\Gamma\backslash S$ is homogeneous then every $\gamma \in \Gamma$ is of constant displacement. The hard part is the converse.

Since the Homogeneity Conjecture was proved for S Riemannian symmetric, some other cases of the conjecture have been proved. The latest cases are those of Corollaries 5.7 and 7.5:

Theorem 8.1. *Let $\widetilde{M} \to M$ be an isotropy–splitting fibration and let $\widetilde{M} \to \Gamma\backslash\widetilde{M}$ be a Riemannian covering. Suppose that $\operatorname{rank} K = \operatorname{rank} G$, or that M is a Riemannian symmetric space, or that $\Gamma \subset I^0(\widetilde{M})$. Then $\Gamma\backslash\widetilde{M}$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement.*

Now we try to describe the broader mathematical context of Theorem 8.1. There are five lines of research there: (i) decreasing the number of case by case verifications of [29], (ii) dealing with nonpositive curvature and bounded isometries, (iii) additional special cases where S is compact, (iv) Killing vector fields of constant length, and (v) extension of these results from Riemannian to Finsler manifolds.

Concerning case by case verifications. My proof [29] of the Homogeneity Conjecture for symmetric spaces involved a certain amount of case by case verification. Some of this was simplified later by Freudenthal [17] and Ozols ([22], [23], [24]) with the restriction that Γ be contained in the identity component of the isometry group.

Nonpositive curvature and bounded isometries. This approach was implicit in the treatment of symmetric spaces of noncompact type in [27] and [29], and extended in [31] to all Riemannian manifolds S of non-positive sectional curvature. The idea is to apply an isometry γ of bounded displacement to a geodesic σ and see that σ and $\gamma(\sigma)$ bound a flat totally geodesic strip in S. This was extended later by Druetta [16] to manifolds without focal points. Further evidence for the Homogeneity Conjecture was developed by Dotti, Miatello and the author in [15] for Riemannian manifolds that admit a transitive semisimple group of isometries that has no compact factor, and by the author in [34] for Riemannian manifolds that admit a transitive exponential solvable group of isometries. Here also see [30].

Killing vector fields of constant length. The infinitesimal version of isometries of constant displacement is that of Killing vector fields of constant length. This topic seems to have been initiated by Berestovskii and Nikonorov in ([1], [2], [3]), and was further developed by Nikonorov ([20], [21]), and by Podestà, myself and Xu ([37], [38], [39]). Also see Corollary 3.5 above.

Finsler extensions. If (M, F) is a Finsler symmetric space, say $M = G/K$ where G is the identity component of the (Finsler) isometry group, then there is a G-invariant Riemannian metric ds^2 on M such that (M, ds^2) is Riemannian symmetric with the same geodesics as (M, F). See [33, Theorem 11.6.1] and the discussion in [33, §11.6]. Deng and I proved the Homogeneity Conjecture for Finsler symmetric spaces in [8] by reduction to the Riemannian case. Further, Deng and others in his school, especially Xu, have done a lot on isometries of constant displacement and on Killing vector fields of constant length; for example see [9], [10], [11], [12], [13] and [14]. The arguments involving reduction to the Riemannian case usually depend on very technical computations. In [8], for example, one has to prove the Berwald condition in order to get around the lack of a de Rham decomposition for Finsler manifolds.

9. Isotropy–splitting fibrations over noncompact spaces

In this section we examine our theory of isotropy–splitting fibrations $\pi : \widetilde{M} \to M$ in the setting in which the base M is noncompact. For example M could be the noncompact dual of one of the Riemannian symmetric spaces of Section 2, or a certain variation for the 3-symmetric and 5-symmetric spaces.

Here is the basic problem with noncompact base manifolds. Recall that $M = G/K$ and $\widetilde{M} = G/K_1$ carry the normal metrics defined by the negative $-\kappa$ of the Killing form of $\mathfrak{g}$. Then $-\kappa$ cannot be definite on $\mathfrak{k}_1^\perp$, for $\mathfrak{k}_1$ cannot be a maximal compactly embedded subalgebra. So we are forced to either restrict attention to the setting of compact Riemannian manifolds M and $\widetilde{M}$, or expand attention to the situation where $\widetilde{M}$ is a noncompact pseudo-Riemannian manifold. At that point we modify the compact manifold definition (1.1) for isotropy–splitting fibrations $\pi : \widetilde{M} \to M$, replacing "compact" by "reductive" and dealing with the lack of a general de Rham decomposition for pseudo-Riemannian manifolds:

$$\begin{aligned}
&G \text{ is a connected real reductive linear algebraic group with } G_{\mathbb{C}} \text{ simply connected,}\\
&K = K_1K_2 \text{ where the } K_i \text{ are closed connected reductive algebraic subgroups of } G \text{ such that}\\
&\qquad\text{(i) } K = (K_1 \times K_2)/(K_1 \cap K_2),\ \text{(ii) } \mathfrak{k}_2 \perp \mathfrak{k}_1 \text{ and (iii) } \dim \mathfrak{k}_1 \neq 0 \neq \dim \mathfrak{k}_2\,,\\
&\text{the centralizers } Z_G(K_1) = Z_{K_1}\widetilde{K_2} \text{ and } Z_G(K_2) = Z_{K_2}\widetilde{K_1} \text{ with } K_1 = \widetilde{K_1}^0 \text{ and } K_2 = \widetilde{K_2}^0\,, \text{ and}\\
&M = G/K \text{ and } \widetilde{M} = G/K_1 \text{ are normal pseudo-Riemannian homogeneous spaces of } G.
\end{aligned} \tag{9.1}$$

As before we may assume that the metrics on M and $\widetilde{M}$ are the normal pseudo-Riemannian metrics defined by the negative of the Killing form of G.

9.1. Noncompact Riemannian symmetric base

A particularly interesting case is where $M = G/K$ is an irreducible Riemannian symmetric space of noncompact type. We list all such isotropy–splitting fibrations $\pi : \widetilde{M} \to M$, given by $G/K_1 \to G/K$ with $\operatorname{rank} K = \operatorname{rank} G$. There of course $M = G/K$ is the noncompact dual of one of the fibrations of Section 2 over a compact symmetric space of nonzero Euler characteristic. The ones with hermitian symmetric space base are characterized by $K = SK'$ where S is a circle group and $K' = [K, K]$ is semisimple. The corresponding fibrations are

$$\begin{aligned}
&G/K' \to G/K \text{ circle bundle over a bounded symmetric domain and}\\
&G/S \to G/K \text{ principal } K'\text{-bundle over bounded symmetric domain.}
\end{aligned}$$

In addition, if $\mathfrak{g} = \mathfrak{su}(s, t)$ then $\mathfrak{k}' = \mathfrak{su}(s) \oplus \mathfrak{su}(t)$, leading to fibrations

$$SU(s,t)/SU(s) \to SU(s,t)/S(U(s)U(t)) \text{ and } SU(s,t)/U(s) \to SU(s,t)/S(U(s)U(t)),$$
$$SU(s,t)/SU(t) \to SU(s,t)/S(U(s)U(t)) \text{ and } SU(s,t)/U(t) \to SU(s,t)/S(U(s)U(t)).$$

When the base $M = G/K$ is a nonhermitian symmetric space, K is simple except in the cases

$$G/K = SO^0(s,t)/SO(s)SO(t) \text{ with } 2 < s \leqq t \text{ and } st \text{ even},$$
$$G/K = Sp(s,t)/Sp(s)Sp(t) \text{ with } 1 \leqq s \leqq t,$$
$$G/K = G_{2,A_1A_1}/A_1A_1\,, G/K = F_{4,A_1C_3}/A_1C_3\,, E_{6,A_1A_5}/A_1A_5\,, E_{7,A_1D_6}/A_1D_6 \text{ or } E_{8,A_1E_7}/A_1E_7\,.$$

In the SO cases, G/K is a quaternion-Kaehler symmetric space for $s = 3$ and for $s = 4$. In the Sp cases G/K is a quaternion-Kaehler symmetric space for $s = 1$. In the exceptional group cases G/K always is a quaternion-Kaehler symmetric space.

Finally we list the isotropy–splitting fibrations $\pi : \widetilde{M} \to M$, given by $G/K_1 \to G/K$ with $\operatorname{rank} K < \operatorname{rank} G$. There $M = G/K$ is the noncompact dual of an odd dimensional real Grassmann manifold $SO(2s+2+2t)/[SO(2s+1)SO(1+2t)]$ with $s,t > 1$, leading to the fibrations

$$G/K = SO^0(2s+1,1+2t)/SO(2s+1) \to SO^0(2s+1,1+2t)/[SO(2s+1)SO(1+2t)] \text{ and}$$
$$G/K = SO^0(2s+1,1+2t)/SO(1+2t) \to SO^0(2s+1,1+2t)/[SO(2s+1)SO(1+2t)]$$

with $s,t > 1$.

9.2. Compact Riemannian dual fibration

Given (G, K_1, K_2) as in (9.1), there is a Cartan involution θ of G that preserves each K_i and restricts on it to a Cartan involution. That defines the compact Cartan dual triple (G^u, K_1^u, K_2^u) and the compact Riemannian isotropy–splitting fibration $\pi^u : \widetilde{M^u} \to M^u$, given by $G^u/K_1^u \to G^u/K_1^uK_2^u$, as in (1.1). Several pseudo-Riemannian isotropy–splitting fibrations $\pi : \widetilde{M} \to M$ can define the same $\pi^u : \widetilde{M^u} \to M^u$. We say that $\pi : \widetilde{M} \to M$ is *associated* to $\pi^u : \widetilde{M^u} \to M^u$.

One special case is the one where G, K_1 and K_2 each is the underlying real structure of a complex Lie group. Then $G = G^u_{\mathbb{C}}$, $K_1 = (K_1)^u_{\mathbb{C}}$ and $K_2 = (K_2)^u_{\mathbb{C}}$.

For each of the isotropy–splitting fibrations $\pi^u : \widetilde{M^u} \to M^u$ that satisfies (1.1) we find all associated pseudo-Riemannian isotropy–splitting fibrations $\pi : \widetilde{M} \to M$ in the tables of [35] and [36]. Here, for example, are the ones for $n_0 = 3$, corresponding to the nearly-Kaehler base spaces of Section 2.3, taken from [36, Table 7.13].

- For $\widetilde{M^u} \to M^u$ where $M^u = F_4/A_2A_2$, M can be M^u or one of

$$F_{4,B_4}/[SU(1,2)SU(3)]\ , F_{4,C_1C_3}/[SU(3)SU(1,2)],$$
$$F_{4,C_1C_3}/[SU(1,2)SU(1,2)] \text{ or } F_4^{\mathbb{C}}/[SL(3;\mathbb{C})SL(3;\mathbb{C})].$$

- For $\widetilde{M^u} \to M^u$ where $M^u = E_6/A_2A_2A_2$, M can be M^u or one of

$$E_6/A_1A_5/[SU(1,2)SU(3)SU(3)],\ E_6/A_1A_5/[SU(1,2)SU(1,2)SU(3)],$$
$$E_{6,D_5T^1}/[SU(1,2)SU(1,2)SU(3)],\ \text{ or } E_6^{\mathbb{C}}/[SL(3;\mathbb{C})SL(3;\mathbb{C})SL(3;\mathbb{C})].$$

- For $\widetilde{M^u} \to M^u$ where $M^u = E_7/A_2A_5$, M can be M^u or one of

$$E_{7,A_1D_6}/[SU(1,2)SU(1,5)],\ E_{7,A_1D_6}/[SU(3)SU(2,4)],\ E_{7,A_1D_6}/[SU(1,2)SU(2,4)],$$
$$E_{7,E_6T^1}/[SU(1,2)SU(1,5)],\ E_{7,E_6T^1}/[SU(3)SU(3,3)],\ \text{or}\ E_7^{\mathbb{C}}/[SL(3;\mathbb{C})SL(6;\mathbb{C})].$$

- For $\widetilde{M^u} \to M^u$ where $M^u = E_8/A_2E_6$, M can be M^u or one of

$$E_{8,A_1E_7}/[SU(1,2)E_6],\ E_{8,A_1E_7}/[SU(1,2)E_{6,D_5T^1}],$$
$$E_{8,A_1E_7}/[SU(3)E_{6,A_1A_5}],\ \text{or}\ E_8^{\mathbb{C}}/[SL(3;\mathbb{C})E_6^{\mathbb{C}}].$$

9.3. Isometries and Killing vector fields

We use the notation (9.1). As in the compact case we have

$$\widetilde{G} := G \times r(K_2) \text{ is connected and algebraic, and acts on } \widetilde{M} \text{ by } (g, r(k_2)) : xK_1 \mapsto gxk_2^{-1}K_1\,. \tag{9.2}$$

Theorem 3.4 extends to the pseudo-Riemannian setting as follows.

Proposition 9.3. *If $\widetilde{M}$ is irreducible then the isometry group of $\widetilde{M}$ has identity component $I^0(\widetilde{M}) = \widetilde{G}$.*

Proof. Evidently $\widetilde{G}$ acts by isometries on $\widetilde{M}$, and by hypothesis $\widetilde{M}$ is an affine algebraic variety. Now suppose that $\widetilde{G} \subsetneqq L$ where L is a closed connected algebraic subgroup of $I(\widetilde{M})$. Let L^{red} denote a maximal reductive subgroup of L that contains $\widetilde{G}$, so $\widetilde{G} \subset L^{red}$. The compact real forms $\widetilde{G}^u \subset L^{red,u}$, and Theorem 3.4 ensures that $\widetilde{G}^u = I^0(\widetilde{M}^u) = L^{red,u}$. Thus $\widetilde{G}$ is a maximal reductive subgroup of L.

Let H denote the isotropy subgroup of L at $1K_1$. It contains $\widetilde{K_1} := K_1 \times \{(k_2, r(k_2)) \mid k_2 \in K_2\}$, the isotropy subgroup of $\widetilde{G}$ at $1K_1$, and $\widetilde{K_1}$ is its maximal reductive subgroup. Let L^{unip} denote the unipotent radical of L. Since L^{unip} is a normal subgroup of L its orbits satisfy $L^{unip}(gK_1) = gL^{unip}(K_1)$. Thus $\widetilde{M} \to L^{unip}\backslash\widetilde{M}$ would be a fiber space if the L^{unip}-orbits on $\widetilde{M}$ were closed submanifolds. To get around that problem let N denote the categorical quotient $L^{unip}\backslash\backslash\widetilde{M}$. We can view it as the base space of the fibration whose fibers are the closures of the L^{unip}-orbits on $\widetilde{M}$. The (transitive) action of $\widetilde{G}$ on $\widetilde{M}$ descends to a smooth transitive action of $\widetilde{G}$ on N. The action of L descends as well. Write $N = \widetilde{G}/Q$ where Q contains $\widetilde{K_1}$. Then $\widetilde{K_1}L^{unip}$ is the isotropy subgroup of L on N and $Q = \widetilde{G} \cap (\widetilde{K_1}L^{unip}) = \widetilde{K_1}$. This says $\widetilde{M} \to L^{unip}\backslash\backslash\widetilde{M}$ is one to one. In other words the action of L^{unip} on $\widetilde{M}$ is trivial. As L acts effectively by its definition, $L^{unip} = \{1\}$. Now $L = L^{red} = \widetilde{G}$. □

In the pseudo-Riemannian setting we don't have a good notion for the displacement of an isometry, but we still have its infinitesimal analog. We define *constant length* for a vector field to mean constant inner product with itself relative to the invariant pseudo-Riemannian metric. Fix a Cartan involution θ of $\widetilde{G}$ that preserves G, K_1, K and $\widetilde{K_1}$. We say that an element $\xi \in \widetilde{\mathfrak{g}}$ is *elliptic* if $d\theta(\mathrm{Ad}(g)\xi) = \mathrm{Ad}(g)\xi$ for some $g \in G$, *hyperbolic* if $d\theta(\mathrm{Ad}(g)\xi) = -\mathrm{Ad}(g)\xi$, for some $g \in G$. In other words ξ is elliptic if all the eigenvalues of $\mathrm{ad}\,(\xi)$ are pure imaginary, hyperbolic if all the eigenvalues of ξ are real. Here is the noncompact base analog of Corollary 3.5:

Corollary 9.4. *Suppose that the pseudo-Riemannian manifold $\widetilde{M} = G/K_1$ is irreducible.*

(1) *The algebra of all Killing vector fields on $\widetilde{M}$ is $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus dr(\mathfrak{k}_2)$.*
(2) *The set of all constant length Killing vector fields on $\widetilde{M}$ is $\mathfrak{l} \oplus dr(\mathfrak{k}_2)$ where $\mathfrak{l} = \{\xi \in \mathfrak{g} \mid \xi$ defines a constant length Killing vector field on $\widetilde{M}\}$.*
(3) *$\mathfrak{l} = \{\xi \in \mathfrak{g} \mid \xi$ defines a constant length Killing vector field on $M\}$.*

(4) *If* $\operatorname{rank} K = \operatorname{rank} G$ *and* $\xi \in \mathfrak{l}$ *then* ξ *is* $\mathrm{Ad}(G)$*-conjugate to an element of* $\mathfrak{k}$ *and the corresponding Killing vector field has norm* $||\xi_{gK_1}|| = 0$ *at every point* $gK_1 \in \widetilde{M}$*. In particular* $\mathfrak{l}$ *does not contain a nonzero elliptic element nor a nonzero hyperbolic element.*

(5) *If* $\operatorname{rank} K = \operatorname{rank} G$ *and* K *is compact, then* $\mathfrak{l} = 0$*, so* $dr(\mathfrak{k}_2)$ *is the set of all constant length Killing vector fields on* $\widetilde{M}$*.*

Proof. The first statement is immediate from Theorem 9.3, so we turn to the second and third.

If ξ is a Killing vector field on $\widetilde{M}$ then $\xi \in \mathfrak{g} \oplus dr(\mathfrak{k}_2)$. Decompose $\xi = \xi' + \xi''$ with $\xi' \in \mathfrak{g}$ and $\xi'' \in dr(\mathfrak{k}_2)$. Then ξ'' has constant length because the corresponding vector field on $\widetilde{M}$ is invariant under the transitive isometry group G. The vector fields of ξ' and ξ'' are orthogonal at $1K_1$. It follows that they are orthogonal at every point of $\widetilde{M}$ because ξ' is orthogonal to the fibers of $\widetilde{M} \to M$ at every point of $\widetilde{M}$. If ξ has constant length now ξ' also has constant length. That proves the second statement. In the argument just above, ξ and ξ' define the same Killing vector field on M. The third statement follows.

Now suppose $\operatorname{rank} K = \operatorname{rank} G$ and let $\xi \in \mathfrak{l}$. Using a Cartan involution of $\mathfrak{g}$ we write $\xi = \xi_{ell} + \xi_{hyp}$ where ξ_{ell} is elliptic and ξ_{hyp} is hyperbolic. Recall that we are using the negative of the Killing form of $\mathfrak{g}$ for the pseudo-Riemannian metrics both on $\widetilde{M}$ and M. If the square length $||\xi||^2 > 0$ on $\widetilde{M}$ then ξ_{ell} never vanishes on M, contradicting $\operatorname{rank} K = \operatorname{rank} G$. If $||\xi||^2 < 0$ then ξ_{hyp} never vanishes on M, contradicting $\operatorname{rank} K = \operatorname{rank} G$. Thus $||\xi||^2 = 0$ on $\widetilde{M}$, and thus on M. As above ξ has a zero on M, in other words some $\mathrm{Ad}(G)$-conjugate of ξ belongs to $\mathfrak{k}$. Thus $\xi \in \mathfrak{k}$ with length $||\xi|| = 0$ at every point of $\widetilde{M}$ In particular, if K is compact then ξ is elliptic, so $\xi = 0$ and $dr(\mathfrak{k}_2)$ is the set of all constant length Killing vector fields on $\widetilde{M}$. $\square$

Corollary 9.5. *Suppose that* $\widetilde{M}$ *is irreducible. Then every isometry of* $\widetilde{M}$ *sends fiber to fiber in the isotropy–split fibration* $\widetilde{M} \to M$*, and thus induces an isometry of* M*.*

Proof. In view of Corollary 9.4(2) and 9.4(4), and because we have a Cartan involution θ of $\widetilde{G}$ such that $\theta(K_2) = K_2$, the tangent space to the fiber $r(K_2)(gK_1)$ is the span of all vector fields $\xi_e + \xi_h$ where $\xi_e \in dr(\mathfrak{k}_2)$ is elliptic and $\xi_h \in dr(\mathfrak{k}_2)$ is hyperbolic. $\square$

Now we can look for the full isometry group of $\widetilde{M}$. In fact the result is very close to results in [15, Section 2], and the way we use it is contained in [15, Section 2], but the argument here is closer to the structure of the isotropy–splitting fibration. Let

$$\mathrm{Out}(G, K_1) = \{\alpha \in \mathrm{Out}(G) \mid \alpha(K_1) = K_1 \text{ and } \alpha|_{K_1} \in \mathrm{Out}(K_1)\} \subset \mathrm{Out}(G, K). \tag{9.6}$$

We define two subgroups of isometry groups as in (3.11) by

$$G^\dagger = \bigcup\nolimits_{\alpha \in \mathrm{Out}(G,K_1)} G\alpha \subset I(M) \text{ and } \widetilde{G}^\dagger = \bigcup\nolimits_{\alpha \in \mathrm{Out}(G,K_1)\,, \beta \in \mathrm{Out}(G,K_2)} G\alpha \cdot r(K_2)\beta \subset I(\widetilde{M}). \tag{9.7}$$

As before, $g\alpha$ acts on M by $xK \mapsto g\alpha(x)K$ and on $\widetilde{M}$ by $xK_1 \mapsto g\alpha(x)K_1$, and $r(k_2)\beta$ acts on $\widetilde{M}$ by $xK_1 \mapsto x\beta(k_2)^{-1}K_1$.

Theorem 9.8. *Let* $\pi : \widetilde{M} \to M$ *be an isotropy–split fibration as in* (9.1)*. Suppose* $\operatorname{rank} K = \operatorname{rank} G$*. Then the identity component* $I^0(\widetilde{M}) = \widetilde{G}$ *and the full isometry group* $I(\widetilde{M}) = \widetilde{G}^\dagger$*.*

Proof. As $\widetilde{G}$ is a reductive linear algebraic group every component of $\mathrm{Aut}(\widetilde{G})$ contains an elliptic element. Thus every component of $\mathrm{Aut}(\widetilde{G})$ has an element in common with $\mathrm{Aut}(\widetilde{G}^u)$. Corollary 9.5 carries this down to M. That gives injections $I(\widetilde{M})/I^0(\widetilde{M}) \hookrightarrow I(\widetilde{M^u})/I^0(\widetilde{M^u})$ and $I(M)/I^0(M) \hookrightarrow I(M^u)/I^0(M^u)$, so Lemmas 3.9 and 3.10 extend to our pseudo-Riemannian setting. Our assertions follow by combining Theorem 3.12 with Proposition 9.3. □

While don't have a notion of constant displacement here, we can at least study homogeneity for pseudo-Riemannian coverings by $\widetilde{M}$.

Corollary 9.9. *Let $\pi : \widetilde{M} \to M$ be an isotropy–split fibration as in (9.1). Suppose* $\operatorname{rank} K = \operatorname{rank} G$. *Let p denote the projection $\widetilde{G}^\dagger \to G^\dagger$ of $I(\widetilde{M})$ into $I(M)$.*

(1) *Let $\gamma \in I(\widetilde{M})$ such that $p(\gamma)$ is elliptic and the centralizer of γ is transitive on $\widetilde{M}$. Then $p(\gamma) \in Z_G$.*

(2) *Consider a pseudo-Riemannian covering $\widetilde{M} \to \Gamma\backslash\widetilde{M}$ such that $p(\Gamma)$ has compact closure in $I(M)$ (for example such that $p(\Gamma)$ is finite). Then $\Gamma\backslash\widetilde{M}$ is homogeneous if and only if $p(\Gamma) \subset Z_G$.*

Proof. Let J denote the centralizer of $p(\gamma)$ in $G^\dagger$. Then $\mathfrak{g} = \mathfrak{j} + \mathfrak{k}$. As $p(\gamma)$ is semisimple $\mathfrak{g}^u = \mathfrak{j}^u + \mathfrak{k}^u$ so J^u is transitive on M^u. Also, $p(\gamma) \in G^u$ because it is elliptic. Now $p(\gamma)$ has constant displacement on M^u and the assertion follows from Theorem 5.6. □

9.4. Isotropy–Split fibrations over odd indefinite symmetric spaces

We now deal with the cases where $\operatorname{rank} K < \operatorname{rank} G$ and $M = G/K$ is a pseudo-Riemannian symmetric space. Here we follow the lines of Section 7.

According to the classification, the only compact irreducible Riemannian symmetric spaces G^u/K^u with $\operatorname{rank} K < \operatorname{rank} G$ are

$$SU(n)/SO(n), SU(2n)/Sp(n), SO(2s+2+2t)/[SO(2s+1) \times SO(1+2t)], E_6/F_4, E_6/Sp(4),$$
$$(K^u \times K^u)/diag(K^u).$$

The only ones of these spaces for which K^u splits are the odd dimensional oriented real Grassmann manifolds $SO(2s+2+2t)/[SO(2s+1) \times SO(1+2t)]$ with $s, t \geqq 1$. Thus we look at

$$\begin{aligned} &\pi : \widetilde{M} \to M \text{ given by } G/K_1 \to G/K_1K_2 \text{ where } G = SO^0(2s+1, 1+2t), \\ &K_1 = SO^0(u, v) \text{ and } K_2 = SO^0(a, b) \end{aligned} \tag{9.10}$$

with conditions $u + a = 2s + 1$ and $v + b = 1 + 2t$ for the signature of $\mathbb{R}^{2s+1,1+2t}$, and $u + v = 2s + 1$ and $a + b = 1 + 2t$ for $K_1^u = SO(2s+1)$ and $K_2^u = SO(1+2t)$. Here a determines u, v and b, so in fact

$$K_1 = SO^0(2s+1-a, a) \text{ and } K_2 = SO^0(a, 1+2t-a) \text{ for } 0 \leqq a \leqq \min(2s+1, 1+2t).$$

The symmetry of the pseudo-Riemannian symmetric space M is $\rho = \operatorname{Ad}\left(\begin{smallmatrix} I_{2s+1} & 0 \\ 0 & -I_{1+2t} \end{smallmatrix}\right)$. If a is even then K_1 has outer automorphism $\sigma_{1,a} = \operatorname{Ad}\left(\begin{smallmatrix} I_{2s} & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and K_2 has outer automorphism $\sigma_{2,a} = \operatorname{Ad}\left(\begin{smallmatrix} -1 & 0 \\ 0 & I_{2t} \end{smallmatrix}\right)$; then ρ and $\sigma_{2,a}$ belong to the same component of $O(2s+1, 1+2t)$. If a is odd then K_1 has outer automorphism $\tau_{1,a} = \operatorname{Ad}\left(\begin{smallmatrix} -1 & 0 \\ 0 & I_{2s} \end{smallmatrix}\right)$ and K_2 has outer automorphism $\tau_{2,a} = \operatorname{Ad}\left(\begin{smallmatrix} I_{2t} & 0 \\ 0 & -1 \end{smallmatrix}\right)$; then ρ and $\tau_{1,a}$ belong to the same component of $O(2s+1, 1+2t)$. Combining this with Corollary 9.5, we have

Proposition 9.11. *The full isometry group $I(\widetilde{M}) = (O(2s+1, 1+2t) \times r(SO(a, 1+2t-a)))$.*

The proof of Corollary 9.9 is valid for our isotropy–split fibrations (9.10), except that we reduce to Theorem 7.4 instead of Theorem 5.6. Thus:

Corollary 9.12. *Let $\pi : \widetilde{M} \to M$ be one of the isotropy–split fibrations (9.10). Let p denote the projection $\widetilde{G}^\dagger \to G^\dagger$ from $I(\widetilde{M})$ to $O(2s+1, 1+2t)$.*

(1) *If $\gamma \in I(\widetilde{M})$ such that $p(\gamma)$ is elliptic and the centralizer of γ is transitive on $\widetilde{M}$. Then $p(\gamma) = \pm I$.*

(2) *Consider a pseudo-Riemannian covering $\widetilde{M} \to \Gamma\backslash\widetilde{M}$ such that $p(\Gamma)$ has compact closure (for example such that $p(\Gamma)$ is finite or $p(\Gamma) \subset K$). Then $\Gamma\backslash\widetilde{M}$ is homogeneous if and only if $p(\Gamma) \subset \{\pm I\}$.*

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