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Stepwise Square Integrability for Nilradicals of Parabolic Subgroups and Maximal Amenable Subgroups

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Abstract. In a series of recent papers ([19], [20], [22], [23]) we extended the notion of square integrability, for representations of nilpotent Lie groups, to that of stepwise square integrability. There we discussed a number of applications based on the fact that nilradicals of minimal parabolic subgroups of real reductive Lie groups are stepwise square integrable. In Part I we prove stepwise square integrability for nilradicals of arbitrary parabolic subgroups of real reductive Lie groups. This is technically more delicate than the case of minimal parabolics. We further discuss applications to Plancherel formulae and Fourier inversion formulae for maximal exponential solvable subgroups of parabolics and maximal amenable subgroups of real reductive Lie groups. Finally, in Part II, we extend a number of those results to (infinite dimensional) direct limit parabolics. These extensions involve an infinite dimensional version of the Peter-Weyl Theorem, construction of a direct limit Schwartz space, and realization of that Schwartz space as a dense subspace of the corresponding L^2 space.
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Part I: Finite Dimensional Theory

1. Stepwise Square Integrable Representations

There is a very precise theory of square integrable representations of nilpotent Lie groups due to Moore and the author [9]. It is based on the Kirillov's general representation theory [4] for nilpotent Lie groups, in which he introduced coadjoint orbit theory to the subject. When a nilpotent Lie group has square integrable representations its representation theory, Plancherel and Fourier inversion formulae, and other aspects of real analysis, become explicit and transparent.

Somewhat later it turned out that many familiar nilpotent Lie groups have foliations, in fact semidirect product towers composed of subgroups that have

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square integrable representations. These include nilradicals of minimal parabolic subgroups, e.g. the group of strictly upper triangular real or complex matrices. All the analytic benefits of square integrability carry over to stepwise square integrable nilpotent Lie groups.

In order to indicate our results here we must recall the notions of square integrability and stepwise square integrability in sufficient detail to carry them over to nilradicals of arbitrary parabolic subgroups of real reductive Lie groups.

A connected simply connected Lie group N with center Z is called *square integrable*, or is said to *have square integrable representations*, if it has unitary representations π whose coefficients $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ satisfy $|f_{u,v}| \in L^2(N/Z)$. C.C. Moore and the author worked out the structure and representation theory of these groups [9]. If N has one such square integrable representation then there is a certain polynomial function $\mathrm{Pf}(\lambda)$ on the linear dual space $\mathfrak{z}^*$ of the Lie algebra of Z that is key to harmonic analysis on N. Here $\mathrm{Pf}(\lambda)$ is the Pfaffian of the antisymmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$ given by $b_\lambda(x,y) = \lambda([x,y])$. The square integrable representations of N are the π_λ (corresponding to coadjoint orbits $\mathrm{Ad}^*(N)\lambda$) where $\lambda \in \mathfrak{z}^*$ with $\mathrm{Pf}(\lambda) \neq 0$, Plancherel almost irreducible unitary representations of N are square integrable, and, up to an explicit constant, $|\mathrm{Pf}(\lambda)|$ is the Plancherel density on the unitary dual $\widehat{N}$ at π_λ. Concretely,

Theorem 1.1. [9] *Let N be a connected simply connected nilpotent Lie group that has square integrable representations. Let Z be its center and $\mathfrak{v}$ a vector space complement to $\mathfrak{z}$ in $\mathfrak{n}$, so $\mathfrak{v}^* = \{\gamma \in \mathfrak{n}^* \mid \gamma|_\mathfrak{z} = 0\}$. If f is a Schwartz class function $N \to \mathbb{C}$ and $x \in N$ then*

$$f(x) = c \int_{\mathfrak{z}^*} \Theta_{\pi_\lambda}(r_x f)|\mathrm{Pf}(\lambda)|d\lambda \tag{1.1}$$

where $c = d!2^d$ with $2d = \dim \mathfrak{n}/\mathfrak{z}$, $r_x f$ is the right translate $(r_x f)(y) = f(yx)$, and Θ is the distribution character

$$\Theta_{\pi_\lambda}(f) = c^{-1}|\mathrm{Pf}(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi)d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N). \tag{1.2}$$

Here f_1 is the lift $f_1(\xi) = f(\exp(\xi))$ of f from N to $\mathfrak{n}$, $\widehat{f_1}$ is its classical Fourier transform, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$, *and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to* $\mathrm{Ad}^*(N)\lambda$.

More generally, we will consider the situation where

$$\begin{aligned} N = &L_1 L_2 \dots L_{m-1} L_m \text{ where} \\ &\text{(a) each factor } L_r \text{ has unitary reps with coefficients in } L^2(L_r/Z_r), \\ &\text{(b) each } N_r := L_1 L_2 \dots L_r \text{ is normal in } N \text{ with } N_r = N_{r-1} \rtimes L_r\,, \\ &\text{(c) if } r \geqq s \text{ then } [\mathfrak{l}_r, \mathfrak{z}_s] = 0 \end{aligned} \tag{1.3}$$

The conditions of (1.3) are sufficient to construct the representations of interest to us here, but not sufficient to compute the Pfaffian that is the Plancherel density.

For that, in the past we used the *strong computability condition*

$$\text{Decompose } \mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r \text{ and } \mathfrak{n} = \mathfrak{s} + \mathfrak{v} \text{ (vector space direct) where } \mathfrak{s} = \oplus\, \mathfrak{z}_r \text{ and } \mathfrak{v} = \oplus\, \mathfrak{v}_r;\ \text{then } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v}_s \text{ for } r > s. \tag{1.4}$$

The problem is that the strong computability condition (1.4) can fail for some non-minimal real parabolics, but we will see that, for the Plancherel density, we only need the *weak computability condition*

$$\text{Decompose } \mathfrak{l}_r = \mathfrak{l}'_r \oplus \mathfrak{l}''_r, \text{ direct sum of ideals, where } \mathfrak{l}''_r \subset \mathfrak{z}_r \text{ and } \mathfrak{v}_r \subset \mathfrak{l}'_r;\ \text{then } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}''_s + \mathfrak{v}_s \text{ for } r > s. \tag{1.5}$$

where we retain $\mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r$ and $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$.

In the setting of (1.3), (1.4) and (1.5) it is useful to denote

$$\begin{aligned}
&\text{(a) } d_r = \tfrac{1}{2}\dim(\mathfrak{l}_r/\mathfrak{z}_r) \text{ so } \tfrac{1}{2}\dim(\mathfrak{n}/\mathfrak{s}) = \textstyle\sum d_i\,, \text{ and } c = 2^{\sum d_i} \cdot d_1! d_2! \dots d_m!\\
&\text{(b) } b_{\lambda_r} : (x,y) \mapsto \lambda_r([x,y]) \text{ viewed as a bilinear form on } \mathfrak{l}_r/\mathfrak{z}_r\\
&\text{(c) } S = Z_1 Z_2 \dots Z_m = Z_1 \times \dots \times Z_m \text{ where } Z_r \text{ is the center of } L_r\\
&\text{(d) } P: \text{ polynomial } P(\lambda) = \mathrm{Pf}(b_{\lambda_1})\mathrm{Pf}(b_{\lambda_2})\dots\mathrm{Pf}(b_{\lambda_m}) \text{ on } \mathfrak{s}^*\\
&\text{(e) } \mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid P(\lambda) \neq 0\}\\
&\text{(f) } \pi_\lambda \in \widehat{N} \text{ where } \lambda \in \mathfrak{t}^*: \text{ irreducible unitary representation, as follows.}
\end{aligned} \tag{1.6}$$

Construction 1.2. [20] Given $\lambda \in \mathfrak{t}^*$, in other words $\lambda = \lambda_1 + \dots + \lambda_m$ where $\lambda_r \in \mathfrak{z}_r$ with each $\mathrm{Pf}(b_{\lambda_r}) \neq 0$, we construct $\pi_\lambda \in \widehat{N}$ by recursion on m. If $m = 1$ then π_λ is a square integrable representation of $N = L_1$. Now assume $m > 1$. Then we have the irreducible unitary representation $\pi_{\lambda_1+\dots+\lambda_{m-1}}$ of $L_1 L_2 \dots L_{m-1}$. and (1.3(c)) shows that L_m stabilizes the unitary equivalence class of $\pi_{\lambda_1+\dots+\lambda_{m-1}}$. Since L_m is topologically contractible the Mackey obstruction vanishes and $\pi_{\lambda_1+\dots+\lambda_{m-1}}$ extends to an irreducible unitary representation $\pi^\dagger_{\lambda_1+\dots+\lambda_{m-1}}$ on N on the same Hilbert space. View the square integrable representation π_{λ_m} of L_m as a representation of N whose kernel contains $L_1 L_2 \dots L_{m-1}$. Then we define $\pi_\lambda = \pi^\dagger_{\lambda_1+\dots+\lambda_{m-1}} \widehat{\otimes} \pi_{\lambda_m}$.

Definition 1.3. The representations π_λ of (1.6(f)), constructed just above, are the *stepwise square integrable* representations of N relative to the decomposition (1.3). If N has stepwise square integrable representations relative to (1.3) we will say that N is *stepwise square integrable.*

Remark 1.4. Construction 1.2 of the stepwise square integrable representations π_λ uses (1.3(c)), $[\mathfrak{l}_r, \mathfrak{z}_s] = 0$ for $r > s$, so that L_r stabilizes the unitary equivalence class of $\pi_{\lambda_1+\dots+\lambda_{r-1}}$. The condition (1.4), $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v}$ for $r > s$, enters the picture in proving that the polynomial P of (1.6(d)) is the Pfaffian $\mathrm{Pf} = \mathrm{Pf}_\mathfrak{n}$ of b_λ on $\mathfrak{n}/\mathfrak{s}$. However we don't need that, and the weaker (1.5) is sufficient to show that P is the Plancherel density. See Theorem 1.6 below.

Lemma 1.5. [20] *Assume that N has stepwise square integrable representations. Then Plancherel measure is concentrated on the set $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$ of all stepwise square integrable representations.*

Theorem 1.1 extends to the stepwise square integrable setting, as follows.

Theorem 1.6. *Let N be a connected simply connected nilpotent Lie group that satisfies* (1.3) *and* (1.5). *Then Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ, then the coefficient $f_{u,v}(x) = \langle u, \pi_\nu(x)v\rangle$ satisfies*

$$||f_{u,v}||^2_{L^2(N/S)} = \frac{||u||^2||v||^2}{|P(\lambda)|}. \tag{1.7}$$

The distribution character Θ_{π_λ} of π_λ satisfies

$$\Theta_{\pi_\lambda}(f) = c^{-1}|P(\lambda)|^{-1}\int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi)d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N) \tag{1.8}$$

where $\mathcal{C}(N)$ is the Schwartz space, f_1 is the lift $f_1(\xi) = f(\exp(\xi))$, $\widehat{f_1}$ is its classical Fourier transform, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$, and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to $\mathrm{Ad}^*(N)\lambda$. The Plancherel formula on N is

$$f(x) = c\int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|P(\lambda)|d\lambda \text{ for } f \in \mathcal{C}(N). \tag{1.9}$$

Theorem 1.6 is proved in [20] for groups N that satisfy (1.3) together with (1.4). We will need it for (1.3) together with the somewhat less restrictive (1.5). The only point where the argument needs a slight modification is in the proof of (1.7). The action of L_m on $\mathfrak{l}_1 + \dots + \mathfrak{l}_{m-1}$ is unipotent, so there is an L_m-invariant measure preserving decomposition $N_m/S_m = (L_1/Z_1) \times \dots \times (N_m/Z_m)$. The case $m = 1$ is the property $|f_{u,v}|^2_{L^2(L_1/Z_1)} = \frac{||u||^2||v||^2}{|\mathrm{Pf}(\lambda)|} < \infty$ of coefficients of square integrable representations. By induction on m, $|f_{u,v}|^2_{L^2(N_{m-1}/S_{m-1}))} = \frac{||u||^2||v||^2}{|\mathrm{Pf}(\lambda_1)\dots\mathrm{Pf}(\lambda_{m-1})|}$ for N_{m-1}. Let $\pi^\dagger$ be the extension of $\pi \in \widehat{N_{m-1}}$ to N_m. Let

$u, v \in \mathcal{H}_{\pi_{\lambda_1+\dots\lambda_{m-1}}}$ and write v_y for $\pi^\dagger_{\lambda_1+\dots+\lambda_{m-1}}(y)v$. Let $u', v' \in \mathcal{H}_{\pi_{\lambda_m}}$.

$$
\begin{aligned}
||f_{u\otimes u',v\otimes v'}||^2_{L^2(N/S)} &= \int_{N/S} |\langle u, \pi^\dagger_{\lambda_1+\dots\lambda_{m-1}}(xy)v\rangle|^2 |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 d(xyS_m) \\
&= \int_{L_m/Z_m} |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 \left(\int_{N_{m-1}/S_{m-1}} |\langle u, \pi^\dagger_{\lambda_1+\dots\lambda_{m-1}}(xy)v\rangle|^2 d(xS_{m-1})\right) d(yZ_m) \\
&= \int_{L_m/Z_m} |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 \left(\int_{N_{m-1}/S_{m-1}} |\langle u, \pi^\dagger_{\lambda_1+\dots\lambda_{m-1}}(x)v_y\rangle|^2 d(xS_{m-1})\right) d(yZ_m) \\
&= \int_{L_m/Z_m} |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 \left(\int_{N_{m-1}/S_{m-1}} |\langle u, \pi_{\lambda_1+\dots\lambda_{m-1}}(x)v_y\rangle|^2 d(xS_{m-1})\right) d(yZ_m) \\
&= \tfrac{||u||^2||v_y||^2}{|\mathrm{Pf}(\lambda_1)\dots\mathrm{Pf}(\lambda_{m-1})|} \int_{N_m/Z_m} |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 d(yZ_m) \\
&= \tfrac{||u||^2||v||^2}{|\mathrm{Pf}(\lambda_1)\dots\mathrm{Pf}(\lambda_{m-1})|} \int_{N_m/Z_m} |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 d(yZ_m) = \frac{||u\otimes u'||^2||v\otimes v'||^2}{|\mathrm{Pf}(\lambda_1)\dots\mathrm{Pf}(\lambda_m)|} < \infty.
\end{aligned}
$$

Thus Theorem 1.6 is valid as stated.

The first goal of this note is to show that if N is the nilradical of a parabolic subgroup Q of a real reductive Lie group, then N is stepwise square integrable, specifically that it satisfies (1.3) and (1.5), so that Theorem 1.6 applies to it. That is Theorem 4.5. The second goal is to examine applications to Fourier analysis on the parabolic Q and several important subgroups, such as the maximal split solvable subgroups and the maximal amenable subgroup of Q. The third goal is to extend all these results to direct limit parabolics in a certain class of infinite dimensional real reductive Lie groups.

In Section 2 we recall the restricted root machinery used in [20] to show that nilradicals of minimal parabolics are stepwise square integrable. In Section 3 we make a first approximation to refine that machinery to apply it to general parabolics. That is enough to see that those parabolics satisfy (1.3), and to construct their stepwise square integrable representations. But it not quite enough to compute the Plancherel density. Then in Section 4 we introduce an appropriate modification of the earlier stepwise square integrable machinery. We prove (1.5) in general and use the result to compute the Plancherel density and verify the estimates and inversion formula of Theorem 1.6 for arbitrary parabolic subgroups of real reductive Lie groups. The main result is Theorem 4.5.

In Section 5 we apply Theorem 4.5 to obtain explicit Plancherel and Fourier inversion formulae for the maximal exponential solvable subgroups AN in real parabolic subgroups $Q = MAN$, following the lines of the minimal parabolic case studied in [22]. The key point here is computation of the Dixmier-Pukánszky operator D for the group AN. Recall that D is a pseudo-differential operator that compensates lack of unimodularity in AN.

There are technical obstacles to extending our results to non-minimal parabolics $Q = MAN$, many involving the orbit types for noncompact reductive groups M, but in Section 6 we do carry out the extension to the maximal amenable subgroups $(M \cap K)AN$. This covers all the maximal amenable subgroups of G that satisfy a certain technical condition [8].

That ends Part I: Finite Dimensional Theory. We go on to Part II: Infinite Dimensional Theory.

In Section 7 we discuss infinite dimensional direct limits of nilpotent Lie groups and the setup for studying direct limits of stepwise square integrable representations. Then in Section 8 we introduce the machinery of propagation, which will allow us to deal with nilradicals of direct limit parabolics.

In Section 9 we apply this machinery to an L^2 space for the direct limit nilradicals. This L^2 space is formed using the formal degree inherent in stepwise square integrable representations, and it is not immediate that its elements are functions. But we also introduce a limit Schwartz space, based on matrix coefficients of C^∞ vectors for stepwise square integrable representations. It is a well defined LF (limit of Fréchet) space, sitting naturally in the L^2 space, and we can view that L^2 space as its Hilbert space completion. That is Proposition 9.8. We follow it with a fairly explicit Fourier Inversion Formula, Theorem 9.10.

In Section 10 we work out the corresponding results for the maximal exponential locally solvable subgroup AN of the direct limit parabolic $Q = MAN$. We have to be careful about the Schwartz space and the lack of a Dixmier-Pukánszky operator in the limit, but the results of Section 9 to extend from N to AN. See Proposition 10.4 and Theorem 10.6. In Section 11 we develop similar results for the maximal lim-compact subgroup U of M, carefully avoiding the analytic complications that would result from certain classes of Type II and Type III representations.

In Section 12 we fit the results of Sections 9 and 11 together for an analysis of the L^2 space, the Schwartz space, and the Fourier Inversion formula, for the limit group UN in the parabolic $Q = MAN$. Finally, in Section 13, we combine the results of Sections 10 and 12 for the corresponding results on the maximal amenable subgroup UAN of the limit parabolic Q. See Proposition 13.2 and Theorem 13.4.

2. Specialization to Minimal Parabolics

In order to prove our result for nilradicals of arbitrary parabolics we need to study the construction that gives the decomposition $N = L_1 L_2 \dots L_m$ of 1.3 and the form of the Pfaffian polynomials for the individual the square integrable layers L_r .

Let G be a connected real reductive Lie group, $G = KAN$ an Iwasawa decompsition, and $Q = MAN$ the corresponding minimal parabolic subgroup. Complete $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Now we have root systems

- $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$: roots of $\mathfrak{g}_\mathbb{C}$ relative to $\mathfrak{h}_\mathbb{C}$ (ordinary roots),
- $\Delta(\mathfrak{g}, \mathfrak{a})$: roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ (restricted roots),
- $\Delta_0(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid 2\alpha \notin \Delta(\mathfrak{g}, \mathfrak{a})\}$ (nonmultipliable restricted roots).

The choice of $\mathfrak{n}$ is the same as the choice of a positive restricted root systen

$\Delta^+(\mathfrak{g}, \mathfrak{a})$. Define

$$\begin{aligned} &\beta_1 \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximal positive restricted root and} \\ &\beta_{r+1} \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximum among the roots of } \Delta^+(\mathfrak{g}, \mathfrak{a}) \\ &\qquad\qquad \text{orthogonal to all } \beta_i \text{ with } i \leqq r \end{aligned} \tag{2.1}$$

The resulting roots (we usually say *root* for *restricted root*) β_r, $1 \leqq r \leqq m$, are mutually strongly orthogonal, in particular mutually orthogonal, and each $\beta_r \in \Delta_0(\mathfrak{g}, \mathfrak{a})$. For $1 \leqq r \leqq m$ define

$$\begin{aligned} &\Delta_1^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \beta_1 - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\} \text{ and} \\ &\Delta_{r+1}^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \setminus (\Delta_1^+ \cup \dots \cup \Delta_r^+) \mid \beta_{r+1} - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\}. \end{aligned} \tag{2.2}$$

We know [20, Lemma 6.1] that if $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ then either $\alpha \in \{\beta_1, \dots, \beta_m\}$ or α belongs to exactly one of the sets Δ_r^+.

The layers are are the

$$\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum\nolimits_{\Delta_r^+} \mathfrak{g}_\alpha \text{ for } 1 \leqq r \leqq m \tag{2.3}$$

Denote

$$\begin{aligned} &s_{\beta_r} \text{ is the Weyl group reflection in } \beta_r, \\ &\sigma_r : \Delta(\mathfrak{g}, \mathfrak{a}) \to \Delta(\mathfrak{g}, \mathfrak{a}) \text{ by } \sigma_r(\alpha) = -s_{\beta_r}(\alpha). \end{aligned} \tag{2.4}$$

Then σ_r leaves β_r fixed and preserves Δ_r^+. Further, if $\alpha, \alpha' \in \Delta_r^+$ then $\alpha + \alpha'$ is a (restricted) root if and only if $\alpha' = \sigma_r(\alpha)$, and in that case $\alpha + \alpha' = \beta_r$.

From this it follows [20, Theorem 6.11] that $N = L_1 L_2 \dots L_m$ satisfies (1.3) and (1.4), so it has stepwise square integrable representations. Further [20, Lemma 6.4] the L_r are Heisenberg groups in the sense that if $\lambda_r \in \mathfrak{z}_r^*$ with $\mathrm{Pf}_{\mathfrak{l}_r}(\lambda_r) \neq 0$ then $\mathfrak{l}_r / \ker \lambda_r$ is an ordinary Heisenberg group of dimension $\dim \mathfrak{v}_r + 1$.

3. Intersection with an Arbitrary Real Parabolic

Every parabolic subgroup of G is conjugate to a parabolic that contains the minimal parabolic $Q = MAN$. Let Ψ denote the set of simple roots for the positive system $\Delta^+(\mathfrak{g}, \mathfrak{a})$. Then the parabolic subgroups of G that contain Q are in one to one correspondence with the subsets $\Phi \subset \Psi$, say $Q_\Phi \leftrightarrow \Phi$, as follows. Denote $\Psi = \{\psi_i\}$ and set

$$\begin{aligned} \Phi^{red} &= \left\{\alpha = \sum\nolimits_{\psi_i \in \Psi} n_i \psi_i \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid n_i = 0 \text{ whenever } \psi_i \notin \Phi\right\} \\ \Phi^{nil} &= \left\{\alpha = \sum\nolimits_{\psi_i \in \Psi} n_i \psi_i \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid n_i > 0 \text{ for some } \psi_i \notin \Phi\right\}. \end{aligned} \tag{3.1}$$

Then, on the Lie algebra level, $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi + \mathfrak{a}_\Phi + \mathfrak{n}_\Phi$ where

$$\begin{aligned} &\mathfrak{a}_\Phi = \{\xi \in \mathfrak{a} \mid \psi(\xi) = 0 \text{ for all } \psi \in \Phi\} = \Phi^\perp, \\ &\mathfrak{m}_\Phi + \mathfrak{a}_\Phi \text{ is the centralizer of } \mathfrak{a}_\Phi \text{ in } \mathfrak{g}, \text{ so } \mathfrak{m}_\Phi \text{ has root system } \Phi^{red}, \text{ and} \\ &\mathfrak{n}_\Phi = \sum\nolimits_{\alpha \in \Phi^{nil}} \mathfrak{g}_\alpha, \text{ nilradical of } \mathfrak{q}_\Phi, \text{ sum of the positive } \mathfrak{a}_\Phi\text{-root spaces.} \end{aligned} \tag{3.2}$$

Since $\mathfrak{n} = \sum_r \mathfrak{l}_r$, as given in (2.3) we have

$$\mathfrak{n}_\Phi = \sum\nolimits_r (\mathfrak{n}_\Phi \cap \mathfrak{l}_r) = \sum\nolimits_r \Big((\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi) + \sum\nolimits_{\Delta_r^+} (\mathfrak{g}_\alpha \cap \mathfrak{n}_\Phi) \Big) . \qquad (3.3)$$

As $\mathrm{ad}\,(\mathfrak{m})$ is irreducible on each restricted root space, if $\alpha \in \{\beta_r\} \cup \Delta_r^+$ then $\mathfrak{g}_\alpha \cap \mathfrak{n}_\Phi$ is 0 or all of $\mathfrak{g}_\alpha$.

Lemma 3.1. *Suppose $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi = 0$. Then $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = 0$.*

Proof. Since $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi = 0$, the root β_r has form $\sum_{\psi \in \Phi} n_\psi \psi$ with each $n_\psi \geqq 0$ and $n_\psi = 0$ for $\psi \notin \Phi$. If $\alpha \in \Delta_r^+$ it has form $\sum_{\psi \in \Psi} \ell_\psi \psi$ with $0 \leqq \ell_\psi \leqq n_\psi$ for each $\psi \in \Psi$. In particular $\ell_\psi = 0$ for $\psi \notin \Phi$. Now every root space of $\mathfrak{l}_r$ is contained in $\mathfrak{m}_\Psi$. In particular $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = 0$. ■

Remark 3.2. We can define a partial order on $\{\beta_i\}$ by: $\beta_{i+1} \succ \beta_i$ when the set of positive roots of which β_{i+1} is a maximum is contained in the corresponding set for β_i. This is only a consideration when one further disconnects the Dynkin diagram by deleting a node at which $-\beta_i$ attaches, which doesn't happen for type A. If $\beta_s \succ \beta_r$ in this partial order, and $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi = 0$, then $\mathfrak{g}_{\beta_s} \cap \mathfrak{n}_\Phi = 0$ as well, so $\mathfrak{l}_s \cap \mathfrak{n}_\Phi = 0$.

Lemma 3.3. *Suppose $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi \neq 0$. Define $J_r \subset \Delta_r^+$ by $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \mathfrak{g}_{\beta_r} + \sum_{J_r} \mathfrak{g}_\alpha$. Decompose $J_r = J_r' \cup J_r''$ (disjoint) where $J_r' = \{\alpha \in J_r \mid \sigma_r \alpha \in J_r\}$ and $J_r'' = \{\alpha \in J_r \mid \sigma_r \alpha \notin J_r\}$. Then $\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha$ belongs to a single $\mathfrak{a}_\Phi$-root space in $\mathfrak{n}_\Phi$, i.e. $\alpha|_{\mathfrak{a}_\Phi} = \beta_r|_{\mathfrak{a}_\Phi}$, for every $\alpha \in J_r''$.*

Proof. Two restricted roots $\alpha = \sum_\Psi n_i \psi_i$ and $\alpha' = \sum_\Psi \ell_i \psi_i$ have the same restriction to $\mathfrak{a}_\Phi$ if and only if $n_i = \ell_i$ for all $\psi_i \notin \Phi$. Now suppose $\alpha \in J_r''$ and $\alpha' = \sigma_r \alpha$. Then $n_i > 0$ for some $\psi_i \notin \Phi$ but $\ell_i = 0$ for all $\psi_i \notin \Phi$. Thus α and $\beta_r = \alpha + \sigma_r \alpha$ have the same ψ_i-coefficient $n_i = n_i + \ell_i$ for every $\psi_i \notin \Phi$. In other words the corresponding restricted root spaces are contained in the same $\mathfrak{a}_\Phi$-root space. ■

Lemma 3.4. *Suppose $\mathfrak{l}_r \cap \mathfrak{n}_\Phi \neq 0$. Then the algebra $\mathfrak{l}_r \cap \mathfrak{n}_\Phi$ has center $\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha$, and $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = (\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha) + (\sum_{J_r'} \mathfrak{g}_\alpha))$. Further, $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \left(\sum_{J_r''} \mathfrak{g}_\alpha\right) \oplus \left(\mathfrak{g}_{\beta_r} + \left(\sum_{J_r'} \mathfrak{g}_\alpha\right)\right)$ direct sum of ideals.*

Proof. This is immediate from the statements and proofs of Lemmas 3.1 and 3.3. ■

Following the cascade construction (2.1) it will be convenient to define sets of simple restricted roots

$$\Psi_1 = \Psi \text{ and } \Psi_{s+1} = \{\psi \in \Psi \mid \langle \psi, \beta_i \rangle = 0 \text{ for } 1 \leqq i \leqq s\}. \qquad (3.4)$$

Note that Ψ_r is the simple root system for $\{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \alpha \perp \beta_i \text{ for } i < r\}$.

Lemma 3.5. *If $r > s$ then $[\mathfrak{l}_r \cap \mathfrak{n}_\Phi\, ,\, \mathfrak{g}_{\beta_s} + \sum_{J_s''} \mathfrak{g}_\alpha] = 0$.*

Proof. Suppose that $\alpha \in J''_s$. Express α and $\sigma_s\alpha$ as sums of simple roots, say $\alpha = \sum n_i\psi_i$ and $\sigma_s\alpha = \sum \ell_i\psi_i$. Then, $\ell_i = 0$ for all $\psi_i \in \Psi_s \cap \Phi^{nil}$ and $\beta_s = \sum(n_i + \ell_i)\psi_i$. In other words the coefficient of ψ_i is the same for α and β_s whenever $\psi_i \in \Psi_s \cap \Phi^{nil}$. Now let $\gamma \in (\{\beta_r\} \cup \Delta_r^+) \cap \Phi^{nil}$ where $r > s$, and express $\gamma = \sum c_i\psi_i$. Then $c_{i_0} > 0$ for some $\beta_{i_0} \in (\Psi_r \cap \Phi^{nil})$. Note $\Psi_r \subset \Psi_s$, so $c_{i_0} > 0$ for some $\beta_{i_0} \in (\Psi_s \cap \Phi^{nil})$. Also, $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}_s$ because $r > s$. If $\gamma + \alpha$ is a root then its ψ_{i_0}-coefficient is greater than that of β_s, which is impossible. Thus $\gamma + \alpha$ is not a root. The lemma follows. ∎

We look at a particular sort of linear functional on $\sum_r \bigl(\mathfrak{g}_{\beta_s} + \sum_{J''_s}\mathfrak{g}_\alpha\bigr)$. Choose $\lambda_r \in \mathfrak{g}^*_{\beta_r}$ such that b_{λ_r} is nondegenerate on $\sum_r \sum_{J'_r} \mathfrak{g}_\alpha$. Set $\lambda = \sum \lambda_r$. We know that (1.3(c)) holds for the nilradical of the minimal parabolic $\mathfrak{q}$ that contains $\mathfrak{q}_\Phi$. By Lemma 3.5 it follows that $b_\lambda(\mathfrak{l}_r, \mathfrak{l}_s) = \lambda([\mathfrak{l}_r, \mathfrak{l}_s] = 0$ for $r > s$. For this particular type of λ, the bilinear form b_λ has kernel $\sum_r \bigl(\mathfrak{g}_{\beta_s} + \sum_{J''_s}\mathfrak{g}_\alpha\bigr)$ and is nondegenerate on $\sum_r \sum_{J'_r} \mathfrak{g}_\alpha$.

At this point, the decomposition $N_\Phi = (L_1 \cap N_\Phi)(L_2 \cap N_\Phi)\dots(L_m \cap N_\Phi)$ satisfies the first two conditions of (1.3):

(a) each factor $L_r \cap N_\Phi$ has unitary representations with coefficients
in $L^2((L_r \cap N_\Phi)/(center))$, and

(b) each $N_r \cap N_\Phi := (L_1 \cap N_\Phi)\dots(L_r \cap N_\Phi)$ is a normal subgroup of N_Φ
with $N_r \cap N_\Phi = (N_{r-1} \cap N_\Phi) \rtimes (L_r \cap N_\Phi)$ semidirect.

With Lemma 3.5 this is enough to carry out Construction 1.2 of our representations π_λ of N_Φ. However it is not enough for (1.3(c)) and (1.5). For that we will group the $L_r \cap N_\Phi$ in such a way that (1.5) is immediate and (1.3(c)) follows from Lemma 3.5. This will be done in the next section.

4. Extension to Arbitrary Parabolic Nilradicals

In this section we address (1.3(c)) and (1.5), completing the proof that N_Φ has a decomposition that leads to stepwise square integrable representations.

We start with some combinatorics. Denote sets of indices as follows. q_1 is the first index of (1.3) (usually 1) such that $\beta_{q_1}|_{\mathfrak{a}_\Phi} \neq 0$; define

$$I_1 = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_1}|_{\mathfrak{a}_\Phi}\}.$$

Then q_2 is the first index of (1.3) such that $q_2 \notin I_1$ and $\beta_{q_2}|_{\mathfrak{a}_\Phi} \neq 0$; define

$$I_2 = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_2}|_{\mathfrak{a}_\Phi}\}.$$

Continuing, q_k is the first index of (1.3) such that $q_k \notin (I_1 \cup \dots \cup I_{k-1})$ and $\beta_{q_k}|_{\mathfrak{a}_\Phi} \neq 0$; define

$$I_k = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_k}|_{\mathfrak{a}_\Phi}\}$$

as long as possible. Write ℓ for the last index k that leads to a nonempty set I_k. Then, in terms of the index set of (1.3), $I_1 \cup \dots \cup I_\ell$ consists of all the indices i for which $\beta_i|_{\mathfrak{a}_\Phi} \neq 0$.

For $1 \leqq j \leqq \ell$ define

$$\mathfrak{l}_{\Phi,j} = \sum\nolimits_{i \in I_j} (\mathfrak{l}_i \cap \mathfrak{n}_\Phi) = \Bigl(\sum\nolimits_{i \in I_j} \mathfrak{l}_i\Bigr) \cap \mathfrak{n}_\Phi \text{ and } \mathfrak{l}^\dagger_{\Phi,j} = \sum\nolimits_{k \geqq j} \mathfrak{l}_{\Phi,k}\,. \tag{4.1}$$

Lemma 4.1. *If $k \geqq j$ then $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{l}_{\Phi,j}$. For each index j, $\mathfrak{l}_{\Phi,j}$ and $\mathfrak{l}^\dagger_{\Phi,j}$ are subalgebras of $\mathfrak{n}_\Phi$ and $\mathfrak{l}_{\Phi,j}$ is an ideal in $\mathfrak{l}^\dagger_{\Phi,j}$.*

Proof. As we run along the sequence $\{\beta_1, \beta_2, \dots\}$ the coefficients of the simple roots are weakly decreasing, so in particular the coefficients of the roots in $\Psi \setminus \Phi$ are weakly decreasing. If $r \in I_k$, $s \in I_j$ and $k > j$ now $r > s$. Using $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}_s$ (and thus $[(\mathfrak{l}_r \cap \mathfrak{n}_\Phi), (\mathfrak{l}_s \cap \mathfrak{n}_\Phi)] \subset \mathfrak{l}_s \cap \mathfrak{n}_\Phi$) for $r > s$ it follows that $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{l}_{\Phi,j}$ for $k > j$.

Now suppose $k = j$. If $r = s$ then $[\mathfrak{l}_r, \mathfrak{l}_r] = \mathfrak{g}_{\beta_r}$, so we may assume $r > s$, and thus $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}_s \subset \mathfrak{l}_{\Phi,j}$. It follows that $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{l}_{\Phi,j}$ for $k = j$.

Now it is immediate that $\mathfrak{l}_{\Phi,j}$ and $\mathfrak{l}^\dagger_{\Phi,j}$ are subalgebras of $\mathfrak{n}_\Phi$ and $\mathfrak{l}_{\Phi,j}$ is an ideal in $\mathfrak{l}^\dagger_{\Phi,j}$. ■

Lemma 4.2. *If $k > j$ then $[\mathfrak{l}_{\Phi,k}\,, \mathfrak{l}_{\Phi,j}] \cap \sum_{i \in I_j} \mathfrak{g}_{\beta_i} = 0$.*

Proof. This is implicit in Theorem 1.6, which gives (1.5), but we give a direct proof for the convenience of the reader. Let $\mathfrak{g}_\gamma \subset \mathfrak{l}_{\Phi,k}$ and $\mathfrak{g}_\alpha \subset \mathfrak{l}_j$ with $[\mathfrak{g}_\gamma, \mathfrak{g}_\alpha] \cap \sum_{i \in I_j} \mathfrak{g}_{\beta_i} \neq 0$. Then $[\mathfrak{g}_\gamma, \mathfrak{g}_\alpha] = \mathfrak{g}_{\beta_i}$ where $\mathfrak{g}_\gamma \subset \mathfrak{l}_r$ and $\mathfrak{g}_\alpha \subset \mathfrak{l}_i$, so $\mathfrak{g}_\gamma = \mathfrak{g}_{\beta_i - \alpha} \subset \mathfrak{l}_r \cap \mathfrak{l}_i = 0$. That contradiction proves the lemma. ■

Given $r \in I_j$ we use the notation of Lemma 3.3 to decompose

$$\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \mathfrak{l}'_r + \mathfrak{l}''_r \text{ where } \mathfrak{l}'_r = \mathfrak{g}_{\beta_r} + \sum\nolimits_{J'_r} \mathfrak{g}_\alpha \text{ and } \mathfrak{l}''_r = \sum\nolimits_{J''_r} \mathfrak{g}_\alpha\,. \tag{4.2}$$

Here J'_r consists of roots $\alpha \in \Delta^+_r$ such that $\mathfrak{g}_\alpha + \mathfrak{g}_{\beta_r - \alpha} \subset \mathfrak{n}_\Phi$, and J''_r consists of roots $\alpha \in \Delta^+_r$ such that $\mathfrak{g}_\alpha \subset \mathfrak{n}_\Phi$ but $\mathfrak{g}_{\beta_r - \alpha} \not\subset \mathfrak{n}_\Phi$. For $1 \leqq j \leqq \ell$ define

$$\mathfrak{z}_{\Phi,j} = \sum\nolimits_{i \in I_j} (\mathfrak{g}_{\beta_i} + \mathfrak{l}''_i) \tag{4.3}$$

and decompose

$$\mathfrak{l}_{\Phi,j} = \mathfrak{l}'_{\Phi,j} + \mathfrak{l}''_{\Phi,j} \text{ where } \mathfrak{l}'_{\Phi,j} = \sum\nolimits_{i \in I_j} \mathfrak{l}'_i \text{ and } \mathfrak{l}''_{\Phi,j} = \sum\nolimits_{i \in I_j} \mathfrak{l}''_i\,. \tag{4.4}$$

Lemma 4.3. *Recall $\mathfrak{l}^\dagger_{\Phi,j} = \sum_{k \geqq j} \mathfrak{l}_{\Phi,k}$ from (4.1). For each j, both $\mathfrak{z}_{\Phi,j}$ and $\mathfrak{l}''_{\Phi,j}$ are central ideals in $\mathfrak{l}^\dagger_{\Phi,j}$, and $\mathfrak{z}_{\Phi,j}$ is the center of $\mathfrak{l}_{\Phi,j}$.*

Proof. Lemma 3.3 shows that $\alpha|_{\mathfrak{a}_\Phi} = \beta_i|_{\mathfrak{a}_\Phi}$ whenever $i \in I_j$ and $\mathfrak{g}_\alpha \subset \mathfrak{l}''_{\Phi,j}$. If $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}''_i] \neq 0$ it contains some $\mathfrak{g}_\delta$ such that $\mathfrak{g}_\delta \subset \mathfrak{l}_{\Phi,j}$ and at least one of the coefficients of δ along roots of $\Psi \setminus \Phi$ is greater than that of β_i. As $\mathfrak{g}_\delta \subset \mathfrak{l}_i$ that is impossible. Thus $\mathfrak{l}''_{\Phi,j}$ is a central ideal in $\mathfrak{l}^\dagger_{\Phi,j}$. The same is immediate for $\mathfrak{z}_{\Phi,j} = \sum_{i \in I_j} (\mathfrak{g}_{\beta_i} + \mathfrak{l}''_i)$. In particular $\mathfrak{z}_{\Phi,j}$ is central in $\mathfrak{l}_{\Phi,j}$. But the center of $\mathfrak{l}_{\Phi,j}$ can't be any larger, by definition of $\mathfrak{l}'_{\Phi,j}$. ■

Decompose

$$\mathfrak{n}_\Phi = \mathfrak{z}_\Phi + \mathfrak{v}_\Phi \text{ where } \mathfrak{z}_\Phi = \sum_j \mathfrak{z}_{\Phi,j}\ ,\ \mathfrak{v}_\Phi = \sum_j \mathfrak{v}_{\Phi,j} \text{ and } \mathfrak{v}_{\Phi,j} = \sum_{i\in I_j}\sum_{\alpha\in J'_i} \mathfrak{g}_\alpha\,. \tag{4.5}$$

Then Lemma 4.3 gives us (1.5) for the $\mathfrak{l}_{\Phi,j}$: $\mathfrak{l}_{\Phi,j} = \mathfrak{l}'_{\Phi,j} \oplus \mathfrak{l}''_{\Phi,j}$ with $\mathfrak{l}''_{\Phi,j} \subset \mathfrak{z}_{\Phi,j}$ and $\mathfrak{v}_{\Phi,j} \subset \mathfrak{l}'_{\Phi,j}$.

Lemma 4.4. *For generic $\lambda_j \in \mathfrak{z}^*_{\Phi,j}$ the kernel of b_{λ_j} on $\mathfrak{l}_{\Phi,j}$ is just $\mathfrak{z}_{\Phi,j}$, in other words b_{λ_j} is is nondegenerate on $\mathfrak{v}_{\Phi,j} \simeq \mathfrak{l}_{\Phi,j}/\mathfrak{z}_{\Phi,j}$. In particular $L_{\Phi,j}$ has square integrable representations.*

Proof. From the definition of $\mathfrak{l}'_{\Phi,j}$, the bilinear form b_{λ_j} on $\mathfrak{l}_{\Phi,j}$ annihilates the center $\mathfrak{z}_{\Phi,j}$ and is nondegenerate on $\mathfrak{v}_{\Phi,j}$. Thus the corresponding representation π_{λ_j} of $L_{\Phi,j}$ has coefficients that are square integrable modulo its center. ■

Now we come to our first main result:

Theorem 4.5. *Let G be a real reductive Lie group and Q a real parabolic subgroup. Express $Q = Q_\Phi$ in the notation of* (3.1) *and* (3.2)*. Then its nilradical N_Φ has decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ that satisfies the conditions of* (1.3) *and* (1.5) *as follows. The center $Z_{\Phi,j}$ of $L_{\Phi,j}$ is the analytic subgroup for $\mathfrak{z}_{\Phi,j}$ and*

$$\begin{aligned} &\text{(a) each } L_{\Phi,j} \text{ has unitary reps with coefficients in } L^2(L_{\Phi,j}/Z_{\Phi,j}),\\ &\text{(b) each } N_{\Phi,j} := L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,j} \text{ is a normal subgroup of } N_\Phi\\ &\qquad\qquad \text{with } N_{\Phi,j} = N_{\Phi,j-1} \rtimes L_{\Phi,j} \text{ semidirect, and}\\ &\text{(c) } [\mathfrak{l}_{\Phi,k},\mathfrak{z}_{\Phi,j}] = 0 \text{ and } [\mathfrak{l}_{\Phi,k},\mathfrak{l}_{\Phi,j}] \subset \mathfrak{v}_{\Phi,j} + \mathfrak{l}''_{\Phi,j} \text{ for } k > j. \end{aligned} \tag{4.6}$$

In particular N_Φ has stepwise square integrable representations relative to the decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$, and the results of Theorem 1.6, *specifically* (1.7), (1.8) *and* (1.9), *hold for N_Φ.*

Proof. Statement (a) is the content of Lemma 4.4, and statement (b) follows from Lemma 4.1. The first part of (c), $[\mathfrak{l}_{\Phi,k},\mathfrak{z}_{\Phi,j}] = 0$ for $k > j$, is contained in Lemma 4.3. The second part, $[\mathfrak{l}_{\Phi,k},\mathfrak{l}_{\Phi,j}] \subset \mathfrak{v}_\Phi + \mathfrak{l}''_{\Phi,j}$ for $k > j$, follows from Lemma 4.2. Now Theorem 1.6 applies. ■

5. The Maximal Exponential-Solvable Subgroup $A_\Phi N_\Phi$

In this section we extend the considerations of [22, §4] from minimal parabolics to the exponential-solvable subgroups $A_\Phi N_\Phi$ of real parabolics $Q_\Phi = M_\Phi A_\Phi N_\Phi$. It turns out that the of Plancherel and Fourier inversion formulae of N_Φ go through, with only small changes, to the non-unimodular solvable group $A_\Phi N_\Phi$. We follow the development in [22, §4].

Let H be a separable locally compact group of type I. Then [6, §1] the Fourier inversion formula for H has form

$$f(x) = \int_{\widehat{H}} \operatorname{trace} \pi(D(r_x f))\, d\mu_H(\pi) \tag{5.1}$$

where D is an invertible positive self adjoint operator on $L^2(H)$, conjugation semi-invariant of weight equal to that of the modular function δ_H, $r_x f$ is the right translate $y \mapsto f(yx)$, and μ is a positive Borel measure on the unitary dual $\widehat{H}$. When H is unimodular, D is the identity and (5.1) reduces to the usual Fourier inversion formula for H. In general the semi-invariance of D compensates any lack of unimodularity. See [6, §1] for a detailed discussion including a discussion of the domains of D and $D^{1/2}$. Here $D \otimes \mu$ is unique up to normalization of Haar measure, but (D, μ) is not unique, except of course when we fix one of them, such as in the unimodular case when we take $D = 1$. Given such a pair (D, μ) we refer to D as a *Dixmier-Pukánszky operator* and to μ as the associated Plancherel measure.

One goal of this section is to describe a "best" choice of the Dixmier-Pukánszky operator for $A_\Phi N_\Phi$ in terms of the decomposition $N_\Phi = L_{\Phi,1} L_{\Phi,2} \dots L_{\Phi,\ell}$ that gives stepwise square integrable representations of N_Φ.

Let $\delta = \delta_{Q_\Phi}$ denote the modular function of Q_Φ. Its kernel contains $M_\Phi N_\Phi$ because $\mathrm{Ad}(M_\Phi)$ is reductive with compact center and $\mathrm{Ad}(N_\Phi)$ is unipotent. Thus $\delta(man) = \delta(a)$, and if $\xi \in \mathfrak{a}_\Phi$ then $\delta(\exp(\xi)) = \exp(\mathrm{trace}\,(\mathrm{ad}\,(\xi)))$. Note that δ also is the modular function for $A_\Phi N_\Phi$.

Lemma 5.1. *Let $\xi \in \mathfrak{a}_\Phi$. Then each $\dim \mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j}$ is even, and*

(i) *the trace of* $\mathrm{ad}\,(\xi)$ *on* $\mathfrak{l}_{\Phi,j}$ *is* $\frac{1}{2}\dim(\mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j})\beta_{j_0}(\xi)$ *for any* $j_0 \in I_j$,

(ii) *the trace of* $\mathrm{ad}\,(\xi)$ *on* $\mathfrak{n}_\Phi$, *on* $\mathfrak{a}_\Phi + \mathfrak{n}_\Phi$ *and on* $\mathfrak{q}_\Phi$ *is* $\frac{1}{2}\sum_j(\dim \mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j})\beta_{j_0}(\xi)$,

(iii) *the determinant of* $\mathrm{Ad}(\exp(\xi))$ *on* $\mathfrak{n}_\Phi$, *on* $\mathfrak{a}_\Phi + \mathfrak{n}_\Phi$,
and on $\mathfrak{q}_\Phi$, *is* $\prod_j \exp(\beta_{j_0}(\xi))^{\frac{1}{2}(\dim \mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j})}$.

Proof. We use the notation of (4.2), (4.3) and (4.4). It is immediate that $\dim \mathfrak{l}_r + \dim(\mathfrak{g}_{\beta_r} + \mathfrak{l}''_r)$ is even. Sum over $r \in I_j$ to see that $\dim \mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j}$ is even.

The trace of $\mathrm{ad}\,(\xi)$ on $\mathfrak{l}_r \cap \mathfrak{n}_\Phi$ is $(\dim \mathfrak{g}_{\beta_r})\beta_r(\xi)$ on $\mathfrak{g}_{\beta_r}$, and we add $\frac{1}{2}\sum_{\alpha \in J'_r}(\dim \mathfrak{g}_\alpha)\beta_r(\xi)$ for the pairs $\mathfrak{g}_\alpha, \mathfrak{g}'_\alpha \in \Delta_r^+ \cap \Phi^{nil}$ that pair into $\mathfrak{g}_{\beta_r}$, plus $\sum_{\alpha \in J''_r}(\dim \mathfrak{g}_\alpha)\beta_r(\xi)$ since $\alpha \in J''_r$ implies $\alpha|_{\mathfrak{a}_\Phi} = \beta_r|_{\mathfrak{a}_\Phi}$. Now the trace of $\mathrm{ad}\,(\xi)$ on $\mathfrak{l}_r \cap \mathfrak{n}_\Phi$ is

$$(\tfrac{1}{2}\dim \mathfrak{g}_{\beta_r} + \tfrac{1}{2}\dim \mathfrak{l}'_r + \dim \mathfrak{l}''_r)\beta_r(\xi) = \tfrac{1}{2}\dim(\mathfrak{l}_r \cap \mathfrak{n}_\Phi) + \dim(\mathfrak{g}_{\beta_r} + \mathfrak{l}''_r)\beta_r(\xi).$$

Summing over $r \in I_j$ we arrive at assertion (i). Then sum over j for (ii) and exponentiate for (iii). ∎

We reformulate Lemma 5.1 as

Lemma 5.2. *The modular function* $\delta = \delta_{Q_\Phi}$ *of* $Q_\Phi = M_\Phi A_\Phi N_\Phi$ *is*

$$\delta(man) = \prod\nolimits_j \exp(\beta_{j_0}(\log a))^{\frac{1}{2}(\dim \mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j})}.$$

The modular function $\delta_{A_\Phi N_\Phi} = \delta|_{A_\Phi N_\Phi}$ *, and* $\delta_{U_\Phi A_\Phi N_\Phi} = \delta|_{U_\Phi A_\Phi N_\Phi}$ *where* U_Φ *is a maximal compact subgroup of* M_Φ *.*

Consider semi-invariance of the polynomial P of (1.6(d)), which by definition is the product of factors $\mathrm{Pf}_{\mathfrak{l}_{\Phi,j}}$. Using (4.5) and Lemma 4.4, calculate with bases of the $\mathfrak{v}_{\Phi,j}$ as in [22, Lemma 4.4] to arrive at

Lemma 5.3. *Let* $\xi \in \mathfrak{a}_\Phi$ *and* $a = \exp(\xi) \in A_\Phi$ *. Then*

$$\mathrm{ad}\,(\xi)P = \left(\tfrac{1}{2}\sum\nolimits_j \dim(\mathfrak{l}_{\Phi,j}/\mathfrak{z}_{\Phi,j})\beta_{j_0}(\xi)\right) P$$

and

$$\mathrm{Ad}(a)P = \left(\prod\nolimits_j (\exp(\beta_{j_0}(\xi)))^{\frac{1}{2}\sum_j \dim(\mathfrak{l}_{\Phi,j}/\mathfrak{z}_{\Phi,j})}\right) P.$$

Definition 5.4. The *quasi-center* of $\mathfrak{n}_\Phi$ is $\mathfrak{s}_\Phi = \sum_j \mathfrak{z}_{\Phi,j}$. Fix a basis $\{e_t\}$ of $\mathfrak{s}_\Phi$ consisting of ordinary root vectors, $e_t \in \mathfrak{g}_{\alpha_t}$. The *quasi-center determinant* relative to the choice of $\{e_t\}$ is the function $\mathrm{Det}_{\mathfrak{s}_\Phi}(\lambda) = \prod_t \lambda(e_t)$ on $\mathfrak{s}_\Phi^*$.

Let $a \in A_\Phi$ and compute

$$(\mathrm{Ad}(a)\mathrm{Det}_{\mathfrak{s}_\Phi})(\lambda) = \mathrm{Det}_{\mathfrak{s}_\Phi}(\mathrm{Ad}^*(a)^{-1}\lambda) = \prod\nolimits_t \lambda(\mathrm{Ad}(a)e_t).$$

Each $e_t \in \mathfrak{z}_{\Phi,j}$ is multiplied by $\exp(\beta_{j_0}(\log a))$. So

$$(\mathrm{Ad}(a)\mathrm{Det}_{\mathfrak{s}_\Phi})(\lambda) = \left(\prod\nolimits_j \exp(\beta_{j_0}(\log a))^{\dim \mathfrak{z}_{\Phi,j}}\right)\mathrm{Det}_{\mathfrak{s}_\Phi}(\lambda).$$

Now

Lemma 5.5. *If* $\xi \in \mathfrak{a}_\Phi$ *then* $\mathrm{Ad}(\exp(\xi))\mathrm{Det}_{\mathfrak{s}_\Phi} = \left(\prod_j \exp(\beta_{j_0}(\xi))^{\dim \mathfrak{z}_{\Phi,j}}\right)\mathrm{Det}_{\mathfrak{s}_\Phi}$ *where* $j_0 \in I_j$ *.*

Combining Lemmas 5.1, 5.2 and 5.5 we have

Proposition 5.6. *The product* $P \cdot \mathrm{Det}_{\mathfrak{s}_\Phi}$ *is an* $\mathrm{Ad}(Q_\Phi)$*-semi-invariant polynomial on* $\mathfrak{s}_\Phi^*$ *of degree* $\frac{1}{2}(\dim \mathfrak{n}_\Phi + \dim \mathfrak{s}_\Phi)$ *and of weight equal to the weight of the modular function* δ_{Q_Φ} *.*

Denote $V_\Phi = \exp(\mathfrak{v}_\Phi)$ and $S_\Phi = \exp(\mathfrak{s}_\Phi)$. Then $V_\Phi \times S_\Phi \to N_\Phi$, by $(v, s) \mapsto vs$, is an analytic diffeomorphism. Define

$$D_0 : \text{ Fourier transform of } P \cdot \mathrm{Det}_{\mathfrak{s}_\Phi} \text{ acting on } A_\Phi N_\Phi = A_\Phi V_\Phi S_\Phi \text{ by acting on the } S_\Phi \text{ variable.} \tag{5.2}$$

Theorem 5.7. *The operator D_0 of* (5.2) *is an invertible self-adjoint differential operator of degree $\frac{1}{2}(\dim \mathfrak{n}_\Phi + \dim \mathfrak{s}_\Phi)$ on $L^2(A_\Phi N_\Phi)$ with dense domain the Schwartz space $\mathcal{C}(A_\Phi N_\Phi)$, and $D := D_0^{1,2}(D_0^{1/2})^*$ is a well defined invertible positive self-adjoint operator of the same degree $\frac{1}{2}(\dim \mathfrak{n}_\Phi + \dim \mathfrak{s}_\Phi)$ on $L^2(A_\Phi N_\Phi)$ with dense domain $\mathcal{C}(A_\Phi N_\Phi)$. In particular D is a Dixmier-Pukánszky operator on $A_\Phi N_\Phi$ with domain equal to the space of rapidly decreasing C^∞ functions.*

Proof. Since it is the Fourier transform of a real polynomial, D_0 is a differential operator that is self-adjoint on $L^2(A_\Phi N_\Phi)$ with dense domain $\mathcal{C}(A_\Phi N_\Phi)$. Thus D is well defined, and is positive and self-adjoint as asserted. Now it remains only to see that D (and thus D_0) are invertible.

Invertibility of D comes out of Dixmier's theory of quasi-Hilbert algebras [2] as applied by Kleppner and Lipsman to group extensions. Specifically, [5, §6] leads to a Dixmier-Pukánszky operator, there called M. The quasi-Hilbert algebra in question is defined on [5, pp. 481–482], the relevant transformations M and Υ are specified in [12, Theorem 1], and invertibility of M is shown in [2, pp. 293–294]. Unwinding the definitions of M and Υ in [5, §6] one sees that the Dixmier-Pukánszky operator M of [5] is the same as our operator D. That completes the proof. ∎

The action of $\mathfrak{a}_\Phi$ on $\mathfrak{z}_{\Phi,j}$ is scalar, $\mathrm{ad}\,(\alpha)\zeta = \beta_{j_0}(\alpha)\zeta$ where (as before) $j_0 \in I_j$. So the isotropy algebra $(\mathfrak{a}_\Phi)_\lambda$ is the same at every $\lambda \in \mathfrak{t}_\Phi^*$, given by $(\mathfrak{a}_\Phi)_\lambda = \{\alpha \in \mathfrak{a}_\Phi \mid \text{ every } \beta_{j_0}(\alpha) = 0\}$. Thus the (A_Φ)-stabilizer on $\mathfrak{t}_\Phi^*$ is

$$A'_\Phi := \{\exp(\alpha) \mid \text{ every } \beta_{j_0}(\alpha) = 0\}, \text{ independent of choice of } \lambda \in \mathfrak{t}_\Phi^*. \tag{5.3}$$

Given $\lambda \in \mathfrak{t}_\Phi^*$, in other words given a stepwise square integrable representation π_λ where $\lambda \in \mathfrak{s}_\Phi^*$, we write $\pi_\lambda^\dagger$ for the extension of π_λ to a representation of $A'_\Phi N_\Phi$ on the same Hilbert space. That extension exists because A'_Φ is a vector group, thus contractible to a point, so $H^2(A'_\Phi;\mathbb{C}') = H^2(point;\mathbb{C}') = \{1\}$, and the Mackey obstruction vanishes. Now the representations of $A_\Phi N_\Phi$ corresponding to π_λ are the

$$\pi_{\lambda,\xi} := \mathrm{Ind}\,_{A'_\Phi N_\Phi}^{A_\Phi N_\Phi}(\exp(i\xi) \otimes \pi_\lambda^\dagger) \tag{5.4}$$

where $\xi \in (\mathfrak{a}'_\Phi)^*$ and $\exp(i\xi) : \exp(\alpha) := \exp(i\xi(\alpha))$ for $\alpha \in \mathfrak{a}'_\Phi$. Note also that

$$\pi_{\lambda,\xi} \cdot \mathrm{Ad}(an) = \pi_{\mathrm{Ad}^*(a)\lambda,\xi} \text{ for } a \in A_\Phi \text{ and } n \in N_\Phi. \tag{5.5}$$

The resulting Plancherel formula (5.1), $f(x) = \int_{\widehat{H}} \mathrm{trace}\,\pi(D(r_x f))d\mu_H(\pi)$, where $H = A_\Phi N_\Phi$, is

Theorem 5.8. *Let $Q_\Phi = M_\Phi A_\Phi N_\Phi$ be a parabolic subgroup of the real reductive Lie group G. Let D denote the Dixmier-Pukánszky operator of* (5.2). *Let $\pi_{\lambda,\xi} \in \widehat{A_\Phi N_\Phi}$ as described in* (5.4) *and let $\Theta_{\pi_{\lambda,\xi}} : h \mapsto \mathrm{trace}\,\pi_{\lambda,\xi}(h)$ denote its distribution character. Then $\Theta_{\pi_{\lambda,\xi}}$ is a tempered distribution. If $f \in \mathcal{C}(A_\Phi N_\Phi)$ and $x \in A_\Phi N_\Phi$ then*

$$f(x) = c\int_{(\mathfrak{a}'_\Phi)^*}\left(\int_{\mathfrak{s}_\Phi^*/\mathrm{Ad}^*(A_\Phi)} \Theta_{\pi_{\lambda,\xi}}(D(r_x f))|\mathrm{Pf}(\lambda)|d\lambda\right)d\phi$$

where $c = 2^{d_1+\dots+d_m} d_1!d_2!\dots d_m!$ *as in* (1.6a) *and* m *is the number of factors* L_r *in* N_n .

Proof. We compute along the lines of the computation of [7, Theorem 2.7] and [5, Theorem 3.2].

$$\begin{aligned}
&\operatorname{trace}\pi_{\lambda,\phi}(Dh)\\
&\quad= \int_{x\in A_\Phi/A'_\Phi} \delta(x)^{-1}\operatorname{trace}\int_{N_\Phi A'_\Phi}(Dh)(x^{-1}nax)\cdot(\pi_\lambda^\dagger\otimes\exp(i\phi))(na)\,dn\,da\,dx\\
&\quad= \int_{x\in A_\Phi/A'_\Phi} \operatorname{trace}\int_{N_\Phi A'_\Phi}(Dh)(nx^{-1}ax)\cdot(\pi_\lambda^\dagger\otimes\exp(i\phi))(xnx^{-1}a)\,dn\,da\,dx.
\end{aligned}$$

Now

$$\begin{aligned}
&\int_{(\mathfrak{a}'_\Phi)^*}\operatorname{trace}\pi_{\lambda,\phi}(Dh)\,d\phi\\
&= \int_{\widehat{A'_\Phi}}\int_{x\in A_\Phi/A'_\Phi}\operatorname{trace}\int_{N_\Phi A'_\Phi}(Dh)(nx^{-1}ax)(\pi_\lambda^\dagger\otimes\exp(i\phi))(xnx^{-1}a)\,dn\,da\,dx\;d\phi\\
&= \int_{x\in A_\Phi/A'_\Phi}\int_{\widehat{A'_\Phi}}\operatorname{trace}\int_{N_\Phi A'_\Phi}(Dh)(nx^{-1}ax)(\pi_\lambda^\dagger\otimes\exp(i\phi))(xnx^{-1}a)\,dn\,da\;d\phi\;dx\\
&= \int_{x\in A_\Phi/A'_\Phi}\operatorname{trace}\int_{N_\Phi}(Dh)(n)\pi_\lambda^\dagger(xnx^{-1})dn\,dx\\
&= \int_{x\in A_\Phi/A'_\Phi}\operatorname{trace}\int_{N_\Phi}(Dh)(n)(\mathrm{Ad}(x^{-1})\cdot\pi_\lambda^\dagger)(n)dn\,dx\\
&= \int_{x\in A_\Phi/A'_\Phi}\operatorname{trace}\,(\mathrm{Ad}(x^{-1})\cdot\pi_\lambda^\dagger)(Dh))\,dx \qquad (5.6)\\
&= \int_{x\in A_\Phi/A'_\Phi}(\mathrm{Ad}(x^{-1})\cdot\pi_\lambda^\dagger)_*(D)\;\operatorname{trace}\,(\mathrm{Ad}(x^{-1})\cdot\pi_\lambda^\dagger)(h)dx\\
&= \int_{x\in A_\Phi/A'_\Phi}(\pi_\lambda^\dagger)_*(\mathrm{Ad}(x)\cdot D)\operatorname{trace}\,(\mathrm{Ad}(x^{-1})\cdot\pi_\lambda^\dagger)(h)\,dx\\
&= \int_{x\in A_\Phi/A'_\Phi}\delta_{A_\Phi N_\Phi}(x)\operatorname{trace}\,(\mathrm{Ad}(x^{-1})\cdot\pi_\lambda^\dagger)(h)\,dx\\
&= \int_{\lambda'\in\mathrm{Ad}^*(A_\Phi)\lambda}\operatorname{trace}\pi_{\lambda'}^\dagger(h)|\mathrm{Pf}(\lambda')|d\lambda'.
\end{aligned}$$

Summing over $\overline{\lambda} = \mathrm{Ad}^*(A_\Phi)(\lambda) \in \mathfrak{t}^*/\mathrm{Ad}^*(A_\Phi)$ we now have

$$\begin{aligned}
&\int_{\overline{\lambda}\in\mathfrak{t}^*_\Phi/\mathrm{Ad}^*(A_\Phi)}\left(\int_{(\mathfrak{a}'_\Phi)^*}\operatorname{trace}\pi_{\lambda,\phi}(Dh)\,d\phi\right)d\overline{\lambda}\\
&\qquad= \int_{\overline{\lambda}\in\mathfrak{t}^*_\Phi/\mathrm{Ad}^*(A_\Phi)}\left(\int_{\lambda'\in\mathrm{Ad}^*(A_\Phi)\lambda}\operatorname{trace}\pi_{\lambda'}^\dagger(h)|\mathrm{Pf}(\lambda')|d\lambda'\right)d\overline{\lambda} \qquad (5.7)\\
&\qquad= \int_{\lambda\in\mathfrak{s}^*_\Phi}\operatorname{trace}\pi_\lambda(h)|\mathrm{Pf}(\lambda)|d\lambda = h(1).
\end{aligned}$$

If $h = r_x f$ then $h(1) = f(x)$ and the theorem follows. ∎

6. The Maximal Amenable Subgroup $U_\Phi A_\Phi N_\Phi$

In this section we extend our results on N_Φ and $A_\Phi N_\Phi$ to the maximal amenable subgroups

$$E_\Phi := U_\Phi A_\Phi N_\Phi \text{ where } U_\Phi \text{ is a maximal compact subgroup of } M_\Phi .$$

Of course if $\Phi = \emptyset$, i.e. if Q_Φ is a minimal parabolic, then $U_\Phi = M_\Phi$. We start by recalling the classification of maximal amenable subgroups in real reductive Lie groups.

Recall the definition. A *mean* on a locally compact group H is a linear functional μ on $L^\infty(H)$ of norm 1 and such that $\mu(f) \geqq 0$ for all real-valued $f \geqq 0$. H is *amenable* if it has a left-invariant mean. There are more than a dozen useful equivalent conditions. Solvable groups and compact groups are amenable, as are extensions of amenable groups by amenable subgroups. In particular if U_Φ is a maximal compact subgroup of M_Φ then $E_\Phi := U_\Phi A_\Phi N_\Phi$ is amenable.

We'll need a technical condition [8, p. 132]. Let H be the group of real points in a linear algebraic group whose rational points are Zariski dense, let A be a maximal $\mathbb{R}$-split torus in H, let $Z_H(A)$ denote the centralizer of A in H, and let H^0 be the algebraic connected component of the identity in H. Then H is *isotropically connected* if $H = H^0 \cdot Z_H(A)$. More generally we will say that a subgroup $H \subset G$ is *isotropically connected* if the algebraic hull of $\mathrm{Ad}_G(H)$ is isotropically connected. The point is Moore's theorem

Proposition 6.1. [8, Theorem 3.2]. *The groups $E_\Phi := U_\Phi A_\Phi N_\Phi$ are maximal amenable subgroups of G. They are isotropically connected and self-normalizing. As Φ runs over the $2^{|\Psi|}$ subsets of Ψ the E_Φ are mutually non-conjugate. An amenable subgroup $H \subset G$ is contained in some E_Φ if and only if it is isotropically connected.*

Now we need some notation and definitions. If $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ we denote

$$[\alpha]_\Phi = \{\gamma \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \gamma|_{\mathfrak{a}_\Phi} = \alpha|_{\mathfrak{a}_\Phi}\} \text{ and } \mathfrak{g}_{[\Phi,\alpha]} = \sum\nolimits_{\gamma \in [\alpha]_\Phi} \mathfrak{g}_\gamma . \tag{6.1}$$

Recall [17, Theorem 8.3.13] that the various $\mathfrak{g}_{[\Phi,\alpha]}$, $\alpha \notin \Phi^{red}$, are $\mathrm{ad}\,(\mathfrak{m}_\Phi)$-invariant and are absolutely irreducible as $\mathrm{ad}\,(\mathfrak{m}_\Phi)$-modules.

Definition 6.2. The decomposition $N_\Phi = L_{\Phi,1} L_{\Phi,2} \dots L_{\Phi,\ell}$ of Theorem 4.5 is *invariant* if each $\mathrm{ad}\,(\mathfrak{m}_\Phi)\mathfrak{z}_{\Phi,j} \subset \mathfrak{z}_{\Phi,j}$, equivalently if each $\mathrm{Ad}(M_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$, in other words whenever $\mathfrak{z}_{\Phi,j} = \mathfrak{g}_{[\Phi,\beta_{j_0}]}$. The decomposition $N_\Phi = L_{\Phi,1} L_{\Phi,2} \dots L_{\Phi,\ell}$ is *weakly invariant* if each $\mathrm{Ad}(U_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$.

Here are four special cases. (1) If Φ is empty, i.e. if Q_Φ is a minimal parabolic subgroup, then the decomposition $N_\Phi = L_{\Phi,1} L_{\Phi,2} \dots L_{\Phi,\ell}$ is invariant.

(2) If $|\Psi \setminus \Phi| = 1$, i.e. if Q_Φ is a maximal parabolic subgroup, then $N_\Phi = L_{\Phi,1}$ is invariant.

(3) Let $G = SL(6;\mathbb{R})$ with simple roots $\Psi = \{\psi_1, \dots, \psi_5\}$ in the usual order and $\Phi = \{\psi_1, \psi_4, \psi_5\}$. Then $\beta_1 = \psi_1 + \dots + \psi_5$, $\beta_2 = \psi_2 + \psi_3 + \psi_4$ and $\beta_3 = \psi_3$. Note $\beta_1|_{\mathfrak{a}_\Phi} = \beta_2|_{\mathfrak{a}_\Phi} \neq \beta_3|_{\mathfrak{a}_\Phi} = (\psi_3 + \psi_4)|_{\mathfrak{a}_\Phi}$. Thus $\mathfrak{n}_\Phi = \mathfrak{l}_{\Phi,1} + \mathfrak{l}_{\Phi_2}$ with $\mathfrak{l}_{\Phi,1} = (\mathfrak{l}_1 + \mathfrak{l}_2) \cap \mathfrak{n}_\Phi$ and $\mathfrak{l}_{\Phi_2} = \mathfrak{g}_{\beta_3}$. Now $\mathfrak{g}_{[\Phi,\beta_3]} \neq \mathfrak{z}_{\Phi,2}$ so the decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is not invariant.

(4) In the example just above, $[\beta_3] = \{\psi_3, \psi_3 + \psi_4, \psi_3 + \psi_4 + \psi_5\}$. The semisimple part $[\mathfrak{m}_\Phi, \mathfrak{m}_\Phi]$ of $\mathfrak{m}_\Phi$ is direct sum of $\mathfrak{m}_1 = \mathfrak{sl}(2;\mathbb{R})$ with simple root ψ_1 and $\mathfrak{m}_{4,5} = \mathfrak{sl}(3;\mathbb{R})$ with simple roots ψ_4 and ψ_5. The action of $[\mathfrak{m}_\Phi, \mathfrak{m}_\Phi]$ on $\mathfrak{g}_{[\beta_3]}$ is trivial on $\mathfrak{m}_1$ and the usual (vector) representation of $\mathfrak{m}_{4,5}$. That remains irreducible on the maximal compact $\mathfrak{so}(3)$ in $\mathfrak{m}_{4,5}$. It follows that here the decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is not weakly invariant.

Lemma 6.3. *Let $F = \exp(i\mathfrak{a}) \cap K$. Then F is an elementary abelian 2-group of cardinality $\leqq 2^{\dim \mathfrak{a}}$. In particular, F is finite, and if $x \in F$ then $x^2 = 1$. Further, F is central in M_Φ (thus also in U_Φ), $U_\Phi = FU_\Phi^0$, $E_\Phi = FE_\Phi^0$ and $M_\Phi = FM_\Phi^0$.*

Proof. Let θ be the Cartan involution of G for which $K = G^\theta$. If $x \in F$ then $x = \theta(x) = x^{-1}$ so $x^2 = 1$. Now F is an elementary abelian 2-group of cardinality $\leqq 2^{\dim \mathfrak{a}}$, in particular F is finite.

Let G_u denote the compact real form of $G_\mathbb{C}$ such that $G \cap G_u = K$, and let $A_{\Phi,u}$ denote the torus subgroup $\exp(i\mathfrak{a}_\Phi)$. The centralizer $M_{\Phi,u}A_{\Phi,u} = Z_{G_u}(A_{\Phi,u})$ is connected. It has a maximal torus $C_{\Phi,u}B_{\Phi,u}A_{\Phi,u}$ corresponding to

$$\mathfrak{h}_\mathbb{C} = \mathfrak{c}_{\Phi,\mathbb{C}} + \mathfrak{b}_{\Phi,\mathbb{C}} + \mathfrak{a}_{\Phi,\mathbb{C}} \tag{6.2}$$

where $\mathfrak{c}_\Phi$ is a Cartan subalgebra of $\mathfrak{u}_\Phi$, $\mathfrak{c}_\Phi + \mathfrak{b}_\Phi$ is a Cartan subalgebra of $\mathfrak{m}_\Phi$ and $\mathfrak{b}_\Phi + \mathfrak{a}_\Phi = \mathfrak{a}$. The complexification $M_{\Phi,\mathbb{C}}A_{\Phi,\mathbb{C}} = Z_{G_\mathbb{C}}(A_{\Phi,\mathbb{C}})$ is connected and has connected Cartan subgroup $C_{\Phi,\mathbb{C}}B_{\Phi,\mathbb{C}}A_{\Phi,\mathbb{C}}$. Now every component of $M_\Phi A_\Phi = (M_{\Phi,\mathbb{C}}A_{\Phi,\mathbb{C}}) \cap G$ contains an element of $\exp(\mathfrak{c}_\Phi + i\mathfrak{b}_\Phi + i\mathfrak{a}_\Phi)$. Thus every component of its maximal compact subgroup U_Φ contains an element of $\exp(i\mathfrak{b}_\Phi + i\mathfrak{a}_\Phi) = \exp(i\mathfrak{a})$. This proves $U_\Phi \subset FU_\Phi^0$. But $F \subset M_\Phi A_\Phi$, and is finite and central there, so $F \subset U_\Phi$. Now $U_\Phi = FU_\Phi^0$. It follows that $M_\Phi = FM_\Phi^0$. As E_Φ is the semidirect product of U_Φ with an exponential solvable (thus topologically contractible) group it also follows that $E_\Phi = FE_\Phi^0$. ∎

Notice that the parabolic Q_Φ is cuspidal (in the sense of Harish-Chandra) if and only if $\mathfrak{b}_\Phi = 0$, in other words if and only if M_Φ has discrete series representations. The cuspidal parabolics are the ones used to construct standard tempered representations of real reductive Lie groups.

Lemma 6.4. *The action of F on $\mathfrak{s}_\Phi^*$ is trivial.*

Proof. We know that the action of F is trivial on each $\mathfrak{z}_j^*$ [22, Proposition 3.6]. The action of M_Φ is absolutely irreducible on every $\mathfrak{a}_\Phi$-root space [17, Theorem 8.13.3]. Recall $\mathfrak{z}_{\Phi,j} = \sum_{I_j}(\mathfrak{g}_{\beta_i} + \mathfrak{l}_i'')$ where $\mathfrak{l}_i'' = \sum_{J_i''} \mathfrak{g}_\alpha$ from (4.2) and (4.3). Using Lemma 3.3 we see that the action of F is trivial on each $\mathfrak{g}_{\beta_i} + \mathfrak{l}_i''$, thus trivial on $\mathfrak{z}_{\Phi,j}$, and thus trivial on their sum $\mathfrak{s}_\Phi$, and finally by duality is trivial on $\mathfrak{s}_\Phi^*$. ∎

When $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is weakly invariant we can proceed more or less as in [22]. Set

$$\mathfrak{r}_\Phi^* = \{\lambda \in \mathfrak{s}_\Phi^* \mid P(\lambda) \neq 0 \text{ and } \mathrm{Ad}(U_\Phi)\lambda \text{ is a principal } U_\Phi\text{-orbit on } \mathfrak{s}_\Phi^*\}. \qquad (6.3)$$

If $c \neq 0$ and $\lambda \in \mathfrak{r}_\Phi^*$ then $c\lambda \in \mathfrak{r}_\Phi^*$. Thus we obtain $\mathfrak{r}_\Phi^*$ by scaling the set of all λ in a unit sphere s of $\mathfrak{s}_\Phi^*$ (for any norm) such that $\mathrm{Ad}(U_\Phi)\lambda$ is a principal U_Φ-orbit on s. Thus, as in the case of compact group actions on compact spaces, $\mathfrak{r}_\Phi^*$ is dense, open and U_Φ-invariant in $\mathfrak{s}_\Phi^*$. By definition of principal orbit the isotropy subgroups of U_Φ at the various points of $\mathfrak{r}_\Phi^*$ are conjugate, and we take a measurable section σ to $\mathfrak{r}_\Phi^* \to \mathrm{Ad}^*(U_\Phi)\backslash\mathfrak{r}_\Phi^*$ on whose image all the isotropy subgroups are the same,

$$U'_\Phi: \text{ isotropy subgroup of } U_\Phi \text{ at } \sigma(U_\Phi(\lambda)), \text{ independent of } \lambda \in \mathfrak{r}_\Phi^*. \qquad (6.4)$$

Lemma 6.4 says that $U'_\Phi = F(U'_\Phi \cap U^0_\Phi)$ In view of Lemma 6.4 the principal isotropy subgroups U'_Φ are specified by the work of W.-C. and W.-Y. Hsiang [3] on the structure and classification of principal orbits of compact connected linear groups. With a glance back at (5.3) we have

$$U'_\Phi A'_\Phi: \text{ isotropy subgp of } U_\Phi A_\Phi \text{ at } \sigma(U_\Phi A_\Phi(\lambda)), \text{ independent of } \lambda \in \mathfrak{r}_\Phi^*. \quad (6.5)$$

The first consequence, as in [22, Proposition 3.3], is

Theorem 6.5. *Suppose that $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is weakly invariant. Let $f \in \mathcal{C}(U_\Phi N_\Phi)$ Given $\lambda \in \mathfrak{r}_\Phi^*$ let $\pi_\lambda^\dagger$ denote the extension of π_λ to a representation of $U'_\Phi N_\Phi$ on the space of π_λ. Then the Plancherel density at $\mathrm{Ind}_{U'_\Phi N_\Phi}^{U_\Phi N_\Phi}(\pi_\lambda^\dagger \otimes \mu')$, $\mu' \in \widehat{U'_\Phi}$, is $(\dim \mu')|P(\lambda)|$ and the Plancherel Formula for $U_\Phi N_\Phi$ is*

$$f(un) = c\int_{\mathfrak{r}_\Phi^*/\mathrm{Ad}^*(U_\Phi)} \sum\nolimits_{\mu' \in \widehat{U'_\Phi}} \mathrm{trace}\left(\left(\mathrm{Ind}_{U'_\Phi N_\Phi}^{U_\Phi N_\Phi}(\pi_\lambda^\dagger \otimes \mu')\right)(r_{un}f)\right)\cdot \dim(\mu')\cdot|P(\lambda)|d\lambda$$

where $c = 2^{d_1+\dots+d_m} d_1!d_2!\dots d_m!$, from (1.6).

Combining Theorems 5.8 and 6.5 we come to

Theorem 6.6. *Let $Q_\Phi = M_\Phi A_\Phi N_\Phi$ be a parabolic subgroup of the real reductive Lie group G. Let U_Φ be a maximal compact subgroup of M_Φ, so $E_\Phi := U_\Phi A_\Phi N_\Phi$ is a maximal amenable subgroup of Q_Φ. Suppose that the decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is weakly invariant. Given $\lambda \in \mathfrak{r}_\Phi^*$, $\phi \in \mathfrak{a}'_\Phi$ and $\mu' \in \widehat{U'_\Phi}$ denote*

$$\pi_{\lambda,\phi,\mu'} = \mathrm{Ind}_{U'_\Phi A'_\Phi N_\Phi}^{U_\Phi A_\Phi N_\Phi}(\pi_\lambda^\dagger \otimes e^{i\phi} \otimes \mu') \in \widehat{E_\Phi}.$$

Let $\Theta_{\pi_{\lambda,\phi,\mu'}} : h \mapsto \mathrm{trace}\,\pi_{\lambda,\phi,\mu'}(h)$ denote its distribution character. Then $\Theta_{\pi_{\lambda,\phi,\mu'}}$ is a tempered distribution on the maximal amenable subgroup E_Φ. If $f \in \mathcal{C}(E_\Phi)$ then

$$f(x) = c\int_{(\mathfrak{a}'_\Phi)^*}\left(\int_{\mathfrak{r}_\Phi^*/\mathrm{Ad}^*(U_\Phi A_\Phi)}\left(\sum\nolimits_{\mu' \in \in \widehat{U'_\Phi}} \Theta_{\pi_{\lambda,\phi,\mu'}}(D(r_x f))\dim(\mu')\right)|P(\lambda)|d\lambda\right)d\phi$$

where $c = (\frac{1}{2\pi})^{\dim \mathfrak{a}'_\Phi/2}\, 2^{d_1+\dots+d_m} d_1!d_2!\dots d_m!$.

Proof. Theorem 13.4 extends this result to certain direct limit parabolics, and the calculation in the proof of Theorem 13.4 specializes to give the proof of Theorem 6.6. ∎

When weak invariance fails we replace the $\mathfrak{z}_{\Phi,j}$ by the larger

$$\mathfrak{g}_{[\Phi,\beta_j]} = \sum_{\alpha\in[\beta_j]_\Phi} \mathfrak{g}_\alpha \text{ where } [\beta_j]_\Phi = \{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a}) \mid \alpha|_{\mathfrak{a}_\Phi} = \beta_{j_0}|_{\mathfrak{a}_\Phi}\}, \tag{6.6}$$

for any $j_0 \in I_j$, as in (6.1). Note that $\mathfrak{g}_{[\Phi,\beta_j]}$ is an irreducible $\mathrm{Ad}(M^0_\Phi)$-module. We need to show that we can replace $\mathfrak{s}_\Phi = \sum \mathfrak{z}_{\Phi,j}$ by $\widetilde{\mathfrak{s}_\Phi} := \sum_j \mathfrak{g}_{[\Phi,\beta_j]}$ in our Plancherel formulae. The key is

Lemma 6.7. *Let* $\lambda_j \in \mathfrak{g}^*_{[\Phi,\beta_j]}$. *Split* $\mathfrak{g}_{[\Phi,\beta_j]} = \mathfrak{z}_{\Phi,j} + \mathfrak{w}_{\Phi,j}$ *where* $\mathfrak{w}_{\Phi,j} = \mathfrak{g}_{[\Phi,\beta_j]} \cap \mathfrak{v}_\Phi$ *is the sum of the* $\mathfrak{g}_\alpha$ *that occur in* $\mathfrak{g}_{[\Phi,\beta_j]}$ *but not in* $\mathfrak{z}_{\Phi,j}$. *Then the Pfaffian* $\mathrm{Pf}_j(\lambda_j) = \mathrm{Pf}_j(\lambda_j|_{\mathfrak{z}_{\Phi,j}})$.

Proof. Write $\lambda_j = \lambda_{\mathfrak{z},j} + \lambda_{\mathfrak{w},j}$ where $\lambda_{\mathfrak{z},j}(\mathfrak{w}_{\Phi,j}) = 0 = \lambda_{\mathfrak{w},j}(\mathfrak{z}_{\Phi,j})$. Let $\mathfrak{g}_\gamma, \mathfrak{g}_\delta \subset \mathfrak{l}_{\Phi,j}$ with $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] \neq 0$. Then $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] \subset \mathfrak{l}_{\Phi,j}$, so $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] \cap \mathfrak{w}_{\Phi,j} = 0$, in particular $\lambda_{\mathfrak{w},j}([\mathfrak{g}_\gamma, \mathfrak{g}_\delta]) = 0$. In other words $\lambda_j([\mathfrak{g}_\gamma, \mathfrak{g}_\delta]) = \lambda_j|_{\mathfrak{z}_{\Phi,j}}([\mathfrak{g}_\gamma, \mathfrak{g}_\delta])$. Now $b_{\lambda_j|_{\mathfrak{z}_{\Phi,j}}} = b_{\lambda_j}$, so their Pfaffians are the same. ∎

In order to extend Theorems 6.5 and 6.6 we now need only make some trivial changes to (6.3), (6.4), (6.5) and the measurable section:

- $\widetilde{\mathfrak{r}_\Phi}^* = \{\lambda \in \widetilde{\mathfrak{s}_\Phi}^* \mid P(\lambda) \neq 0 \text{ and } \mathrm{Ad}(U_\Phi)\lambda \text{ is a principal } U_\Phi\text{-orbit on } \widetilde{\mathfrak{s}_\Phi}^*\}$.
- $\widetilde{\sigma}$: measurable section to $\widetilde{\mathfrak{r}_\Phi}^* \to \widetilde{\mathfrak{r}_\Phi}^* \backslash U_\Phi$ on whose image all the isotropy subgroups are the same.
- U'_Φ : isotropy subgroup of U_Φ at $\widetilde{\sigma}(U_\Phi(\lambda))$, independent of $\lambda \in \widetilde{\mathfrak{r}_\Phi}^*$.
- $U'_\Phi A'_\Phi$: isotropy subgroup of $U_\Phi A_\Phi$ at $\widetilde{\sigma}(U_\Phi A_\Phi(\lambda))$, independent of $\lambda \in \widetilde{\mathfrak{r}_\Phi}^*$.

The result is

Theorem 6.8. *In Theorems* 6.5 *and* 6.6 *one can omit the requirement that* $N_\Phi = L_{\Phi,1} L_{\Phi,2} \dots L_{\Phi,\ell}$ *be weakly invariant.*

PART II: INFINITE DIMENSIONAL THEORY

7. Direct Limits of Nilpotent Lie Groups

In this section we describe the basic outline for direct limits of stepwise square integrable representations of simply connected nilpotent Lie groups. Later we will specialize these constructions to nilradicals $N_{\Phi,\infty} = \varinjlim N_{\Phi,n}$ of parabolic subgroups $Q_{\Phi,\infty} = \varinjlim Q_{\Phi,n}$ in our real reductive Lie groups $G_\infty = \varinjlim G_n$. In order to do that we will need to adjust the ordering in the decompositions (1.3) so that they fit together as n increases. We do that by reversing the indices and keeping the L_r constant as n goes to infinity. Thus, we suppose that

$$\{N_n\} \text{ is a strict direct system of connected nilpotent Lie groups,} \tag{7.1}$$

in other words the connected simply connected nilpotent Lie groups N_n have the property that N_n is a closed analytic subgroup of N_ℓ for all $\ell \geqq n$. As usual, Z_r denotes the center of L_r . For each n, we require that

$$\begin{aligned}
&N_n = L_1 L_2 \cdots L_{m_n} \text{ where}\\
&\quad \text{(a) } L_r \text{ is a closed analytic subgroup of } N_n \text{ for } 1 \leqq r \leqq m_n \text{ and}\\
&\quad \text{(b) each } L_r \text{ has unitary representations with coefficients in } L^2(L_r/Z_r).\\
&\quad \text{(x) } L_{p,q} = L_{p+1}L_{p+2}\cdots L_q \text{ for } p < q \text{ and}\\
&\qquad\qquad N_{\ell,n} = L_{m_\ell+1}L_{m_\ell+2}\cdots L_{m_n} = L_{m_\ell,m_n} \text{ for } \ell < n;\\
&\quad \text{(c) } N_{\ell,n} \text{ is normal in } N_n \text{ and } N_n = N_r \ltimes N_{r,n} \text{ semidirect product,}\\
&\quad \text{(d) } \mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r \text{ and } \mathfrak{n}_n = \mathfrak{s}_n + \textstyle\bigoplus_{r\leqq m_n} \mathfrak{v}_r \text{ where } \mathfrak{s}_n = \bigoplus_{r\leqq m_n} \mathfrak{z}_r;\\
&\qquad \text{then } [\mathfrak{l}_r, \mathfrak{z}_s] = 0 \text{ and } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}''_s + \mathfrak{v} \text{ for } r < s \text{ where}\\
&\qquad \mathfrak{l}_r = \mathfrak{l}'_r \oplus \mathfrak{l}''_r \text{ direct sum of ideals with } \mathfrak{l}''_r \subset \mathfrak{z}_r \text{ and } \mathfrak{v}_r \subset \mathfrak{l}'_r
\end{aligned} \tag{7.2}$$

With this we can follow the lines of the constructions in [20, Section 5] as indicated in §1 above. Denote

$$P_n(\gamma_n) = \mathrm{Pf}_1(\lambda_1)\mathrm{Pf}_2(\lambda_2)\cdots \mathrm{Pf}_{m_n}(\lambda_{m_n})\,, \lambda_r \in \mathfrak{z}_r^* \text{ and } \gamma_n = \lambda_1 + \cdots + \lambda_{m_n} \tag{7.3}$$

and the nonsingular set

$$\mathfrak{t}_n^* = \{\gamma_n \in \mathfrak{s}_n^* \mid P_n(\gamma_n) \neq 0\}. \tag{7.4}$$

When $\gamma_n \in \mathfrak{t}_n^*$ the stepwise square integrable representation $\pi_{\gamma_n} \in \widehat{N_n}$ is defined as in Construction 1.2, but with the indices reversed: $\pi_{\lambda_1+\cdots+\lambda_{m+1}} = \pi^\dagger_{\lambda_1+\cdots+\lambda_m} \widehat{\otimes} \pi_{\lambda_{m+1}}$ with representation space $\mathcal{H}_{\pi_{\lambda_1+\cdots+\lambda_{m+1}}} = \mathcal{H}_{\pi_{\lambda_1+\cdots+\lambda_m}} \widehat{\otimes} \mathcal{H}_{\pi_{\lambda_{m+1}}}$.

The parameter space for our representations of the direct limit Lie group $N = \varinjlim N_n$ is

$$\mathfrak{t}^* = \left\{ \gamma = (\gamma_\ell) \in \mathfrak{s}^* = \varprojlim \mathfrak{s}_\ell^* \,\middle|\, \gamma_\ell = \lambda_1 + \cdots + \lambda_{m_\ell} \in \mathfrak{t}_\ell^* \text{ for all } \ell \right\}. \tag{7.5}$$

The closed normal subgroups $N_{n,n+1}$ and $N_{n,\infty}$ satisfy $N_n \cong N_{n+1}/N_{n,n+1} \cong N/N_{n,\infty}$. Let $\gamma \in \mathfrak{t}^*$ and denote

$$\begin{aligned}
\pi_{\gamma,n}&: \text{ the stepwise square integrable } \pi_{\lambda_1+\cdots+\lambda_{m_n}} \in \widehat{N_n}\\
\pi_{\gamma,n,n+1}&: \text{ the stepwise square integrable } \pi_{\lambda_{m_n+1}+\cdots+\lambda_{m_{n+1}}} \in \widehat{N_{n,n+1}}
\end{aligned} \tag{7.6}$$

Using $N_n \cong N_{n+1}/N_{n,n+1}$ we lift $\pi_{\gamma,n}$ to a representation $\pi^\dagger_{\gamma,n}$ of N_{n+1} whose kernel contains $N_{n,n+1}$ and we extend $\pi_{\gamma,n,n+1}$ to a representation $\pi^\dagger_{\gamma,n,n+1}$ of N_{n+1} on the same representation space $\mathcal{H}_{\pi_{\gamma,n,n+1}}$. Then we define

$$\pi_{\gamma,n+1} = \pi^\dagger_{\gamma,n} \otimes \pi^\dagger_{\gamma,n,n+1} \in \widehat{N_{n+1}}\,. \tag{7.7}$$

The representation space is the projective (jointly continuous) tensor product $\mathcal{H}_{\pi_{\gamma,n+1}} = \mathcal{H}_{\pi_{\gamma,n}} \widehat{\otimes} \mathcal{H}_{\pi_{\gamma,n,n+1}}$ where $\mathcal{H}_{\pi_{\gamma,n,n+1}} = \mathcal{H}_{\pi_{\lambda,m_n+1}} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{H}_{\pi_{\lambda,m_{n+1}}}$. Choose a C^∞ unit vector $e_{n+1} \in \mathcal{H}_{\pi_{\gamma,n,n+1}}$. Then

$$v \mapsto v \otimes e_{n+1} \text{ is an } N_n\text{-equivariant isometry } \mathcal{H}_{\pi_{\gamma,n}} \hookrightarrow \mathcal{H}_{\pi_{\gamma,n+1}} \tag{7.8}$$

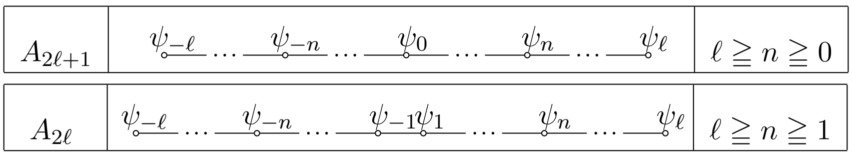
exhibits $\pi_{\gamma,n}$ as the restriction $\pi_{\gamma,n+1}|_{N_n}$ on the subspace $(\mathcal{H}_{\pi_{\gamma,n}} \otimes e_{n+1})$ of $\mathcal{H}_{\pi_{\gamma,n+1}}$.

Lemma 7.1. *The maps just described, define direct system $\{(\pi_{\gamma,n}, \mathcal{H}_{\pi_{\gamma,n}})\}$ of irreducible stepwise square integrable unitary representations, and thus define an irreducible unitary representation $\pi_\gamma = \varinjlim \pi_{\gamma,n}$ of $N = \varinjlim N_n$ on the Hilbert space $\mathcal{H}_{\pi_\gamma} = \varinjlim \mathcal{H}_{\pi_{\gamma,n}}$.*

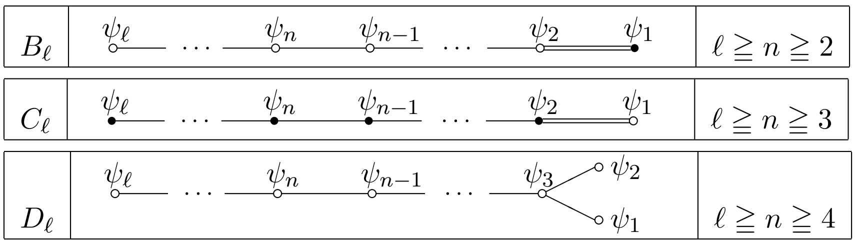
The representations π_γ described in Lemma 7.1 are the *limit stepwise square integrable* representations of N. Corollary 9.9 will show that the unitary equivalence class of π_γ is independent of the choice of the C^∞ unit vectors e_n.

8. Direct Limit Structure of Parabolics and some Subgroups

We adapt the constructions Section 7 to limits of nilradicals of parabolic subgroups. That requires some alignment of root systems so that the direct limit respects the restricted root structures, in particular the strongly orthogonal root structures, of the N_n. We enumerate the set $\Psi_n = \Psi(\mathfrak{g}_n, \mathfrak{a}_n)$ of nonmultipliable simple restricted roots so that, in the Dynkin diagram, for type A we spread from the center of the diagram. For types B, C and D, ψ_1 is the *right* endpoint. In other words for $\ell \geqq n$ Ψ_ℓ is constructed from Ψ_n adding simple roots to the *left* end of their Dynkin diagrams. Thus



(8.1)



(8.2)

We describe this by saying that G_ℓ *propagates* G_n. For types B, C and D this is the same as the notion of propagation in [10] and [11].

The direct limit groups obtained this way are $SL(\infty;\mathbb{C})$, $SO(\infty;\mathbb{C})$, $Sp(\infty;\mathbb{C})$, $SL(\infty;\mathbb{R})$, $SL(\infty;\mathbb{H})$, $SU(\infty,q)$ with $q \leqq \infty$, $SO(\infty,q)$ with $q \leqq \infty$, $Sp(\infty,q)$ with $q \leqq \infty$, $Sp(\infty;\mathbb{R})$ and $SO^*(2\infty)$.

Let $\{G_n\}$ be a direct system of real semisimple Lie groups in which G_ℓ propagates G_n for $\ell \geqq n$. Then the corresponding simple restricted root systems satisfy $\Psi_n \subset \Psi_\ell$ as indicated in (8.1) and (8.2). Consider conditions on a family $\Phi = \{\Phi_n\}$ of subsets $\Phi_n \subset \Psi_n$ such that $G_n \hookrightarrow G_\ell$ maps the corresponding parabolics $Q_{\Phi,n} \hookrightarrow Q_{\Phi,\ell}$. Then we have

$$Q_{\Phi,\infty} := \varinjlim Q_{\Phi,n} \text{ inside } G_\infty := \varinjlim G_n \,. \tag{8.3}$$

Express $Q_{\Phi,n} = M_{\Phi,n}A_{\Phi,n}N_{\Phi,n}$ and $Q_{\Phi,\ell} = M_{\Phi,\ell}A_{\Phi,\ell}N_{\Phi,\ell}$. Then $M_{\Phi,n} \hookrightarrow M_{\Phi,\ell}$ is equivalent to $\Phi_n \subset \Phi_\ell$, $A_{\Phi,n} \hookrightarrow A_{\Phi,\ell}$ is implicit in the condition that G_ℓ propagates G_n, and $N_{\Phi,n} \hookrightarrow N_{\Phi,\ell}$ is equivalent to $(\Psi_n \backslash \Phi_n) \subset (\Psi_\ell \backslash \Phi_\ell)$. As before let $U_{\Phi,n}$ denote a maximal compact subgroup of $M_{\Phi,n}$; we implicitly assume that $U_{\Phi,n} \hookrightarrow U_{\Phi,\ell}$ whenever $M_{\Phi,n} \hookrightarrow M_{\Phi,\ell}$.

We will extend some of our results from the finite dimensional setting to these subgroups of $Q_{\Phi,\infty}$.

$$\begin{aligned}
&\text{(a) } N_{\Phi,\infty} := \varinjlim N_{\Phi,n} \text{ maximal locally unipotent subgroup,}\\
&\qquad\qquad\qquad \text{requiring } (\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell),\\
&\text{(b) } A_{\Phi,\infty} := \varinjlim A_{\Phi,n}\,,\\
&\text{(c) } U_{\Phi,\infty} := \varinjlim U_{\Phi,n} \text{ maximal lim-compact subgroup, requiring } \Phi_n \subset \Phi_\ell,\\
&\text{(d) } U_{\Phi,\infty}N_{\Phi,\infty} := \varinjlim U_{\Phi,n}N_{\Phi,n}\,,\\
&\qquad\qquad\qquad \text{requiring } \Phi_n \subset \Phi_\ell \text{ and } (\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell).
\end{aligned} \tag{8.4}$$

To study these we will need to extend some notation from the finite dimensional setting to the system $\{\mathfrak{g}_n\}$. For $\alpha \in \Delta^+(\mathfrak{g}_n, \mathfrak{a}_n)$ we denote

$$[\alpha]_{\Phi,n} = \{\delta \in \Delta^+(\mathfrak{g}_n, \mathfrak{a}_n) \mid \delta|_{\mathfrak{a}_{\Phi,n}} = \alpha|_{\mathfrak{a}_{\Phi,n}}\} \text{ and } \mathfrak{g}_{\Phi,n,\alpha} = \sum\nolimits_{\delta \in [\alpha]_{\Phi,n}} \mathfrak{g}_\delta \,. \tag{8.5}$$

The adjoint action of $\mathfrak{m}_{\Phi,n}$ on $\mathfrak{g}_{\Phi,n,\alpha}$ is absolutely irreducible [17, Theorem 8.3.13]; $\mathfrak{g}_{\Phi,n,\alpha}$ is the sum of the root spaces for roots $\delta = \sum_{\psi \in \Psi_n} n_\psi(\delta)\psi \in \Delta^+(\mathfrak{g}_n, \mathfrak{a}_n)$ such that $n_\psi(\delta) = n_\psi(\alpha)$ for all $\psi \in \Psi_n \backslash \Phi_n$, in other words the same coefficients along $\Psi_n \setminus \Phi_n$ in $\sum_{\psi \in \Psi_n} n_\psi(\cdot)\psi$. The following lemma is immediate.

Lemma 8.1. *Let $n \leqq \ell$ and assume the condition $(\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell)$ of* (8.4)(a) *for $N_{\Phi,\infty}$. Then $\mathfrak{g}_{\Phi,n,\alpha} \subset \mathfrak{g}_{\Phi,\ell,\alpha}$. In particular we have the joint $\mathfrak{a}_{\Phi,\infty}$-eigenspaces $\mathfrak{g}_{\Phi,\infty,\alpha} = \varinjlim_n \mathfrak{g}_{\Phi,n,\alpha}$ in $\mathfrak{n}_{\Phi,\infty}$.*

We will also say something about representations, but not about Fourier inversion, for the

$$\begin{aligned}
&A_{\Phi,\infty}N_{\Phi,\infty} := \varinjlim A_{\Phi,n}N_{\Phi,n} \text{ maximal exponential solvable subgroup,}\\
&\qquad\qquad \text{where } (\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell) \text{ for } n \leqq \ell, \text{ and for the}\\
&E_{\Phi,\infty} := \varinjlim E_{\Phi,n} \text{ maximal amenable subgroup,}\\
&\qquad\qquad \text{where } \Phi_n \subset \Phi_\ell \text{ and } (\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell) \text{ for } n \leqq \ell\,.
\end{aligned} \tag{8.6}$$

Here $E_{\Phi,n} = U_{\Phi,n}A_{\Phi,n}N_{\Phi,n}$, so $E_{\Phi,\infty} = U_{\Phi,\infty}A_{\Phi,\infty}N_{\Phi,\infty}$. The difficulty with Fourier inversion for the two limit groups of (8.6) is that we don't have an explicit Dixmier-Pukánszky operator.

Start with $N_{\Phi,\infty}$. For that we must assume $(\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell)$. In view of the propagation assumption on the G_n the maximal set of strongly orthogonal non-multipliable roots in $\Delta^+(\mathfrak{g}_n, \mathfrak{a}_n)$ is increasing in n. It is obtained by cascading up (we reversed the indexing from the finite dimensional setting) has form $\{\beta_1, \dots, \beta_{r_n}\}$ in $\Delta^+(\mathfrak{g}_n, \mathfrak{a}_n)$. Following ideas of Section 4 we partition $\{\beta_1, \dots, \beta_{r_n}\} = \bigcup_k \bigcup_{I_{n,k}} \beta_i$ where $I_{n,k}$ consists of the indices i for which the β_i have a given restriction to $\mathfrak{a}_{\Phi,n}$ and belong to $\Delta^+(\mathfrak{g}_n, \mathfrak{a}_n)$. Note that $I_{n,k}$ can increase as n increases. This happens in some cases where the Φ stop growing, i.e. where there is an index n_0 such that $\Phi_n = \Phi_{n_0} \neq \emptyset$ for $n \geqq n_0$. That is the case when $\Delta(\mathfrak{g}_n, \mathfrak{a}_n)$ is of type A_n with each $\Psi = \{\psi_1\}$. Thus we also denote $I_{\infty,k} = \bigcup_n I_{n,k}$. As in (4.1), following the idea of $\mathfrak{l}_j = \mathfrak{z}_j + \mathfrak{v}_j$, we define

$$\begin{aligned}
\mathfrak{l}_{\Phi,n,j} &= \sum\nolimits_{i \in I_{n,j}} (\mathfrak{l}_i \cap \mathfrak{n}_{\Phi,n}), \text{ the } \beta_j \text{ part of } \mathfrak{n}_{\Phi,n}, \\
\mathfrak{l}_{\Phi,\infty,j} &= \sum\nolimits_{i \in I_{\infty,j}} (\mathfrak{l}_i \cap \mathfrak{n}_{\Phi}), \text{ the } \beta_j \text{ part of } \mathfrak{n}_{\Phi,\infty}, \\
\mathfrak{z}_{\Phi,\infty,j} &= \sum\nolimits_n \mathfrak{z}_{\Phi,n,j}, \ \mathfrak{s}_{\Phi,\infty} = \sum\nolimits_j \mathfrak{z}_{\Phi,\infty,j} \text{ and } \mathfrak{v}_{\Phi,\infty} = \sum\nolimits_{n,j} \mathfrak{v}_{\Phi,n,j},
\end{aligned} \tag{8.7}$$

so $\mathfrak{n}_{\Phi,\infty} = \mathfrak{s}_{\Phi,\infty} + \mathfrak{v}_{\Phi,\infty}$. We'll also use $\mathfrak{s}_{\Phi,n} = \sum_j \mathfrak{z}_{\Phi,n,j}$ and $\mathfrak{v}_{\Phi,n} = \sum_j \mathfrak{v}_{\Phi,n,j}$, so $\mathfrak{n}_{\Phi,n} = \mathfrak{s}_{\Phi,n} + \mathfrak{v}_{\Phi,n}$.

$L_{\Phi,n,j}$ denotes the analytic subgroup with Lie algebra $\mathfrak{l}_{\Phi,n,j}$ and $L_{\Phi,\infty,j} = \varinjlim_n L_{\Phi,n,j}$ has Lie algebra $\mathfrak{l}_{\Phi,\infty,j}$. We have this set up so that

$$N_{\Phi,\infty} = \varinjlim_n N_{\Phi,n} = \varinjlim_j L_{\Phi,\infty,j} = \varinjlim_j \varinjlim_n L_{\Phi,n,j}. \tag{8.8}$$

9. Representations of the Limit Groups I: $N_{\Phi,\infty}$

In this section we indicate the limit stepwise square integrable representations $\pi_{\Phi,\gamma} = \varinjlim \pi_{\Phi,\gamma_n}$ of the direct limit group $N_{\Phi,\infty} = \varinjlim N_{\Phi,n}$. The parameter space for the stepwise square integrable representations of the $N_{\Phi,n}$ is given by $\mathfrak{t}^*_{\Phi,n} = \{\gamma_n \in \mathfrak{s}^*_{\Phi,n} \mid P(\gamma_n) \neq 0\}$ where $\gamma_n = \sum_1^{m_n} \lambda_j$ and $P(\gamma_n)$ is the product of the Pfaffians $P_j(\lambda_j)$. Note that $\gamma_\ell|_{\mathfrak{s}_{\Phi,n}} = \gamma_n$ for $\ell > n$. The parameter space for the $\pi_{\Phi,\gamma}$ is $\mathfrak{t}^*_{\Phi,\infty} = \{(\gamma_n) \in \mathfrak{s}^*_{\Phi,\infty} \mid \text{ each } \gamma_n \in \mathfrak{t}^*_{\Phi,n}\}$ where $\mathfrak{s}^*_{\Phi,\infty} = \varprojlim \mathfrak{s}_{\Phi,n}$. The stepwise square integrable representations π_{γ_n} were obtained recursively in Construction 1.2, from square integrable representations of the L_r, $r \leqq m_n$, and in Lemma 7.1 we described method of construction of their direct limits $\pi_{\Phi,\gamma}$.

As noted before we must assume the condition $(\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell)$ of (8.4)(a), so that $\{N_{\Phi,n}\}$ is a direct system, in order to work with $N_{\Phi,\infty}$. Then we have the decompositions (8.7) and (8.8). With those in mind we will build up the parameter space for direct limits of stepwise square integrable representations of $N_{\Phi,\infty}$ in two steps. First,

Lemma 9.1. *If $\lambda \in \mathfrak{g}^*_{\Phi,\infty,\beta_j}$ the antisymmetric bilinear form b_λ on $\mathfrak{n}_{\Phi,\infty,j}/\mathfrak{z}_{\Phi,\infty,j}$ satisfies $b_\lambda = b_{\lambda|_{\mathfrak{z}_{\Phi,\infty,j}}}$.*

Proof. Let n be sufficiently large that $\mathfrak{g}_{\beta_j} \subset \mathfrak{l}_{\Phi,n,j}$. Apply Lemma 8.1 to each $\mathfrak{g}_{\Phi,\ell,\beta_j}$ with $\ell \geqq n$. That gives $(b_\lambda)|_{\mathfrak{n}_{\Phi,\ell}/\mathfrak{s}_{\Phi,\ell}} = (b_{\lambda|_{\mathfrak{z}_{\Phi,\ell,j}}})|_{\mathfrak{n}_{\Phi,\ell}/\mathfrak{s}_{\Phi,\ell}}$. As ℓ increases the additional brackets go into $\mathfrak{n}_{\Phi,\ell}$ and thus into the kernel of b_λ. ■

Second, we define the β_j part of the parameter space. In view of Lemma 9.1 we need only look at the $\lambda_j = (\lambda_{n,j}) \in \varprojlim_n \mathfrak{z}^*_{\Phi,n,j}$ that belong to

$$\mathfrak{t}^*_{\Phi,\infty,j} = \left\{ (\lambda_{n,j}) \mid \lambda_{n,j} \in \mathfrak{z}^*_{\Phi,n,j} \text{ with } P_{\mathfrak{l}_{\Phi,n,j}}(\lambda_{n,j}) \neq 0 \text{ for } n \geqq n(\lambda_j) \right\} \tag{9.1}$$

where $n(\lambda_j)$ is the first index n such that $\lambda_{n,j} \in \mathfrak{z}^*_{\Phi,n,j}$. We start this way because of the possibility that the $\mathfrak{z}_{\Phi,n,j}$ could grow, for fixed j, if the multiplicity of β_j as a joint eigenvalue of $\mathrm{ad}\,(\mathfrak{a}_{\Phi,n})$, increases as n increases. Third,

$$\mathfrak{t}^*_{\Phi,\infty} = \left\{ \gamma = (\gamma_n) \in \varprojlim_n \mathfrak{s}^*_{\Phi,n} \,\middle|\, \text{ every } \gamma_n \in \mathfrak{t}^*_{\Phi,n} \right\}. \tag{9.2}$$

Fix $\gamma = (\gamma_j) \in \mathfrak{t}^*_{\Phi,\infty}$. As in Construction 1.2 and Lemma 7.1 we have the limit stepwise square integrable representation $\pi_{\Phi,\lambda_j,\infty}$ of $L_{\Phi,\infty,j}$. Apply Construction 1.2 and Lemma 7.1 to the $\pi_{\Phi,\lambda_j,\infty}$ as j increases, obtaining the limit stepwise square integrable representation $\pi_{\Phi,\gamma,\infty} \in \widehat{N_{\Phi,\infty}}$.

Theorem 9.2. *Assume the condition* $(\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell)$ *of* (8.4)(a), *so that* $\{N_{\Phi,n}\}$ *is a direct system and* $N_{\Phi,\infty} = \varinjlim N_{\Phi,n}$ *is well defined. Let* $\gamma = (\gamma_n) \in \mathfrak{t}^*_{\Phi,\infty}$ *and* $\pi_{\Phi,\gamma,\infty} = \varinjlim \pi_{\Phi,\gamma,n}$ *as in* Lemma 7.1. *View* $\mathcal{H}_{\pi_{\Phi,\gamma,\infty}} = \varinjlim \mathcal{H}_{\pi_{\Phi,\gamma,n}}$ *in the category of Hilbert spaces and partial isometries. Let* $u, v \in \mathcal{H}_{\pi_{\Phi,\gamma,\ell}} \subset \mathcal{H}_{\pi_{\Phi,\gamma,\infty}}$. *Then the coefficient function* $f_{\pi_{\Phi,\gamma,\infty};\, u,v}(x) = \langle u, \pi_{\Phi,\gamma,\infty}(x) v\rangle$ *satisfies*

$$\left|\left| f_{\pi_{\Phi,\gamma,\infty};\, u,v}|_{N_{\Phi,\ell}} \right|\right|^2_{L^2(N_{\Phi,\ell}/S_{\Phi,\ell})} = \tfrac{||u||^2 ||v||^2}{|P_\ell(\gamma_\ell)|} \tag{9.3}$$

Proof. Let $u = \bigotimes u_j$ and $v = \bigotimes v_j$ where $u_j, v_j \in \mathcal{H}_{\pi_{\Phi,\gamma_j,\infty}}$, the representation spaces of the $\pi_{\Phi,\gamma_j,\infty}$. We know from stepwise square integrability that the coefficients satisfy

$$\left|\left| f_{\pi_{\Phi,\gamma_j,\infty};\, u_j,v_j}|_{N_n} \right|\right|^2_{L^2(L_{\Phi,n,j}/Z_{\Phi,n,j})} = \tfrac{||u_j||^2 ||v_j||^2}{|P_{\mathfrak{l}_{\Phi,n,j}}(\gamma_j)|} \text{ for } n \gg 0.$$

In other words,

$$\left|\left| f_{\pi_{\Phi,\gamma_j,\infty};\, u_j,v_j}|_{N_n} \right|\right|^2_{L^2(L_{\Phi,\infty,j}/Z_{\Phi,\infty,j})} = \tfrac{||u_j||^2 ||v_j||^2}{|P_{\mathfrak{l}_{\Phi,\infty,j}}(\gamma_j)|}.$$

Taking the product over j we have (9.3) for decomposable u and v. Decomposable vectors are dense in $\mathcal{H}_{\pi_{\Phi,\gamma,\infty}}$ so (9.3) follows from the decomposable case by continuity. ∎

Now we continue as in [23, Sections 3, 4 & 5]. The first step is the rescaling implicit in Theorem 9.2, specifically in (9.3), which holds in our situation with only the obvious changes. Recall $N_{\Phi,a,b} = L_{\Phi,m_a+1} \dots L_{\Phi,m_b} = L_{\Phi,m_a+1,m_n}$, and $N_{\Phi,a,\infty} = \varinjlim_b N_{\Phi,a,b}$, so $N_{\Phi,\infty} = N_{\Phi,n} \ltimes N_{\Phi,n,\infty}$.

Proposition 9.3. *Let* $\gamma \in \mathfrak{t}^*_{\Phi,\infty}$ *and* $\ell > n$ *so that* $\gamma_\ell|_{\mathfrak{s}_{\Phi,n}} = \gamma_n$. *Then* $\pi_{\Phi,\gamma,\ell}|_{N_{\Phi,n}}$ *is an infinite multiple of* $\pi_{\Phi,\gamma,n}$. *Split* $\mathcal{H}_{\pi_{\Phi,\gamma,\ell}} = \mathcal{H}'\widehat{\otimes}\mathcal{H}''$ *where* $\mathcal{H}' = \mathcal{H}_{\pi_{\Phi,\gamma,n}}$, *and where* $\mathcal{H}'' = \mathcal{H}_{\pi_{\Phi,\lambda,m_n+1}}\widehat{\otimes}\cdots\widehat{\otimes}\mathcal{H}_{\pi_{\Phi,\lambda,m_\ell}}$ *with* $\gamma_\ell = \lambda_1 + \cdots + \lambda_{m_\ell}$. *Choose a* C^∞ *unit vector* $e \in \mathcal{H}''$, *so*

$$\mathcal{H}_{\pi_{\Phi,\gamma,n}} \hookrightarrow \mathcal{H}_{\pi_{\Phi,\gamma,\ell}} \textit{ by } v \mapsto v \otimes e \tag{9.4}$$

is an N_n-equivariant isometric injection that sends C^∞ vectors to C^∞ vectors. If $u, v \in \mathcal{H}_{\pi_{\Phi,\gamma,n}}$ then

$$||f_{\pi_{\Phi,\gamma,\ell}; u\otimes e, v\otimes e}||^2_{L^2(N_{\Phi,\ell}/S_{\Phi,\ell})} = \tfrac{|P_n(\gamma,n)|}{|P_\ell(\gamma,\ell)|}||f_{\pi_{\Phi,\gamma,n}; u\otimes e, v\otimes e}||^2_{L^2(N_{\Phi,n}/S_{\Phi,n})} \tag{9.5}$$

Given $\gamma \in \mathfrak{t}^*_{\Phi,\infty}$ consider the unitary character $\zeta_\gamma = \exp(2\pi i\gamma)$ on $S_{\Phi,\infty}$, given by $\zeta_\gamma(\exp(\xi)) = e^{2\pi i\gamma(\xi)}$ for $\xi \in \mathfrak{s}_{\Phi,\infty}$. The corresponding Hilbert space is

$$L^2(N_{\Phi,\infty}/S_{\Phi,\infty}; \zeta_\gamma) = \varinjlim L^2(N_{\Phi,n}/S_{\Phi,n}; \zeta_{\gamma_n})$$

where

$$L^2(N_{\Phi,n}/S_{\Phi,n}; \zeta_{\gamma_n}) = \\ \{f : N_{\Phi,n} \to \mathbb{C} \mid f(gx) = \zeta_\gamma(x)^{-1} f(g) \text{ and } |f| \in L^2(N_{\Phi,n}/S_{\Phi,n}) \text{ for } x \in S_{\Phi,n}\}.$$

The finite linear combinations of the coefficients $f_{\pi_{\Phi,\gamma,n}; u,v}$, where $u, v \in \mathcal{H}_{\pi_{\Phi,\gamma,n}}$, are dense in $L^2(N_{\Phi,n})$. That gives us a $N_{\Phi,n} \times N_{\Phi,n}$ equivariant Hilbert space isomorphism

$$L^2(N_{\Phi,n}/S_{\Phi,n}; \zeta_{\gamma_n}) \cong \mathcal{H}_{\pi_{\Phi,\gamma,n}} \widehat{\otimes} \mathcal{H}^*_{\pi_{\Phi,\gamma,n}}.$$

The stepwise square integrable group $N_{\Phi,n}$ satisfies

$$L^2(N_{\Phi,n}) = \int_{\gamma_n \in \mathfrak{t}^*_{\Phi,n}} \mathcal{H}_{\pi_{\Phi,\gamma,n}} \widehat{\otimes} \mathcal{H}^*_{\pi_{\Phi,\gamma,n}} |P_n(\gamma)| d\gamma_n .$$

That expands functions on $N_{\Phi,\infty} = N_{\Phi,1}N_{\Phi,2}\dots$ that depend only on the first m_n factors. To increase the number of factors we must deal the renormalization implicit in (9.5). Reformulate (9.5):

$$p_{\gamma,n,\ell} : f_{\pi_{\Phi,\gamma,\ell}; u\otimes u', v\otimes v'} \mapsto \langle u', v'\rangle \tfrac{|P_n(\gamma_n)|}{|P_\ell(\gamma_\ell)|} f_{\pi_{\Phi,\gamma,n}; u,v} \tag{9.6}$$

is the orthogonal projection dual to $\mathcal{H}_{\pi_{\Phi,\gamma,n}} \hookrightarrow \mathcal{H}_{\pi_{\Phi,\gamma,\ell}}$. These maps sum over (γ_n, γ_ℓ) to a Hilbert space projection $p_{\ell,n} = \left(\int_{\gamma_\ell \in \mathfrak{s}_{\Phi,\ell}} p_{\gamma,n,\ell} d\gamma'\right)$,

$$p_{\ell,n} : L^2(N_{\Phi,\ell}) \to L^2(N_{\Phi,n}) \text{ for } \ell \geqq n. \tag{9.7}$$

The maps (9.7) define an inverse system in the category of Hilbert spaces and partial isometries:

$$L^2(N_{\Phi,1}) \xleftarrow{p_{2,1}} L^2(N_{\Phi,2}) \xleftarrow{p_{3,2}} L^2(N_{\Phi,3}) \xleftarrow{p_{4,3}} \ \dots \ \longleftarrow L^2(N_\Phi) \tag{9.8}$$

where the projective limit $L^2(N_\Phi) := \varprojlim\{L^2(N_{\Phi,n}), p_{\ell,n}\}$ is taken in that category. We now have the Hilbert space projective limit

$$L^2(N_\Phi) := \varprojlim\{L^2(N_{\Phi,n}), p_{\ell,n}\}. \tag{9.9}$$

Because of the renormalizations in (9.6), the elements of $L^2(N_\Phi)$ do not have an immediate interpretation as functions on N_Φ. We address that problem by looking at the Schwartz space.

The Schwartz space considerations of [23, Section 5] extend to our setting with only obvious modifications, so we restrict our discussion to the relevant definitions and results.

Given $\gamma = (\gamma_n) \in \mathfrak{t}^*_{\Phi,\infty}$ we have the unitary character $\zeta_\gamma = \exp(2\pi i\gamma)$ on $S_{\Phi,\infty} = \varinjlim S_{\Phi,n}$. Express $\zeta_\gamma = (\zeta_{\gamma_n}) \in \varprojlim \widehat{S_{\Phi,n}}$. The corresponding *relative Schwartz space* $\mathcal{C}((N_{\Phi,n}/S_{\Phi,n}); \zeta_{\gamma_n})$ consists of all functions $f \in C^\infty(N_{\Phi,n})$ such that

$$\begin{aligned} &f(xs) = \zeta_{\gamma_n}(s)^{-1} f(x) \text{ for } x \in N_{\Phi,n}\,, s \in S_{\Phi,n}\,, \text{ and } |q(g)p(D)f| \text{ bounded} \\ &\text{on } N_{\Phi,n}/S_{\Phi,n} \text{ for all polynomials } p, q \text{ on } N_{\Phi,n}/S_{\Phi,n} \text{ and all } D \in \mathcal{U}(\mathfrak{n}_{\Phi,n}). \end{aligned} \tag{9.10}$$

The corresponding *limit Schwartz space* $\mathcal{C}((N_{\Phi,\infty}/S_{\Phi,\infty}); \zeta_\gamma) = \varprojlim \mathcal{C}(N_{\Phi,n}/S_{\Phi,n}; \zeta_{\gamma_n})$, consisting of all functions $f \in C^\infty(N_{\Phi,\infty})$ such that

$$\begin{aligned} &f(xs) = \zeta_{\gamma}(s)^{-1} f(x) \text{ for } x \in N_{\Phi,\infty}\,, s \in S_{\Phi,\infty}\,, \text{ and } |q(g)p(D)f| \text{ bounded} \\ &\text{on } N_{\Phi,\infty}/S_{\Phi,\infty} \text{ for all polynomials } p, q \text{ on } N_{\Phi,\infty}/S_{\Phi,\infty}, \text{ all } D \in \mathcal{U}(\mathfrak{n}_{\Phi,\infty}). \end{aligned} \tag{9.11}$$

As expected, $\mathcal{C}(N_{\Phi,n}/S_{\Phi,n}; \zeta_{\gamma_n})$ is a nuclear Fréchet space and it is dense in $L^2(N_{\Phi,n}/S_{\Phi,n}; \zeta_{\gamma_n})$, and [23, Theorem 5.7] and its corollaries go through in our setting as follows.

Theorem 9.4. *Let $\gamma = (\gamma_n) \in \mathfrak{t}^*_{\Phi,\infty}$. Let $n > 0$ and let u and v be C^∞ vectors for the stepwise square integrable representation $\pi_{\Phi,\gamma,n}$ of $N_{\Phi,n}$. Then the coefficient function $f_{\pi_{\Phi,\gamma,n};u,v}$ belongs to the relative Schwartz space $\mathcal{C}((N_{\Phi,n}/S_{\Phi,n}); \zeta_{\gamma_n})$, and the coefficient function $f_{\pi_{\Phi,\gamma,\infty};u,v}$ belongs to the limit relative Schwartz space $\mathcal{C}(N_{\Phi,\infty}/S_{\Phi,\infty}; \zeta_\gamma)$.*

Corollary 9.5. *Let $\gamma = (\gamma_n) \in \mathfrak{t}^*_{\Phi,\infty}$. Let $n > 0$ and let u and v be C^∞ vectors for the stepwise square integrable representation $\pi_{\Phi,\gamma,n}$ of $N_{\Phi,n}$. Then the coefficient function $f_{\pi_{\Phi,\gamma,n};u,v} \in L^1(N_{\Phi,n}/S_{\Phi,n}; \zeta_{\gamma_n})$, and the coefficient function $f_{\pi_{\Phi,\gamma,\infty};u,v} \in \varprojlim L^1(N_{\Phi,n}/S_{\Phi,n}; \zeta_{\gamma_n})$.*

In fact the argument shows

Corollary 9.6. *Let L be a connected simply connected nilpotent Lie group, Z its center, and $\lambda \in \mathfrak{z}^*$ such that π_λ is a square integrable (mod Z) representation of L. Let $\zeta = e^{2\pi i\lambda} \in \widehat{Z}$ and let u and v be C^∞ vectors for π_λ . Then the coefficient $f_{\pi_\lambda;u,v} \in L^1(L/Z, \zeta_\lambda)$.*

A norm $|\xi|$ on $\mathfrak{n}_{\Phi,n}$ corresponds to a norm $||\exp(\xi)|| := ||\xi||$ on $N_{\Phi,n}$. Thus classical Schwartz space $\mathcal{C}(\mathfrak{n}_{\Phi,n})$ on the real vector space $\mathfrak{n}_{\Phi,n}$, corresponds to the Schwartz space $\mathcal{C}(N_{\Phi,n})$, which thus is defined by seminorms

$$\nu_{k,D,n}(f) = \sup_{x \in N_{\Phi,n}} |(1 + |x|^2)^k (Df)(x)|. \tag{9.12}$$

Here k is a positive integer, and $D \in \mathcal{U}(\mathfrak{n}_{\Phi,n})$ is a differential operator acting on the left on $N_{\Phi,n}$. Since $\exp : \mathfrak{n}_{\Phi,n} \to N_{\Phi,n}$ is a polynomial diffeomorphism, $f \mapsto f \cdot \exp$ is a topological isomorphism of $\mathcal{C}(N_{\Phi,n})$ onto $\mathcal{C}(\mathfrak{n}_{\Phi,n})$:

$$\mathcal{C}(N_{\Phi,n}) = \{f \in C^\infty(N_{\Phi,n}) \mid f \circ \exp \in \mathcal{C}(\mathfrak{n}_{\Phi,n})\}. \tag{9.13}$$

We now define the Schwartz space

$$\mathcal{C}(N_{\Phi,\infty}) = \{f \in C^\infty(N_{\Phi,\infty}) \mid f|_{N_{\Phi,n}} \in \mathcal{C}(N_{\Phi,n}) \text{ for } n \gg 0\} = \varprojlim \mathcal{C}(N_{\Phi,n}) \quad (9.14)$$

where the inverse limit is taken in the category of complete locally convex topological vector spaces and continuous linear maps. Since $\mathcal{C}(N_{\Phi,n})$ is defined by the seminorms (9.12), the same follows for $\mathcal{C}(N_{\Phi,\infty})$. In other words,

Lemma 9.7. *The Schwartz space $\mathcal{C}(N_{\Phi,\infty})$ consists of all $f \in C^\infty(N_{\Phi,\infty})$ such that, for all $n > 0$, $\nu_{k,D,n}(f)$ is bounded for all integers $k > 0$ all $D \in \mathcal{U}(\mathfrak{n}_{\Phi,n})$. Here the seminorms $\nu_{k,D,n}$ are given by* (9.12).

Every $f \in \mathcal{C}(N_{\Phi,n})$ is a limit in $\mathcal{C}(N_{\Phi,n})$ of finite linear combinations of the functions $f_{\gamma_n}(x) = \int_{S_{\Phi,n}} f(xs)\zeta_{\gamma_n}(s)ds$ in $\mathcal{C}((N_{\Phi,n}/S_{\Phi,n}), \zeta_{\gamma_n})$. Specifically, denote $\varphi_x(\gamma_n) := f_{\gamma_n}(x)$. Then φ_x is a multiple of the Fourier transform $\mathcal{F}_{\Phi,n}(\ell(x)^{-1}f)|_{S_{\Phi,n}})$.

The inverse Fourier transform $\mathcal{F}^{-1}_{\Phi,n}(\varphi_x)$ reconstructs f from the f_{γ_n}. Since the relative Schwartz space $\mathcal{C}((N_{\Phi,}/S_{\Phi,n}); \zeta_{\gamma_n})$ is dense in $L^2(N_{\Phi,n}/S_{\Phi,n}, \zeta_{\gamma_n})$ and the the set of finite linear combinations of coefficients $f_{\pi_{\Phi,\gamma,n};u,v}$ (where u, v are C^∞ vectors) is dense in $\mathcal{C}(N_{\Phi,n}/S_{\Phi,n}, \zeta_{\gamma_n})$, now every $f \in \mathcal{C}(N_{\Phi,n})$ is a Schwartz wave packet along $\mathfrak{s}^*_{\Phi,n}$ of coefficients of the various $\pi_{\Phi,\gamma,n}$, u and v smooth. Now we combine the inverse system (9.8) and its Schwartz space analog.

$$\begin{tikzcd}
\mathcal{C}(N_{\Phi,1}) \arrow[d, hook, "r_1"] & \mathcal{C}(N_{\Phi,2}) \arrow[l, "q_{2,1}"'] \arrow[d, hook, "r_2"] & \mathcal{C}(N_{\Phi,3}) \arrow[l, "q_{3,2}"'] \arrow[d, hook, "r_3"] & \dots \arrow[l, "q_{4,3}"'] \arrow[d] & \mathcal{C}(N_\Phi) = \varprojlim \mathcal{C}(N_{\Phi,n}) \arrow[l] \arrow[d, hook, "r_\infty"] \\
L^2(N_{\Phi,1}) & L^2(N_{\Phi,2}) \arrow[l, "p_{2,1}"'] & L^2(N_{\Phi,3}) \arrow[l, "p_{3,2}"'] & \dots \arrow[l, "p_{4,3}"'] & L^2(N_\Phi) = \varprojlim L^2(N_{\Phi,n}) \arrow[l]
\end{tikzcd} \quad (9.15)$$

The $r_n : \mathcal{C}(N_{\Phi,n}) \hookrightarrow L^2(N_{\Phi,n})$ are continuous injections with dense image, so $r_\infty : \mathcal{C}(N_{\Phi,\infty}) \hookrightarrow L^2(N_{\Phi,\infty})$ is a continuous injection with dense image. Putting all this together as in the minimal parabolic case [23, Section 5], we have proved

Proposition 9.8. *Assume* (8.4)(a), *so that $\{N_{\Phi,n}\}$ is a direct system and $N_{\Phi,\infty} = \varinjlim N_{\Phi,n}$ is well defined. Define $r_\infty : \mathcal{C}(N_{\Phi,\infty}) \hookrightarrow L^2(N_{\Phi,\infty})$ as in the commutative diagram* (9.15). *Then $L^2(N_{\Phi,\infty})$ is a Hilbert space completion of $\mathcal{C}(N_{\Phi,\infty})$. In particular r_∞ defines a pre-Hilbert space structure on $\mathcal{C}(N_{\Phi,\infty})$ with completion $L^2(N_{\Phi,\infty})$.*

As in [23, Corollary 5.17], $\mathcal{C}(N_{\Phi,\infty})$ is independent of the choices made in the construction of $L^2(N_{\Phi,\infty})$, so

Corollary 9.9. *The limit Hilbert space $L^2(N_{\Phi,\infty}) = \varprojlim\{L^2(N_{\Phi,n}), p_{\ell,n}\}$ of* (9.15) *, and the left/right regular representation of $N_{\Phi,\infty} \times N_{\Phi,\infty}$ on $L^2(N_{\Phi,\infty})$, are independent of the choice of vectors e in* (9.4).

Recall the notation

- $\mathfrak{t}^*_{\Phi,\infty} := \varprojlim \mathfrak{t}^*_{\Phi,n} = \{\gamma = (\gamma_n) \mid \gamma_n \in \mathfrak{t}^*_{\Phi,n} \text{ and if } \ell \geqq n \text{ then } \gamma_\ell|_{\mathfrak{s}_{\Phi,n}} = \gamma_n\}$.

- if $\gamma = (\gamma_n) \in \mathfrak{t}^*_{\Phi,\infty}$ then $\pi_{\Phi,\gamma,\infty} = \varprojlim \pi_{\Phi,\gamma_n}$ is constructed as in Section 7,
- The distribution characters $\Theta_{\pi_{\Phi,\gamma,n}}$ on the $N_{\Phi,n}$ are given by (1.2), and
- $\mathcal{C}(N_{\Phi,\infty}) = \varprojlim \mathcal{C}(N_{\Phi,n}) = \{f = (f_n) \mid f_n \in \mathcal{C}(N_{\Phi,n})\,, f_\ell|_{N_{\Phi,n}} = f_n \text{ for } \ell \geqq n\}$.

As for minimal parabolics [23, Section 6], the limit Fourier inversion formula is

Theorem 9.10. *Suppose that $N_{\Phi,\infty} = \varinjlim N_{\Phi,n}$ where $\{N_{\Phi,n}\}$ satisfies* (7.2)*. Let $f = (f_n) \in \mathcal{C}(N_{\Phi,\infty})$ and $x \in N_{\Phi,\infty}$. Then $x \in N_{\Phi,n}$ for some n and*

$$f(x) = c_n \int_{\mathfrak{t}^*_{\Phi,n}} \Theta_{\pi_{\Phi,\gamma,n}}(r_x f)|\mathrm{Pf}_{\mathfrak{n}_{\Phi,n}}(\gamma_n)|d\gamma_n \tag{9.16}$$

where $c_n = 2^{d_1+\dots+d_m} d_1!d_2!\dots d_m!$ as in (1.6a) *and m is the number of factors L_r in $N_{\Phi,n}$.*

Proof. By Theorem 1.6, $f(x) = f_n(x) = c_n \int_{\mathfrak{t}^*_{\Phi,n}} \Theta_{\pi_{\Phi,\gamma,n}}(r_x f)|\mathrm{Pf}_{\mathfrak{n}_{\Phi,n}}(\gamma_n)|\,d\gamma_n$. ■

10. Representations of the Limit Groups II: $A_{\Phi,\infty}N_{\Phi,\infty}$

We extend some of the results of Section 9 to the maximal exponential (locally) solvable subgroup $A_{\Phi,\infty}N_{\Phi,\infty}$.

The first step is to locate the $A_{\Phi,\infty}$-stabilizer of a limit square integrable representation π_γ of $N_{\Phi,\infty}$. Following (5.3) we set

$$A'_{\Phi,\infty} = \{\exp(\xi) \mid \xi \in \mathfrak{a}_{\Phi,\infty} \text{ and every } \beta_j(\xi) = 0\}. \tag{10.1}$$

Lemma 10.1. *If $\gamma = (\gamma_n) \in \mathfrak{t}^*_{\Phi,\infty}$ then $A'_{\Phi,\infty}$ is the stabilizer of π_γ in $A_{\Phi,\infty}$.*

Proof. Recall the J''_r from Lemma 3.3. Then Lemma 3.4 tells us that, for each r_0 , $\mathfrak{l}_{\Phi,r_0}$ has center

$$\mathfrak{z}_{\Phi,r_0} = \sum\nolimits_{\beta_r|\mathfrak{a}_\Phi = \beta_{r_0}|\mathfrak{a}_\Phi} \left(\mathfrak{g}_{\beta_r} + \sum\nolimits_{J''_r} \mathfrak{g}_\alpha\right),$$

and Lemma 3.3 then says that $\mathfrak{z}_{\Phi,r}$ is an $\mathrm{ad}\,(\mathfrak{a}_\Phi)$ eigenspace on $\mathfrak{g}$. Thus the $\mathrm{ad}^*(\mathfrak{a}_\Phi)$-stabilizer of γ is given by $\beta_r(\mathfrak{a}_{\Phi,\infty}) = 0$ for all r. ■

Lemma 5.1 shows that our methods cannot yield a Dixmier-Pukánszky operator for $A_{\Phi,\infty}N_{\Phi,\infty}$ nor for $U_{\Phi,\infty}A_{\Phi,\infty}N_{\Phi,\infty}$, but we do have such operators D_n for the $A_{\Phi,n}N_{\Phi,n}$ and the $U_{\Phi,n}A_{\Phi,n}N_{\Phi,n}$.

Let $\gamma = (\gamma_n) \in \mathfrak{t}^*_{\Phi,\infty}$. Then $\pi_{\Phi,\gamma,\infty}$ extends from $N_{\Phi,\infty}$ to a representation $\pi^\dagger_{\Phi,\gamma,\infty}$ of $A'_{\Phi,\infty}N_{\Phi,\infty}$ with the same representation space, because every $\pi_{\Phi,\gamma,n}$ extends that way from $N_{\Phi,n}$ to $A'_{\Phi,n}N_{\Phi,n}$. The representations of $A'_{\Phi,n}N_{\Phi,n}$ corresponding to γ_n are the $\exp(2\pi i\xi|_{\mathfrak{a}'_{\Phi,n}}) \otimes \pi^\dagger_{\Phi,\gamma,n}$. The representation of $A_{\Phi,\infty}N_{\Phi,\infty}$ and the $A_{\Phi,n}N_{\Phi,n}$, corresponding to γ and $\xi = (\xi_n) \in (\mathfrak{a}'_{\Phi,\infty})^*$, is the

$$\pi_{\Phi,\gamma,\xi,\infty} := \varinjlim \pi_{\Phi,\gamma,\xi,n} \text{ where } \pi_{\Phi,\gamma,\xi,n} = \mathrm{Ind}_{A'_{\Phi,n}N_{\Phi,n}}^{A_{\Phi,n}N_{\Phi,n}} \left(\exp(2\pi i\xi_n) \otimes \pi^\dagger_{\Phi,\gamma,n}\right). \tag{10.2}$$

If $\dim(A_{\Phi,\infty}/A'_{\Phi,\infty}) < \infty$, (10.2) says $\pi_{\Phi,\gamma,\xi,\infty} = \mathrm{Ind}\,_{A'_{\Phi,\infty}N_{\Phi,\infty}}^{A_{\Phi,\infty}N_{\Phi,\infty}}(\exp(2\pi i\xi)\otimes\pi^\dagger_{\Phi,\gamma,\infty})$, because then one can integrate over $A_{\Phi,\infty}/A'_{\Phi,\infty}$. Or in general one may view (10.2) as an interpretation of $\pi_{\Phi,\gamma,\xi,\infty} = \mathrm{Ind}\,_{A'_{\Phi,\infty}N_{\Phi,\infty}}^{A_{\Phi,\infty}N_{\Phi,\infty}}(\exp(2\pi i\xi)\otimes\pi^\dagger_{\Phi,\gamma,\infty})$.

Lemma 10.2. *Let $\gamma,\gamma' \in \mathfrak{t}^*_{\Phi,\infty}$. Then the representations $\pi_{\Phi,\gamma,\xi,\infty}$ and $\pi_{\Phi,\gamma',\xi',\infty}$ are equivalent if and only if both $\xi' = \xi$ and $\gamma' \in \mathrm{Ad}^*(A_{\Phi,\infty})(\gamma)$. Express $\gamma = (\gamma_n)$ with $\gamma_n = \sum_{j=1}^{m_n}\gamma_{n,j}$ where $\gamma_{n,j} \in \mathfrak{z}^*_j$. Then $\mathrm{Ad}^*(A_{\Phi,\infty})(\gamma)$ consists of all $\left(\sum_{j=1}^{m_n} c_j\gamma_{n,j}\right)$ with every $c_j > 0$.*

Proof. The Mackey little group method implies $\pi_{\Phi,\gamma,\xi,n} \simeq \pi_{\Phi,\gamma',\xi',n}$ just when $\xi' = \xi$ and $\gamma'_n \in \mathrm{Ad}^*(A_{\Phi,n})(\gamma_n)$. The first assertion follows. The second is because the action of $\mathrm{Ad}^*(A_{\Phi,\infty})$ on $\gamma_{n,j}$ is multiplication by an arbitrary positive real $c_j = \exp(i\beta_j(\alpha))$ for $\alpha \in \mathfrak{a}_{\Phi,\infty}$. ∎

The representation space $\mathcal{H}_{\pi_{\Phi,\gamma,\xi,\infty}}$ of $\pi_{\Phi,\gamma,\xi,\infty} \in (A_{\Phi,\infty}N_{\Phi,\infty})\widehat{\ }$ is the same as that of $N_{\Phi,\infty}$, except for the unitary character $\exp(2\pi i\xi)$. We thus obtain

$$L^2((A_{\Phi,n}N_{\Phi,n}/A'_{\Phi,n}S_{\Phi,n});(\exp(2\pi i\xi)\otimes\zeta_n)) \cong (\mathcal{H}_{\pi_{\Phi,\gamma,\xi,n}}\widehat{\otimes}\mathcal{H}^*_{\pi_{\Phi,\gamma,\xi,n}}).$$

Summing over $\mathfrak{t}^*_{\Phi,n}$ and $\mathfrak{a}_{\Phi,n}/\mathfrak{a}'_{\Phi,n}$ we proceed as in Section 9; then

$$L^2(A_{\Phi,n}N_{\Phi,n}) = \int_{\mathfrak{a}_{\Phi,n}/\mathfrak{a}'_{\Phi,n}}\int_{\mathfrak{t}^*_{\Phi,n}} (\mathcal{H}_{\pi_{\Phi,\gamma,\xi,n}}\widehat{\otimes}\mathcal{H}^*_{\pi_{\Phi,\gamma,\xi,n}})|P_n(\gamma_n)|d\gamma_n\,d\xi$$

so, as in (9.8) and (9.9),

$$L^2(A_{\Phi,\infty}N_{\Phi,\infty}) = \varinjlim \int_{\mathfrak{a}_{\Phi,n}/\mathfrak{a}'_{\Phi,n}}\int_{\mathfrak{t}^*_{\Phi,n}} (\mathcal{H}_{\pi_{\Phi,\gamma,\xi,n}}\widehat{\otimes}\mathcal{H}^*_{\pi_{\Phi,\gamma,\xi,n}})|P_n(\gamma_n)|d\gamma_n\,d\xi.$$

Since the base spaces of the unitary line bundles

$$A_{\Phi,n}N_{\Phi,n} \to A_{\Phi,n}N_{\Phi,n}/A'_{\Phi,n}S_{\Phi,n} \text{ and } N_{\Phi,n} \to N_{\Phi,n}/S_{\Phi,n}$$

are similar, we modify (9.10) for the *relative Schwartz space*

$$\mathcal{C}((A_{\Phi,n}N_{\Phi,n}/A'_{\Phi,n}S_{\Phi,n});\exp(2\pi i\xi)\otimes\zeta_{\gamma_n})$$

to consist of all functions $f \in C^\infty(A_{\Phi,n}N_{\Phi,n})$ such that

$$\begin{aligned} f(xas) = &\exp(-2\pi i\xi(\log a))\zeta_{\gamma_n}(s)^{-1}f(x) \text{ for } x \in N_{\Phi,n}, a \in A'_{\Phi,n}, s \in S_{\Phi,n})\\ &\text{with } |q(g)p(D)f| \text{ bounded on } A_{\Phi,n}N_{\Phi,n}/A'_{\Phi,n}S_{\Phi,n}\\ &\text{for all polynomials } p,q \text{ on } N_{\Phi,n}/S_{\Phi,n} \text{ and all } D \in \mathcal{U}(\mathfrak{n}_{\Phi,n}).\end{aligned} \tag{10.3}$$

The corresponding *limit relative Schwartz space* is

$$\begin{aligned}\mathcal{C}((A_{\Phi,\infty}N_{\Phi,\infty}/A'_{\Phi,\infty}S_{\Phi,\infty});&(\exp(2\pi i\xi)\otimes\zeta_\gamma))\\ &= \varprojlim \mathcal{C}((A_{\Phi,n}N_{\Phi,n}/A'_{\Phi,n}S_{\Phi,n});(\exp(2\pi i\xi)\otimes\zeta_{\gamma_n})),\end{aligned}$$

consisting of all functions $f \in C^\infty(A_{\Phi,\infty}N_{\Phi,\infty})$ such that

$$\begin{aligned} f(xas) = \exp(-2\pi i\xi(\log a))\zeta_\gamma(s)^{-1}f(x) \text{ for } x \in N_{\Phi,\infty}, a \in A'_{\Phi,\infty} \text{ and} \\ s \in S_{\Phi,\infty}\,, \text{ and } |q(g)p(D)f| \text{ is bounded on } A_{\Phi,\infty}N_{\Phi,\infty}/A'_{\Phi,\infty}S_{\Phi,\infty} \\ \text{for all polynomials } p, q \text{ on } N_{\Phi,\infty}/S_{\Phi,\infty} \text{ and all } D \in \mathcal{U}(\mathfrak{n}_{\Phi,\infty}). \end{aligned} \tag{10.4}$$

Theorem 9.4 and Corollaries 9.5 and 9.6 hold for our groups $A_{\Phi,\bullet}N_{\Phi,\bullet}$ here with essentially no change, so we will not repeat them.

We use Casselman's extension [1, p. 4] of the classical definition for seminorms and Schwartz space (which we used for $\mathfrak{n}_{\Phi,n}$). First, we have seminorms on the $\mathfrak{a}_{\Phi,n} + \mathfrak{n}_{\Phi,n}$ as in (9.12) as follows. Fix a continuous norm $||\varphi||$ on $A_{\Phi,n}N_{\Phi,n}$ such that

$$\begin{aligned} ||1_{A_{\Phi,n}N_{\Phi,n}}|| = 1, ||x|| = ||x^{-1}|| \geqq 1 \text{ for all } x, \text{ and} \\ ||x||/||y|| \leqq ||xy|| \leqq ||x||\,||y|| \text{ for all } x, y. \end{aligned} \tag{10.5}$$

That gives seminorms

$$\nu_{k,D,n}(f) = \sup\nolimits_{x \in A_{\Phi,n}N_{\Phi,n}} ||x||^k|Df(x)| (k > 0 \text{ and } D \in \mathcal{U}(\mathfrak{a}_{\Phi,n} + \mathfrak{n}_{\Phi,n})). \tag{10.6}$$

That defines the Schwartz space $\mathcal{C}(A_{\Phi,n}N_{\Phi,n})$ as in (9.13):

$$\mathcal{C}(A_{\Phi,n}N_{\Phi,n}) = \{f \in C^\infty(A_{\Phi,n}N_{\Phi,n}) \mid \nu_{k,D,n}(f) < \infty\} \tag{10.7}$$

for all $k > 0$ and $D \in \mathcal{U}(\mathfrak{a}_{\Phi,n} + \mathfrak{n}_{\Phi,n})$. Finally we define $\mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty})$ to be the inverse limit in the category of locally convex topological vector spaces and continuous linear maps, as in (9.14):

$$\begin{aligned} &\mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty}) = \\ &\quad \{f \in C^\infty(A_{\Phi,\infty}N_{\Phi,\infty}) \mid f|_{A_{\Phi,n}N_{\Phi,n}} \in \mathcal{C}(A_{\Phi,n}N_{\Phi,n})\} = \varprojlim \mathcal{C}(A_{\Phi,n}N_{\Phi,n}). \end{aligned} \tag{10.8}$$

Lemma 10.3. ([1, Proposition 1.1]) *The Schwartz space $\mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty})$ consists of all functions $f \in C^\infty(A_{\Phi,\infty}N_{\Phi,\infty})$ such that, for all $n > 0$, $\nu_{k,D,n}(f) < \infty$ for all integers $k > 0$ and all $D \in \mathcal{U}(\mathfrak{a}_{\Phi,n} + \mathfrak{n}_{\Phi,n})$. Here $\nu_{k,D,n}$ is given by* (10.6). *The $\mathcal{C}(A_{\Phi,n}N_{\Phi,n})$ are nuclear Fréchet spaces and $\mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty})$ is an LF space. The left/right actions of $(A_{\Phi,n}N_{\Phi,n} \times A_{\Phi,n}N_{\Phi,n})$ on $\mathcal{C}(A_{\Phi,n}N_{\Phi,n})$ and of $(A_{\Phi,\infty}N_{\Phi,\infty} \times A_{\Phi,\infty}N_{\Phi,\infty})$ on $\mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty})$ are continuous.*

As for the $N_{\Phi,n}$, if $f \in \mathcal{C}(A_{\Phi,n}N_{\Phi,n})$ it is a limit in $\mathcal{C}(A_{\Phi,n}N_{\Phi,n})$ of finite linear combinations of the $f_{\xi,\gamma,n}(x) = \int_{A_{\Phi,n}}\int_{S_{\Phi,n}} f(xas)\exp(2\pi i\xi(\log a))\zeta_{\gamma_n}(s)ds da$ in $\mathcal{C}((A_{\Phi,n}N_{\Phi,n}/A'_{\Phi,n}S_{\Phi,n}), (\exp(2\pi i\xi)\zeta_{\gamma_n}))$. Specifically, denote $\varphi_x(\xi, \gamma_n) := f_{\xi,\gamma_n}(x)$. Then φ_x is a multiple of the classical Fourier transform $\mathcal{F}_{\Phi,n}(\ell(x)^{-1}f)|_{A_{\Phi,n}S_{\Phi,n}})$, and the inverse Fourier transform $\mathcal{F}^{-1}_{\Phi,n}(\varphi_x)$ reconstructs f from the f_{ξ,γ_n}.

The relative Schwartz space $\mathcal{C}((A_{\Phi,n}N_{\Phi,n}/A'_{\Phi,n}S_{\Phi,n}), (\exp(2\pi i\xi)\zeta_{\gamma_n}))$ is dense in $L^2((A_{\Phi,n}N_{\Phi,n}/A'_{\Phi,n}S_{\Phi,n}), (\exp(2\pi i\xi)\zeta_{\gamma_n}))$. Finite linear combinations of coefficients of the $\pi_{\Phi,\gamma,\xi,n}$ along C^∞ vectors form dense subset of $\mathcal{C}((A_{\Phi,n}N_{\Phi,n}/A'_{\Phi,n}S_{\Phi,n}), (\exp(2\pi i\xi)\zeta_{\gamma_n}))$. So every $f \in \mathcal{C}(A_{\Phi,n}N_{\Phi,n})$ is a Schwartz wave packet along $\mathfrak{a}^*_{\Phi,n} + \mathfrak{s}^*_{\Phi,n}$ of coefficients of the various $\pi_{\Phi,\gamma,\xi,n}$, and the corresponding inverse systems fit together as in (9.15):

$$\begin{array}{ccccccc}
\mathcal{C}(A_{\Phi,1}N_{\Phi,1}) & \xleftarrow{q_{2,1}} & \mathcal{C}(A_{\Phi,2}N_{\Phi,2}) & \xleftarrow{q_{3,2}} & \dots & \longleftarrow & \mathcal{C}(A_\Phi N_\Phi) = \varprojlim \mathcal{C}(A_{\Phi,n}N_{\Phi,n}) \\
\downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & \downarrow & & \downarrow{\scriptstyle r_\infty} \\
L^2(A_{\Phi,1}N_{\Phi,1}) & \xleftarrow{p_{2,1}} & L^2(A_{\Phi,2}N_{\Phi,2}) & \xleftarrow{p_{3,2}} & \dots & \longleftarrow & L^2(A_\Phi N_\Phi) = \varprojlim L^2(A_{\Phi,n}N_{\Phi,n})
\end{array} \tag{10.9}$$

As before, the r_n are continuous injections with dense image, and it follows that $r_\infty : \mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty}) \hookrightarrow L^2(A_{\Phi,\infty}N_{\Phi,\infty})$ is a continuous injection with dense image. As in Proposition 9.8 we conclude

Proposition 10.4. *Assume* (8.4)(a)*, so* $\{A_{\Phi,n}N_{\Phi,n}\}$ *is a direct system and* $A_{\Phi,\infty}N_{\Phi,\infty} = \varinjlim A_{\Phi,n}N_{\Phi,n}$ *is well defined. Let*

$$r_\infty : \mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty}) \hookrightarrow L^2(A_{\Phi,\infty}N_{\Phi,\infty})$$

as in (10.9)*. Then* $L^2(A_{\Phi,\infty}N_{\Phi,\infty})$ *is a Hilbert space completion of* $\mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty})$*. In particular* r_∞ *defines a pre-Hilbert space structure on* $\mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty})$ *with completion* $L^2(A_{\Phi,\infty}N_{\Phi,\infty})$*.*

Corollary 10.5. *The Hilbert space* $L^2(A_{\Phi,\infty}N_{\Phi,\infty}) = \varprojlim\{L^2(A_{\Phi,n}N_{\Phi,n}), p_{\ell,n}\}$ *of* (10.9) *, and the left/right regular representation of* $(A_{\Phi,\infty}N_{\Phi,\infty}) \times (A_{\Phi,\infty}N_{\Phi,\infty})$ *on* $L^2(A_{\Phi,\infty}N_{\Phi,\infty})$*, are independent of the choice of* C^∞ *unit vectors* e *in the inclusions* $\mathcal{H}_{\pi,\gamma,\xi,n} \hookrightarrow \mathcal{H}_{\pi,\gamma,\xi,\ell}$ *,* $\ell \geqq n$*, by* $v \mapsto v \otimes e$*.*

The distribution characters $\Theta_{\pi_{\Phi,\gamma,\xi,n}} = \exp(2\pi i\xi)\Theta_{\pi_{\Phi,\gamma,n}}$ where $\Theta_{\pi_{\Phi,\gamma,n}}$ is given by (1.2). The limit Schwartz space $\mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty}) = \varprojlim \mathcal{C}(A_{\Phi,n}N_{\Phi,n})$ consists of all $f = (f_n)$ where each $f_n \in \mathcal{C}(A_{\Phi,n}N_{\Phi,n})$. As in the case of minimal parabolics [23, Section 6], the limit Fourier inversion formula is

Theorem 10.6. *Suppose that* $(\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell)$ *for* $\ell \geqq n$*, so that* $A_{\Phi,\infty}N_{\Phi,\infty} = \varinjlim A_{\Phi,n}N_{\Phi,n}$ *is well defined. Let* D_n *be a Dixmier-Pukánszky operator for* $A_{\Phi,n}N_{\Phi,n}$ *. Let* $f = (f_n) \in \mathcal{C}(A_{\Phi,n}N_{\Phi,n})$ *and* $x \in A_{\Phi,\infty}N_{\Phi,\infty}$ *. Then* $x \in A_{\Phi,n}N_{\Phi,n}$ *for some* n *and*

$$f(x) = c_n \int_{\xi \in (\mathfrak{a}'_{\Phi,n})^*} \int_{\mathfrak{s}^*_{\Phi,n}/\mathrm{Ad}^*(A_{\Phi,n})} \Theta_{\pi_{\Phi,\gamma,\xi,n}}(D_n(r_x f)) |\mathrm{Pf}_{\mathfrak{n}_n}(\gamma_n)| d\gamma_n d\xi \tag{10.10}$$

where $c_n = (\frac{1}{2\pi})^{\dim \mathfrak{a}'_\Phi/2}\, 2^{d_1+\cdots+d_{m_n}} d_1! d_2! \dots d_{m_n}!$ *as in* (1.6a) *and* m_n *is the number of factors* L_r *in* $N_{\Phi,n}$*.*

Proof. Apply Theorem 5.8 to $A_{\Phi,n}N_{\Phi,n}$. ∎

11. Representations of the Limit Groups III: $U_{\Phi,\infty}$

We are going to study highest weight limit representations of $U_{\Phi,\infty} = \varinjlim U_{\Phi,n}$. These are the representations for which there is an explicit Peter-Weyl Theorem

[18, Theorem 4.3]. We restrict our attention to highest weight representations of $U_{\Phi,\infty}$ for a good reason: as noted in papers ([13], [14], [15] and [16]) of Strătilă and Voiculescu, irreducible unitary representations of $U(\infty)$ and other lim-compact groups can be extremely complicated, even of Type III. This is summarized in [21, Section 9].

Recall from Section 8 that G_ℓ propagates G_n for $\ell \geqq n$. In particular $\Phi = (\Phi_n)$ where Φ_n is the simple root system for $(\mathfrak{m}_{\Phi,n} + \mathfrak{a}_{\Phi,n})_{\mathbb{C}}$ and $\Phi_n \subset \Phi_\ell$ for $\ell \geqq n$. It is implicit that the maximal compact subgroups $K_m \subset G_m$ satisfy $K_n \subset K_\ell$ for $\ell \geqq n$, so $K := \varinjlim K_n$ is a maximal lim–compact subgroup of $G := \varinjlim G_n$. We decompose the Cartan subalgebras of $\mathfrak{g}_{\Phi,n}$ and $\mathfrak{g}_{\Phi,\infty}$ along the lines of the proof of Lemma 6.3, as follows:

$$\mathfrak{h}_n = \mathfrak{c}_{\Phi,n} + \mathfrak{b}_{\Phi,n} + \mathfrak{a}_{\Phi,n} \text{ and } \mathfrak{h}_\infty = \mathfrak{c}_{\Phi,\infty} + \mathfrak{b}_{\Phi,\infty} + \mathfrak{a}_{\Phi,\infty} \tag{11.1}$$

where $\mathfrak{a}_{\Phi,n}$ is as before, $\mathfrak{c}_{\Phi,n}+\mathfrak{b}_{\Phi,n}$ is a Cartan subalgebra of $\mathfrak{m}_{\Phi,n}$, $\mathfrak{b}_{\Phi,n}+\mathfrak{a}_{\Phi,n} = \mathfrak{a}_n$, and $\mathfrak{c}_{\Phi,n}$ is a Cartan subalgebra of $\mathfrak{u}_{\Phi,n}$. Then $\mathfrak{a}_{\Phi,\infty} = \varinjlim \mathfrak{a}_{\Phi,n}$, $\mathfrak{b}_{\Phi,\infty} = \varinjlim \mathfrak{b}_{\Phi,n}$, and $\mathfrak{c}_{\Phi,\infty} = \varinjlim \mathfrak{c}_{\Phi,n}$. Further, $\mathfrak{a}_{\Phi,n} = \mathfrak{b}_{\Phi,n} + \mathfrak{a}_{\Phi,n}$ and $\mathfrak{c}_{\Phi,n} = \mathfrak{h}_n \cap \mathfrak{k}_n$. Notice that $C_{\Phi,n} := \exp(\mathfrak{c}_{\Phi,n})$ is a maximal torus in $U^0_{\Phi,n}$.

We define a simple root system for $\mathfrak{u}_{\Phi,n}$ along the lines of an idea of Borel and de Siebenthal. Let $\{\mathfrak{m}^{(i)}_{\Phi,n}\}$ be the simple ideals in $\mathfrak{m}_{\Phi,n}$ and let $\{\Phi^{(i)}_n\}$ denote the corresponding subsets of Φ_n . If every root in $\Phi^{(i)}_n$ is compact we set $\Sigma^{(i)}_n = \Phi^{(i)}_n$. Otherwise $\Phi^{(i)}_n$ contains just one noncompact root, say $\alpha^{(i)}_n$. let $\beta^{(i)}_n$ denote the maximal root of $\mathfrak{m}^{(i)}_{\Phi,n}$. If $\alpha^{(i)}_n$ has coefficient 1 as a summand of $\beta^{(i)}_n$ we set $\Sigma^{(i)}_n = \Phi^{(i)}_n \setminus \{\alpha^{(i)}_n\}$. If it has coefficient 2 as a summand of $\beta^{(i)}_n$ we set $\Sigma^{(i)}_n = (\Phi^{(i)}_n \setminus \{\alpha^{(i)}_n\}) \cup \{-\beta^{(i)}_n\}$. Now $\Sigma_n := \bigcup \Sigma^{(i)}_n$ is a simple root system $\mathfrak{u}_{\Phi,n}$ and for its semisimple part $[\mathfrak{u}_{\Phi,n}, \mathfrak{u}_{\Phi,n}]$.

Lemma 11.1. *If $\ell \geqq n$ then $\Sigma_{\Phi,n} \subset \Sigma_{\Phi,\ell}$. Thus $\Sigma_\Phi := \bigcup \Sigma_{\Phi,n}$ is a simple root system for the semisimple part $[\mathfrak{u}_{\Phi,\infty}, \mathfrak{u}_{\Phi,\infty}] := \varinjlim [\mathfrak{u}_{\Phi,n}, \mathfrak{u}_{\Phi,n}]$ of $\mathfrak{u}_{\Phi,\infty}$.*

Proof. If $\alpha \in \Sigma_{\Phi,n}$ is not simple as a root of $\mathfrak{u}_{\Phi,n+1}$, then, as a linear combination of roots in Φ_{n+1} , it must involve a root from $\Phi_{n+1} \setminus \Phi_n$. That contradicts the fact that $\alpha \in \Delta((\mathfrak{m}_{\Phi,n})_{\mathbb{C}}, (\mathfrak{c}_{\Phi,n} + \mathfrak{b}_{\Phi,n})_{\mathbb{C}})$. ■

As in Lemma 6.3 we define $F_n = \exp(i\mathfrak{a}_n) \cap K_n$. It is an elementary abelian 2-subgroup of $U_{\Phi,n}$, central in both $U_{\Phi,n}$ and $M_{\Phi,n}$, and has the properties

$$U_{\Phi,n} = F_n U^0_{\Phi,n}\,, M_{\Phi,n} = F_n M^0_{\Phi,n}\,, \text{ and } E_{\Phi,n} = F_n E^0_{\Phi,n}\,.$$

Further, $F_n C_{\Phi,n}$ is a Cartan subgroup of $U_{\Phi,n}$. Passing to the limit, we define

$$F = \varinjlim F_n = \exp(i\mathfrak{a}) \cap K\,, \text{ and } C_{\Phi,\infty} = \varinjlim C_{\Phi,n}$$

so that

$$U_{\Phi,\infty} = F U^0_{\Phi,\infty}\,, \text{ and } F C_{\Phi,\infty} \text{ is a lim-compact Cartan subgroup of } U_{\Phi,\infty}\,.$$

Definition 11.2. Let $\lambda_n \in \mathfrak{c}^*_{\Phi,n}$. Then λ_n is *integral* if $\exp(2\pi i\lambda_n)$ is a well defined unitary character on the torus $C_{\Phi,n}$, and λ_n is *dominant integral* if it is integral and $\langle\lambda_n,\alpha\rangle \geqq 0$ for every $\alpha \in \Sigma_{\Phi,n}$. Write $\Lambda_{\Phi,n}$ for the set of dominant integral weights in $\mathfrak{c}^*_{\Phi,n}$.

Let $\lambda = (\lambda_n) \in \mathfrak{c}^*_{\Phi,\infty}$. Then λ is *integral* if $\exp(2\pi i\lambda)$ is a well defined unitary character on the torus $C_{\Phi,\infty}$, in other words if each λ_n is integral. And λ is *dominant integral* if it is integral and $\langle\lambda,\alpha\rangle \geqq 0$ for every $\alpha \in \Sigma_\Phi$, in other words if each λ_n is dominant integral. Write $\Lambda_{\Phi,\infty}$ for the set of all dominant integral weights in $\mathfrak{c}^*_{\Phi,\infty}$.

Each $\lambda_n \in \Lambda_{\Phi,n}$ is the highest weight of an irreducible unitary representation $\mu_{\lambda,n}$ of $U^0_{\Phi,n}$. Let $\mathcal{H}_{\lambda_n}$ denote the representation space and $u_{\lambda,n}$ a highest weight unit vector. Now let $\lambda = (\lambda_n) \in \Lambda_{\Phi,\infty}$. Then $u_{\lambda,n} \mapsto u_{\lambda,\ell}$ defines a $U^0_{\Phi,n}$-equivariant isometric injection $\mathcal{H}_{\lambda_n} \hookrightarrow \mathcal{H}_{\lambda_\ell}$. Thus λ defines a direct limit highest weight unitary representation

$$\mu_\lambda = \varinjlim \mu_{\lambda,n} \in \widehat{U^0_\Phi} \text{ with representation space } \mathcal{H}_\lambda = \varinjlim \mathcal{H}_{\lambda_n} .$$

Different choices of $\{u_{\lambda,n}\}$ lead to equivalent representations. Here recall [11, Theorem 5.10] that if $\ell \geqq n$ then $\mu_{\lambda,\ell}|_{U^0_{\Phi,n}}$ contains $\mu_{\lambda,n}$ with multiplicity 1, so there is no ambiguity (beyond phase changes $u_{\lambda,n} \mapsto e^{i\epsilon_n}u_{\lambda,n}$) about the inclusion $\mathcal{H}_{\lambda_n} \hookrightarrow \mathcal{H}_{\lambda_\ell}$. Now denote

$$\begin{aligned} \Xi_{\Phi,n} &= \left\{\mu_{\lambda,n,\varphi} := \varphi \otimes \mu_{\lambda,n} \,\middle|\, \varphi \in \widehat{F}\,, \lambda_n \in \Lambda_{\Phi,n} \text{ and } \varphi|_{F\cap U^0_{\Phi,n}} = \mu_{\lambda,n}|_{F\cap U^0_{\Phi,n}} \right\}, \\ \Xi_{\Phi,\infty} &= \left\{\mu_{\lambda,\varphi} := \varphi \otimes \mu_\lambda \,\middle|\, \varphi \in \widehat{F}\,, \lambda = (\lambda_n) \in \Lambda_{\Phi,\infty} \text{ and } \varphi|_{F\cap U^0_\Phi} = \mu_\lambda|_{F\cap U^0_\Phi} \right\}. \end{aligned} \tag{11.2}$$

Lemma 11.1 shows that the direct system $\{U^0_{\Phi,n}\}$ is strict and is parabolic in the sense of [18, Eq. 4.2]. Thus we have the Peter-Weyl Theorem for parabolic direct limits [18, Theorem 4.3], and it follows immediately for the system $\{U_{\Phi,n}\}$. Rescaling matrix coefficients with the Frobenius-Schur orthogonality relations as in (9.5) and (9.6) we obtain Hilbert space projections $p_{\ell,n} : L^2(U_{\Phi,\ell}) \to L^2(U_{\Phi,n})$ and an inverse system

$$L^2(U_{\Phi,1}) \xleftarrow{p_{2,1}} L^2(U_{\Phi,2}) \xleftarrow{p_{3,2}} L^2(U_{\Phi,3}) \xleftarrow{p_{4,3}} \;...\; \longleftarrow L^2(U_{\Phi,\infty}) \tag{11.3}$$

in the category of Hilbert spaces and projections, where the projective limit $L^2(U_{\Phi,\infty}) := \varprojlim\{L^2(U_{\Phi,n}), p_{\ell,n}\}$ is taken in that category. We now have the Hilbert space projective limit

$$L^2(N_{\Phi,\infty}) := \varprojlim\{L^2(N_{\Phi,n}), p_{\ell,n}\} = \sum\nolimits_{\mu_{\lambda,\varphi}\in\Xi_{\Phi,\infty}} \mathcal{H}_\lambda \widehat{\otimes} \mathcal{H}^*_\lambda \text{ orthogonal direct sum.} \tag{11.4}$$

The left/right representation of $U_{\Phi,\infty} \times U_{\Phi,\infty}$ on $L^2(U_{\Phi,\infty})$ is multiplicity-free, preserves each summand $\mathcal{H}_\lambda\widehat{\otimes}\mathcal{H}^*_\lambda$, and acts on $\mathcal{H}_\lambda\widehat{\otimes}\mathcal{H}^*_\lambda$ by the irreducible representation of highest weight (λ,λ^*). The connection with matrix coefficients is

$$\begin{array}{ccccccc} \mathcal{C}(U_{\Phi,1}) & \xleftarrow{q_{2,1}} & \mathcal{C}(U_{\Phi,2}) & \xleftarrow{q_{3,2}} & \dots & \longleftarrow & \mathcal{C}(U_{\Phi,\infty}) = \varprojlim \mathcal{C}(U_{\Phi,n}) \\ \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & \downarrow & & \downarrow{\scriptstyle r_\infty} \\ L^2(U_{\Phi,1}) & \xleftarrow{p_{2,1}} & L^2(U_{\Phi,2}) & \xleftarrow{p_{3,2}} & \dots & \longleftarrow & L^2(U_{\Phi,\infty}) = \varprojlim L^2(U_{\Phi,n}) \end{array} \tag{11.5}$$

as in (9.15). As in Proposition 9.8 this realizes the limit space $L^2(U_{\Phi,\infty})$ as a Hilbert space completion of the Schwartz space $\mathcal{C}(U_{\Phi,\infty})$, and because of compactness the latter in turn is the projective limit of spaces $\mathcal{C}(U_{\Phi,n}) = C^\infty(U_{\Phi,n})$. The Fourier inversion formula for $U_{\Phi,\infty}$ is given stepwise as in Theorem 9.10.

12. Representations of the Limit Groups IV: $U_{\Phi,\infty}N_{\Phi,\infty}$

We combine some of the results of Sections 9 and 11, extending them to the subgroup $U_{\Phi,\infty}N_{\Phi,\infty}$. In view of the discussion culminating in (8.4) we assume that the direct system $\{G_n\}$ of real semisimple Lie groups satisfies

$$\begin{gathered}\text{if } \ell \geqq n \text{ then } \Phi_n \subset \Phi_\ell \text{ and } (\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell) \text{ so that} \\ U_{\Phi,\infty} := \varinjlim U_{\Phi,n} \text{ and } U_{\Phi,\infty}N_{\Phi,\infty} := \varinjlim U_{\Phi,n}N_{\Phi,n} \text{ exist.}\end{gathered} \tag{12.1}$$

We also extend Definition 6.2:

Definition 12.1. The direct limit groups $N_{\Phi,\infty} = L_{\Phi,1}L_{\Phi,2}L_{\Phi,3}\dots$ is *weakly invariant* if each $\mathrm{Ad}(U_{\Phi,\infty})\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$.

We'll need a variation on Lemma 6.3. Recall the maximal lim-compact subgroup $K = \varinjlim K_n$.

Lemma 12.2. *Let $F_n = \exp(i\mathfrak{a}_{\Phi,n}) \cap K_n$ and $F = \exp(i\mathfrak{a}_{\Phi,\infty}) \cap K$. Then $F = \varinjlim F_n$ is contained in $U_{\Phi,\infty}$ and is central in $M_{\Phi,\infty}$; if $x \in F$ then $x^2 = 1$, $U_{\Phi,\infty} = FU^0_{\Phi,\infty}$; and $M_{\Phi,\infty} = FM^0_{\Phi,\infty}$.*

Proof. Lemma 6.3 contains the corresponding results for the F_n. It follows that F is a subgroup of $U_{\Phi,\infty}$ central in $M_{\Phi,\infty}$, that describes the components as stated, and in which every element has square 1. ∎

Lemma 12.3. *The action of* $\mathrm{Ad}(F)$ *on $\mathfrak{s}^*_{\Phi,\infty}$ is trivial.*

Proof. Lemma 6.4 shows that $\mathrm{Ad}(F_\ell)$ is trivial on $\mathfrak{s}^*_{\Phi,n}$ whenever $\ell \geqq n$. ∎

Now suppose that $N_{\Phi,\infty} = L_{\Phi,1}L_{\Phi,2}L_{\Phi,3}\dots$ is weakly invariant. We continue as in Section 6.

$$\mathfrak{r}^*_{\Phi,\infty} = \{(\gamma_n) \in \mathfrak{t}^*_{\Phi,\infty} \mid \text{ each } \mathrm{Ad}^*(U_{\Phi,n})\gamma_n \text{ is a principal } U_{\Phi,n}\text{-orbit on } \mathfrak{s}^*_{\Phi,n}\}. \tag{12.2}$$

It is dense, open and $\mathrm{Ad}^*(U_{\Phi,\infty})$-invariant in $\mathfrak{s}^*_{\Phi,\infty}$. Let σ be a measurable section to $\mathfrak{r}^*_{\Phi,\infty} \to \mathrm{Ad}^*(U_{\Phi,\infty})\backslash\mathfrak{r}^*_{\Phi,\infty}$ on whose image all the isotropy subgroups are the same. We use the notation

$$U'_{\Phi,\infty}: \text{ isotropy subgp of } U_{\Phi,\infty} \text{ at } \sigma(\mathrm{Ad}^*(U_{\Phi,\infty})(\gamma)), \text{ independent of } \gamma \in \mathfrak{r}^*_{\Phi,\infty}. \tag{12.3}$$

As a bonus, in view of Lemma 10.1, the isotropy subgroup of $U_{\Phi,\infty}A_{\Phi,\infty}$ at $\mathrm{Ad}^*(a)\sigma(\mathrm{Ad}^*(U_{\Phi,\infty})(\gamma))$ is $U'_{\Phi,\infty}A'_{\Phi,\infty}$, independent of $a \in A_{\Phi,\infty}$ and $\gamma \in \mathfrak{r}^*_{\Phi,\infty}$. Note that $U'_{\Phi,\infty} = \varinjlim U'_{\Phi,n}$ where $U'_{\Phi,n}$ is the isotropy subgroup of $U_{\Phi,n}$ at

$\sigma(\mathrm{Ad}^*(U_{\Phi,\infty})(\gamma))_n$, independent of $\gamma \in \mathfrak{r}^*_{\Phi,\infty}$. Given $\mu' \in \widehat{U'_{\Phi,\infty}}$, say $\mu' = \varinjlim \mu'_n$ where $\mu'_n \in \widehat{U'_{\Phi,n}}$, and γ is in the image of σ, we have representations

$$\begin{aligned} &\pi_{\Phi,\gamma,\mu',n} := \mathrm{Ind}\,_{U'_{\Phi,n}N_{\Phi,n}}^{U_{\Phi,n}N_{\Phi,n}}(\mu'_n \otimes \pi_{\Phi,\gamma,n}) \text{ and} \\ &\qquad \pi_{\Phi,\gamma,\mu',\infty} = \mathrm{Ind}\,_{U'_{\Phi,\infty}N_{\Phi,\infty}}^{U_{\Phi,\infty}N_{\Phi,\infty}}(\mu' \otimes \pi_{\Phi,\gamma,\infty}) := \varinjlim \pi_{\Phi,\gamma,\mu',n} . \end{aligned} \tag{12.4}$$

To be precise here, μ' must be a cocycle representation of $U'_{\Phi,\infty}$ where the cocycle ε is the inverse of the Mackey obstruction to extending $\pi_{\Phi,\gamma,\infty}$ to a representation of $U'_{\Phi,\infty}N_{\Phi,\infty}$.

As in (10.3) the *relative Schwartz space* $\mathcal{C}((U_{\Phi,n}N_{\Phi,n}/U'_{\Phi,n}S_{\Phi,n}), \mu'_n \otimes \zeta_{\gamma_n})$ consists of all functions $f \in C^\infty(U_{\Phi,n}N_{\Phi,n})$ such that

$$\begin{aligned} &f(xus) = \mu'_n(u)^{-1}\zeta_{\gamma_n}(s)^{-1}f(x) \ (x \in N_{\Phi,n}, u \in U'_{\Phi,n} , s \in S_{\Phi,n}), \text{ and} \\ &\quad |q(g)p(D)f| \text{ is bounded on } U_{\Phi,n}N_{\Phi,n}/U'_{\Phi,n}S_{\Phi,n} \text{ for all} \\ &\quad\ \text{polynomials } p, q \text{ on } N_{\Phi,n}/S_{\Phi,n} \text{ and all } D \in \mathcal{U}(\mathfrak{u}_{\Phi,n} + \mathfrak{n}_{\Phi,n}). \end{aligned} \tag{12.5}$$

The corresponding *limit relative Schwartz space* is

$$\begin{aligned} &\mathcal{C}((U_{\Phi,\infty}N_{\Phi,\infty}/U'_{\Phi,\infty}S_{\Phi,\infty}),(\mu' \otimes \zeta_\gamma)) \\ &\qquad\qquad = \varprojlim \mathcal{C}((U_{\Phi,n}N_{\Phi,n}/U'_{\Phi,n}S_{\Phi,n}), (\mu'_n \otimes \zeta_{\gamma_n})), \end{aligned}$$

consisting of all functions $f \in C^\infty(U_{\Phi,\infty}N_{\Phi,\infty})$ such that

$$\begin{aligned} &f(xus) = \mu'(u)^{-1}\zeta_\gamma(s)^{-1}f(x)\,(x \in N_{\Phi,\infty}, u \in U'_{\Phi,\infty}, s \in S_{\Phi,\infty} , \text{ and} \\ &\quad |q(g)p(D)f| \text{ is bounded on } U_{\Phi,\infty}N_{\Phi,\infty}/U'_{\Phi,\infty}S_{\Phi,\infty} \text{ for all} \\ &\quad\ \text{polynomials } p, q \text{ on } N_{\Phi,\infty}/S_{\Phi,\infty} \text{ and all } D \in \mathcal{U}(\mathfrak{u}_{\Phi,\infty} + \mathfrak{n}_{\Phi,\infty}). \end{aligned} \tag{12.6}$$

Theorem 9.4 and Corollaries 9.5 and 9.6 hold for our groups $U_{\Phi,n}N_{\Phi,n}$ here with essentially no change, so we will not repeat them.

Following the discussion in Section 10 for $\mathcal{C}(A_{\Phi,n}N_{\Phi,n})$ and $\mathcal{C}(A_{\Phi,\infty}N_{\Phi,\infty})$ we define seminorms

$$\nu_{k,D,n}(f) = \sup_{x \in U_{\Phi,n}N_{\Phi,n}} ||x||^k |Df(x)| \tag{12.7}$$

for all $k > 0$ and $D \in \mathcal{U}(\mathfrak{u}_{\Phi,n} + \mathfrak{n}_{\Phi,n})$. As in (9.13) that defines the Schwartz space

$$\begin{aligned} &\mathcal{C}(U_{\Phi,n}N_{\Phi,n}) = \\ &\quad \{f \in C^\infty(U_{\Phi,n}N_{\Phi,n})\,|\,\nu_{k,D,n}(f) < \infty \text{ for } k > 0, D \in \mathcal{U}(\mathfrak{u}_{\Phi,n} + \mathfrak{n}_{\Phi,n})\}\,. \end{aligned} \tag{12.8}$$

Finally we define $\mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty})$ to be the inverse limit in the category of locally convex topological vector spaces and continuous linear maps, as in (9.14):

$$\begin{aligned} \mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty}) &= \varprojlim \mathcal{C}(U_{\Phi,n}N_{\Phi,n}) \\ &= \left\{f \in C^\infty(U_{\Phi,\infty}N_{\Phi,\infty}) \,\middle|\, f|_{U_{\Phi,n}N_{\Phi,n}} \in \mathcal{C}(U_{\Phi,n}N_{\Phi,n})\right\}. \end{aligned} \tag{12.9}$$

Then we have

Lemma 12.4. *The Schwartz space $\mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty})$ consists of all functions $f \in C^\infty(U_{\Phi,\infty}N_{\Phi,\infty})$ such that $\nu_{k,D,n}(f) < \infty$ for all integers $k > 0$ and all $D \in \mathcal{U}(\mathfrak{u}_{\Phi,\infty} + \mathfrak{n}_{\Phi,\infty})$. Here $\nu_{k,D,n}$ is given by* (12.7)*. The $\mathcal{C}(U_{\Phi,n}N_{\Phi,n})$ are nuclear Fréchet spaces and $\mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty})$ is an LF space. The left/right actions of $(U_{\Phi,n}N_{\Phi,n}) \times (U_{\Phi,n}N_{\Phi,n})$ on $\mathcal{C}(U_{\Phi,n}N_{\Phi,n})$ and of $(U_{\Phi,\infty}N_{\Phi,\infty}) \times (U_{\Phi,\infty}N_{\Phi,\infty})$ on $\mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty})$ are continuous.*

We construct $L^2(U_{\Phi,\infty}N_{\Phi,\infty}) := \varprojlim L^2(U_{\Phi,n}N_{\Phi,n})$ along the lines of Section 9. Let $\gamma = (\gamma_n) \in \mathfrak{r}^*_{\Phi,\infty}$ such that γ is in the image of σ. Consider $\mu' = \varinjlim \mu'_n \in \widehat{U'_{\Phi,\infty}}$ where (i) $\mu'_n \in \widehat{U'_{\Phi,n}}$ and (ii) $\mathcal{H}_{\mu'_n} \subset \mathcal{H}_{\mu'_\ell}$ from a map $u_n \mapsto u_\ell$ of highest weight unit vectors, for $\ell \geqq n$.

Every $f \in \mathcal{C}(U_{\Phi,n}N_{\Phi,n})$ is a Schwartz wave packet along $(\mathfrak{u}'_{\Phi,n})^* + \mathfrak{s}^*_{\Phi,n}$ of coefficients of the various $\pi_{\Phi,\gamma,\mu',n}$, and the corresponding inverse systems fit together as in (9.15):

$$\begin{array}{ccccccc}
\mathcal{C}(U_{\Phi,1}N_{\Phi,1}) & \xleftarrow{q_{2,1}} & \mathcal{C}(U_{\Phi,2}N_{\Phi,2}) & \xleftarrow{q_{3,2}} & \cdot & \longleftarrow & \mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty}) = \varprojlim \mathcal{C}(U_{\Phi,n}N_{\Phi,n}) \\
\downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & \downarrow & & \downarrow{\scriptstyle r_\infty} \\
L^2(U_{\Phi,1}N_{\Phi,1}) & \xleftarrow{p_{2,1}} & L^2(U_{\Phi,2}N_{\Phi,2}) & \xleftarrow{p_{3,2}} & \cdot & \longleftarrow & L^2(U_{\Phi,\infty}N_{\Phi,\infty}) = \varprojlim L^2(U_{\Phi,n}N_{\Phi,n})
\end{array} \tag{12.10}$$

The map $r_\infty : \mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty}) \hookrightarrow L^2(U_{\Phi,\infty}N_{\Phi,\infty})$ is a continuous injection with dense image, properties inherited from the r_n. As in Proposition 9.8 we conclude

Proposition 12.5. *Assume* (8.4)(d)*, so that $U_{\Phi,\infty}N_{\Phi,\infty} = \varinjlim U_{\Phi,n}N_{\Phi,n}$ is well defined. Define $r_\infty : \mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty}) \hookrightarrow L^2(U_{\Phi,\infty}N_{\Phi,\infty})$ as in the commutative diagram* (12.10)*. Then $L^2(U_{\Phi,\infty}N_{\Phi,\infty})$ is a Hilbert space completion of $\mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty})$. In particular r_∞ defines a pre-Hilbert space structure on $\mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty})$ with completion $L^2(U_{\Phi,\infty}N_{\Phi,\infty})$.*

As in [23, Corollary 5.17] $\mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty})$ is independent of the choices we made in the construction of $L^2(U_{\Phi,\infty}N_{\Phi,\infty})$, so

Corollary 12.6. *The Hilbert space $L^2(U_{\Phi,\infty}N_{\Phi,\infty}) = \varprojlim\{L^2(U_{\Phi,n}N_{\Phi,n}), p_{\ell,n}\}$ of* (12.10) *, and the left/right regular representation of $(U_{\Phi,\infty}N_{\Phi,\infty}) \times (U_{\Phi,\infty}N_{\Phi,\infty})$ on $L^2(U_{\Phi,\infty}N_{\Phi,\infty})$, are independent of the choice of vectors $\{e\}$ in* (9.4)*and highest weight unit vectors $\{u_n\}$.*

The limit Fourier inversion formula is

Theorem 12.7. *Given $\pi_{\Phi,\gamma,\mu_{\lambda,\varphi,n}} \in \widehat{U_{\Phi,n}N_{\Phi,n}}$, let $\Theta_{\pi_{\Phi,\gamma,\mu_{\lambda,\varphi,n}}}$ denote its distribution character. Then $\Theta_{\pi_{\Phi,\gamma,\mu_{\lambda,\varphi,n}}}$ is tempered. Let $f \in \mathcal{C}(U_{\Phi,\infty}N_{\Phi,\infty})$ and $x \in U_{\Phi,\infty}N_{\Phi,\infty}$. Then $x \in U_{\Phi,n}N_{\Phi,n}$ for some n and*

$$f(x) = c_n \int_{\gamma_n \in \mathfrak{t}^*_{\Phi,n}} \sum\nolimits_{\mu'_n \in \widehat{U'_{\Phi,n}}} \Theta_{\pi_{\Phi,\gamma,\mu',n}}(r_x f) \deg(\mu') |\mathrm{Pf}_{\mathfrak{n}_n}(\gamma_n)| d\gamma_n \tag{12.11}$$

where $c_n = 2^{d_1+\dots+d_{m_n}} d_1! d_2! \dots d_{m_n}!$ as in (1.6a) *and m_n is the number of factors L_r in $N_{\Phi,n}$.*

Proof. We adapt the computation of [7, Theorem 2.7]. Let $h = r_x f$ and apply [5, Theorem 3.2].

$$\begin{aligned}\operatorname{trace}\pi_{\Phi,\gamma,\mu',n}(h) &= \int_{x\in U_{\Phi,n}/U'_{\Phi,n}} \operatorname{trace}\int_{yu\in N_{\Phi,n}U'_{\Phi,n}} h(x^{-1}yux)\cdot(\pi_{\Phi,\gamma,n}\otimes\mu'_n)(yu)\,dy\,du\,dx\\ &= \int_{x\in U_{\Phi,n}/U'_{\Phi,n}} \operatorname{trace}\int_{N_{\Phi,n}U'_{\Phi,n}} h(yx^{-1}ux)\cdot(\pi_{\Phi,\gamma,n}\otimes\mu'_n)(xyx^{-1}u)\,dy\,du\,dx.\end{aligned}$$

Now

$$\begin{aligned}&\sum_{\widehat{U'_{\Phi,n}}} \operatorname{trace}\pi_{\Phi,\gamma,\mu',n}(h)\deg\mu'_n\\ &\quad= \sum_{\widehat{U'_{\Phi,n}}}\int_{x\in U_{\Phi,n}/U'_{\Phi,n}} \operatorname{trace}\int_{N_{\Phi,n}U'_{\Phi,n}} h(yx^{-1}ux)(\pi_{\Phi,\gamma,\mu',n})(xyx^{-1}u)\,dy\,du\,dx\,\deg\mu'_n\\ &\quad= \int_{x\in U_{\Phi,n}/U'_{\Phi,n}}\sum_{\widehat{U'_{\Phi,n}}} \operatorname{trace}\int_{N_{\Phi,n}U'_{\Phi,n}} h(yx^{-1}ux)(\pi_{\Phi,\gamma,\mu',n})(xyx^{-1}u)\,dy\,du\,\deg\mu'_n\,dx\\ &\quad= \int_{x\in U_{\Phi,n}/U'_{\Phi,n}} \operatorname{trace}\int_{N_{\Phi,n}} h(y)\pi_{\Phi,\gamma,\mu',n}(xyx^{-1})dy\,dx\\ &\quad= \int_{x\in U_{\Phi,n}/U'_{\Phi,n}} \operatorname{trace}\int_{N_{\Phi,n}} h(y)(x^{-1}\cdot\pi_{\Phi,\gamma,\mu',n})(y)dy\,dx\\ &\quad= \int_{x\in U_{\Phi,n}/U'_{\Phi,n}} \operatorname{trace}\left((x^{-1}\cdot\pi_{\Phi,\gamma,\mu',n})(h)\right)dx\\ &\quad= \int_{\mathrm{Ad}^*(U_{\Phi,n})\gamma} \operatorname{trace}\pi_{\Phi,\gamma,\mu'_n}(h)|\mathrm{Pf}(\gamma_n)|d\gamma_n.\end{aligned}$$

Summing over the the space of $U_{\Phi,n}$-orbits on $\mathfrak{s}^*_{\Phi,n}$ we now have

$$\begin{aligned}&\int_{U_{\Phi,n}\backslash\mathfrak{s}^*_{\Phi,n}}\sum_{\widehat{U'_{\Phi,n}}} \operatorname{trace}\pi_{\Phi,\gamma,\mu',n}(h)\deg\mu'_n|\mathrm{Pf}(\gamma_n)|d\gamma_n\\ &\qquad= \int_{U_{\Phi,n}\backslash\mathfrak{s}^*_{\Phi,n}} \operatorname{trace}\pi_{\Phi,\gamma,\mu',n}(h)|\mathrm{Pf}(\gamma_n)|d\gamma_n\\ &\qquad= \int_{\mathfrak{s}^*_{\Phi,n}} \operatorname{trace}\pi_{\Phi,\gamma,n}(h)|\mathrm{Pf}(\gamma_n)|d\gamma_n = h(1) = f(x)\,.\end{aligned}$$

That completes the proof. ∎

13. Representations of the Limit Groups V: $U_{\Phi,\infty}A_{\Phi,\infty}N_{\Phi,\infty}$

We extend some of the results of Sections 10 and 12 to the maximal amenable subgroups $E_{\Phi,\infty} := U_{\Phi,\infty}A_{\Phi,\infty}N_{\Phi,\infty}$ of G. Here we are using amenability of the $E_{\Phi,n} := U_{\Phi,n}A_{\Phi,n}N_{\Phi,n}$.

As in Definition 6.2 the decomposition $N_{\Phi,\infty} = L_{\Phi,1}L_{\Phi,2}\dots$ is *invariant* if each $\operatorname{ad}(\mathfrak{m}_{\Phi,\infty})\mathfrak{z}_{\Phi,\infty,j} = \mathfrak{z}_{\Phi,\infty,j}$, in other words if each $N_{\Phi,n} = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,m_n}$ is invariant. Similarly $N_{\Phi,\infty} = L_{\Phi,1}L_{\Phi,2}\dots$ is *weakly invariant* if each $\operatorname{ad}(\mathfrak{u}_{\Phi,\infty})\mathfrak{z}_{\Phi,\infty,j} = \mathfrak{z}_{\Phi,\infty,j}$, i.e. if each $N_{\Phi,n} = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,m_n}$ is weakly invariant.

Recall the principal orbit set $\mathfrak{r}^*_{\Phi,\infty}$ from (12.2) and the measurable section $\sigma : \mathrm{Ad}^*(U_{\Phi,\infty})\backslash\mathfrak{r}^*_{\Phi,\infty} \to \mathfrak{r}^*_{\Phi,\infty}$ on whose image all the isotropy subgroups of $\mathrm{Ad}^*(U_{\Phi,\infty})$ are the same. Note that σ is $\mathrm{Ad}^*(A_{\Phi,\infty})$-equivariant, so we may view it as a section to $\mathfrak{r}^*_{\Phi,\infty} \to \mathrm{Ad}^*(U_{\Phi,\infty}A_{\Phi,\infty})\backslash\mathfrak{r}^*_{\Phi,\infty}$ on whose image all the isotropy subgroups of $\mathrm{Ad}^*(U_{\Phi,\infty}A_{\Phi,\infty})$ are the same. Following (6.5), (10.1) and (12.3), and as remarked just after (12.3), that common isotropy subgroup is independent of $\gamma \in \mathfrak{r}^*_{\Phi,\infty}$ and is given by

$$U'_{\Phi,\infty}A'_{\Phi,\infty}: \text{ isotropy of } U_{\Phi,\infty}A_{\Phi,\infty} \text{ at } \sigma(\mathrm{Ad}^*(U_{\Phi,\infty}A_{\Phi,\infty}))(\gamma). \tag{13.1}$$

Let $\gamma \in \mathfrak{t}^*_{\Phi,\infty}$ be in the image of σ. Then $\pi_{\Phi,\gamma,\infty}$ extends to a representation $\pi^\dagger_{\Phi,\gamma,\infty}$ of $U'_{\Phi,\infty}A'_{\Phi,\infty}N_{\Phi,\infty}$ on the same space $\mathcal{H}_{\pi_{\Phi,\gamma_\infty}}$. Given $\mu' \in \widehat{U'_{\Phi,\infty}}$ and $\xi' = (\xi'_n) \in (\mathfrak{a}'_{\Phi,\infty})^*$ the corresponding representation of $E_{\Phi,\infty} := U_{\Phi,\infty}A_{\Phi,\infty}N_{\Phi,\infty}$ is induced from $E'_{\Phi,\infty} := U'_{\Phi,\infty}A'_{\Phi,\infty}N_{\Phi,\infty}$ as follows.

$$\begin{aligned}\pi_{\Phi,\gamma,\xi',\mu',\infty} = \varinjlim \pi_{\Phi,\gamma,\xi',\mu',n} \text{ where } \pi_{\Phi,\gamma,\xi',\mu',n} = \mathrm{Ind}_{E'_{\Phi,n}}^{E_{\Phi,n}} \left(\mu'_n \otimes \exp(2\pi i \xi'_n) \otimes \pi^\dagger_{\Phi,\gamma,n}\right),\\ \text{in other words } \pi_{\Phi,\gamma,\xi',\mu',n} = \mathrm{Ind}_{U'_{\Phi,n}A'_{\Phi,n}N_{\Phi,n}}^{U_{\Phi,n}A_{\Phi,n}N_{\Phi,n}} \left(\mu'_n \otimes \exp(2\pi i \xi'_n) \otimes \pi^\dagger_{\Phi,\gamma,n}\right).\end{aligned} \tag{13.2}$$

As in Section 10, $\mathcal{C}((U_{\Phi,n}A_{\Phi,n}N_{\Phi,n}/U'_{\Phi,n}A'_{\Phi,n}S_{\Phi,n}), (\mu'_n \otimes \exp(2\pi i \xi'_n) \otimes \zeta_{\gamma_n}))$ consists of all functions $f \in C^\infty(U_{\Phi,n}A_{\Phi,n}N_{\Phi,n})$ such that

$$\begin{aligned} &f(xuas) = \mu'_n(u)^{-1}\exp(-2\pi i \xi'(\log a))\zeta_{\gamma_n}(s)^{-1}f(x)\\ &\quad (x \in N_{\Phi,n}, u \in U'_{\Phi,n}, a \in A'_{\Phi,n}, s \in S_{\Phi,n}), \text{ and } |q(g)p(D)f| \text{ is}\\ &\quad \text{bounded on } U_{\Phi,n}A_{\Phi,n}N_{\Phi,n}/U'_{\Phi,n}A'_{\Phi,n}S_{\Phi,n} \text{ for all polynomials}\\ &\quad p, q \text{ on } A_{\Phi,n}N_{\Phi,n}/A'_{\Phi,n}S_{\Phi,n} \text{ and all } D \in \mathcal{U}(\mathfrak{u}_{\Phi,n} + \mathfrak{a}_{\Phi,n} + \mathfrak{n}_{\Phi,n}).\end{aligned} \tag{13.3}$$

That is the *relative Schwartz space*. The corresponding *limit relative Schwartz space* is

$$\begin{aligned}&\mathcal{C}((U_{\Phi,\infty}A_{\Phi,\infty}N_{\Phi,\infty}/U'_{\Phi,\infty}A'_{\Phi,\infty}S_{\Phi,\infty}), (\mu' \otimes \exp(2\pi i \xi') \otimes \zeta_\gamma))\\ &\qquad = \varprojlim \mathcal{C}((U_{\Phi,n}A_{\Phi,n}N_{\Phi,n}/U'_{\Phi,n}A'_{\Phi,n}S_{\Phi,n}), (\mu'_n \otimes \exp(2\pi i \xi'_n) \otimes \zeta_{\gamma_n})).\end{aligned} \tag{13.4}$$

Again, Theorem 9.4 and Corollaries 9.5 and 9.6 hold *mutatis mutandis* for the groups $E_{\Phi,n}$ so we won't repeat them. We extend the definition (12.7) of seminorms on $U_{\Phi,n}N_{\Phi,n}$ to $E_{\Phi,n} = U_{\Phi,n}A_{\Phi,n}N_{\Phi,n}$:

$$\nu_{k,D,n}(f) = \sup_{x \in E_{\Phi,n}} ||x||^k |Df(x)| \ (k > 0,\ D \in \mathfrak{e}_{\Phi,n}\ ,\ f \in C^\infty(E_{\Phi,n})). \tag{13.5}$$

That defines the Schwartz space $\mathcal{C}(E_{\Phi,n})$:

$$\mathcal{C}(E_{\Phi,n}) = \{f \in C^\infty(E_{\Phi,n})\,|\nu_{k,D,n}(f) < \infty \text{ for } k > 0 \text{ and } D \in \mathcal{U}(\mathfrak{e}_{\Phi,n})\}\,. \tag{13.6}$$

Finally we define $\mathcal{C}(E_{\Phi,\infty})$ to be the inverse limit in the category of locally convex topological vector spaces and continuous linear maps,

$$\mathcal{C}(E_{\Phi,\infty}) = \left\{f \in C^\infty(E_{\Phi,\infty})\,\middle|\, f|_{E_{\Phi,n}} \in \mathcal{C}(E_{\Phi,n})\right\} = \varprojlim \mathcal{C}(E_{\Phi,n}). \tag{13.7}$$

As before

Lemma 13.1. *The Schwartz space $\mathcal{C}(E_{\Phi,\infty})$ consists of all $f \in C^\infty(E_{\Phi,\infty})$ such that $\nu_{k,D,n}(f) < \infty$ for all integers $k > 0$ and all $D \in \mathcal{U}(\mathfrak{e}_{\Phi,\infty})$. Here $\nu_{k,D,n}$ is given by* (13.5). *The $\mathcal{C}(E_{\Phi,n})$ are nuclear Fréchet spaces and $\mathcal{C}(E_{\Phi,\infty})$ is an LF space. The left/right actions of $(E_{\Phi,n}) \times (E_{\Phi,n})$ on $\mathcal{C}(E_{\Phi,n})$ and of $(E_{\Phi,\infty}) \times (E_{\Phi,\infty})$ on $\mathcal{C}(E_{\Phi,\infty})$ are continuous.*

We construct $L^2(E_{\Phi,\infty}) := \varprojlim L^2(E_{\Phi,n})$ as before. Let $\gamma = (\gamma_n) \in \mathfrak{r}^*_{\Phi,\infty}$ be in the image of σ. Consider $\mu' = (\mu'_n) \in \widehat{U'_{\Phi,\infty}}$ and $\xi = (\xi_n) \in \mathfrak{a}_{\Phi,n}$, For $\ell \geqq n$ we consider the maps on representation spaces given by $\mathcal{H}_{\pi_{\Phi,\gamma,\xi,\mu',n}} \subset \mathcal{H}_{\pi_{\Phi,\gamma,\xi,\mu',\ell}}$ from maps $u_n \mapsto u_\ell$ of highest weight unit vectors.

Every $f \in \mathcal{C}(E_{\Phi,n})$ is a Schwartz wave packet along $(\mathfrak{u}'_{\Phi,n})^*\mathfrak{a}'_{\Phi,n} + \mathfrak{s}^*_{\Phi,n}$ of coefficients of the various $\pi_{\Phi,\gamma,\xi,\mu',n}$. The corresponding inverse systems fit together as in (9.15):

$$\begin{array}{ccccccc}
\mathcal{C}(E_{\Phi,1}) & \xleftarrow{q_{2,1}} & \mathcal{C}(E_{\Phi,2}) & \xleftarrow{q_{3,2}} & \dots & \longleftarrow & \mathcal{C}(E_{\Phi,\infty}) = \varprojlim \mathcal{C}(E_{\Phi,n}) \\
\downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & \downarrow & & \downarrow{\scriptstyle r_\infty} \\
L^2(E_{\Phi,1}) & \xleftarrow{p_{2,1}} & L^2(E_{\Phi,2}) & \xleftarrow{p_{3,2}} & \dots & \longleftarrow & L^2(E_{\Phi,\infty}) = \varprojlim L^2(E_{\Phi,n})
\end{array} \tag{13.8}$$

The r_n are continuous injections with dense image, so $r_\infty : \mathcal{C}(E_{\Phi,\infty}) \hookrightarrow L^2(E_{\Phi,\infty})$ is a continuous injection with dense image. As in Proposition 9.8 we conclude

Proposition 13.2. *Assume* (8.4)(d), *so that* $E_{\Phi,\infty} = \varinjlim E_{\Phi,n}$ *is well defined. Define* $r_\infty : \mathcal{C}(E_{\Phi,\infty}) \hookrightarrow L^2(E_{\Phi,\infty})$ *as in* (13.8). *Then* $L^2(E_{\Phi,\infty})$ *is a Hilbert space completion of* $\mathcal{C}(E_{\Phi,\infty})$. *In particular* r_∞ *defines a pre-Hilbert space structure on* $\mathcal{C}(E_{\Phi,\infty})$ *with completion* $L^2(E_{\Phi,\infty})$.

As in [23, Corollary 5.17] $\mathcal{C}(E_{\Phi,\infty})$ is independent of the choices we made in the construction of $L^2(E_{\Phi,\infty})$, so

Corollary 13.3. *The Hilbert space* $L^2(E_{\Phi,\infty}) = \varprojlim\{L^2(E_{\Phi,n}), p_{\ell,n}\}$ *of* (13.8), *and the left/right regular representation of* $E_{\Phi,\infty} \times E_{\Phi,\infty}$ *on it, are independent of the choice of vectors* $\{e\}$ *in* (9.4) *and highest weight unit vectors* $\{u_n\}$.

The limit Fourier inversion formula is

Theorem 13.4. *Given* $\pi_{\Phi,\gamma,\xi,\mu_{\lambda,\varphi,n}} \in \widehat{E_{\Phi,n}}$ *let* $\Theta_{\pi_{\Phi,\gamma,\xi,\mu_{\lambda,\varphi,n}}}$ *denote its distribution character. Then* $\Theta_{\pi_{\Phi,\gamma,\xi,\mu_{\lambda,\varphi,n}}}$ *is a tempered distribution. Let* $f \in \mathcal{C}(E_{\Phi,\infty})$ *and* $x \in E_{\Phi,\infty}$. *Then* $x \in E_{\Phi,n}$ *for some* n *and*

$$f(x) = c_n \int_{\gamma_n \in \mathfrak{t}^*_{\Phi,n}} \int_{\xi \in \mathfrak{a}'_{\Phi,n}} \sum\nolimits_{\mu'_n \in \widehat{U'_{\Phi,n}}} \Theta_{\pi_{\Phi,\gamma,\xi,\mu',n}}(r_x f) \deg(\mu') |\mathrm{Pf}_{\mathfrak{n}_n}(\gamma_n)| d\xi\, d\gamma_n \tag{13.9}$$

where $c_n = (\frac{1}{2\pi})^{\dim \mathfrak{a}'_\Phi/2}\ 2^{d_1+\cdots+d_{m_n}} d_1! d_2! \dots d_{m_n}!$ *and* m_n *is the number of factors* L_r *in* $N_{\Phi,n}$.

Proof. We combine the ideas in the proofs of Theorems 5.8 and 12.7. In an attempt to keep the notation under control we write U''_n for $U_{\Phi,n}/U'_{\Phi,n}$ and A''_n for $A_{\Phi,n}/A'_{\Phi,n}$, and more generally we drop the subscript Φ. We write δ for the modular function of Q_Φ . Let $h = r_x f$. Using [5, Theorem 3.2],

$$\begin{aligned}\operatorname{trace}\pi_{\gamma,\xi,\mu',n}(Dh) &= \int_{x\in U''_nA''_n} \delta^{-1}(x)\ \operatorname{trace}\int_{yau\in N_nA'_nU'_n} (Dh)(x^{-1}yaux)\times \\ &\qquad\qquad \times(\pi^\dagger_{\gamma,n}\otimes\exp(2\pi i\xi)\mu'_n)(yau)\,dy\,da\,du\,dx \\ &= \int_{x\in U''A''} \operatorname{trace}\int_{N_nA'_nU'_n} (Dh)(yx^{-1}aux)\times \\ &\qquad\qquad \times(\pi^\dagger_{\gamma,n}\otimes\exp(2\pi i\xi)\mu'_n)(xyx^{-1}u)\,dy\,da\,du\,dx.\end{aligned}$$

Now $\sum_{\widehat{U'_n}}\int_{\widehat{A'_n}} \operatorname{trace}\pi_{\gamma,\xi,\mu',n}(Dh)d\xi\deg\mu'_n$

$$\begin{aligned}&= \sum_{\widehat{U'_n}}\int_{\widehat{A'_n}}\int_{x\in U''_nA''_n} \operatorname{trace}\int_{N_nU'_nA'_n} (Dh)(yx^{-1}aux)\times \\ &\qquad \times(\pi^\dagger_{\gamma,n}\otimes\exp(2\pi i\xi)\mu'_n)(xyx^{-1}au)\,dy\,da\,du\,dx\,d\xi\ \deg\mu'_n \\ &= \int_{x\in U''_nA''_n}\sum_{\widehat{U'_n}}\int_{\widehat{A'_n}} \operatorname{trace}\int_{N_nU'_n} (Dh)(yx^{-1}aux)\times \\ &\qquad \times(\pi^\dagger_{\gamma,n}\otimes\exp(2\pi i\xi)\mu'_n)(xyx^{-1}au)\,dy\,da\,du\,d\xi\ \deg\mu'_n\,dx \\ &= \int_{x\in U''_nA''_n} \operatorname{trace}\int_{N_n} (Dh)(y)\pi^\dagger_{\gamma,n}(xyx^{-1})dy\,dx \\ &= \int_{x\in U''_nA''_n} \operatorname{trace}\int_{N_n} (Dh)(y)(\operatorname{Ad}^*(x)^{-1}\cdot\pi^\dagger_{\gamma,n})(y)dy\,dx \\ &= \int_{x\in U''_nA''_n} \operatorname{trace}\left((\operatorname{Ad}^*(x)^{-1}\cdot\pi^\dagger_{\gamma,n})(Dh)\right)dx \\ &= \int_{x\in U''_nA''_n} (\operatorname{Ad}^*(x)^{-1}\cdot\pi_{\gamma,n})_*(D)\ \operatorname{trace}\left(\operatorname{Ad}(x)^{-1}\cdot\pi^\dagger_{\gamma,n}\right)(h)\,dx \\ &= \int_{x\in U''_nA''_n} (\operatorname{Ad}^*(x)D)\ \operatorname{trace}\left(\operatorname{Ad}(x)^{-1}\cdot\pi^\dagger_{\gamma,n}\right)(h)\,dx \\ &= \int_{x\in U''_nA''_n} \delta(x)\operatorname{trace}\left(\operatorname{Ad}(x)^{-1}\cdot\pi^\dagger_{\gamma,n}\right)(h)\,dx \\ &= \int_{\gamma'_n\in\operatorname{Ad}^*(U_nA_n)\gamma_n} \operatorname{trace}\pi^\dagger_{\gamma',n}(h)|\operatorname{Pf}(\gamma'_n)|d\gamma'_n.\end{aligned}$$

Summing over the the space of U_nA_n-orbits on $\mathfrak{s}^*_n$ we now have

$$\begin{aligned}&\int_{\gamma_n\in\operatorname{Ad}^*(U_nA_n)\backslash\mathfrak{s}^*_n}\left(\sum_{\widehat{U'_n}}\int_{\widehat{A'_n}} \operatorname{trace}\pi_{\gamma,\xi,\mu',n}(Dh)d\xi\deg\mu'_n\right)d\gamma_n \\ &\qquad = \int_{\gamma_n\in\operatorname{Ad}^*(U_nA_n)\backslash\mathfrak{s}^*_n}\left(\int_{\gamma'_n\in\operatorname{Ad}^*(U_nA_n)\gamma_n} \operatorname{trace}\pi^\dagger_{\gamma'_n}(h)|\operatorname{Pf}(\gamma'_n)|d\gamma'_n\right)d\gamma_n \\ &\qquad = \int_{\gamma_n\in\mathfrak{s}^*_n} \operatorname{trace}\pi_{\gamma_n}(h)|\operatorname{Pf}(\gamma_n)|d\gamma_n = h(1) = f(x)\,.\end{aligned}$$

That completes the proof. ∎

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