Stepwise Square Integrable Representations: The Concept and Some Consequences

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Abstract There are some new developments on Plancherel formula and growth of matrix coefficients for unitary representations of nilpotent Lie groups. These have several consequences for the geometry of weakly symmetric spaces and analysis on parabolic subgroups of real semisimple Lie groups, and to (infinite dimensional) locally nilpotent Lie groups. Many of these consequences are still under development. In this note I'll survey a few of these new aspects of representation theory for nilpotent Lie groups and parabolic subgroups.

1 Introduction

There is a well developed theory of square integrable representations of nilpotent Lie groups [17]. It is based on the general representation theory of Kirillov [12] for connected nilpotent real Lie groups. A connected simply connected Lie group N with center Z is called *square integrable* if it has unitary representations π whose coefficients $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ satisfy $|f_{u,v}| \in \mathcal{L}^2(N/Z)$. If N has one such square integrable representation then there is a certain polynomial function $P(\gamma)$ on the linear dual space $\mathfrak{z}^*$ of the Lie algebra of Z that is key to harmonic analysis on N. Here $P(\gamma)$ is the Pfaffian of the antisymmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$ given by $b_\lambda(x, y) = \lambda([x, y])$ where $\gamma = \lambda|_{\mathfrak{z}}$. The square integrable representations of N are certain easily-constructed representations π_γ where $\gamma \in \mathfrak{z}^*$ with $P(\gamma) \neq 0$, Plancherel almost irreducible unitary representations of N are square integrable, and up to an explicit constant $|P(\gamma)|$ is the Plancherel density of the unitary dual $\widehat{N}$ at π_λ. This theory has some interesting analytic consequences [26].

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More recently there was a serious extension of that theory [27]. Under certain conditions, the nilpotent Lie group N has a decomposition into subgroups that have square integrable representations, and the Plancherel formula then is synthesized explicitly in terms of the Plancherel formulae of those subgroups. In particular the extended theory applies to nilradicals of minimal parabolic subgroups [27]. With a minor technical adjustment it has just been extended to nilradicals of arbitrary real parabolics [32]. The consequences include explicit Plancherel and Fourier inversion formulas. Applications include analysis on minimal parabolic subgroups [28] and, more generally, on maximal amenable subgroups of parabolics [32], They also include analysis on commutative spaces, i.e. on Gelfand pairs [31]. We sketch some of these developments. Due to constraints of time and space we pass over many aspects of operator theory and orbit geometry, for example those described in [2–4], related to stepwise square integrable representations.

In Sect. 2 we recall the basic facts [17], with a few extensions, on square integrable representations of nilpotent Lie groups. In Sect. 3 we recall the concept and main results for stepwise square integrable nilpotent Lie group.

In Sect. 4 we show how nilradicals of minimal parabolic subgroups have the required decomposition for stepwise square integrability. This is a construction based on concept of strongly orthogonal restricted roots.

In Sect. 5 we indicate the consequences for homogeneous compact nilmanifolds, and in Sect. 6 we mention the application to analysis on commutative nilmanifolds.

In Sect. 7 we start the extension of stepwise square integrability results from the nilradical N of a minimal parabolic $P = MAN$ to various subgroups that contain N. This section concentrates on the subgroup MN and takes advantage of principal orbit theory. That gives a sharp simplification to the Plancherel and Fourier Inversion formulae. In Sect. 8 we look at P and its subgroup AN. They are not unimodular, so we introduce the Dixmier–Pukánszky operator D whose semi-invariance balances that of the modular function. It is a key point for the Plancherel and Fourier Inversion formulae.

Sections 9 and 10 are a short discussion of work in progress on the extension of results from minimal parabolics to parabolics in general. There are two places where matters diverge from the minimal parabolic case. First, there is a technical adjustment to the definition of stepwise square integrable representation, caused by the fact that in the non-minimal case the restricted roots need not form a root system. Second, again for technical reasons, the explicit Plancherel Formula only comes through for the maximal amenable subgroups UAN of G, and not for all of the parabolic.

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2 Square Integrable Representations

Let G be a unimodular locally compact group with center Z, and let π be an irreducible unitary representation. We associate the central character $\chi_\pi \in \widehat{Z}$ by $\pi(z) = \chi_\pi(x) \cdot 1$ for $z \in Z$. Consider a matrix coefficient $f_{u,v} : x \mapsto \langle u, \pi(x)v\rangle$. Then $|f_{u,v}|$ is a well defined function on G/Z. Fix Haar measures μ_G on G, μ_Z on Z and $\mu_{G/Z}$ on G/Z such that $d\mu_G = d\mu_Z\, d\mu_{G/Z}$. The following results are well known.

Theorem 2.1 *The following conditions on $\pi \in \widehat{G}$ are equivalent.*

(1) There exist nonzero $u, v \in \mathcal{H}_\pi$ with $|f_{u,v}| \in \mathcal{L}^2(G/Z)$.
(2) $|f_{u,v}| \in \mathcal{L}^2(G/Z)$ for all $u, v \in \mathcal{H}_\pi$.
(3) π is a discrete summand of the representation $\mathrm{Ind}_Z^G(\chi_\pi)$.

Theorem 2.2 *If the conditions of* Theorem 2.1 *are satisfied for an irreducible $\pi \in \widehat{G}$, then there is a number* $\deg \pi > 0$ *such that*

$$\int_{G/Z} f_{u,v}(x)\overline{f_{u',v'}(x)}d\mu_{G/Z}(xZ) = \tfrac{1}{\deg \pi}\langle u, u'\rangle\overline{\langle v v'\rangle} \tag{1}$$

for all $u, u', v, v' \in \mathcal{H}_\pi$. If $\pi_1, \pi_2 \in \widehat{G}$ are inequivalent and satisfy the conditions of Theorem 2.1, *and $\chi_{\pi_1} = \chi_{\pi_2}$, then*

$$\int_{G/Z} \langle u, \pi_1(x)v\rangle\overline{\langle u', \pi_2(x)v'\rangle}d\mu_{G/Z}(xZ) = 0 \tag{2}$$

for all $u, v \in \mathcal{H}_{\pi_1}$ and all $u', v' \in \mathcal{H}_{\pi_2}$.

The main results of [17] shows exactly how this works for nilpotent Lie groups.

Theorem 2.3 *Let N be a connected simply connected Lie group with center Z, $\mathfrak{n}$ and $\mathfrak{z}$ their Lie algebras, and $\mathfrak{n}^*$ the linear dual space of $\mathfrak{n}$. Let $\lambda \in \mathfrak{n}^*$ and let π_λ denote the irreducible unitary representation attached to* $\mathrm{Ad}^*(N)\lambda$ *by the Kirillov theory* [12]. *Then the following conditions are equivalent.*

(1) π_λ satisfies the conditions of Theorem 2.1.
(2) The coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \{\nu \in \mathfrak{n}^* \mid \nu|_\mathfrak{z} = \lambda|_\mathfrak{z}\}$.
(3) The bilinear form $b_\lambda(x, y) = \lambda([x, y])$ on $\mathfrak{n}/\mathfrak{z}$ is nondegenerate.
(4) The universal enveloping algebra $\mathcal{U}(\mathfrak{z})$ is the center of $\mathcal{U}(\mathfrak{n})$.

The Pfaffian polynomial $\mathrm{Pf}\,(b_\lambda)$ *is a polynomial function $P(\lambda|_\mathfrak{z})$ on $\mathfrak{z}^*$, and the set of representations π_λ for which these conditions hold, is parameterized by the set $\{\gamma \in \mathfrak{z}^* \mid P(\gamma) \neq 0\}$ (which is empty or Zariski open in $\mathfrak{z}^*$).*

We will say that the connected simply connected Lie group N is *square integrable* if there exists $\lambda \in \mathfrak{n}^*$ such that $P(\lambda|_\mathfrak{z}) \neq 0\}$. For convenience we will sometimes write $P(\lambda)$ for $P(\lambda|_\mathfrak{z})$ and π_γ for π_λ where $\gamma = \lambda|_\mathfrak{z}$.

Theorem 2.4 *Let N be a square integrable connected simply connected Lie group with center Z. Then Plancherel measure on $\widehat{N}$ is concentrated on $\{\pi_\lambda \mid P(\lambda) \neq 0\}$, and there the Plancherel measure is given by the measure $|P(\lambda)d\lambda|$ on $\mathfrak{z}^*$ and the formal degree* $\deg \pi_\lambda = |P(\lambda|_{\mathfrak{z}})|$.

Given $\gamma \in \mathfrak{z}^*$ with $P(\gamma) \neq 0$ and a Schwartz class ($\mathcal{C}(N)$) function f on N we write $\mathcal{O}(\gamma)$ for the co-adjoint orbit $\mathrm{Ad}^*(N)\gamma = \gamma + \mathfrak{z}^\perp$, f_γ for the restriction of $f \cdot \exp$ to $\mathcal{O}(\gamma)$, and $\widehat{f_\gamma}$ for the Fourier transform of f_γ on $\mathcal{O}(\gamma)$.

Theorem 2.5 *Let N be a square integrable connected simply connected Lie group with center Z and $f \in \mathcal{C}(N)$. If $\gamma \in \mathfrak{z}^*$ with $P(\gamma) \neq 0$ then the distribution character of π_γ is given by*

$$\Theta_{\pi_\gamma}(f) = \mathrm{trace} \int_N f(x)\pi_\gamma(x)d\mu_G(x) = c^{-1}|P(\gamma)|^{-1} \int_{\nu \in \mathcal{O}(\gamma)} \widehat{f_\gamma}\, d\nu \qquad (3)$$

where $c = d!2^d$ and $d = \dim(\mathfrak{n}/\mathfrak{z})/2$ and $d\nu$ is ordinary Lebesgue measure on the affine space $\mathcal{O}(\gamma)$. The Fourier Inversion formula for N is

$$f(x) = c \int_{\mathfrak{z}^*} \Theta_\gamma(r_x f)|P(\gamma)|\, d\gamma \text{ where } (r_x f)(y) = f(yx) \text{ (right translate).} \qquad (4)$$

There also are multiplicity results on $\mathcal{L}^2(N/\Gamma)$ where N is square integrable and Γ is a discrete co-compact subgroup, but they are the same as in the stepwise square integrable case, so we postpone their description.

3 Stepwise Square Integrability

In order to go beyond square integrable nilpotent groups, we suppose that the connected simply connected nilpotent Lie group decomposes as

$$\begin{aligned} &N = L_1 L_2 \dots L_{m-1} L_m \text{ where} \\ &\quad \text{(a) each } L_r \text{ has unitary representations with coeff in } \mathcal{L}^2(L_r/Z_r), \\ &\quad \text{(b) } N_r := L_1 L_2 \dots L_r \text{ is normal in } N \text{ with } N_r = N_{r-1} \rtimes L_r\,, \qquad (5) \\ &\quad \text{(c) } [\mathfrak{l}_r, \mathfrak{z}_s] = 0 \text{ and } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v} \text{ for } r > s \text{ with } \mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r \\ &\qquad \text{where } \mathfrak{n} = \mathfrak{s} + \mathfrak{v},\ \mathfrak{s} = \oplus \mathfrak{z}_r \text{ and } \mathfrak{v} = \oplus \mathfrak{v}_r\,. \end{aligned}$$

We will use the following notation.

$$\begin{aligned}
&\text{(a) } d_r = \tfrac{1}{2}\dim(\mathfrak{l}_r/\mathfrak{z}_r) \text{ so } \tfrac{1}{2}\dim(\mathfrak{n}/\mathfrak{s}) = d_1 + \cdots + d_m\,,\\
&\qquad\qquad \text{and } c = 2^{d_1+\cdots+d_m} d_1!d_2!\ldots d_m!\\
&\text{(b) } b_{\lambda_r} : (x, y) \mapsto \lambda([x, y]) \text{ viewed as a bilinear form on } \mathfrak{l}_r/\mathfrak{z}_r\\
&\text{(c) } S = Z_1 Z_2 \ldots Z_m = Z_1 \times \cdots \times Z_m \text{ where } Z_r \text{ is the center of } L_r\\
&\text{(d) } P:\ \text{polynomial } P(\lambda) = \text{Pf}\,(b_{\lambda_1})\text{Pf}\,(b_{\lambda_2})\ldots\text{Pf}\,(b_{\lambda_m}) \text{ on } \mathfrak{s}^*\\
&\text{(e) } \mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid P(\lambda) \neq 0\}\\
&\text{(f) } \pi_\lambda \in \widehat{N} \text{ for } \lambda \in \mathfrak{s}^* \text{ with } P(\lambda) \neq 0, \text{ irreducible unitary representation}\\
&\qquad\qquad \text{of } N = L_1 L_2 \ldots L_m \text{ constructed as follows.}
\end{aligned} \tag{6}$$

Start with the representation $\pi_{\lambda_1} \in \widehat{N_1}$ specified by $\lambda_1 \in \mathfrak{z}_1^*$ with $\text{Pf}\,(b_{\lambda_1}) \neq 0$. Choose an invariant polarization $\mathfrak{p}_1' \subset \mathfrak{n}_2$ for the linear functional $\lambda_1' \in \mathfrak{n}_2^*$ that agrees with λ_1 on $\mathfrak{n}_1$ and vanishes on $\mathfrak{l}_2$. Since L_r centralizes S_{r-1}, $\text{ad}^*(\mathfrak{l}_2)(\lambda_1')|_{\mathfrak{z}_1+\mathfrak{l}_2} = 0$, so $\mathfrak{p}_1' = \mathfrak{p}_1 + \mathfrak{l}_2$ where $\mathfrak{p}_1$ is an invariant polarization for $\lambda_1 \in \mathfrak{n}_1^*$. The associated representations are $\pi_{\lambda_1}' \in \widehat{N_2}$ and $\pi_{\lambda_1} \in \widehat{N_1}$. Note that $N_2/P_1' = N_1/P_1$, so the representation spaces $\mathcal{H}_{\pi_{\lambda_1}'} = \mathcal{L}^2(N_2/P_1') = \mathcal{L}^2(N_1/P_1) = \mathcal{H}_{\pi_{\lambda_1}}$. In other words, π_{λ_1}' extends π_{λ_1} to a unitary representation of N_2 on the same Hilbert space $\mathcal{H}_{\pi_{\lambda_1}}$, and $d\pi_{\lambda_1'}(\mathfrak{z}_2) = 0$. Now the Mackey Little Group method gives us

Lemma 3.1 *The irreducible unitary representations of N_2, whose restrictions to N_1 are multiples of π_{λ_1}, are the $\pi_{\lambda_1}' \widehat{\otimes} \gamma$ where $\gamma \in \widehat{L_2} = \widehat{N_2/N_1}$.*

Given $\lambda_2 \in \mathfrak{z}_2^*$ with $\text{Pf}\,(b_{\lambda_2}) \neq 0$ we have $\pi_{\lambda_2} \in \widehat{L_2}$ with coefficients in $\mathcal{L}^2(L_2/Z_2)$. In the notation of Lemma 3.1 we define

$$\pi_{\lambda_1+\lambda_2} \in \widehat{N_2} \text{ by } \pi_{\lambda_1+\lambda_2} = \pi_{\lambda_1}' \widehat{\otimes} \pi_{\lambda_2}\,. \tag{7}$$

Proposition 3.2 *The coefficients $f_{z,w}(xy) = \langle z, \pi_{\lambda_1+\lambda_2}(xy)w\rangle$ of $\pi_{\lambda_1+\lambda_2}$ belong to $\mathcal{L}^2(N_2/S_2)$, in fact satisfy $||f_{z,w}||^2_{\mathcal{L}^2(N_r/S_r)} = \frac{||z||^2||w||^2}{\deg(\pi_{\lambda_1})\ldots\deg(\pi_{\lambda_r})}$.*

Proposition 3.2 starts a recursion using $N_{r+1} = N_r \rtimes L_{r+1}$. We fix nonzero $\lambda_i \in \mathfrak{z}_i^*$ for $1 \leqq i \leqq r+1$, and we start with the representation $\pi_{\lambda_1+\cdots+\lambda_r}$ constructed step by step from the square integrable representations $\pi_{\lambda_i} \in \widehat{L_i}$ for $1 \leqq i \leqq r$. The representation space $\mathcal{H}_{\pi_{\lambda_1+\cdots+\lambda_r}} = \mathcal{H}_{\pi_{\lambda_1}} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{H}_{\pi_{\lambda_r}}$. The coefficients of $\pi_{\lambda_1+\cdots+\lambda_r}$ have absolute value in $\mathcal{L}^2(N_r/S_r)$. They satisfy

$$||f_{z,w}||^2_{\mathcal{L}^2(N_r/S_r)} = \frac{||z||^2||w||^2}{\deg(\pi_{\lambda_1})\ldots\deg(\pi_{\lambda_r})}\,. \tag{8}$$

Then $\pi_{\lambda_1+\cdots+\lambda_r}$ extends to a representation $\pi_{\lambda_1+\cdots+\lambda_r}'$ of L_{r+1} on the same Hilbert space $\mathcal{H}_{\pi_{\lambda_1+\cdots+\lambda_r}}$, and it satisfies $d\pi_{\lambda_1+\cdots+\lambda_r}'(\mathfrak{z}_{r+1}) = 0$. As in Lemma 3.1,

Lemma 3.3 *The irreducibles $\pi \in \widehat{N_{r+1}}$, whose restrictions to N_r are multiples of $\pi_{\lambda_1+\cdots+\lambda_r}$, are the $\pi'_{\lambda_1+\cdots+\lambda_r} \widehat{\otimes} \gamma$ where $\gamma \in \widehat{L_{r+1}} = \widehat{N_{r+1}/N_r}$.*

As in Proposition 3.2, define $\pi_{\lambda_1+\cdots+\lambda_{r+1}} = \pi'_{\lambda_1+\cdots+\lambda_r} \widehat{\otimes} \pi_{\lambda_{r+1}}$. Then

Proposition 3.4 *The coefficients $f_{z,w}(x_1 \dots x_{r+1}) = \langle z, \pi_{\lambda_1+\cdots+\lambda_{r+1}}(x_1 x_2 \cdots x_{r+1}) w\rangle$ of $\pi_{\lambda_1+\cdots+\lambda_{r+1}}$ belong to $\mathcal{L}^2(N_{r+1}/S_{r+1})$, in fact satisfy*

$$||f_{z,w}||^2_{\mathcal{L}^2(N_{r+1}/S_{r+1})} = \frac{||z||^2||w||^2}{\deg(\pi_{\lambda_1})\dots\deg(\pi_{\lambda_{r+1}})}\,.$$

Since $\deg \pi_{\lambda_r} = |\mathrm{Pf}\,(b_{\lambda_r})|$, Proposition 3.4 is the recursion step for our construction. Passing to the end case $r+1 = m$ we see that Plancherel measure is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. Using (5)(c) to see that conjugation by elements of L_s has no effect on the $\mathrm{Pf}\,(b_{\lambda_r})$ for $r < s$, we arrive at

Theorem 3.5 *Let N be a connected simply connected nilpotent Lie group that satisfies (5). Then Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*, P(\lambda) \neq 0\}$. If $\lambda \in \mathfrak{t}^*$, $P(\lambda) \neq 0$ and $u, v \in \mathcal{H}_{\pi_\lambda}$, then the coefficient $f_{u,v}(x) = \langle u, \pi_\nu(x)v\rangle$ satisfies*

$$||f_{u,v}||^2_{\mathcal{L}^2(N/S)} = ||u||^2||v||^2/|P(\lambda)|\,. \tag{9}$$

The distribution character $\Theta_{\pi_\lambda} : f \mapsto \mathrm{trace} \int_G f(x)\pi(x)dx$ of π_λ is given by

$$\Theta_{\pi_\lambda}(f) = c^{-1}|P(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f_\lambda}(\xi) d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N) \tag{10}$$

where $\mathcal{C}(N)$ is the Schwartz space, $\mathcal{O}(\lambda) = \mathrm{Ad}^*(N)\lambda = \mathfrak{s}^\perp + \lambda$, f_λ is the lift $f_\lambda(\xi) = f(\exp(\xi))$, $\widehat{f_\lambda}$ is its classical Fourier transform, and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{s}^\perp$ to $\mathrm{Ad}^*(N)\lambda$. Further,

$$f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|P(\lambda)|d\lambda \text{ for } f \in \mathcal{C}(N). \tag{11}$$

Definition 3.6 The representations π_λ of (6)(f) are the *stepwise square integrable* representations of N relative to (5). ◇

The left action $(l(x)f)(g) = f(x^{-1}g)$ and the right action $(r(y)f)(g) = f(gy)$ of N on functions carries over to coefficients of π as $l(x)r(y)f_{u,v} = f_{\pi(x)u,\pi(y)v}$. If $\pi = \pi_\lambda$ stepwise square integrable, $u, v \in \mathcal{H}_{\pi_\lambda}$ are C^∞ vectors, and if Φ and Ψ belong to the universal enveloping algebra $\mathcal{U}(\mathfrak{n})$, then $l(\Phi)r(\Psi)f_{u,v} = f_{d\pi(\Psi)u,d\pi(\Phi)v}$ is just another coefficient, C^∞ and $\mathcal{L}^2(N/S)$. If $\zeta_\lambda \in \widehat{S}$ is the quasicentral character of π_λ it follows that $f_{u,v}$ belongs to the relative Schwartz space $\mathcal{C}(N/S, \zeta_\lambda)$. In particular it follows that $|f_{u,v}| \in \mathcal{L}^p(N/S)$ for all $p \geqq 1$. Taking Schwartz class wave packets over S of coefficient functions of stepwise square integrable representations of N one can express the Plancherel formula of Theorem 3.5 in terms of coefficient functions.

4 Nilradicals of Minimal Parabolics

Fix a real simple Lie group G, an Iwasawa decomposition $G = KAN$, and a minimal parabolic subgroup $Q = MAN$ in G. Let $m = \text{rank}_{\mathbb{R}} G = \dim_{\mathbb{R}} A$. As usual, write $\mathfrak{k}$ for the Lie algebra of K, $\mathfrak{a}$ for the Lie algebra of A, and $\mathfrak{n}$ for the Lie algebra of N. Complete $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Now we have root systems

$$\begin{aligned} &\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{: roots of } \mathfrak{g}_{\mathbb{C}} \text{ relative to } \mathfrak{h}_{\mathbb{C}} \text{ (ordinary roots)},\\ &\Delta(\mathfrak{g}, \mathfrak{a}) \text{: roots of } \mathfrak{g} \text{ relative to } \mathfrak{a} \text{ (restricted roots)},\\ &\Delta_0(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid 2\alpha \notin \Delta(\mathfrak{g}, \mathfrak{a})\} \text{ (nonmultipliable)}. \end{aligned} \tag{12}$$

Here $\Delta(\mathfrak{g}, \mathfrak{a})$ and $\Delta_0(\mathfrak{g}, \mathfrak{a})$ are root systems in the usual sense. Any positive root system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \subset \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ defines positive systems

$$\begin{aligned} &\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\gamma|_{\mathfrak{a}} \mid \gamma \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ and } \gamma|_{\mathfrak{a}} \neq 0\},\\ &\Delta_0^+(\mathfrak{g}, \mathfrak{a}) = \Delta_0(\mathfrak{g}, \mathfrak{a}) \cap \Delta^+(\mathfrak{g}, \mathfrak{a}). \end{aligned} \tag{13}$$

We can (and do) choose $\Delta^+(\mathfrak{g}, \mathfrak{h})$ so that

$$\begin{aligned} &\mathfrak{n} \text{ is the sum of the positive restricted root spaces and}\\ &\text{if } \gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ and } \gamma|_{\mathfrak{a}} \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ then } \gamma \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}). \end{aligned} \tag{14}$$

Two roots are called *strongly orthogonal* if their sum and their difference are not roots. Then they are orthogonal. The Kostant cascade construction is

$$\begin{aligned} &\beta_1 \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximal positive restricted root and}\\ &\beta_{r+1} \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximum among the roots of } \Delta^+(\mathfrak{g}, \mathfrak{a})\\ &\text{that are orthogonal to all } \beta_i \text{ with } i \leqq r \end{aligned} \tag{15}$$

Then the β_r are mutually strongly orthogonal. Each $\beta_r \in \Delta_0^+(\mathfrak{g}, \mathfrak{a})$, and β_1 is unique because $\Delta(\mathfrak{g}, \mathfrak{a})$ is irreducible. For $1 \leqq r \leqq m$ define

$$\begin{aligned} &\Delta_1^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \beta_1 - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\} \text{ and}\\ &\Delta_{r+1}^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \setminus (\Delta_1^+ \cup \cdots \cup \Delta_r^+) \mid \beta_{r+1} - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\}. \end{aligned} \tag{16}$$

Lemma 4.1 *If $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$, either $\alpha \in \{\beta_1, \ldots, \beta_m\}$ or α belongs to just one Δ_r^+.*

Lemma 4.2 *$\Delta_r^+ \cup \{\beta_r\} = \{\alpha \in \Delta^+ \mid \alpha \perp \beta_i$ for $i < r$ and $\langle \alpha, \beta_r \rangle > 0\}$. In particular, $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}_t$ where $t = \min\{r, s\}$.*

Lemma 4.1 shows that the Lie algebra $\mathfrak{n}$ of N is the direct sum of its subspaces

$$\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum\nolimits_{\Delta_r^+} \mathfrak{g}_\alpha \text{ for } 1 \leqq r \leqq m \tag{17}$$

and Lemma 4.2 shows that $\mathfrak{n}$ has an increasing foliation by ideals

$$\mathfrak{n}_r = \mathfrak{l}_1 + \mathfrak{l}_2 + \cdots + \mathfrak{l}_r \text{ for } 1 \leqq r \leqq m. \tag{18}$$

Now we will see that the corresponding group level decomposition $N = L_1 L_2 \dots L_m$ and the semidirect product decompositions $N_r = N_{r-1} \rtimes L_r$ satisfy (5). Denote

$$\begin{aligned} &s_{\beta_r} \text{ is the Weyl group reflection in } \beta_r \text{ and} \\ &\sigma_r : \Delta(\mathfrak{g}, \mathfrak{a}) \to \Delta(\mathfrak{g}, \mathfrak{a}) \text{ by } \sigma_r(\alpha) = -s_{\beta_r}(\alpha). \end{aligned} \tag{19}$$

Note that $\sigma_r(\beta_s) = -\beta_s$ for $s \neq r$, $+\beta_s$ if $s = r$. If $\alpha \in \Delta_r^+$ we still have $\sigma_r(\alpha) \perp \beta_i$ for $i < r$ and $\langle \sigma_r(\alpha), \beta_r \rangle > 0$. If $\sigma_r(\alpha) < 0$ then $\beta_r - \sigma_r(\alpha) > \beta_r$ contradicting maximality of β_r. Thus, using Lemma 4.2, $\sigma_r(\Delta_r^+) = \Delta_r^+$.

Lemma 4.3 *If $\alpha \in \Delta_r^+$ then $\alpha + \sigma_r(\alpha) = \beta_r$. (It is possible that $\alpha = \sigma_r(\alpha) = \frac{1}{2}\beta_r$ when $\frac{1}{2}\beta_r$ is a root.). If $\alpha, \alpha' \in \Delta_r^+$ and $\alpha + \alpha' \in \Delta(\mathfrak{g}, \mathfrak{a})$ then $\alpha + \alpha' = \beta_r$.*

Lemma 4.4 *Let $\mathfrak{n}$ be a nilpotent Lie algebra, $\mathfrak{z}$ its center, and $\mathfrak{v}$ a vector space complement to $\mathfrak{z}$ in $\mathfrak{n}$. Suppose that $\mathfrak{v} = \mathfrak{u} + \mathfrak{u}'$, $\mathfrak{u} = \sum \mathfrak{u}_a$ and $\mathfrak{u}' = \sum \mathfrak{u}'_a$, and $\mathfrak{z} = \sum \mathfrak{z}_b$ with $\dim \mathfrak{z}_b = 1$ in such a way that* (i) *each $[\mathfrak{u}_a, \mathfrak{u}_a] = 0 = [\mathfrak{u}'_a, \mathfrak{u}'_a]$,* (ii) *if $a_1 \neq a_2$ then $[\mathfrak{u}_{a_1}, \mathfrak{u}'_{a_2}] = 0$ and* (iii) *for each a there is a nondegenerate pairing $\mathfrak{u}_a \otimes \mathfrak{u}'_a \to \mathfrak{z}_{b_a}$, by $u \otimes u' \mapsto [u, u']$. Then $\mathfrak{n}$ is a direct sum of Heisenberg algebras $\mathfrak{z}_{b_a} + \mathfrak{u}_a + \mathfrak{u}'_a$ and the commutative algebra that is the sum of the remaining $\mathfrak{z}_b$.*

Now one runs through a number of special situations: (1) If $\mathfrak{g}$ is the split real form of $\mathfrak{g}_{\mathbb{C}}$ then each L_r has square integrable representations. (2) If $\mathfrak{g}$ is simple but not absolutely simple then each L_r has square integrable representations. (3) If G is the quaternion special linear group $SL(n; \mathbb{H})$ then L_1 has square integrable representations. (4) If G is the group E_{6,F_4} of collineations of the Cayley projective plane then L_1 has square integrable representations. (5) The group L_1 has square integrable representations. (6) If $\mathfrak{g}$ is absolutely simple then each L_r has square integrable representations. Putting these together, Theorem 3.5 applies to nilradicals of minimal parabolic subgroups:

Theorem 4.5 *Let G be a real reductive Lie group, $G = KAN$ an Iwasawa decomposition, $\mathfrak{l}_r$ and $\mathfrak{n}_r$ the subalgebras of $\mathfrak{n}$ defined in (17) and (18), and L_r and N_r the corresponding analytic subgroups of N. Then the L_r and N_r satisfy (5). In particular, Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ, then the coefficient $f_{u,v}(x) = \langle u, \pi_\lambda(x)v \rangle$ satisfies $||f_{u,v}||^2_{\mathcal{L}^2(N/S)} = \frac{||u||^2||v||^2}{|P(\lambda)|}$. The distribution character Θ_{π_λ} of π_λ satisfies $\Theta_{\pi_\lambda}(f) = c^{-1}|P(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f}_\lambda(\xi) d\nu_\lambda(\xi)$ for $f \in \mathcal{C}(N)$. Here $\mathcal{C}(N)$ is the Schwartz space, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{s}^\perp + \lambda$, f_γ is the lift $f_\gamma(\xi) = f(\exp(\xi))$ to $\mathfrak{s}^\perp + \lambda$, $\widehat{f}_\gamma$ is its classical Fourier transform, and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{s}^\perp$ to $\mathrm{Ad}^*(N)\lambda$. The Plancherel formula on N is $f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|P(\lambda)| d\lambda$ for $f \in \mathcal{C}(N)$.*

5 Compact Nilmanifolds

Here are the basic facts on discrete uniform (i.e. co-compact) subgroups of connected simply connected nilpotent Lie groups. See [21, Chap. 2] for an exposition.

Proposition 5.1 *The following are equivalent.*

- *N has a discrete subgroup Γ with N/Γ compact.*
- *$N \cong N_{\mathbb{R}}$ where $N_{\mathbb{R}}$ is the group of real points in a unipotent linear algebraic group defined over the rational number field $\mathbb{Q}$.*
- *$\mathfrak{n}$ has a basis $\{\xi_j\}$ for which the coefficients $c^k_{i,j}$ in $[\xi_i, \xi_j] = \sum c^k_{i,j}\xi_k$ are rational numbers.*

Under those conditions let $\mathfrak{n}_{\mathbb{Q}}$ denote the rational span of $\{\xi_j\}$ and let $\mathfrak{n}_{\mathbb{Z}}$ be the integral span. Then $\exp(\mathfrak{n}_{\mathbb{Z}})$ generates a discrete subgroup $N_{\mathbb{Z}}$ of $N = N_{\mathbb{R}}$ and $N_{\mathbb{R}}/N_{\mathbb{Z}}$ is compact. Conversely, if Γ is a discrete co-compact subgroup of N then the $\mathbb{Z}$-span of $\exp^{-1}(\Gamma)$ is a lattice in $\mathfrak{n}$ for which any generating set $\{\xi_j\}$ is a basis of $\mathfrak{n}$ such that the coefficients $c^k_{i,j}$ in $[\xi_i, \xi_j] = \sum c^k_{i,j}\xi_k$ are rational numbers.

The conditions of Proposition 5.1 hold for the nilpotent groups studied in Sect. 4; there one can choose the basis $\{\xi_j\}$ of $\mathfrak{n}$ so that the $c^k_{i,j}$ are integers.

The basic facts on square integrable representations that occur in compact quotients N/Γ, as described in [17, Theorem 7], are

Proposition 5.2 *Let N be a connected simply connected nilpotent Lie group that has square integrable representations, and let Γ a discrete co-compact subgroup. Let Z be the center of N and normalize the volume form on $\mathfrak{n}/\mathfrak{z}$ by normalizing Haar measure on N so that $N/Z\Gamma$ has volume 1. Let P be the corresponding Pfaffian polynomial on $\mathfrak{z}^*$. Note that $\Gamma \cap Z$ is a lattice in Z and $\exp^{-1}(\Gamma \cap Z)$ is a lattice (denote it Λ) in $\mathfrak{z}$. That defines the dual lattice Λ^* in $\mathfrak{z}^*$. Then a square integrable representation π_λ occurs in $\mathcal{L}^2(N/\Gamma)$ if and only if $\lambda \in \Lambda^*$, and in that case π_λ occurs with multiplicity $|P(\lambda)|$.*

Definition 5.3 Let $N = N_{\mathbb{R}}$ be defined over $\mathbb{Q}$ as in Proposition 5.1, so we have a fixed rational form $N_{\mathbb{Q}}$. We say that a connected Lie subgroup $L \subset N$ is *rational* if $L \cap N_{\mathbb{Q}}$ is a rational form of L, in other words if $\mathfrak{l} \cap \mathfrak{n}_{\mathbb{Q}}$ contains a basis of $\mathfrak{l}$. We say that a decomposition (5) is *rational* if the subgroups L_r and N_r are rational. ◇

The following is immediate from this definition.

Lemma 5.4 *Let N be defined over $\mathbb{Q}$ as in* Proposition 5.1 *with rational structure defined by a discrete co-compact subgroup Γ. If the decomposition* (5) *is rational then each $\Gamma \cap Z_r$ in Z_r, each $\Gamma \cap L_r$ in L_r, each $\Gamma \cap S_r$ in S_r, and each $\Gamma \cap N_r$ in N_r, is a discrete co-compact subgroup defining the same rational structure as the one defined by its intersection with $N_{\mathbb{Q}}$.*

Now assume that N and Γ satisfy the rationality conditions of Lemma 5.4. Then for each r, $Z_r \cap \Gamma$ is a lattice in the center Z_r of L_r, and $\Lambda_r := \log(Z_r \cap \Gamma)$ is a lattice in its Lie algebra $\mathfrak{z}_r$. That defines the dual lattice Λ_r^* in $\mathfrak{z}_r^*$. We normalize the Pfaffian polynomials on the $\mathfrak{z}_r^*$, and thus the polynomial P on $\mathfrak{s}^*$, by requiring that the $N_r/(S_r \cdot (N_r \cap \Gamma))$ have volume 1.

Theorem 5.5 *Let $\lambda \in \mathfrak{t}^*$. Then a stepwise square integrable representation π_λ of N occurs in $\mathcal{L}^2(N/\Gamma)$ if and only if each $\lambda_r \in \Lambda_r^*$, and in that case the multiplicity of π_λ on $\mathcal{L}^2(N/\Gamma)$ is $|P(\lambda)|$.*

6 Commutative Spaces

A commutative space $X = G/K$, or equivalently a Gelfand pair (G, K), consists of a locally compact group G and a compact subgroup K such that the convolution algebra $\mathcal{L}^1(K\backslash G/K)$ is commutative. When G is a connected Lie group it is equivalent to say that the algebra $\mathcal{D}(G, K)$ of G-invariant differential operators on G/K is commutative. We say that the commutative space G/K is a commutative nilmanifold if it is a nilmanifold in the sense that some nilpotent analytic subgroup N of G acts transitively. When G/K is connected and simply connected it follows that N is the nilradical of G, that N acts simply transitively on G/K, and that G is the semidirect product group $N \rtimes K$, so that $G/K = (N \rtimes K)/K$. In this section we study the role of square integrability and stepwise square integrability for commutative nilmanifolds $G/K = (N \rtimes K)/K$.

The cases where G/K and (G, K) are *irreducible* in the sense that $[\mathfrak{n}, \mathfrak{n}]$ (which must be central) is the center of $\mathfrak{n}$ and K acts irreducibly on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, have been classified by E.B. Vinberg [22, 23]. See [26, Sect. 13.4B] for the Lie algebra structure $\mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$. The classification of commutative nilmanifolds is based on Vinberg's work and was completed by O. Yakimova in [34, 35].

It turns out that almost all commutative manifolds correspond to nilpotent groups that are square integrable. The exceptions are those with a certain direct factor, and in those cases the nilpotent group is stepwise square integrable in two steps, so in those cases the Plancherel formula follows directly from the general result above. See [31] for the details.

7 Minimal Parabolics: Subgroup MN

Fix an Iwasawa decomposition $G = KAN$ for a simple Lie group G and the minimal parabolic subgroup $Q = MAN$. As usual, write $\mathfrak{k}$ for the Lie algebra of K, $\mathfrak{a}$ for the Lie algebra of A, $\mathfrak{m}$ for the Lie algebra of M, and $\mathfrak{n}$ for the Lie algebra of N. Complete $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then we have root systems $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$,

$\Delta(\mathfrak{g}, \mathfrak{a})$ and $\Delta_0(\mathfrak{g}, \mathfrak{a})$ described in (12). M is the centralizer of A in K. Write ^{0} for identity component; then $Q^0 = M^0AN$.

Recall the Pf-nonsingular set $\mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid \text{Pf}\,(b_\lambda) \neq 0\}$ of (6)(e); so $\text{Ad}^*(M)\mathfrak{t}^* = \mathfrak{t}^*$. Further, if $\lambda \in \mathfrak{t}^*$ and $c \neq 0$ then $c\lambda \in \mathfrak{t}^*$, in fact $\text{Pf}\,(b_{c\lambda}) = c^{\dim(\mathfrak{n}/\mathfrak{s})/2}\,\text{Pf}\,(b_\lambda)$.

Fix an M-invariant inner product (μ, ν) on $\mathfrak{s}^*$. So $\text{Ad}^*(M)$ preserves each sphere $\mathfrak{s}_t^* = \{\lambda \in \mathfrak{s}^* \mid (\lambda, \lambda) = t^2\}$. Two orbits $\text{Ad}^*(M)\mu$ and $\text{Ad}^*(M)\nu$ are of the *same orbit type* if the isotropy subgroups M_μ and M_ν are conjugate, and an orbit is *principal* if all nearby orbits are of the same type. Since M and $\mathfrak{s}_t^*$ are compact, there are only finitely many orbit types of M on $\mathfrak{s}_t^*$, there is only one principal orbit type, and the union of the principal orbits forms a dense open subset of $\mathfrak{s}_t^*$ whose complement has codimension $\geqq 2$. See [5, Chap. 4, Sect. 3] for a complete treatment of this material, or [10, Part II, Chap. 3, Sect. 1] modulo references to [5], or [18, Chap. 5] for a basic treatment, still with some references to [5].

The action of M on $\mathfrak{s}^*$ commutes with dilation so the structural results on the $\mathfrak{s}_t$ also hold on $\mathfrak{s}^* = \bigcup_{t>0} \mathfrak{s}_t^*$. Define the Pf-nonsingular principal orbit set as follows:

$$\mathfrak{u}^* = \{\lambda \in \mathfrak{t}^* \mid \text{Ad}^*(M)\lambda \text{ is a principal } M\text{-orbit on } \mathfrak{s}^*\}. \tag{20}$$

Now principal orbit set $\mathfrak{u}^*$ is a dense open set with complement of codimension $\geqq 2$ in $\mathfrak{s}^*$. If $\lambda \in \mathfrak{u}^*$ and $c \neq 0$ then $c\lambda \in \mathfrak{u}^*$ with isotropy $M_{c\lambda} = M_\lambda$. If $\lambda \in \mathfrak{u}_t^* := \mathfrak{u}^* \cap \mathfrak{s}_t^*$, so $\text{Ad}^*(M)\lambda$ is a Pf-nonsingular principal orbit of M on the sphere $\mathfrak{s}_t^*$, then $\text{Ad}^*(M^0)\lambda$ is a principal orbit of M^0 on $\mathfrak{s}_t^*$. Principal orbit isotropy subgroups of compact connected linear groups are studied in [11] and the possibilities for the isotropy $(M^0)_\lambda$ are essentially known. The following lets us go from $(M^0)_\lambda$ to M_λ.

Proposition 7.1 ([28]) *Suppose that G is connected and linear. Then $M = FZ_GM^0$ where Z_G is the center of G, $F = (\exp(i\mathfrak{a}) \cap K)$ is an elementary abelian 2-group, and $Ad^*(F)$ acts trivially on $\mathfrak{s}^*$. If $\lambda \in \mathfrak{u}^*$ then the isotropy $M_\lambda = FZ_G(M^0)_\lambda$.*

Thus the groups M_λ are specified by the work of W.-C. and W.-Y. Hsiang [11].

Given $\lambda \in \mathfrak{u}^*$ the stepwise square integrable representation $\pi_\lambda \in \widehat{N}$ one proves that the Mackey obstruction $\varepsilon \in H^2(M_\lambda; U(1))$ is trivial, and in fact that π_λ extends to a unitary representation $\pi_\lambda^\dagger$ of $N \rtimes M_\lambda$ on the representation space of π_λ.

Each $\lambda \in \mathfrak{u}^*$ now defines classes

$$\mathcal{E}(\lambda) := \left\{\pi_\lambda^\dagger \otimes \gamma \mid \gamma \in \widehat{M_\lambda}\right\},\ \mathcal{F}(\lambda) := \left\{\text{Ind}\,_{NM_\lambda}^{NM}(\pi_\lambda^\dagger \otimes \gamma) \mid \pi_\lambda^\dagger \otimes \gamma \in \mathcal{E}(\lambda)\right\} \tag{21}$$

of irreducible unitary representations of $N \rtimes M_\lambda$ and NM. The Mackey little group method, plus the fact that the Plancherel density on $\widehat{N}$ is polynomial on $\mathfrak{s}^*$, and $\mathfrak{s}^* \setminus \mathfrak{u}^*$ has measure 0 in $\mathfrak{t}^*$, gives us

Proposition 7.2 *Plancherel measure for NM is concentrated on $\bigcup_{\lambda\in\mathfrak{u}^*} \mathcal{F}(\lambda)$, equivalence classes of irreducible representations $\eta_{\lambda,\gamma} := \text{Ind}\,_{NM_\lambda}^{NM}(\pi_\lambda^\dagger \otimes \gamma)$ such that $\pi_\lambda^\dagger \otimes \gamma \in \mathcal{E}(\lambda)$ and $\lambda \in \mathfrak{u}^*$. Further*

$$\eta_{\lambda,\gamma}|_N = \left(\mathrm{Ind}_{NM_\lambda}^{NM}(\pi_\lambda^\dagger \otimes \gamma)\right)\Big|_N = \int_{M/M_\lambda} (\dim \gamma)\, \pi_{\mathrm{Ad}^*(m)\lambda}\, d(mM_\lambda).$$

There is a Borel section σ to $\mathfrak{u}^* \to \mathfrak{u}^*/\mathrm{Ad}^*(M)$ that picks out an element in each M-orbit so that M has the same isotropy subgroup at each of those elements. In other words in each M-orbit on $\mathfrak{u}^*$ we measurably choose an element $\lambda = \sigma(\mathrm{Ad}^*(M)\lambda)$ such that those isotropy subgroups M_λ are all the same. Let us denote

$$M_\diamond\text{: isotropy subgroup of } M \text{ at } \sigma(\mathrm{Ad}^*(M)\lambda) \text{ for every } \lambda \in \mathfrak{u}^* \tag{22}$$

We replace M_λ by $M_\diamond$, independent of $\lambda \in \mathfrak{u}^*$, in Proposition 7.2. That lets us assemble to representations of Proposition 7.2 for a Plancherel Formula, as follows. Since M is compact, we have the Schwartz space $\mathcal{C}(NM)$ just as in the discussion of $\mathcal{C}(N)$ between (6) and Theorem 3.5, except that the pullback $\exp^* \mathcal{C}(NM) \neq \mathcal{C}(\mathfrak{n}+\mathfrak{m})$. The same applies to $\mathcal{C}(NA)$ and $\mathcal{C}(NAM)$.

Proposition 7.3 *Let $f \in \mathcal{C}(NM)$ and write $(f_m)(n) = f(nm) = ({}_n f)(m)$ for $n \in N$ and $m \in M$. The Plancherel density at $\mathrm{Ind}_{NM_\diamond}^{NM}(\pi_\lambda^\dagger \otimes \gamma)$ is $(\dim \gamma)|\mathrm{Pf}\,(b_\lambda)|$ and the Plancherel Formula for NM is*

$$f(nm) = c \int_{\mathfrak{u}^*/\mathrm{Ad}^*(M)} \sum_{\mathcal{F}(\lambda)} \mathrm{trace}\, \eta_{\lambda,\gamma}({}_n f_m) \cdot \dim(\gamma) \cdot |\mathrm{Pf}\,(b_\lambda)| d\lambda$$

where $c = 2^{d_1+\cdots+d_m} d_1! d_2! \ldots d_m!$, from (6)*, as in* Theorem 3.5.

8 Minimal Parabolics: MAN and AN

Let G be a separable locally compact group of type I. Then [14, Sect. 1] the Plancherel formula for G has form

$$f(x) = \int_{\widehat{G}} \mathrm{trace}\, \pi(D(r(x)f))d\mu_G(\pi) \tag{23}$$

where D is an invertible positive self adjoint operator on $L^2(G)$, conjugation-semi-invariant of weight equal to the modular function δ_G, and μ is a positive Borel measure on the unitary dual $\widehat{G}$. If G is unimodular then D is the identity and (23) reduces to the usual Plancherel formula. The point is that semi-invariance of D compensates any lack of unimodularity. See [14, Sect. 1] for a detailed discussion. $D \otimes \mu$ is unique (up to normalization of Haar measures) and one tries to find a "best" choice of D. Given any such pair (D, μ) we refer to D as a *Dixmier–Pukánszky Operator* on G and to μ as the associated *Plancherel measure* on $\widehat{G}$. We will construct a Dixmier–Pukánszky Operator from the Pfaffian polynomial $\mathrm{Pf}\,(b_\lambda)$.

Let δ_{AN} and δ_Q denote the modular functions on AN and on $Q = MAN$. As M is compact and $\mathrm{Ad}_Q(N)$ is unipotent on $\mathfrak{p}$, they are determined by their restrictions to A, where they are given by $\delta(\exp(\xi)) = \exp(\mathrm{trace}\,(\mathrm{ad}(\xi)))$ with $\xi = \log a \in \mathfrak{a}$.

Lemma 8.1 *Let* $\xi \in \mathfrak{a}$. *Then* $\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r) \in \mathbb{Z}$ *for* $1 \leqq r \leqq m$ *and*
(i) *the trace of* $\mathrm{ad}(\xi)$ *on* $\mathfrak{l}_r$ *is* $\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)\beta_r(\xi)$,
(ii) *the trace of* $\mathrm{ad}(\xi)$ *on* $\mathfrak{n}$ *and on* $\mathfrak{p}$ *is* $\frac{1}{2}\sum_r(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)\beta_r(\xi)$,
(iii) *the determinant of* $\mathrm{Ad}(\exp(\xi))$ *on* $\mathfrak{n}$ *and on* $\mathfrak{p}$ *is* $\prod_r \exp(\beta_r(\xi))^{\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)}$,
(iv) $\delta_Q(man) = \prod_r \exp(\beta_r(\log a))^{\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)}$ *and* $\delta_{AN} = \delta_Q|_{AN}$.

Now compute

Lemma 8.2 *Let* $\xi \in \mathfrak{a}$ *and* $a = \exp(\xi) \in A$. *Then* $\mathrm{ad}(\xi)\mathrm{Pf} = \left(\frac{1}{2}\sum_r \dim(\mathfrak{l}_r/\,\mathfrak{z}_r)\,\beta_r(\xi)\right)\mathrm{Pf}$ *and* $\mathrm{Ad}(a)\mathrm{Pf} = \left(\prod_r \exp(\beta_r(\xi))^{\frac{1}{2}\dim(\mathfrak{l}_r/\dim \mathfrak{z}_r)}\right)\mathrm{Pf}$.

At this point it is convenient to introduce some notation and definitions.

Definition 8.3 The algebra $\mathfrak{s}$ is the *quasi-center* of $\mathfrak{n}$. The polynomial function $\mathrm{Det}_{\mathfrak{s}^*}(\lambda) := \prod_r(\beta_r(\lambda))^{\dim \mathfrak{g}_{\beta_r}}$ on $\mathfrak{s}^*$ is the *quasi-center determinant.*

For $\xi \in \mathfrak{a}$ and $a = \exp(\xi) \in A$ compute $(\mathrm{Ad}(a)\mathrm{Det}_{\mathfrak{s}^*})(\lambda) = \mathrm{Det}_{\mathfrak{s}^*}(\mathrm{Ad}^*(a^{-1})(\lambda)) = \prod_r(\beta_r(\mathrm{Ad}(a^{-1})^*\lambda))^{\dim \mathfrak{g}_{\beta_r}} = \prod_r(\beta_r(\exp(\beta_r(\xi))\lambda))^{\dim \mathfrak{g}_{\beta_r}}$. In other words,

Lemma 8.4 *Let* $a = \exp(\xi) \in A$. *Then* $\mathrm{Ad}(a)\mathrm{Det}_{\mathfrak{s}^*} = \left(\prod_r \exp(\beta_r(\xi))^{\dim \mathfrak{z}_r}\right)\mathrm{Det}_{\mathfrak{s}^*}$ *where* $\xi = \log a \in \mathfrak{a}$.

Combining Lemmas 8.1, 8.2 and 8.4 we have

Proposition 8.5 *The product* $\mathrm{Pf}\cdot\mathrm{Det}_{\mathfrak{s}^*}$ *is an* $\mathrm{Ad}(MAN)$*-semi-invariant (and thus* $\mathrm{Ad}(AN)$*-semi-invariant) polynomial on* $\mathfrak{s}^*$ *of degree* $\frac{1}{2}(\dim \mathfrak{n} + \dim \mathfrak{s})$ *and of weight equal to the respective modular functions of* Q *and* AN.

From $\mathfrak{n} = \mathfrak{v} + \mathfrak{s}$ we have $N = VS$ where $V = \exp(\mathfrak{v})$ and $S = \exp(\mathfrak{s})$. Now define

$$D:\ \text{Fourier transform of } \mathrm{Pf}\cdot\mathrm{Det}_{\mathfrak{s}^*}, \text{ acting on the } S \text{ variable of } N = VS. \tag{24}$$

Theorem 8.6 *The operator* D *of* (24) *is an invertible self-adjoint differential operator of degree* $\frac{1}{2}(\dim \mathfrak{n} + \dim \mathfrak{s})$ *on* $L^2(MAN)$ *with dense domain* $\mathcal{C}(MAN)$, *and it is* $\mathrm{Ad}(MAN)$*-semi-invariant of weight equal to the modular function* δ_{MAN}. *In other words* $|D|$ *is a Dixmier–Pukánszky Operator on* MAN *with domain equal to the space of rapidly decreasing* C^∞ *functions. This applies as well to* AN.

Since $\lambda \in \mathfrak{t}^*$ has nonzero projection on each summand $\mathfrak{z}_r^*$ of $\mathfrak{s}^*$, and $a \in A$ acts by the positive real scalar $\exp(\beta_r(\log(a)))$ on $\mathfrak{z}_r$,

$$A_\lambda = \exp(\{\xi \in \mathfrak{a} \mid \text{each } \beta_r(\xi) = 0\}),\ \text{independent of } \lambda \in \mathfrak{t}^*. \tag{25}$$

Because of this independence, and using $\mathfrak{a}_\diamond = \{\xi \in \mathfrak{a} \mid \text{ each } \beta_r(\xi) = 0\}$, we define

$$A_\diamond = A_\lambda \text{ for any (and thus for all) } \lambda \in \mathfrak{t}^*. \tag{26}$$

Lemma 8.7 *If $\lambda \in \sigma(\mathfrak{u}^*)$ then the stabilizer $(MA)_\lambda = M_\diamond A_\diamond$.*

There is no problem with the Mackey obstruction:

Lemma 8.8 *Let $\lambda \in \sigma(\mathfrak{u}^*)$. Recall the extension* (before (21)) *$\pi_\lambda^\dagger$ of π_λ to $NM_\diamond$. Then $\pi_\lambda^\dagger$ extends to $\widetilde{\pi}_\lambda \in \widehat{NM_\diamond A_\diamond}$ with the same representation space as π_λ.*

When $\lambda \in \sigma(\mathfrak{u}^*)$, $\widehat{A_\diamond}$ consists of the unitary characters $\exp(i\phi) : a \mapsto e^{i\phi(\log a)}$ with $\phi \in \mathfrak{a}_\diamond^*$. The representations of Q corresponding to λ are the

$$\pi_{\lambda,\gamma,\phi} := \mathrm{Ind}_{NM_\diamond A_\diamond}^{NMA}(\widetilde{\pi}_\lambda \otimes \gamma \otimes \exp(i\phi)) \text{ where } \gamma \in \widehat{M_\diamond} \text{ and } \phi \in \mathfrak{a}_\diamond^* . \tag{27}$$

$\mathrm{Ad}^*(A)$ fixes γ because A centralizes M, and it fixes ϕ because A is commutative, so

$$\pi_{\lambda,\gamma,\phi} \cdot \mathrm{Ad}((ma)^{-1}) = \pi_{\mathrm{Ad}^*(ma)\lambda,\gamma,\phi} \tag{28}$$

Proposition 8.9 *Plancherel measure for Q is concentrated on the set of all $\pi_{\lambda,\gamma,\phi}$ for $\lambda \in \sigma(\mathfrak{u}^*)$, $\gamma \in \widehat{M_\diamond}$ and $\phi \in \mathfrak{a}_\diamond^*$. The equivalence class of $\pi_{\lambda,\gamma,\phi}$ depends only on $(\mathrm{Ad}^*(MA)\lambda, \gamma, \phi)$.*

Representations of AN are the case $\gamma = 1$. In effect, let π'_λ denote the obvious extension $\widetilde{\pi}_\lambda|_{AN}$ of the stepwise square integrable representation π_λ from N to $NA_\diamond$ where $\widetilde{\pi}_\lambda$ is given by Lemma 8.8. Denote

$$\pi_{\lambda,\phi} = \mathrm{Ind}_{NA_\diamond}^{NA}(\pi'_\lambda \otimes \exp(i\phi)) \text{ where } \lambda \in \mathfrak{u}^* \text{ and } \phi \in \mathfrak{a}_\diamond^*. \tag{29}$$

Corollary 8.10 *Plancherel measure for AN is concentrated on the set of all $\pi_{\lambda,\phi}$ for $\lambda \in \mathfrak{u}^*$ and $\phi \in \mathfrak{a}_\diamond^*$. The equivalence class of $\pi_{\lambda,\phi}$ depends only on $(\mathrm{Ad}^*(MA)\lambda, \phi)$.*

A result of C.C. Moore implies

Lemma 8.11 *The Pf-nonsingular principal orbit set $\mathfrak{u}^*$ is a finite union of open $\mathrm{Ad}^*(MA)$-orbits.*

Let $\{\mathcal{O}_1, \dots, \mathcal{O}_v\}$ denote the (open) $\mathrm{Ad}^*(MA)$-orbits on $\mathfrak{u}^*$. Denote $\lambda_i = \sigma(\mathcal{O}_i)$, so $\mathcal{O}_i = \mathrm{Ad}^*(MA)\lambda_i$ and $(MA)_{\lambda_i} = M_\diamond A_\diamond$ for $1 \leqq i \leqq v$. Then Proposition 8.9 becomes

Theorem 8.12 *Plancherel measure for MAN is concentrated on the set (of equivalence classes of) unitary representations $\pi_{\lambda_i,\gamma,\phi}$ for $1 \leqq i \leqq v$, $\gamma \in \widehat{M_\diamond}$ and $\phi \in \mathfrak{a}_\diamond^*$.*

The Plancherel Formula (or Fourier Inversion Formula) for MAN is

Theorem 8.13 *Let $Q = MAN$ be a minimal parabolic subgroup of the real reductive Lie group G. Given $\pi_{\lambda,\gamma,\phi} \in \widehat{MAN}$ as described in (27) let $\Theta_{\pi_{\lambda,\gamma,\phi}} : h \mapsto \mathrm{trace}\, \pi_{\lambda,\gamma,\phi}(h)$ denote its distribution character. Then $\Theta_{\pi_{\lambda,\gamma,\phi}}$ is a tempered distribution. If $f \in \mathcal{C}(MAN)$ then*

$$f(x) = c \sum_{i=1}^{v} \sum_{\gamma \in \widehat{M_\diamond}} \int_{\mathfrak{a}_\diamond^*} \Theta_{\pi_{\lambda_i,\gamma,\phi}}(D(r(x)f))|\mathrm{Pf}\,(b_{\lambda_i})|\dim\gamma\; d\phi$$

where $c > 0$ depends on normalizations of Haar measures.

The Plancherel Theorem for NA follows similar lines. For the main computation in the proof of Theorem 8.13 we omit M and γ. That gives

$$\int_{\mathfrak{a}_\diamond^*} \mathrm{trace}\,\pi_{\lambda_0,\phi}(Dh)\,d\phi = \int_{\mathrm{Ad}^*(A)\lambda_0} \mathrm{trace}\,\pi_\lambda(h)|\mathrm{Pf}\,(b_\lambda)|d\lambda \tag{30}$$

In order to go from an $\mathrm{Ad}^*(A)\lambda_0$ to an integral over $\mathfrak{u}^*$ we use M to parameterize the space of $\mathrm{Ad}^*(A)$-orbits on $\mathfrak{u}^*$. If $\lambda \in \mathfrak{u}^*$ one proves $\mathrm{Ad}^*(A)\lambda \cap \mathrm{Ad}^*(M)\lambda = \{\lambda\}$. That leads to

Proposition 8.14 *Plancherel measure for NA is concentrated on the equivalence classes of representations $\pi_{\lambda,\phi} = \mathrm{Ind}_{NA_\diamond}^{NA}(\pi'_\lambda \otimes \exp(i\phi))$ where $\lambda \in S_i := \mathrm{Ad}^*(M)\lambda_i$, $1 \leqq i \leqq v$, π'_λ extends π_λ from N to $NA_\diamond$ and $\phi \in \mathfrak{a}_\diamond^*$. Representations $\pi_{\lambda,\phi}$ and $\pi_{\lambda',\phi'}$ are equivalent if and only if $\lambda' \in \mathrm{Ad}^*(A)\lambda$ and $\phi' = \phi$. Further, $\pi_{\lambda,\phi}|_N = \int_{a \in A/A_\diamond} \pi_{\mathrm{Ad}^*(a)\lambda} da$.*

Theorem 8.15 *Let $Q = MAN$ be a minimal parabolic subgroup of the real reductive Lie group G. If $\pi_{\lambda,\phi} \in \widehat{AN}$ let $\Theta_{\pi_{\lambda,\phi}} : h \mapsto \mathrm{trace}\,\pi_{\lambda,\phi}(h)$ denote its distribution character. Then $\Theta_{\pi_{\lambda,\phi}}$ is a tempered distribution. If $f \in \mathcal{C}(AN)$ then*

$$f(x) = c \sum_{i=1}^{v} \int_{\lambda \in \mathrm{Ad}*(M)\lambda_i} \int_{\mathfrak{a}_\diamond^*} \mathrm{trace}\,\pi_{\lambda,\phi}(D(r(x)f))|\mathrm{Pf}\,(b_\lambda)|d\lambda d\phi.$$

where $c > 0$ depends on normalizations of Haar measures.

9 Parabolic Subgroups in General: The Nilradical

In Sects. 7 and 8 we studied minimal parabolic subgroups $Q = MAN$ in simple Lie groups, along with certain of their subgroups MN and AN. This section and the next form a glance at more general parabolics. This material is taken from [32], which is a work in progress, and is limited to the part that I've written down. We start with the structure of the nilradical.

The condition (c) of (5) does not always hold for nilradicals of parabolic subgroups. In this section and the next we weaken (5) to

$$\begin{aligned} &N = L_1 L_2 \dots L_{m-1} L_m \text{ where} \\ &\quad \text{(a) each} L_r \text{ has unitary representations with coefficients in } L^2(L_r/Z_r), \\ &\quad \text{(b) each } N_r := L_1 L_2 \dots L_r = N_{r-1} \rtimes L_r \text{ semidirect,} \\ &\quad \text{(c) if } r \geqq s \text{ then } [\mathfrak{l}_r, \mathfrak{z}_s] = 0. \end{aligned} \tag{31}$$

The conditions of (31) are sufficient to construct stepwise square integrable representations, but are not always sufficient to compute the Pfaffian that is the Plancherel density. So we refer to (5) as the *strong computability condition* and make use of the *weak computability condition*

$$\text{Let } \mathfrak{l}_r = \mathfrak{l}'_r \oplus \mathfrak{l}''_r \text{ where } \mathfrak{l}''_r \subset \mathfrak{z}_r \text{ and } \mathfrak{v}_r \subset \mathfrak{l}'_r; \text{ then } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}''_s + \mathfrak{v}_s \text{ for } r > s. \tag{32}$$

where we retain $\mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r$ and $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$.

Consider an arbitrary parabolic subgroup of G. It contains a minimal parabolic $Q = MAN$. Let Ψ denote the set of simple roots for the positive system $\Delta^+(\mathfrak{g}, \mathfrak{a})$. Then the parabolic subgroups of G that contain Q are in one to one correspondence with the subsets $\Phi \subset \Psi$, say $Q_\Phi \leftrightarrow \Phi$, as follows. Denote $\Psi = \{\psi_i\}$ and set

$$\begin{aligned} \Phi^{red} &= \left\{ \alpha = \sum\nolimits_{\psi_i \in \Psi} n_i \psi_i \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid n_i = 0 \text{ whenever } \psi_i \notin \Phi \right\} \\ \Phi^{nil} &= \left\{ \alpha = \sum\nolimits_{\psi_i \in \Psi} n_i \psi_i \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid n_i > 0 \text{ for some } \psi_i \notin \Phi \right\}. \end{aligned} \tag{33}$$

On the Lie algebra level, $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi + \mathfrak{a}_\Phi + \mathfrak{n}_\Phi$ where

$$\begin{aligned} &\mathfrak{a}_\Phi = \{\xi \in \mathfrak{a} \mid \psi(\xi) = 0 \text{ for all } \psi \in \Phi\} = \Phi^\perp, \\ &\mathfrak{m}_\Phi + \mathfrak{a}_\Phi \text{ is the centralizer of } \mathfrak{a}_\Phi \text{ in } \mathfrak{g}, \text{ so } \mathfrak{m}_\Phi \text{ has root system } \Phi^{red}, \text{ and} \\ &\mathfrak{n}_\Phi = \sum\nolimits_{\alpha \in \Phi^{nil}} \mathfrak{g}_\alpha, \text{ nilradical of } \mathfrak{q}_\Phi, \text{ sum of the positive } \mathfrak{a}_\Phi\text{-root spaces.} \end{aligned} \tag{34}$$

Since $\mathfrak{n} = \sum_r \mathfrak{l}_r$, as given in (17) and (18) we have

$$\mathfrak{n}_\Phi = \sum\nolimits_r (\mathfrak{n}_\Phi \cap \mathfrak{l}_r) = \sum\nolimits_r \left((\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi) + \sum\nolimits_{\Delta_r^+} (\mathfrak{g}_\alpha \cap \mathfrak{n}_\Phi) \right). \tag{35}$$

As $\mathrm{ad}(\mathfrak{m})$ is irreducible on each restricted root space, if $\alpha \in \{\beta_r\} \cup \Delta_r^+$ then $\mathfrak{g}_\alpha \cap \mathfrak{n}_\Phi$ is 0 or all of $\mathfrak{g}_\alpha$.

Lemma 9.1 *Suppose* $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi = 0$. *Then* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = 0$.

Lemma 9.2 *Suppose* $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi \neq 0$. *Define* $J_r \subset \Delta_r^+$ *by* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \mathfrak{g}_{\beta_r} + \sum_{J_r} \mathfrak{g}_\alpha$. *Decompose* $J_r = J'_r \cup J''_r$ *where* $J'_r = \{\alpha \in J_r \mid \sigma_r \alpha \in J_r\}$ *and* $J''_r = \{\alpha \in J_r \mid \sigma_r \alpha \notin J_r\}$. *Then* $\mathfrak{g}_{\beta_r} + \sum_{J''_r} \mathfrak{g}_\alpha$ *belongs to a single* $\mathfrak{a}_\Phi$*-root space in* $\mathfrak{n}_\Phi$, *i.e.* $\alpha|_{\mathfrak{a}_\Phi} = \beta_r|_{\mathfrak{a}_\Phi}$, *for every* $\alpha \in J''_r$.

Lemma 9.3 *Suppose* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi \neq 0$. *Then the algebra* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi$ *has center* $\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha$, *and* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = (\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha) + (\sum_{J_r'} \mathfrak{g}_\alpha)$. *Further,* $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \left(\sum_{J_r''} \mathfrak{g}_\alpha\right) \oplus \left(\mathfrak{g}_{\beta_r} + \left(\sum_{J_r'} \mathfrak{g}_\alpha\right)\right)$ *direct sum of ideals.*

It will be convenient to define sets of simple $\mathfrak{a}_\Phi$-roots

$$\Psi_1 = \Psi \text{ and } \Psi_{s+1} = \{\psi \in \Psi \mid \langle \psi, \beta_i \rangle = 0 \text{ for } 1 \leqq i \leqq s\}. \tag{36}$$

Note that Ψ_r is the simple root system for $\{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \alpha \perp \beta_i \text{ for } i < r\}$.

Lemma 9.4 *If* $r > s$ *then* $[\mathfrak{l}_r \cap \mathfrak{n}_\Phi\ ,\ \mathfrak{g}_{\beta_s} + \sum_{J_s''} \mathfrak{g}_\alpha] = 0$.

For our dealings with arbitrary parabolics it is not sufficient to consider linear functionals on $\sum_r \mathfrak{g}_{\beta_r}$. Instead we have to look at linear functionals on $\sum_r \left(\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha\right)$. of the form $\lambda = \sum \lambda_r$ where $\lambda_r \in \mathfrak{g}_{\beta_r}^*$ such that b_{λ_r} is nondegenerate on $\sum_r \sum_{J_r'} \mathfrak{g}_\alpha$. We know that (5)(c) holds for the nilradical of the minimal parabolic $\mathfrak{q}$ that contains $\mathfrak{q}_\Phi$. In view of Lemma 9.4 it follows that $b_\lambda(\mathfrak{l}_r, \mathfrak{l}_s) = \lambda([\mathfrak{l}_r, \mathfrak{l}_s] = 0$ for $r > s$. For this particular type of λ, the bilinear form b_λ has kernel $\sum_r \left(\mathfrak{g}_{\beta_s} + \sum_{J_s''} \mathfrak{g}_\alpha\right)$ and is nondegenerate on $\sum_r \sum_{J_r'} \mathfrak{g}_\alpha$. Then $N_\Phi = (L_1 \cap N_\Phi)(L_2 \cap N_\Phi) \ldots (L_m \cap N_\Phi)$ satisfies the first two conditions of (5). That is enough to carry out the construction of stepwise square integrable representations π_λ of N_Φ, but one needs to do more to deal with Pfaffian polynomials as in (5)(c) and (32).

Let $I_1 = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_1}|_{\mathfrak{a}_\Phi}\}$ where q_1 is the first index of (5) with $\beta_{q_1}|_{\mathfrak{a}_\Phi} \neq 0$. Next, $I_2 = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_2}|_{\mathfrak{a}_\Phi}\}$ where q_2 is the first index of (5) such that $q_2 \notin I_1$ and $\beta_{q_2}|_{\mathfrak{a}_\Phi} \neq 0$. Continuing as long as possible, $I_k = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_k}|_{\mathfrak{a}_\Phi}\}$ where q_k is the first index of (5) such that $q_k \notin (I_1 \cup \cdots \cup I_{k-1})$ and $\beta_{q_k}|_{\mathfrak{a}_\Phi} \neq 0$. Then $I_1 \cup \cdots \cup I_\ell$ consists of all the indices i for which $\beta_i|_{\mathfrak{a}_\Phi} \neq 0$. For $1 \leqq j \leqq \ell$ define

$$\mathfrak{l}_{\Phi,j} = \sum\nolimits_{i \in I_j} (\mathfrak{l}_i \cap \mathfrak{n}_\Phi) = \left(\sum\nolimits_{i \in I_j} \mathfrak{l}_i\right) \cap \mathfrak{n}_\Phi \text{ and } \mathfrak{l}_{\Phi,j}^\dagger = \sum\nolimits_{k \geqq j} \mathfrak{l}_{\Phi,k}\,. \tag{37}$$

Lemma 9.5 *If* $k \geqq j$ *then* $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{l}_{\Phi,j}$. *For each index* j, $\mathfrak{l}_{\Phi,j}$ *and* $\mathfrak{l}_{\Phi,j}^\dagger$ *are subalgebras of* $\mathfrak{n}_\Phi$ *and* $\mathfrak{l}_{\Phi,j}$ *is an ideal in* $\mathfrak{l}_{\Phi,j}^\dagger$.

Lemma 9.6 *If* $k > j$ *then* $[\mathfrak{l}_{\Phi,k}\ ,\ \mathfrak{l}_{\Phi,j}] \cap \sum_{i \in I_j} \mathfrak{g}_{\beta_i} = 0$.

In the notation of Lemma 9.2, if $r \in I_j$ then

$$\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \mathfrak{l}_r' + \mathfrak{l}_r'' \text{ where } \mathfrak{l}_r' = \mathfrak{g}_{\beta_r} + \sum\nolimits_{J_r'} \mathfrak{g}_\alpha \text{ and } \mathfrak{l}_r'' = \sum\nolimits_{J_r''} \mathfrak{g}_\alpha\,. \tag{38}$$

For $1 \leqq j \leqq \ell$ define

$$\mathfrak{z}_{\Phi,j} = \sum\nolimits_{i \in I_j} (\mathfrak{g}_{\beta_i} + \mathfrak{l}_i'') \tag{39}$$

and decompose

$$\mathfrak{l}_{\Phi,j} = \mathfrak{l}'_{\Phi,j} + \mathfrak{l}''_{\Phi,j} \text{ where } \mathfrak{l}'_{\Phi,j} = \sum\nolimits_{i\in I_j} \mathfrak{l}'_i \text{ and } \mathfrak{l}''_{\Phi,j} = \sum\nolimits_{i\in I_j} \mathfrak{l}''_i \,. \tag{40}$$

Lemma 9.7 *Recall* $\mathfrak{l}^\dagger_{\Phi,j} = \sum_{k\geqq j} \mathfrak{l}_{\Phi,k}$ *from* (37). *For each* j, *both* $\mathfrak{z}_{\Phi,j}$ *and* $\mathfrak{l}''_{\Phi,j}$ *are central ideals in* $\mathfrak{l}^\dagger_{\Phi,j}$, *and* $\mathfrak{z}_{\Phi,j}$ *is the center of* $\mathfrak{l}_{\Phi,j}$.

Decompose

$$\mathfrak{n}_\Phi = \mathfrak{z}_\Phi + \mathfrak{v}_\Phi \text{ where } \mathfrak{z}_\Phi = \sum_j \mathfrak{z}_{\Phi,j} \;,\;\; \mathfrak{v}_\Phi = \sum_j \mathfrak{v}_{\Phi,j} \text{ and } \mathfrak{v}_{\Phi,j} = \sum_{i\in I_j} \sum_{\alpha\in J'_i} \mathfrak{g}_\alpha \,. \tag{41}$$

Then Lemma 9.7 gives us (32) for the $\mathfrak{l}_{\Phi,j}$: $\mathfrak{l}_{\Phi,j} = \mathfrak{l}'_{\Phi,j} \oplus \mathfrak{l}''_{\Phi,j}$ with $\mathfrak{l}''_{\Phi,j} \subset \mathfrak{z}_{\Phi,j}$ and $\mathfrak{v}_{\Phi,j} \subset \mathfrak{l}'_{\Phi,j}$.

Lemma 9.8 *For generic* $\lambda_j \in \mathfrak{z}^*_{\Phi,j}$ *the kernel of* b_{λ_j} *on* $\mathfrak{l}_{\Phi,j}$ *is just* $\mathfrak{z}_{\Phi,j}$, *in other words* b_{λ_j} *is nondegenerate on* $\mathfrak{v}_{\Phi,j} \simeq \mathfrak{l}_{\Phi,j}/\mathfrak{z}_{\Phi,j}$. *In particular* $L_{\Phi,j}$ *has square integrable representations.*

Theorem 9.9 *Let* G *be a real reductive Lie group and* Q *a real parabolic subgroup. Express* $Q = Q_\Phi$ *in the notation of* (33) *and* (34). *Then its nilradical* N_Φ *has decomposition* $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ *that satisfies the conditions of* (5) *and* (32) *as follows. The center* $Z_{\Phi,j}$ *of* $L_{\Phi,j}$ *is the analytic subgroup for* $\mathfrak{z}_{\Phi,j}$ *and*

(a) *each* $L_{\Phi,j}$ *has unitary representations with coefficients in* $L^2(L_{\Phi,j}/Z_{\Phi,j})$

(b) *each* $N_{\Phi,j} := L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,j}$ *is a normal subgroup of* N_Φ *with* $N_{\Phi,j} = N_{\Phi,j-1} \rtimes L_{\Phi,j}$ *semidirect,*

(c) $[\mathfrak{l}_{\Phi,k}, \mathfrak{z}_{\Phi,j}] = 0$ *and* $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{v}_{\Phi,j} + \mathfrak{l}''_{\Phi,j}$ *for* $k > j$. (42)

In particular N_Φ *has stepwise square integrable representations relative to the decomposition* $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$.

10 Amenable Subgroups of Semisimple Lie Groups

In this section we apply the results of Sect. 9 to certain important subgroups of the parabolic $Q_\Phi = M_\Phi A_\Phi N_\Phi$, specifically its amenable subgroups $A_\Phi N_\Phi$, $U_\Phi N_\Phi$ and $U_\Phi A_\Phi N_\Phi$ where U_Φ is a maximal compact subgroup of M_Φ.

The theory of the group $U_\Phi N_\Phi$ goes exactly as in Sect. 7. When $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is weakly invariant we can proceed more or less as in [28]. The argument, but not the final result, will make use of

Definition 10.1 The decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ of Theorem 9.9 is *invariant* if each $\mathrm{ad}(\mathfrak{m}_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$, equivalently if each $\mathrm{Ad}(M_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$, in other words whenever $\mathfrak{z}_{\Phi,j} = \mathfrak{g}_{[\Phi,\beta_{j_0}]}$. The decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is *weakly invariant* if each $\mathrm{Ad}(U_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$. ◇

Set

$$\mathfrak{r}_\Phi^* = \{\lambda \in \mathfrak{s}_\Phi^* \mid P(\lambda) \neq 0 \text{ and } \mathrm{Ad}(U_\Phi)\lambda \text{ is a principal } U_\Phi\text{-orbit on } \mathfrak{s}_\Phi^*\}. \tag{43}$$

Then $\mathfrak{r}_\Phi^*$ is dense, open and U_Φ-invariant in $\mathfrak{s}_\Phi^*$. By definition of principal orbit the isotropy subgroups of U_Φ at the various points of $\mathfrak{r}_\Phi^*$ are conjugate, and we take a measurable section σ to $\mathfrak{r}_\Phi^* \to U_\Phi\backslash\mathfrak{r}_\Phi^*$ on whose image all the isotropy subgroups are the same,

$$U'_\Phi : \text{ isotropy subgroup of } U_\Phi \text{ at } \sigma(U_\Phi(\lambda)), \text{ independent of } \lambda \in \mathfrak{r}_\Phi^*. \tag{44}$$

The principal isotropy subgroups U'_Φ are pinned down in [11]. Given $\lambda \in \mathfrak{r}_\Phi^*$ and $\gamma \in \widehat{U'_\Phi}$ let $\pi_\lambda^\dagger$ denote the extension of π_λ to a representation of $U'_\Phi N_\Phi$ on the space of π_λ and define

$$\pi_{\lambda,\gamma} = \mathrm{Ind}_{U'_\Phi N_\Phi}^{U_\Phi N_\Phi}(\gamma \otimes \pi_\lambda^\dagger). \tag{45}$$

The first result in this setting, as in [28, Proposition 3.3], is

Theorem 10.2 *Suppose that $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ as in (31). Then the Plancherel density on $\widehat{U_\Phi N_\Phi}$ is concentrated on the representations $\pi_{\lambda,\gamma}$ of (45), the Plancherel density at $\pi_{\lambda,\gamma}$ is $(\dim\gamma)|P(\lambda)|$, and the Plancherel Formula for $U_\Phi N_\Phi$ is*

$$f(un) = c\int_{\mathfrak{r}_\Phi^*/\mathrm{Ad}^*(U_\Phi)} \sum\nolimits_{\gamma\in\widehat{U'_\Phi}} \mathrm{trace}\,\mathrm{Ind}_{U'_\Phi N_\Phi}^{U_\Phi N_\Phi} r_{un}(f)\cdot\dim(\gamma)\cdot|P(\lambda)|d\lambda$$

where $c = 2^{d_1+\dots+d_\ell}d_1!d_2!\dots d_\ell!$ as in (6).

Recall the notion of amenability. A *mean* on a locally compact group H is a linear functional μ on $L^\infty(H)$ of norm 1 and such that $\mu(f) \geqq 0$ for all real-valued $f \geqq 0$. H is *amenable* if it has a left-invariant mean. Solvable groups and compact groups are amenable, as are extensions of amenable groups by amenable subgroups. In particular $E_\Phi := U_\Phi A_\Phi N_\Phi$ and its closed subgroups are amenable.

We need a technical condition [16, p. 132]. Let H be the group of real points in a linear algebraic group whose rational points are Zariski dense, let A be a maximal $\mathbb{R}$-split torus in H, let $Z_H(A)$ denote the centralizer of A in H, and let H_0 be the algebraic connected component of the identity in H. Then H is *isotropically connected* if $H = H_0 \cdot Z_H(A)$. More generally we will say that a subgroup $H \subset G$ is *isotropically connected* if the algebraic hull of $\mathrm{Ad}_G(H)$ is isotropically connected.

Proposition 10.2 [16, Theorem 3.2]. *The groups $E_\Phi := U_\Phi A_\Phi N_\Phi$ are maximal amenable subgroups of G. They are isotropically connected and self-normalizing.*

The various $\Phi \subset \Psi$ are mutually non-conjugate. An amenable subgroup $H \subset G$ is contained in some E_Φ if and only if it is isotropically connected.

The isotropy subgroups are the same at every $\lambda \in \mathfrak{t}_\Phi^*$,

$$A'_\Phi : \text{ isotropy subgroup of } A_\Phi \text{ at } \lambda \in \mathfrak{r}_\Phi^* . \tag{46}$$

Given a stepwise square integrable representation π_λ where $\lambda \in \mathfrak{s}_\Phi^*$, write $\pi_\lambda^\dagger$ for the extension of π_λ to a representation of $A'_\Phi N_\Phi$ on the same Hilbert space. That extension exists because the Mackey obstruction vanishes. The representations of $A'_\Phi N_\Phi$ corresponding to π_λ are the

$$\pi_{\lambda,\phi} := \mathrm{Ind}_{A'_\Phi N_\Phi}^{A_\Phi N_\Phi}(\exp(i\phi) \otimes \pi_\lambda^\dagger) \text{ where } \phi \in \mathfrak{a}'_\Phi . \tag{47}$$

Note also that

$$\pi_{\lambda,\phi} \cdot \mathrm{Ad}(an) = \pi_{\mathrm{Ad}^*(a)\lambda,\phi} \text{ for } a \in A_\Phi \text{ and } n \in N_\Phi . \tag{48}$$

The resulting formula $f(x) = \int_{\widehat{H}} \mathrm{trace}\, \pi(D(r(x)f))d\mu_H(\pi)$, $H = A_\Phi N_\Phi$, is

Theorem 10.3 *Let $Q_\Phi = M_\Phi A_\Phi N_\Phi$ be a parabolic subgroup of the real reductive Lie group G. Given $\pi_{\lambda,\phi} \in \widehat{A_\Phi N_\Phi}$ as described in (47), its distribution character $\Theta_{\pi_{\lambda,\phi}} : h \mapsto \mathrm{trace}\, \pi_{\lambda,\phi}(h)$ is a tempered distribution. If $f \in \mathcal{C}(A_\Phi N_\Phi)$ then*

$$f(x) = c \int_{(\mathfrak{a}'_\Phi)^*} \left(\int_{\mathfrak{s}_\Phi^*/\mathrm{Ad}^*(A_\Phi)} \Theta_{\pi_{\lambda,\phi}}(D(r(x)f))|\mathrm{Pf}\,(b_\lambda)|d\lambda \right) d\phi$$

where $c = 2^{d_1+\cdots+d_\ell} d_1!d_2!\ldots d_\ell!$.

The representations of $U_\Phi A_\Phi N_\Phi$ corresponding to π_λ are the

$$\pi_{\lambda,\phi,\gamma} := \mathrm{Ind}_{U'_\Phi A'_\Phi N_\Phi}^{U_\Phi A_\Phi N_\Phi}(\gamma \otimes \exp(i\phi) \otimes \pi_\lambda^\dagger) \text{ where } \phi \in \mathfrak{a}'_\Phi \text{ and } \gamma \in \widehat{U'_\Phi} . \tag{49}$$

Combining Theorems 10.2 and 10.3 we arrive at

Theorem 10.4 *Let $Q_\Phi = M_\Phi A_\Phi N_\Phi$ be a parabolic subgroup of the real reductive Lie group G and decompose $N_\Phi = L_{\Phi,1}L_{\Phi,2}\ldots L_{\Phi,\ell}$ as in (31). Then the Plancherel density on $\widehat{U_\Phi A_\Phi N_\Phi}$ is concentrated on the $\pi_{\lambda,\phi,\gamma}$ of (49), the Plancherel density at $\pi_{\lambda,\phi,\gamma}$ is $(\dim \gamma)|P(\lambda)|$, the distribution character $\Theta_{\pi_{\lambda,\phi,\gamma}} : h \mapsto \mathrm{trace}\, \pi_{\lambda,\phi,\gamma}(h)$ is tempered, and if $f \in \mathcal{C}(U_\Phi A_\Phi N_\Phi)$ then*

$$f(x) = c \sum_{\widehat{U'_\Phi}} \int_{(\mathfrak{a}'_\Phi)^*} \left(\int_{\mathfrak{s}_\Phi^*/\mathrm{Ad}^*(U_\Phi A_\Phi)} \Theta_{\pi_{\lambda,\phi,\gamma}}(D(r(x)f))\deg(\gamma)\,|\mathrm{Pf}\,(b_\lambda)|d\lambda \right) d\phi$$

where $c = 2^{d_1+\cdots+d_\ell} d_1!d_2!\ldots d_\ell!$.

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