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CORRECTION

Correction to: Bounded Isometries and Homogeneous Quotients

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A few years ago Profs. Yurii Nikonorov and Ming Xu found a gap [1] in my paper [2] on homogeneity and bounded isometries. In lines 2 and 3 of the third paragraph of the proof of [2, Theorem 2.5] it was only shown that $Ad(sk)$ is bounded, and from that it should have been shown that $Ad(s)$ is bounded. Now Prof. Nikonorov modified my original proof, completing the argument, and agreed to its publication as a correction. See below. The statement of the theorem is unchanged; the modified argument uses a result [3] of Moskowitz.

Theorem ([2, Theorem 2.5]). *Let (M, d) be a metric space on which an exponential solvable Lie group S acts effectively and transitively by isometries. Let $G = I(M, d)$. Then G is a Lie group, any isotropy subgroup K is compact, and $G = SK$. If $g \in G$ is a bounded isometry then g is a central element in S.*

Proof (Yurii Nikonorov [4]). M carries a differentiable manifold structure for which $s \mapsto s(x_0)$ is a diffeomorphism $S \cong M$. In the compact–open topology $G = I(M, d)$ is locally compact and its its action on M is proper [5]. In particular, if $x_0 \in M$ then the isotropy subgroup $K = \{k \in G \mid k(x_0) = x_0\}$ is compact. Further [6, Corollary in §6.3] G is a Lie group. Now S and K are closed subgroups, $G = SK$, and $M = G/K$. The action of G on $M = SK/K = S$ is $(sk) : s' \mapsto s \cdot ks'k^{-1}$. We now assume that $x_0 = 1 \in S$.

Express $g = sk \in G$ with $s \in S$ and $k \in K$. Suppose that g is a bounded isometry of (M, d). So there is a compact set $C \subset S$ such that $\tau(g)(s')s'^{-1} = s[(ks'k^{-1})s'^{-1}] \in C$ for every $s' \in S$. Then $(ks'k^{-1})s'^{-1} \in s^{-1}C$ so the automorphism $a : S \to S$, $a(s') = ks'k^{-1}$ is an automorphism of bounded displacement. By Corollary 1.3 in [3],

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we see that a is trivial. Since the isotropy representation is faithful, it follows that $k = 1$ and $g = s \in S$.

Express $g = s = \exp(\xi)$ where $\xi \in \mathfrak{s} := \mathrm{Lie}(S)$. Decompose $\mathfrak{s}$ as a vector space direct sum $\mathfrak{n} + \mathfrak{a}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$. In a basis respecting that direct sum, $\mathrm{ad}(\xi)|_{\mathfrak{s}}$ has matrix of the form $\left(\begin{smallmatrix} 0 & h \\ 0 & 0 \end{smallmatrix}\right)$. Let N denote the unipotent radical of S, so $\mathfrak{n} := \mathrm{Lie}(N)$. Then the $(1, 1)$ and $(2, 1)$ blocks vanish because g centralizes N by [7, Théorème 1], and the $(2, 2)$ block vanishes because $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{n}$. Note $\left(\begin{smallmatrix} 0 & h \\ 0 & 0 \end{smallmatrix}\right)^2 = 0$, so $\mathrm{Ad}(g)|_{\mathfrak{s}} - I = \mathrm{ad}(\xi)|_{\mathfrak{s}}$. It cannot have relatively compact image unless $h = 0$. Thus $\mathrm{Ad}(g)|_{\mathfrak{s}} = I$, and $g \in (Z_G(S) \cap S) = Z_S$ as asserted. □

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