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Bounded Isometries and Homogeneous Quotients

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Abstract In this paper we give an explicit description of the bounded displacement isometries of a class of spaces that includes the Riemannian nilmanifolds. The class of spaces consists of metric spaces (and thus includes Finsler manifolds) on which an exponential solvable Lie group acts transitively by isometries. The bounded isometries are proved to be of constant displacement. Their characterization gives further evidence for the author's 1962 conjecture on homogeneous Riemannian quotient manifolds. That conjecture suggests that if $\Gamma \backslash M$ is a Riemannian quotient of a connected simply connected homogeneous Riemannian manifold M, then $\Gamma \backslash M$ is homogeneous if and only if each isometry $\gamma \in \Gamma$ is of constant displacement. The description of bounded isometries in this paper gives an alternative proof of an old result of J. Tits on bounded automorphisms of semisimple Lie groups. The topic of constant displacement isometries has an interesting history, starting with Clifford's use of quaternions in non-Euclidean geometry, and we sketch that in a historical note.

1 Introduction

An isometry ρ of a metric space (M, d) is of *constant displacement* if it moves each point the same distance, i.e., if the displacement function $\delta_\rho(x) := d(x, \rho(x))$ is

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constant. Clifford [8] described such isometries for the 3-sphere, using the then-recent discovery of quaternions. Somewhat later Vincent [38] used the term "Clifford translation" for constant displacement isometries of round spheres in his study of spherical space forms $\Gamma \backslash S^n$ with Γ metabelian. Later the author [39–41] used the the term "Clifford translation" in the context of metric spaces, especially Riemannian manifolds, proving

> **Conjecture.** Let M be a connected, simply connected Riemannian homogeneous manifold and let $M \rightarrow \Gamma \backslash M$ be a Riemannian covering. Then $\Gamma \backslash M$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement on M.

for the case where M is a Riemannian symmetric space [42]. In part the argument was case by case, but later Ozols [30–32] gave a general argument for the situation where Γ is a cyclic subgroup of the identity component $I^0(M)$ of the isometry group $I(M)$. Freudenthal [19] discussed the situation where $\Gamma \subset I^0(M)$, and introduced the term *Clifford–Wolf isometry* (CW isometry) for isometries of constant displacement. That seems to be the term in general usage.

Since then there has been a great deal of work on CW isometries and their infinitesimal analogs, Killing vector fields of constant length, in both the Riemannian and the Finsler manifold settings. See [3–5,9–17]. The Conjecture was proved for Finsler symmetric spaces by Deng and the author in [9].

Most of the definitive results on CW isometries are concerned with Riemannian (and later Finsler) symmetric spaces. There we have a full understanding of CW isometries [9,42]. The Conjecture is known for CW isometries on some non-symmetric homogeneous Riemannian manifolds. The homogeneous Riemannian manifolds (M, ds^2) for which the Conjecture is known are: (i) Riemannian symmetric spaces [42], (ii) Riemannian manifolds of non-positive sectional curvature [44] and manifolds without focal points [17], (iii) Riemannian manifolds that admit a transitive semisimple group of isometries that has no compact factor [16], (iv) Stiefel manifolds and some structurally related compact homogeneous Riemannian manifolds [6,7], (v) certain classes of Riemannian normal homogeneous spaces [45,46], and (vi) Riemannian nilmanifolds and Riemannian solvmanifolds (in this paper).

Here we give a complete structure theory for bounded isometries (isometries of bounded displacement) of metric spaces on which an exponential solvable Lie group acts transitively by isometries. We show that all bounded isometries are CW and belong to a certain connected abelian group of CW isometries that is normal in the full isometry group. In the nilmanifold case that normal subgroup is the center of the nilradical of the isometry group, but in other cases it may be smaller. Since it is reduced to the identity in the group AN of an Iwasawa decomposition $G = KAN$, G semisimple, this gives an alternative proof of Tits's theorem [36] that a semisimple Lie group with no compact factor has no nontrivial bounded automorphism.

The class of spaces to which this applies includes Riemannian (and even Finsler) exponential solvmanifolds, in particular Riemannian nilmanifolds. These results prove the Conjecture on homogeneous quotients for those exponential solvmanifolds, and consequently Riemannian nilmanifolds.

In Sect. 2 we work out a complete structure theory for individual bounded isometries of metric spaces (M, d) on which an exponential solvable Lie group S acts transitively by isometries. We first prove that the isometry group $I(M, d)$ is a Lie group and that $I(M, d) = SK$ where K is an isotropy subgroup. This is analogous to the Iwasawa decomposition of a real reductive Lie group. Then we show that every bounded isometry of (M, d) belongs to the center of S. Thus every bounded isometry is CW and that center is a normal subgroup of $I(M, d)$. These results are in Theorem 2.5 and its corollaries.

In Sect. 3 we study quotients $\Gamma\backslash(M, d)$. For locally isometric coverings $\psi : (M, d) \rightarrow \Gamma\backslash(M, d)$ we show that $\Gamma\backslash(M, d)$ is homogeneous if and only if Γ is a discrete group of CW isometries of (M, d). That result is part of Theorem 3.1, which lists several other equivalent conditions. It proves the Conjecture for our class of metric spaces (M, d), in particular for Riemannian (and Finsler) exponential solvmanifolds. One corollary is the infinitesimal version, for Riemannian (and Finsler) exponential solvmanifolds, characterizing the Killing vector fields of constant length.

The arguments for Riemannian nilmanifolds are slightly less complicated because some technical considerations become transparent. The nilmanifold version of our main result is Corollary 3.5.

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1.1 Historical Note

The theory of constant displacement isometries can be traced back to the independent discovery of quaternions by Rodrigues in 1840 [33] and Hamilton in 1844 [20,21]. See [1] for a description. They used quaternions to describe rotations of spheres, but Clifford [8] seems to have introduced their use in differential geometry in his construction[1] of a flat torus in the sphere S^3. In 1890 a paper of Klein [25] introduced group theory per se into the picture. The next year Killing introduced the term "Clifford–Klein space form" for Riemannian manifolds of constant curvature and formulated the "Clifford–Klein space form problem" [26,27] in terms of quotients $\Gamma\backslash M$ where M is a complete simply connected manifold of constant curvature. The classification of spherical space forms $\Gamma\backslash S^n$ was obscured in 1907 by the assertion[2] in the influential Enzyklopädie der Mathematischen Wissenschaft that (in modern terms) if $\Gamma\backslash S^n$ is a spherical space form, n odd, then Γ is a finite group of constant displacement isometries of S^n. This was corrected by Hopf in 1926 [24] and by Threlfall and Seifert in

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[1] In modern terms, the group $\mathbb{H}'$ of unit quaternions, viewed as S^3, acts on itself by left and right translations, $(a, b) : q \mapsto aqb^{-1}$, and one can view the Clifford torus as the orbit of q as a and b each runs over a one-parameter subgroups of $\mathbb{H}'$. Any two such one-parameter groups of transformations of S^3 commute pointwise, so each such one-parameter group acts by isometries of constant displacement on S^3.

[2] Enriques [18, p. 117]: Endlich *läßt sich eine dreidimensionale elliptische Raumform als Ganzes entweder auf den elliptischen oder auf den sphärischen Raum in der Weise abwickeln, daß jedem ihrer Punkte in diesem Raume eine gewisse ganze Anzahl p von (homologen) Punkten entspricht, wo zwei homologe Punkte durch eine sogennante Schiebung von der Länge $\frac{\ell\pi}{p}$ oder $\frac{2\ell\pi}{p}$ durch Deckung gebracht werden können.* Dieses letzte Resultat erstreckt sich auf alle elliptischen Raumformen von ungerader Dimensionenzahl n.

1930 [34,35] with the classification of all spherical space forms $\Gamma \backslash S^3$; see [22]. This was extended in 1947 by Vincent [38] for $\Gamma \backslash S^n$ with Γ metabelian. Vincent introduced the term "Clifford translation" and asked about their relation to binary dihedral and polyhedral groups [38, §10.4].

In 1960 the author formulated and proved the Conjecture for spaces of constant curvature [39], and in 1961 he used that result to answer Vincent's questions [40]. In 1961 the author proved the Conjecture for Riemannian symmetric spaces [42]. Since then, as mentioned earlier in this Introduction, there has been a lot of progress toward the proof of the Conjecture, and this note is a small step in that direction.

2 Bounded Isometries Inside Exponential Solvable Groups

We will follow the convention that Lie groups are denoted by capital Latin letters and their Lie algebras are denoted by the corresponding lower case German letters. Thus, in the definition

Definition 2.1 A solvable Lie group S is *exponential solvable* if the exponential map $\exp : \mathfrak{s} \to S$ is a diffeomorphism. Examples include the simply connected nilpotent Lie groups and the groups AN in the Iwasawa decomposition $G = KAN$ of a real semisimple Lie group.

it will be understood that $\mathfrak{s}$, $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ are the respective Lie algebras of S, G, K, A and N.

We are looking at bounded isometries of metric spaces (M, d) on which an exponential solvable Lie group S acts effectively and transitively (and thus, it will turn out, simply transitively) by isometries. The most interesting case is that of Riemannian exponential solvmanifolds. By *Riemannian exponential solvmanifold* (relative to S) we mean a Riemannian manifold M on which an exponential solvable Lie group S of isometries acts transitively, and the kernel of the action of S is discrete. Then it is easy to see that the action of S on M lifts to a simply transitive action of the universal covering group of S on the universal Riemannian covering space of M. Examples include connected simply connected Riemannian nilmanifolds and (see [2,44] and [23]) connected simply connected Riemannian manifolds of non-positive sectional curvature. However, except for the proof that the isometry group $I(M, d)$ is a Lie group, the arguments are the same for metric spaces as for Riemannian manifolds, so we work in that more general class.

Lemma 2.2 *Let (M, d) be a metric space on which an exponential solvable Lie group S acts effectively and transitively by isometries. Then the action of S on M is simply transitive.*

Proof Let $x_0 \in M$. The isotropy subgroup $S_{x_0} = \{s \in S \mid s(x_0) = x_0\}$ of S preserves all metric balls $B_r(x_0) = \{x \in M \mid d(x, x_0) \leqq r\}$. As S_{x_0} is a closed subgroup of S it is a Lie group, and as the $B_r(x_0)$ are compact, it follows from [37] that S_{x_0} is compact. By the definition of an exponential solvable group, the only compact subgroup of S is $\{1\}$. □

For the rest of the section we fix a metric space (M, d) and an exponential solvable Lie group S acting effectively and transitively by isometries. We may view (M, d) as the group manifold S with a left-invariant metric space structure. The most interesting cases are when d is the distance function of a Riemannian metric ds^2 or a Finsler metric F. In any case we will write $I(M, d)$ for the group of all isometries of (M, d).

We will need an obvious elementary estimate; it is included for completeness.

Lemma 2.3 *Let U be a unipotent group of linear transformations of a real vector space V. Suppose that $v \in V$ is not a fixed point of U. Then $U(v)$ is unbounded, in other words is not contained in a compact subset of V.*

Proof Let $\xi \in \mathfrak{u}$ with $\xi(v) \neq 0$. Then $\exp(t\xi)v = \sum_0^r \frac{1}{n!} t^n \xi^n(v)$ where $\xi^r v \neq 0 = \xi^{r+1} v$. As $t \to \infty$ the $\frac{1}{r!} t^r \xi^r v$ summand dominates the others and is unbounded. $\square$

That is sufficient for our needs if S is nilpotent. But in general we need a sightly less obvious version.

Lemma 2.4 *Let S be an exponential solvable Lie group and $\xi \in \mathfrak{s}$ a non-central element of the Lie algebra. Then $\mathrm{Ad}(S)\xi$ is unbounded.*

Proof Let U be the nilradical of S. If U does not centralize ξ then $\mathrm{Ad}(S)\xi$ is unbounded by Lemma 2.3. Thus we may assume $\mathrm{Ad}(U)\xi = \{\xi\}$. If $s \in S$ now $\mathrm{Ad}(U)\mathrm{Ad}(s)\xi = \mathrm{Ad}(s)\mathrm{Ad}(U)\xi = \mathrm{Ad}(s)\xi$, so $\mathrm{Ad}(U)$ acts trivially on $W := \mathrm{Span}\,\mathrm{Ad}(S)\xi$.

The restriction $\mathrm{Ad}(\mathfrak{s})|_W$ is a commutative linear Lie algebra in which every nonzero element has an eigenvalue with nonzero real part. Thus $\mathrm{Ad}(S)\eta$ is unbounded for some $\eta \in W$, and consequently for some $\eta \in \mathrm{Ad}(S)\xi$. Now $\mathrm{Ad}(S)\xi$ is unbounded. $\square$

Theorem 2.5 *Let (M, d) be a metric space on which an exponential solvable Lie group S acts effectively and transitively by isometries. Let $G = I(M, d)$. Then G is a Lie group, any isotropy subgroup K is compact, and $G = SK$. If $g \in G$ is a bounded isometry then g is a central element in S.*

Proof As noted above, M carries a differentiable manifold structure for which $s \mapsto s(x_0)$ is a diffeomorphism $S \cong M$. As usual $G = I(M, d)$ carries the compact-open topology. The famous theorem of van Dantzig and van der Waerden [37] (or see [28, Corollary4] for an exposition) says that G is locally compact and that its action on M is proper. In particular, if $x_0 \in M$ then the isotropy subgroup $K = \{k \in G \mid k(x_0) = x_0\}$ is compact. Further [29, Corollary in §6.3] G is a Lie group. Now S and K are closed subgroups, $G = SK$, and $M = G/K$.

Express $g = sk$ with $s \in S$ and $k \in K$. If $s = 1 \neq k$ then the differential of k is unbounded on the tangent space to M at x_0, thus unbounded $\mathfrak{s}$, and thus unbounded on (M, d). Thus $s \neq 1$ unless, of course, $g = 1$.

Suppose $s \neq 1$. As K is compact and the displacement function $x \mapsto \delta_g(x)$ is bounded, $\mathrm{Ad}(G)g$ is bounded in G. If $g' = s'k' \in G$ we compute $\mathrm{Ad}(g')g = s'k'skk'^{-1}s'^{-1}$. That is bounded as g' ranges over G, so $\mathrm{Ad}(S)s$ is bounded. Let N be the nilradical of S. Now $\mathrm{Ad}(N)s$ is a bounded unipotent $\mathrm{Ad}(N)$-orbit on $\mathfrak{s}$, which is impossible unless s centralizes N. As in the proof of Lemma 2.4 it follows that s is central in S.

Identify $\mathfrak{s}$ with the tangent space to M at x_0. Suppose $v \in \mathfrak{s}$ with $\mathrm{Ad}(k)v \neq v$ and let C be a compact neighborhood of 0 in $\mathfrak{s}$. As $t \to \infty$, $\mathrm{Ad}(k)(tv)$ must exit $tv + C$, so $\mathrm{Ad}(k)$ is unbounded on $\mathfrak{s}$. Thus k is unbounded on (M, d). That is a contradiction, so $\mathrm{Ad}(k)v = v$ for all $v \in \mathfrak{s}$, in other words k is trivial on the tangent space to M at x_0. As $M = I(M, d)/K$ it follows that $k = 1$.

Summarizing, the bounded isometry g of (M, d) is a central element of S. □

Corollary 2.6 *Let (M, d) be a metric space on which an exponential solvable Lie group S acts effectively and transitively by isometries. Then every bounded isometry of (M, d) is CW.*

Proof Each bounded isometry g is centralized by S, which is transitive on (M, d), so g is CW [39]. □

Corollary 2.7 *Let (M, d) be a metric space on which an exponential solvable Lie group S acts effectively and transitively by isometries. Then the center of S consists of all the CW isometries of (M, d), and it is an abelian normal subgroup of $I(M, d)$.*

Proof If $g \in I(M, d)$ is central in S then it commutes with every element of the transitive group S of isometries, so [39] it is a CW isometry. If $g \in I(M, d)$ is CW then Theorem 2.5 shows that it is a central element of S. □

A *Riemannian nilmanifold* is a connected Riemannian manifold (M, ds^2) on which a nilpotent group N of isometries acts transitively. Then [43, Theorem 4.2] N is the nilpotent radical of the isometry group $I(M, ds^2)$, and $I(M, ds^2)$ is the semidirect product $N \rtimes K$ where K is the isotropy subgroup at a point of M. We abbreviate this situation by writing $G = I(M, ds^2) = N \rtimes K$, so $M = G/K = (N \rtimes K)/K$.

In the case of Riemannian nilmanifolds Lemma 2.2 is obvious, Lemma 2.3 is needed as stated, and the proof of Lemma 2.4 is reduced to its first two sentences. We can skip the first paragraph of the proof of Theorem 2.5. Corollaries 2.6 is unchanged, and Corollary 2.7 is obvious, with $S = N$ nilpotent.

The alternative proof of a result of J. Tits, described in the Introduction, follows from Theorem 2.5 and the Iwasawa decomposition $G = NAK$ of a real reductive Lie group G. There M is a Riemannian symmetric space of noncompact type, $AN = NA$ is exponential solvable, and AN acts transitively by isometries on M.

3 Homogeneous Quotients

We apply Theorem 2.5 to the structure of covering spaces $\psi : (M, d) \to \Gamma\backslash(M, d)$ where (M, d) is a metric space on which an exponential solvable Lie group S acts effectively and transitively by isometries. Recall here [39] that if $\Gamma\backslash(M, d)$ is homogeneous, then Γ consists of CW isometries. We then indicate the simplification for Riemannian nilmanifolds. The following is immediate from Theorem 2.5.

Theorem 3.1 *Let (M, d) be a metric space on which an exponential solvable Lie group S acts effectively and transitively by isometries. Let $x_0 \in M$, let $G = I(M, d)$, and let K denote the isotropy subgroup of G at x_0. Consider a locally isometric covering space $\psi : (M, ds^2) \to \Gamma\backslash(M, ds^2)$. Then the following conditions are equivalent.*

1. Γ *consists of bounded isometries of* (M, d).
2. Γ *consists of CW isometries of* (M, d).
3. *The group* Γ *is a discrete central subgroup* S.
4. $\Gamma \backslash (M, ds^2)$ *is a homogeneous metric space.*
5. $\Gamma \backslash (M, ds^2)$ *is a metric space on which a Lie group* S / Γ *acts transitively, where* S *is exponential solvable and* Γ *is a discrete central subgroup of* S.

Remark 3.2 Theorem 3.1 applies in particular to Riemannian coverings $\psi : (M, ds^2) \rightarrow \Gamma \backslash (M, ds^2)$, to Finsler manifold coverings $\psi : (M, F) \rightarrow \Gamma \backslash (M, F)$, and to nilmanifolds. Thus it tells us how to construct all connected Riemannian nilmanifolds. Start with a connected simply connected Lie group N, say with center Z, and a discrete central subgroup $\Gamma \subset Z$. Fix a positive definite bilinear form b on the Lie algebra $\mathfrak{n}$, and let $\overline{K}$ denote the group of all automorphisms of N that normalize Γ and preserve b. Then b translates around to define an $((N / \Gamma) \rtimes \overline{K})$-invariant Riemannian metric dt^2 on $\Gamma \backslash M = ((N / \Gamma) \rtimes \overline{K}) / \overline{K}$, and $(\Gamma \backslash M, dt^2)$ is a connected Riemannian nilmanifold. Theorem 3.1 says that this construction is exhaustive. □

Another consequence is that coverings of our class of homogeneous metric space quotients, in particular of Riemannian nilmanifolds, always are normal coverings.

Corollary 3.3 *Let* $\varphi : (M_1, d_1) \rightarrow (M_2, d_2)$ *be a locally isometric covering space in which* (M_2, d_2) *is a metric space on which a Lie group* S / Δ_2, S *exponential solvable and* Δ_2 *discrete and central in* S, *acts effectively and transitively by isometries. Then* (M_1, d_1) *is a metric space on which another quotient group* S / Δ_1 *acts effectively and transitively by isometries. Further,* $\Delta_1 \subset \Delta_2$, *and the covering is normal with deck transformation group* Δ_2 / Δ_1.

Proof As described above, the universal covering $\psi_2 : (M, d) \rightarrow (M_2, d_2)$ is given by dividing out with a discrete subgroup Γ_2 of the center of S. As $\varphi : (M_1, d_1) \rightarrow (M_2, d_2)$ is a locally isometric covering, the universal covering $\psi_1 : (M, d) \rightarrow (M_1, d_1)$ is given by dividing out with a subgroup Γ_1 of Γ_2. Since the center of S is abelian, Γ_1 is normal in Γ_2, so φ is the normal locally isometric covering given by dividing out with Γ_2 / Γ_1.

In the case where (M, d) is a Riemannian manifold (M, ds^2) or a Finsler manifold (M, F), every $\xi \in \mathfrak{g}$ defines a Killing vector field ξ^M on (M, d). If ξ^M has bounded length on (M, d) then $\exp(t\xi)$ is a bounded isometry for all real t. Now Theorem 2.5 implies

Corollary 3.4 *Suppose that the metric space* (M, d) *is Riemannian (or Finsler). Let* $M = G/K = SK/K$ *where* $G = I(M, d)$, S *is an exponential solvable Lie group acting transitively on* (M, d), *and* K *is an isotropy subgroup. Let* $\xi \in \mathfrak{g}$ *such that* ξ^M *is a Killing vector field of bounded length on* (M, d). *Then* ξ *belongs to the center of* $\mathfrak{s}$ *and* ξ^M *has constant length on* (M, d).

In the Riemannian nilmanifold setting, the formulation of Theorem 3.1 is a bit less complicated. It becomes

Corollary 3.5 *Let (M, ds^2) be a simply connected Riemannian nilmanifold. Consider a Riemannian covering $\psi : (M, ds^2) \rightarrow \Gamma\backslash(M, ds^2)$. Then the following conditions are equivalent.*

1. Γ *consists of bounded isometries of (M, ds^2).*
2. Γ *consists of CW isometries of (M, ds^2).*
3. $G = I(M, ds^2) = N \rtimes K$ *semidirect product with N nilpotent, and the group Γ is central in N.*
4. $\Gamma\backslash(M, ds^2)$ *is a homogeneous Riemannian manifold.*
5. $\Gamma\backslash(M, ds^2)$ *is a Riemannian nilmanifold.*

Further, every connected Riemannian nilmanifold is isometric to a manifold $\Gamma\backslash(M, ds^2)$ as just described.

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