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Sp(2)/U(1) and a positive curvature problem

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ABSTRACT

A compact Riemannian homogeneous space G/H, with a bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is called positively curved for commuting pairs, if the sectional curvature vanishes for any tangent plane in $T_{eH}(G/H)$ spanned by a linearly independent commuting pair in $\mathfrak{m}$. In this paper, we will prove that on the coset space Sp(2)/U(1), in which U(1) corresponds to a short root, admits positively curved metrics for commuting pairs. B. Wilking recently proved that this Sp(2)/U(1) cannot be positively curved in the general sense. This is the first example to distinguish the set of compact coset spaces admitting positively curved metrics, and that for metrics positively curved only for commuting pairs.

1. Introduction

Let G/H be a compact Riemannian homogeneous space with G compact. With respect to any bi-invariant inner product $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{g}$, there is an invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of the Lie algebra of G, and as usual $\mathfrak{m}$ is identified with the tangent space $T_{eH}(G/H)$.

We call the Riemannian homogeneous space G/H *positively curved for commuting pairs*, if for any linearly independent commuting pair X and Y in $\mathfrak{m}$, the sectional curvature of the tangent plane $\mathrm{span}\{X, Y\} \subset T_{eH}(G/H)$ is positive. This notion contrasts with the traditional algebraic method for the classification of positively curved Riemannian homogeneous spaces [1–4]. In those papers, the method for showing that a compact homogeneous space G/H fails to have strictly positive sectional curvature, is to show that the sectional curvature vanishes for some commuting pair. It was generally accepted that compact coset

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spaces admitting homogeneous metrics positively curved for commuting pairs are exactly the homogeneous Riemannian manifolds of strictly positive sectional curvature.

While trying to generalize these classifications to the Finsler situation [5,6], we found a problem in L. Bérard-Bergery's classification [2] of odd dimensional positively curved Riemannian homogeneous spaces. There is a gap in the argument that the coset space Sp(2)/U(1) (where U(1) corresponds to a short root) cannot be positively curved. After a stratified classification of Cartan subalgebras contained in $\mathfrak{m}$ for this Sp(2)/U(1), we saw that the traditional algebraic method mentioned above cannot be used to exclude Sp(2)/U(1) from the list of positively curved homogeneous spaces. Formally, we have the following main theorem.

Theorem 1.1. *Consider the compact homogeneous space $G/H = \mathrm{Sp}(2)/\mathrm{U}(1)$ in which H corresponds to a short root, with the orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for a bi-invariant inner product. Then there are G-homogeneous Riemannian metrics on it which are positively curved for commuting pairs, i.e. at $o = eH \in G/H$, the sectional curvature $K(o, X \wedge Y) > 0$ for any linearly independent commuting pair X and Y in $\mathfrak{m} = T_oM$.*

After we announced this result, B. Wilking found a way to prove that Sp(2)/U(1) does not admit homogeneous Riemannian metrics of positive curvature (see Theorem 5.1 in Section 5). At the same time as the problem in [2] was fixed, Theorem 5.1, together with the main theorem, provides us the first example of compact homogeneous space that is positively curved for commuting pairs but not positively curved in the general sense. As the traditional algebraic method works well in most other cases, non-positively curved Riemannian homogeneous spaces which are positively curved for commuting pairs may be very rare. We thank Burkhard Wilking and Wolfgang Ziller for several e-mail discussions that led us to this refinement of our original note.

2. The basic setup for Sp(2)/U(1)

Let M be the coset space $G/H = \mathrm{Sp}(2)/\mathrm{U}(1)$, in which H corresponds to a short root. We borrow the following construction from [2] with some minor changes. Any matrix

$$\frac{1}{2}\begin{pmatrix} u+w & v-\lambda \\ v+\lambda & u-w \end{pmatrix}$$

in $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{sp}(2)$ can be identified with a formal row vector (λ, u, v, w), in which the pure imaginary quaternions u, v, and w are viewed as column vectors in $\mathbb{R}^3$ with the more preferred dot and cross products with respect to the standard orthonormal basis $\{e_1, e_2, e_3\}$, instead of quaternion multiplication. For the bi-invariant inner product of $\mathfrak{g}$, the different factors of λ, u, v and w are orthogonal to each other, and the restriction of the bi-invariant inner product to each factor of u, v or w coincides with the standard inner product up to scalar changes. The subalgebra $\mathfrak{h} = \mathrm{Lie}(H) = \mathfrak{u}(1)$ can be identified with the subspace $u = v = w = 0$, i.e. the λ-factor, and its bi-invariant orthogonal complement $\mathfrak{m}$ can be identified with the subspace $\lambda = 0$. For any two vectors $X = (0, u, v, w)$ and $Y = (0, u', v', w')$ in $\mathfrak{m}$, their bracket can be presented as

$$[X, Y] = (v \cdot w' - v' \cdot w, u \times u' + v \times v' + w \times w', u \times v' - u' \times v, u \times w' - u' \times w).$$

Any G-homogeneous metric on M can be defined from an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$. Our presentation of $\mathfrak{m}$ naturally splits, with the u-factor corresponding to the trivial H-representation, and the other two factors each corresponding to the same non-trivial irreducible H-representation, i.e. for $Z = (1, 0, 0, 0) \in \mathfrak{h}$,

$$\mathrm{Ad}(\exp(tZ))(0,0,v,w) = (0,0,\cos(2t)v + \sin(2t)w, -\sin(2t)v + \cos(2t)w).$$

So any $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$ must be of the form $\langle\cdot,\cdot\rangle = \langle\cdot, M\cdot\rangle_{\mathrm{bi}}$, in which the linear isomorphism $M : \mathfrak{m} \to \mathfrak{m}$ satisfies,

$$M(0,u,v,w) = (0, Au, Cv - Bw, Bv + Cw),$$

where A and C are self adjoint, B is skew adjoint, $A > 0$ and $C - \sqrt{-1}B > 0$ (or equivalently $\left(\begin{smallmatrix} C & -B \\ B & C \end{smallmatrix}\right) > 0$). To see this, we use $\mathrm{Ad}(H)$-invariance and the fact that $\mathrm{Ad}(H)$ is trivial on the u-factor and rotates between the v- and w-factors. So $M(0,u,v,w)$ has form $(0, Au, B_1v + B_2w, B_3v + B_4w)$. Since the resulting inner product on $\mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant, the 6×6 matrix $\left(\begin{smallmatrix} B_1 & B_2 \\ B_3 & B_4 \end{smallmatrix}\right)$ commutes with all rotations $\left(\begin{smallmatrix} \cos tI & \sin tI \\ -\sin t & \cos tI \end{smallmatrix}\right)$. It follows that $B_1 = B_4$ and $B_2 = -B_3$. As M is self adjoint and positive definite, $M(0,u,v,w) = (0, Au, Cv - Bw, Bv + Cw)$ with $A > 0$ self adjoint, B skew adjoint, and C self adjoint. Thus the action of M on the v, w 6-plane is given by $\left(\begin{smallmatrix} C & -B \\ B & C \end{smallmatrix}\right) > 0$.

In Bérard-Bergery's argument, he missed the B-term. In later discussion, we only consider small perturbations of the G-normal Riemannian homogeneous metric which corresponds to $M = M_0 = \mathrm{Id}$, so we denote $M_t = I + tL$ for $t \geq 0$, in which $L : \mathfrak{m} \to \mathfrak{m}$ is defined by $L(0,u,v,w) = (0, Au, Cv - Bw, Bv + Cw)$ with A and C self adjoint, and B skew adjoint. For t sufficiently close to 0, the corresponding G-homogeneous metric is denoted as g_t.

3. Proof of the main theorem

With respect to the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, we have linear maps A, B and C defined by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Let $L(0,u,v,w) = (0, Au, Cv - Bw, Bv + Cw)$, $M_t = I + tL$, and g_t the corresponding G-invariant Riemannian metric on M for $t > 0$ sufficiently close to 0.

The sectional curvature $K^{g_t}(o, X \wedge Y)$ of (M, g_t) for the tangent plane $\mathfrak{t} = \mathrm{span}\{X, Y\}$ at $o = eH$ is $K^{g_t}(o, X \wedge Y) = C(X,Y,t)/S(X,Y,t)$ where

$$S(X,Y,t) = g_t(X,X)g_t(Y,Y) - g_t(X,Y)^2$$

and

$$\begin{aligned} C(X,Y,t) = & -\tfrac{3}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle_{g_t} + \tfrac{1}{2}\langle [[Y,X]_{\mathfrak{m}}, Y]_{\mathfrak{m}}, X\rangle_{g_t} + \tfrac{1}{2}\langle [[X,Y]_{\mathfrak{m}}, X]_{\mathfrak{m}}, Y\rangle_{g_t} \\ & + \langle [[X,Y]_{\mathfrak{h}}, X], Y\rangle_{g_t} + \langle U(X,Y,t), U(X,Y,t)\rangle_{g_t} - \langle U(X,X,t), U(Y,Y,t)\rangle_{g_t}. \end{aligned}$$

Here $U : \mathfrak{m} \times \mathfrak{m} \times [0,\epsilon) \to \mathfrak{m}$ is defined by

$$\langle U(X,Y,t), Z\rangle_{g_t} = \tfrac{1}{2}(\langle [Z,X]_{\mathfrak{m}}, Y\rangle_{g_t} + \langle [Z,Y]_{\mathfrak{m}}, X\rangle_{g_t}),$$

or equivalently (see the last section of [2])

$$U(X,Y,t) = \tfrac{1}{2}M_t^{-1}([X, M_tY] + [Y, M_tX]).$$

When $[X,Y] = 0$ and $t = 0$, $K^{g_0}(o, X \wedge Y) = C(X,Y,0) = 0$ by the sectional curvature formula for normal homogeneous spaces [3], and $\frac{d}{dt}C(X,Y,t)|_{t=0} = 0$ because $U(X,Y,0) = 0$. Thus $\frac{d^2}{dt^2}K^{g_t}(o, X \wedge Y)|_{t=0}$ has the same sign (or 0) as $\frac{d^2}{dt^2}C(X,Y,t)|_{t=0}$. Furthermore, when they vanish, $\frac{d^3}{dt^3}K^{g_t}(o,X\wedge Y)|_{t=0}$ has the same sign (or 0) as $\frac{d^3}{dt^3}C(X,Y,t)|_{t=0}$. Direct calculation shows, when $[X,Y]=0$,

$$\tfrac{d^2}{dt^2}\langle U(X,Y,t),U(X,Y,t)\rangle_{g_t}|_{t=0}=\tfrac{1}{2}\langle [X,LY]+[Y,LX],[X,LY]+[Y,LX]\rangle_{\mathrm{bi}},$$

and

$$\begin{aligned}\tfrac{d^2}{dt^2}\langle U(X,X,t),U(Y,Y,t)\rangle_{g_t}|_{t=0}&=2\langle [X,LX],[Y,LY]\rangle_{\mathrm{bi}}=2\langle [[X,LX],Y],LY\rangle_{\mathrm{bi}}\\&=2\langle [X,[LX,Y]],LY\rangle_{\mathrm{bi}}=2\langle [X,LY],[Y,LX]\rangle_{\mathrm{bi}},\end{aligned}$$

thus

$$\tfrac{d^2}{dt^2}C(X,Y,t)|_{t=0}=\tfrac{1}{2}\langle [X,LY]-[Y,LX],[X,LY]-[Y,LX]\rangle_{\mathrm{bi}}. \tag{3.1}$$

Notice that $\frac{1}{S(X,Y)^{1/2}}([X,LY]-[Y,LX])$ depends only on the tangent plane $\mathrm{span}\{X,Y\}$. Thus we have

Lemma 3.2. *If $X,Y\in\mathfrak{m}$ are linearly independent and commute, then $C(X,Y,0)=\frac{d}{dt}C(X,Y,t)|_{t=0}=0$, and $\frac{d^2}{dt^2}C(X,Y,t)|_{t=0}\geqq 0$, with equality if and only if $[X,LY]=[Y,LX]$. Equivalently, for any Cartan subalgebra $\mathfrak{t}\subset\mathfrak{m}$, we have*

$$K^{g_0}(o,\mathfrak{t})=\tfrac{d}{dt}K^{g_t}(o,\mathfrak{t})|_{t=0}=0, \tag{3.3}$$

and

$$\tfrac{d^2}{dt^2}K^{g_t}(o,\mathfrak{t})|_{t=0}\geq 0 \tag{3.4}$$

with equality if an only if $[X,LY]=[Y,LX]$ in where $\mathfrak{t}=\mathrm{span}\{X,Y\}$.

To distinguish between the situations in which $\frac{d^2}{dt^2}C(X,Y,t)|_{t=0}$ is positive or 0, we will prove the following lemma, which is crucial for the proof of the Theorem 1.1.

Lemma 3.5. *Let $X,Y\in\mathfrak{m}$ linearly independent and $\mathfrak{t}=\mathrm{span}\{X,Y\}$. Suppose that $[X,Y]=0$, so $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{t}_0=\mathrm{span}\{(0,0,e_1,0),(0,0,0,e_2)\}$. If $\mathfrak{t}\notin\mathrm{Ad}(H)(\mathfrak{t}_0)$ then*

$$\frac{d^2}{dt^2}C(X,Y,t)\Big|_{t=0}>0,\ \textit{or equivalently}\ \frac{d^2}{dt^2}K^{g_t}(o,\mathfrak{t})\Big|_{t=0}>0. \tag{3.6}$$

If $\mathfrak{t}\in\mathrm{Ad}(H)(\mathfrak{t}_0)$ then

$$\frac{d^2}{dt^2}C(X,Y,t)\Big|_{t=0}=0\ \textit{and}\ \frac{d^3}{dt^3}C(X,Y,t)\Big|_{t=0}>0,$$

or equivalently,

$$\frac{d^2}{dt^2}K^{g_t}(o,\mathfrak{t})\Big|_{t=0}=0\ \textit{and}\ \frac{d^3}{dt^3}K^{g_t}(o,\mathfrak{t})\Big|_{t=0}>0 \tag{3.7}$$

The proof of Lemma 3.5 will be postponed to the next section. We now prove Theorem 1.1, assuming Lemma 3.5.

Denote the set of all Cartan subalgebras of $\mathfrak{g}$ contained in $\mathfrak{m}$ as $\mathcal{C}$, and the set of all tangent planes at $o=eH$ as $\mathcal{G}$. Then $\mathcal{G}$ is a Grassmannian manifold, $\mathcal{C}$ is a compact subvariety. The isotropy subgroup H

has natural $\mathrm{Ad}(H)$-actions on $\mathcal{G}$ which preserve $\mathcal{C}$. It is easy to see, for any valid t, the sectional curvature function $K^{g_t}(o,\cdot)$ is $\mathrm{Ad}(H)$-invariant.

If $\mathfrak{t} \in \mathcal{C}$ is a Cartan subalgebra contained in $\mathfrak{m}$, such that its $\mathrm{Ad}(H)$-orbit does not contain $\mathfrak{t}_0 = \mathrm{span}\{(0,0,e_1,0),(0,0,0,e_2)\}$, then by (3.6) in Lemma 3.5, we can find an open neighborhood $\mathcal{U}$ of $\mathfrak{t}$ in $\mathcal{C}$, and a positive ϵ (sufficiently close to 0, same below), such that for any Cartan subalgebra $\mathfrak{t}' \in \mathcal{U}$ and $t \in (-\epsilon,\epsilon)$, $\frac{d^2}{dt^2}K^{g_t}(o,\mathfrak{t}') > 0$. Together with (3.3) in Lemma 3.2, it indicates for any Cartan subalgebra $\mathfrak{t}' \in \mathcal{U}$ and $t \in (0,\epsilon)$, $K^{g_t}(o,\mathfrak{t}') > 0$.

If $\mathfrak{t} \in \mathcal{C}$ is a Cartan subalgebra contained in $\mathfrak{m}$, such that its $\mathrm{Ad}(H)$-orbit contains $\mathfrak{t}_0$, then by (3.7), we can find an open neighborhood $\mathcal{U}$ of $\mathfrak{t}$ in $\mathcal{C}$, and a positive ϵ, such that for any Cartan subalgebra $\mathfrak{t}' \in \mathcal{U}$ and $t \in (-\epsilon,\epsilon)$, $\frac{d^3}{dt^3}K^{g_t}(o,\mathfrak{t}') > 0$. Together with (3.3) and (3.4) in Lemma 3.2, it indicates for any Cartan subalgebra $\mathfrak{t}' \in \mathcal{U}$ and $t \in (0,\epsilon)$, $K^{g_t}(o,\mathfrak{t}') > 0$.

By the compactness of $\mathcal{C}$, we can find a finite cover for it from the open neighborhoods $\mathcal{U}$ given above, and take a uniform minimum $\epsilon > 0$. Then for any Cartan subalgebra $\mathfrak{t} \in \mathcal{C}$ contained in $\mathfrak{m}$ and $t \in (0,\epsilon)$, $K^{g_t}(o,\mathfrak{t}) > 0$. This completes the proof of Theorem 1.1.

4. Proof of Lemma 3.5

The proof of Lemma 3.5 is an analysis of the Cartan subalgebras of $\mathfrak{g}$ contained in $\mathfrak{m}$. Observe that $\mathcal{C}$ is the union of the following $\mathrm{Ad}(H)$-invariant subsets.

Case I. The Cartan subalgebra $\mathfrak{t}$ is spanned by $X = (0,0,v,w)$ and $Y = (0,0,v',w')$ in $\mathfrak{m}$, it belongs to $\mathcal{C}_1$.

Case II. The tangent plane $\mathfrak{t}$ is spanned by $X = (0,u,v,w)$ and $Y = (0,0,v',w')$ in $\mathfrak{m}$, in which $u \neq 0$, it belongs to $\mathcal{C}_2$.

Case III. The tangent plane $\mathfrak{t}$ is spanned by $X = (0,u,v,w)$ and $Y = (0,u',v',w')$ in $\mathfrak{m}$, in which u and u' are linearly independent, it belongs to $\mathcal{C}_3$.

The two techniques we will use are change of basis in a given $\mathfrak{t}$, and change of $\mathfrak{t}$ in $\mathcal{C}$ by the action of H, to reduce our discussion to several cases with very simple X and Y.

Proof of Lemma 3.5 in Case I. Assume that $X = (0,0,v,w)$ and $Y = (0,0,v',w')$ span the Cartan subalgebra $\mathfrak{t}$.

First, consider the situation where v and w are linearly dependent. Changing basis of $\mathfrak{t}$ by a suitable $\mathrm{Ad}(H)$-action, we can assume $w = 0$. Subtracting a multiple of X from Y we can assume $v' \cdot v = 0$. Since $[X,Y] = 0$ we have $v \times v' = 0$. Thus $v' = 0$. Also from $[X,Y] = 0$, we have $v \cdot w' = 0$. Both v and w' can be normalized to have length 1.

Next, consider the situation that v and w are linearly independent. Because $[X,Y] = 0$, we have

$$v \times v' = -w \times w', \text{ and} \tag{4.1}$$

$$v \cdot w' = v' \cdot w. \tag{4.2}$$

From (4.1), v' and w' are contained in $\mathrm{span}\{v,w\}$. By a suitable $\mathrm{Ad}(H)$-action, we may assume $v \cdot w = 0$. Replacing Y with a suitable linear combination of X and Y, we can assume $v' \cdot v = 0$ as well. If $v' = 0$, it goes back to the last situation, otherwise we can normalize v and v' and assume $|v| = |v'| = 1$. Express $w = b_2 v'$ and $w' = c_1 v + c_2 v'$, with $b_2 \neq 0$. By (4.1) and (4.2), $b_2 = c_1 = \pm 1$. We can further change Y to $\pm Y$ and assume $b_2 = c_1 = 1$. Then

$$X'' = Y + \tfrac{1}{2}(-c_2 \pm \sqrt{c_2^2 + 4X}) = (0,0,v'',w'')$$

where v'' and w'' are linearly independent. Replacing X with X'', we reduce to the last situation.

To summarize, for $\mathfrak{t} \subset \mathcal{C}_1$, we can find a representative $\mathrm{span}\{(0,0,v,0),(0,0,0,w')\}$ in the $\mathrm{Ad}(H)$-orbit of $\mathfrak{t}$, for which $|v| = |w'| = 1$ and $v \cdot w' = 0$.

Now we may suppose $\mathfrak{t}$ is spanned by $X = (0,0,v,0)$ and $Y = (0,0,0,w')$ with $|v| = |w'| = 1$ and $v \cdot w' = 0$. If $\frac{d^2}{dt^2} C(X,Y,t)|_{t=0} = 0$, i.e. $[X, LY] = [Y, LX]$, then

$$w' \cdot Cv = 0, \text{ and} \tag{4.3}$$

$$v \times Bw' = -w' \times Bv. \tag{4.4}$$

From (4.4), B preserves the subspace spanned by v and w', or equivalently $\mathrm{span}\{v, w'\}^\perp$ is an eigenspace of B, which must be $\mathbb{R}e_3$. So $\mathrm{span}\{v, w'\} = \mathrm{span}\{e_1, e_2\}$. Because of (4.3), and the speciality of the chosen C, we must have $\{\pm v, \pm w'\} = \{\pm e_1, \pm e_2\}$, i.e., up to the action of $\mathrm{Ad}(H)$,

$$\mathfrak{t} = \mathrm{span}\{(0,0,e_1,0),(0,0,0,e_2)\}.$$

To summarize, we have $\frac{d^2}{dt^2} C(X,Y,t)|_{t=0} > 0$ when $\mathfrak{t} \in \mathcal{C}_1$ is not contained in the $\mathrm{Ad}(H)$-orbit of $\mathrm{span}\{(0,0,e_1,0),\ (0,0,0,e_2)\}$, and $\frac{d^2}{dt^2} C(X,Y,t)|_{t=0} = 0$, when $\mathfrak{t} \in \mathcal{C}_1$.

Further consider $\frac{d^3}{dt^3} C(X,Y,t)|_{t=0}$, we only need to assume $X = (0,0,e_1,0)$ and $Y = (0,0,0,e_2)$. By direct calculation $[X, LY] = [X, M_t Y] = [Y, LX] = [Y, M_t X] = 0$, and so

$$U(X,Y,t) = 0,$$
$$U(X,X,t) = \big(0, \tfrac{t^2}{1-t^2} e_1 + \tfrac{-t}{1-t^2} e_2, 0, 0\big),$$

and $[Y, M_t Y] = (0, te_1, 0, 0)$. So

$$C(X,Y,t) = -\langle U(X,X), U(Y,Y)\rangle_{g_t} = -\langle U(X,X), [Y, M_t Y]\rangle_{\mathrm{bi}} = \frac{ct^3}{1-t^2},$$

where the constant $c > 0$ comes from the scalar relation between the standard inner product on $\mathbb{R}^3$ and the restriction of the bi-invariant inner product of $\mathfrak{g}$ to the u-factor. Now it is obvious that $\frac{d^3}{dt^3} C(X,Y,t)|_{t=0} > 0$.

Proof of Lemma 3.5 in Case II. Assume that the Cartan subalgebra $\mathfrak{t} \in \mathcal{C}_2$ is spanned by $X = (0,u,v,w)$ and $Y = (0,0,v',w')$ with $u \neq 0$. We normalize u so that $|u| = 1$. Because $[X,Y] = 0$, we have $u \times v' = u \times w' = 0$, i.e. $v', w' \in \mathbb{R}u$. We can apply an element of $\mathrm{Ad}(H)$ and then scale, so that $w' = 0$ and $v' = u$. Using $[X,Y] = 0$ again, we have $v \times v' = 0$ and $v' \cdot w = u \cdot w = 0$. Subtract a suitable multiple of Y from X; we then have $v \cdot v' = 0$, which implies $v = 0$.

To summarize, the $\mathrm{Ad}(H)$-orbit of $\mathfrak{t} \in \mathcal{C}_2$ contains a Cartan that is spanned by $X = (0,u,0,w)$ and $Y = (0,0,u,0)$ with $|u| = 1$ and $u \cdot w = 0$.

If further we have $[X, LY] = [Y, LX]$, then direct calculation shows

$$u \cdot Cw = 0, \tag{4.5}$$

$$w \times Bu + u \times Bw = 0, \tag{4.6}$$

$$u \times (C - A)u = 0, \tag{4.7}$$

$$u \times Bu = 0. \tag{4.8}$$

From (4.7) and (4.8), the unit vector u is a common eigenvector of B, i.e. $u = \pm e_3$, and u is also an eigenvector of $A - C$. But e_3 is not an eigenvector of $A - C$. So in this case we always have $\frac{d^2}{dt^2} K^{g_t}(o, \mathfrak{t})|_{t=0} > 0$.

The proof of Lemma 3.5 in Case III. Let $\mathfrak{t} \in \mathcal{C}_3$ be spanned by $X = (0,u,v,w)$ and $Y = (0,u',v',w')$ with u and u' linearly independent.

We had observed that v, w, v' and w' are all contained in the subspace spanned by u and u'. By $[X,Y]=0$, we have $u \times v' = u' \times v$, from which we see that v and v' are linear combinations of u and u'. Similarly w and w' are linear combinations of u and u'.

Next, consider the situation where v and w are linearly dependent. They cannot both vanish because the u-factor of $[X,Y]$ does not vanish. Acting by a suitable $\mathrm{Ad}(H)$, we can make $w=0$. Subtracting a suitable multiple of X from Y, we have $v \cdot v' = 0$. Then we can find linear combination $Y'' = (0, u'', v'', w'')$ of X and Y to substitute for Y, so that v'' and w'' are also linearly independent and they cannot both vanish. Using a suitable generic $\mathrm{Ad}(H)$ transformation, we reduce to the situation where $\mathfrak{t}$ has basis $X = (0, u, v, \mu_1 v)$ and $Y = (0, u', v', \mu_2 v')$ with the properties (i) u and u' are linearly independent, (ii) v and v' are nonzero vectors in the span of u and u', and (using $[X,Y]=0$) v and v' form another basis of $\mathrm{span}\{u, u'\}$.

Next we go to the general $X = (0, u, v, w)$ and $Y = (0, u', v', w')$ and reduce to the situation above. We may assume that v and w are linearly independent, for otherwise the reduction is immediate. Applying $\mathrm{Ad}(H)$ we can suppose $u \cdot v = 0$. Subtracting a suitable multiple of X from Y, we also have $u \cdot u' = 0$. With suitable scalar changes for X and Y, we normalize u and u' so that $|u| = |u'| = 1$. Denote

$$v = b_2 u', v' = b_1' u + b_2' v, w = c_1 u + c_2 u', \text{ and } w' = c_1' u + c_2' v.$$

Then $[X,Y]=0$ forces

$$b_2' = 0, \text{ and} \tag{4.9}$$

$$b_2 c_2' = b_1' c_1. \tag{4.10}$$

Note that $X'' = X + \lambda Y = (0, u'', v'', w'')$ has linearly dependent entries v'' and w'' if and only if

$$\det \begin{pmatrix} b_1' \lambda & b_2 \\ c_1' \lambda + c_1 & c_2' \lambda + c_2 \end{pmatrix} = b_1' c_2' \lambda^2 + (b_1' c_2 - c_1' b_2)\lambda - b_2 c_1 = 0.$$

By (4.10), the above equation must have a real solution. Substituting the corresponding X'' for X, we reduce the discussion to the case $X = (0, u, v, \mu_1 v)$ and $Y = (0, u', v', \mu_2 v)$, and there $\mathrm{span}\{u, u'\} = \mathrm{span}\{v, v'\}$ is a two dimension subspace in $\mathbb{R}^3$.

If $\mu_1 = \mu_2$, we can apply a suitable element of $\mathrm{Ad}(H)$ to make them vanish. By similar tricks, we can make $u \cdot u' = 0$ and $|u| = |u'| = 1$. There is a real number λ, such that $X'' = X + \lambda Y = (0, u'', v'', 0) = (0, u + \lambda u', v + \lambda v', 0)$ satisfies

$$u'' \cdot v'' = (u' \cdot v')\lambda^2 + (u \cdot v' + v \cdot u')\lambda + u \cdot v = 0,$$

because we can get $u \cdot v + u' \cdot v' = 0$ from $[X,Y]=0$. Replace X with X''; then $u \cdot v = 0$. Subtract a suitable multiple of X from Y; then $u \cdot u' = 0$ again, i.e. v is a scalar multiple of u'. Also, normalize u and u' so that $|u| = |u'| = 1$. Express $v = \nu_1 u'$ and $v' = \nu_2 u + \nu_3 u'$. From $[X,Y]=0$, we get $\nu_1 \nu_2 = 1$ and $\nu_3 = 0$. If we only require $u \cdot u' = 0$ then by suitable scalar changes for Y, we can make $\nu_1 = \nu_2 = 1$.

In this case, we have $X = (0, u, \nu_1 u', 0)$ and $Y = (0, u', \nu_2 u, 0)$, in which $|u| = |u'| = 1$, $u \cdot u' = 0$ and $\nu_1 \nu_2 = 1$. There is another way to present $\mathfrak{t} = \mathrm{span}\{X, Y\}$ in which ν_1 and ν_2 do not appear. Replace Y by $\nu_1 Y$ and u' by $\nu_1 u'$. Then we have $X = (0, u, u', 0)$ and $Y = (0, u', u, 0)$ in where $u \cdot u' = 0$.

If further we have $[X, LY] = [Y, LX]$, then

$$u \cdot B u' = 0, \tag{4.11}$$

$$u \times (A - C)u' = u' \times (A - C)u \tag{4.12}$$

$$u' \times (A - C)u' = u \times (A - C)u, \tag{4.13}$$

$$u \times Bu = u' \times Bu'. \tag{4.14}$$

By (4.14), B preserves $\mathrm{span}\{u, u'\}$, so $\mathrm{span}\{u, u'\} = \mathrm{span}\{e_1, e_2\}$. Then $u \cdot Bu' \neq 0$, contradicting (4.11). So in this case, $\frac{d^2}{dt^2}C(X, Y, t)|_{t=0} > 0$ when X and Y span $\mathfrak{t}$.

If $\mu_1 \neq \mu_2$ then, because $[X, Y] = 0$, $u \times v' = u' \times v$ and $\lambda_2 u \times v' = \lambda_1 u' \times v$, thus $u \times v' = u' \times v = 0$. Applying a suitable element of $\mathrm{Ad}(H)$ we have $X = (0, u, \nu_1 u', 0)$ and $Y = (0, u', \nu_2 u, \nu_3 u)$, with $\nu_1\nu_2 = 1$. By similar argument, we may assume $\nu_1 = \nu_2 = 1$.

If further $[X, LY] = [Y, LX]$ then

$$u \times ((C - A)u - \nu_3 Bu) = u' \times (C - A)u', \tag{4.15}$$

$$u \times (\nu_3(C - A)u + Bu) = u' \times Bu'. \tag{4.16}$$

It follows that Bu, Bu', $(C - A)u$ and $(C - A)u'$ belong to $\mathrm{span}\{u, u'\}$. So $(\mathrm{span}\{u, u'\})^\perp$ consists of the common eigenvectors of B and $C - A$. Thus $(\mathrm{span}\{u, u'\})^\perp = \mathbb{R}e_3$. But e_3 is not an eigenvector of $C - A$. This is a contradiction. That completes the proof of Lemma 3.5. □

As a by-product of the above argument we have the following explicit description for Cartan subalgebras contained in $\mathfrak{m}$, for the space $\mathrm{Sp}(2)/\mathrm{U}(1)$. It may be useful for further study of curvature on that space.

Proposition 4.17. *Let $M = G/H = \mathrm{Sp}(2)/\mathrm{U}(1)$ in where $\mathrm{U}(1)$ corresponds to a short root, and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the corresponding orthogonal decomposition. Then the set $\mathcal{C}$ of all Cartan subalgebras of $\mathfrak{g}$ contained in $\mathfrak{m}$ is the union of four $\mathrm{Ad}(H)$-orbits with the following representatives:*

(1) $\mathrm{span}\{X, Y\}$, *with $X = (0, 0, v, 0)$ and $Y = (0, 0, 0, w')$ such that $|v| = |w'| = 1$ and $v \cdot w' = 0$.*
(2) $\mathrm{span}\{X, Y\}$, *with $X = (0, u, 0, w)$ and $Y = (0, 0, u, 0)$ such that $|u| = 1$ and $u \cdot w = 0$.*
(3) $\mathrm{span}\{X, Y\}$, *with $X = (0, u, u', 0)$ and $Y = (0, u', u, 0)$ such that u and u' are linearly independent and $u \cdot u' = 0$.*
(4) $\mathrm{span}\{X, Y\}$, *with $X = (0, u, u', 0)$ and $Y = (0, u', u, \mu u)$ such that u and u' are linearly independent and $\mu \neq 0$.*

5. This Sp(2)/U(1) cannot be positively curved

With his permission we present the following unpublished theorem of B. Wilking. This theorem came out of discussions of an early version of this note.

Theorem 5.1. *The compact homogeneous space $G/H = \mathrm{Sp}(2)/\mathrm{U}(1)$, in which H corresponds to a short root, does not admit a homogeneous Riemannian metric with all sectional curvatures positive.*

Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the bi-invariant orthogonal decomposition. Any homogeneous Riemannian metric on G/H is one-to-one determined by an $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot, \cdot\rangle = \langle\cdot, M\cdot\rangle_{\mathrm{bi}}$ in which the self adjoint isomorphism $M : \mathrm{m} \to \mathrm{m}$, with respect to $\langle\cdot, \cdot\rangle_{\mathrm{bi}}$, is $\mathrm{Ad}(H)$-invariant and positive definite.

The analytic technique in B. Wilking's proof can be summarized as the following lemma.

Lemma 5.2. *Let G be a compact connected Lie group, H a closed subgroup of G, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ a bi-invariant orthogonal decomposition. Suppose that for any $\mathrm{Ad}(H)$-equivariant linear map $M : \mathfrak{m} \to \mathfrak{m}$, positive definite with respect to the restriction of the bi-invariant inner product to $\mathfrak{m}$, there is an eigenvector $X \in \mathfrak{m}$ for the smallest eigenvalue of M, and another $Z \in \mathfrak{m}$, such that $\{X, Z\}$ is a linearly independent commuting pair. Then G/H does not admit G-homogeneous Riemannian metrics of strictly positive sectional curvature.*

Proof. Any G-homogeneous Riemannian metric is determined by an inner product $\langle\cdot,\cdot\rangle = \langle\cdot, M\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{m}$, where M is a linear map as indicated in the statement of the lemma. We will show the sectional curvature at eH vanishes for the tangent plane spanned by X and $Y = M^{-1}(Z)$, where X and Z are indicated by the lemma. Denote $MX = \lambda X$. Because $\lambda > 0$ is the smallest eigenvalue of M, for any $X' \in \mathfrak{m}$, we have

$$\langle X', M(X')\rangle_{\mathrm{bi}} \geq \lambda\langle X', X'\rangle_{\mathrm{bi}}, \text{ and} \tag{5.3}$$

$$\langle X', M^{-1}X'\rangle_{\mathrm{bi}} \leqq \lambda^{-1}\langle X', X'\rangle_{\mathrm{bi}}. \tag{5.4}$$

Direct calculation shows

$$[X, MY] + [Y, MX] = [X, Z] - \lambda[X, Y] = -\lambda[X, Y]$$

is a vector in $\mathfrak{m}$ (see the last section in [2]), i.e. $[X, Y] \in \mathfrak{m}$. It is easy to see X and Y are linearly independent because $MX = \lambda X$ and $Z = MY$ are linearly independent.

Now apply the sectional curvature formula to the tangent plane spanned by X and Y, i.e. $K(eH, X\wedge Y) = C(X, Y)/S(X, Y)$, in which $S(X, Y) > 0$, and

$$\begin{aligned} C(X, Y) = &-\frac{3}{4}\langle[X, Y]_{\mathfrak{m}}, [X, Y]_{\mathfrak{m}}\rangle + \frac{1}{2}\langle[[Y, X]_{\mathfrak{m}}, Y]_{\mathfrak{m}}, X\rangle \\ &+ \frac{1}{2}\langle[[X, Y]_{\mathfrak{m}}, X]_{\mathfrak{m}}, Y\rangle + \langle[[X, Y]_{\mathfrak{h}}, X], Y\rangle \\ &+ \langle U(X, Y), U(X, Y)\rangle - \langle U(X, X), U(Y, Y)\rangle, \end{aligned} \tag{5.5}$$

where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined by

$$\langle U(X', Y'), Z'\rangle = \frac{1}{2}(\langle[Z', X']_{\mathfrak{m}}, Y'\rangle + \langle[Z', Y']_{\mathfrak{m}}, X'\rangle),$$

or equivalently

$$U(X', Y') = \frac{1}{2}M^{-1}([X', MY'] + [Y', MX']),$$

for any X', Y' and Z' in $\mathfrak{m}$. Because X is an eigenvector of M, $U(X, X) = 0$ and $U(X, Y) = -\frac{1}{2}\lambda M^{-1}([X, Y])$. Because $[X, Y] \in \mathfrak{m}$, we can simplify (5.5) and estimate it as follows,

$$\begin{aligned} C(X, Y) &= -\frac{3}{4}\langle[X, Y], M([X, Y])\rangle_{\mathrm{bi}} + \frac{1}{2}\langle[X, Y], [MX, Y] + [X, MY]\rangle_{\mathrm{bi}} \\ &\quad + \frac{1}{4}\lambda^2\langle M^{-1}([X, Y]), [X, Y]\rangle_{\mathrm{bi}} \\ &= -\frac{3}{4}\langle[X, Y], M([X, Y])\rangle_{\mathrm{bi}} + \frac{1}{2}\lambda\langle[X, Y], [X, Y]\rangle_{\mathrm{bi}} \\ &\quad + \frac{1}{4}\lambda^2\langle M^{-1}([X, Y]), [X, Y]\rangle_{\mathrm{bi}} \\ &\leqq -\frac{3}{4}\lambda\langle[X, Y], [X, Y]\rangle_{\mathrm{bi}} + \frac{1}{2}\lambda\langle[X, Y], [X, Y]\rangle_{\mathrm{bi}} + \frac{1}{4}\lambda\langle[X, Y], [X, Y]\rangle_{\mathrm{bi}} \\ &= 0, \end{aligned} \tag{5.6}$$

in which the inequality makes use of (5.3) and (5.4). This shows $K(eH, X \wedge Y) \leqq 0$. That completes the proof of Lemma 5.2. $\square$

Now back to $G/H = \mathrm{Sp}(2)/\mathrm{U}(1)$ in consideration, and we prove Theorem 5.1.

As mentioned earlier, the $\mathrm{Ad}(H)$-invariant linear map M can be expressed as

$$M(0, u, v, w) = (0, Au, Cv - Bw, Bv + Cw),$$

where u, v and w in $\mathbb{R}^3$ are column vectors. This is the standard presentation of vectors in $\mathfrak{m}$. Here A and $\left(\begin{smallmatrix} C & -B \\ B & C \end{smallmatrix}\right)$ are positive definite matrices. Any eigenvalue of M is either an eigenvalue of A or an eigenvalue of $\left(\begin{smallmatrix} C & -B \\ B & C \end{smallmatrix}\right)$.

If one eigenvalue of A is the smallest eigenvalue of M we can find a nonzero eigenvector $u \in \mathbb{R}^3$ accordingly for A. Then $X = (0, u, 0, 0)$ and $Z = (0, 0, u, 0)$ satisfy the requirement of the lemma.

If one eigenvalue of $\left(\begin{smallmatrix} C & -B \\ B & C \end{smallmatrix}\right)$ is the smallest eigenvalue of M, let $X = (0, 0, v, w)$ denote the corresponding nonzero eigenvector of M. When v and w are linearly dependent, we choose the nonzero vector $Z = (0, v, 0, 0)$ or $Z = (0, w, 0, 0)$ such that X and Z satisfy the requirement of the lemma. When v and w are linearly independent, we can find an element $h \in H$, such that $\mathrm{Ad}(h)X = (0, 0, v', w')$ such that v' and w' are nonzero vectors and $v' \cdot w' = 0$. Take $Z = \mathrm{Ad}(h^{-1})(0, 0, v', 0)$, then X and Z satisfy the requirement of the lemma. This completes the proof of Theorem 5.1.

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