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KILLING VECTOR FIELDS OF CONSTANT LENGTH ON RIEMANNIAN NORMAL HOMOGENEOUS SPACES

MING XU*

College of Mathematics
Tianjin Normal University
Tianjin 300387, P. R. China
mgmgmgxu@163.com

JOSEPH A. WOLF**

Department of Mathematics
University of California
Berkeley, CA 94720–3840, USA
jawolf@math.berkeley.edu

Abstract. Killing vector fields of constant length correspond to isometries of constant displacement. Those in turn have been used to study homogeneity of Riemannian and Finsler quotient manifolds. Almost all of that work has been done for group manifolds or, more generally, for symmetric spaces. This paper extends the scope of research on constant length Killing vector fields to a class of Riemannian normal homogeneous spaces.

1. Introduction

An isometry ρ of a metric space (M,d) is called Clifford–Wolf (CW) if it moves each point the same distance, i.e., if the displacement function $\delta(x) = d(x,\rho(x))$ is constant. W. K. Clifford [4] described such isometries for the 3–sphere, and G. Vincent [19] used the term *Clifford translation* for constant displacement isometries of round spheres in his study of spherical space forms S^n/Γ with Γ metabelian. Later, J. A. Wolf ([20], [21], [22]) extended the use of the term *Clifford translation* to the context of metric spaces, especially Riemannian symmetric spaces. There the point is his theorem [24] that a complete locally symmetric Riemannian manifold M is homogeneous if and only if, in the universal cover $\widetilde{M} \to M = \Gamma\backslash\widetilde{M}$, the covering group Γ consists of Clifford translations. In part, Wolf's argument was case by case, but later V. Ozols ([15], [17], [18]) gave a general argument for the situation where Γ is a cyclic subgroup of the identity component $I^0(\widetilde{M})$ of the isometry group $I(\widetilde{M})$. H. Freudenthal [14] discussed the situation where $\Gamma \subset I^0(\widetilde{M})$, and introduced the term *Clifford–Wolf isometry* (CW) for isometries of constant displacement. That seems to be the term in general usage. More re-

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Corresponding Author: J. A. Wolf, e-mail: jawolf@math.berkeley.edu.

cently, the result [24] for locally symmetric homogeneous Riemannian manifolds was extended to Finsler manifolds by S. Deng and J. A. Wolf [11].

In the setting of non–positive sectional curvature, isometries of bounded displacement are already CW [25]. Further, there has been some work relating CW and homogeneity for pseudo-Riemannian manifolds ([22], [23]).

Recently, V. N. Berestovskii and Yu. G. Nikonorov classified all simply connected Riemannian homogeneous spaces such that the homogeneity can be achieved by CW translations, i.e., CW homogeneous spaces ([1], [2], [3]). Also, S. Deng and M. Xu studied CW isometries and CW homogeneous spaces in Finsler geometry ([5], [6], [7], [8], [9], [10]).

Most of the research on CW translations has been concerned with Riemannian (and later Finsler) symmetric spaces. There we have a full understanding of CW translations ([24] and [11]), but little is known about CW translations on non-symmetric homogeneous Riemannian spaces. For example, there are not many examples of Clifford-Wolf translations on Riemannian normal homogeneous space G/H with G compact simple, except those found on Riemannian symmetric spaces and in some closely related settings (see [12] and [13]).

The infinitesimal version, apparently introduced by V. N. Berestovskii and Yu. G. Nikonorov, is that of Killing vector fields of constant length. We will refer to those Killing vector fields as *Clifford–Killing* vector fields or CK vector fields. They correspond (at least locally) to one parameter local groups of CW isometries. The purpose of this work is to study and classify all CK vector fields on Riemannian normal homogeneous spaces $M = G/H$.

We recall the general definition of Riemannian normal homogeneous spaces. Let G be a connected Lie group and H a compact subgroup, such that $M = G/H$ carries a G–invariant Riemannian metric. Thus the Lie algebra $\mathfrak{g}$ has an $\mathrm{Ad}\,(H)$–invariant direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ where the natural projection $\pi : G \to G/H$ maps $\mathfrak{m}$ onto the tangent space at the base point $o = \pi(\mathbf{e})$, and the Riemannian metric corresponds to a positive definite inner product $\langle\cdot\,,\cdot\rangle_{\mathfrak{m}}$ on $\mathfrak{m}$. The Riemannian manifold M is called *naturally reductive* if the $\mathrm{Ad}\,(H)$–invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ can be chosen so that

$$\langle \mathfrak{pr}_{\mathfrak{m}}[u,v], w\rangle_{\mathfrak{m}} + \langle v, \mathfrak{pr}_{\mathfrak{m}}[u,w]\rangle_{\mathfrak{m}} = 0 \qquad \text{for all } u,v,w \in \mathfrak{m}$$

where $\mathfrak{pr}_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$ is the projection with kernel $\mathfrak{h}$. When $\langle\cdot\,,\cdot\rangle_{\mathfrak{m}}$ is the restriction of an $\mathrm{Ad}\,(G)$–invariant nondegenerate symmetric bilinear form $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{g}$ such that $\mathfrak{h} \perp \mathfrak{m}$, $M = G/H$ is a *Riemannian normal homogeneous space.* In this general definition, Riemannian normal homogeneous space is viewed as a generalization of Riemannian symmetric space, including the non-compact type. If we expect the Riemannian normal homogeneous space to be related to Riemannian isometric submersions, there is another definition of Riemannian normal homogeneous space, which requires $\langle\cdot\,,\cdot\rangle_{\mathfrak{m}}$ to be the restriction of an $\mathrm{Ad}\,(G)$-invariant (bi-invariant) inner product on $\mathfrak{g}$. In this definition, $\mathfrak{g}$ must be compact, i.e., G is quasi-compact, or equivalently the universal cover of G is the product of a compact semi-simple Lie group and a Euclidean space (which can be 0). Note that, in both definitions, the normal homogeneous metric on M depends on G.

In this work, we will only consider the special case that G is a compact connected simple Lie group. Our main result is

Theorem 1.1. *Let G be a compact connected simple Lie group and H a closed subgroup with $0 < \dim H < \dim G$. Fix a normal Riemannian metric on $M = G/H$. Suppose that there is a nonzero vector $v \in \mathfrak{g}$ defining a CK vector field on $M = G/H$. Then M is a complete locally symmetric Riemannian manifold, and its universal Riemannian cover is an odd-dimensional sphere of constant curvature or a Riemannian symmetric space $\mathrm{SU}(2n)/\mathrm{Sp}(n)$.*

It is obvious to see that when $\dim H = \dim G$, M is just a point, and when $\dim H = 0$, M is locally Riemannian symmetric because it is covered by G with the bi-invariant Riemannian metric.

Riemannian normal homogeneity is a much weaker condition than Riemannian symmetry or locally Riemannian symmetric homogeneity. Even in the case where G is a compact connected simple Lie group, every smooth coset space G/H has at least one invariant normal Riemannian metric, while of course the list of Riemannian symmetric spaces G/H is rather short. But Theorem 1.1 provides the same classification result for CK vector fields. It suggests that the existence of nontrivial CK vector fields and CW translations will impose much stronger restrictions on a Riemannian homogeneous space, at least when that space is Riemannian normal homogeneous.

On the other hand, we do not have comprehensive results when G is of noncompact type. When G is compact but not simple, generally speaking, a Riemannian normal homogeneous space M does not have a perfect local decomposition into symmetric spaces. Thus the study of CW translations and CK vector fields in this situation is still open.

The proof of Theorem 1.1 is organized as follows. In Section 2, we summarize the notations and preliminaries for the Riemannian normal homogeneous spaces we will consider. In Section 3, we present some preliminary lemmas, study the CK vector fields at the Cartan subalgebra level, and prove Theorem 1.1 in the easiest situations. In Section 4, we prove Theorem 1.1 when G is an exceptional Lie group. From Section 5 to Section 8, we prove Theorem 1.1 when G is a classical Lie group, i.e., of type $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n$ or $\mathfrak{d}_n$.

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2. Notations about normal homogeneous spaces

Let G be a compact connected simple Lie group and H a closed subgroup with $0 < \dim H < \dim G$. We denote $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. Fix a bi-invariant inner product $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$. It defines a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with $\mathfrak{h} \perp \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The orthogonal projection to each factor is denoted $\mathrm{pr}_{\mathfrak{h}}$ or $\mathfrak{pr}_{\mathfrak{m}}$ respectively. We naturally identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at $o = \pi(\mathbf{e})$ where $\pi : G \to G/H$ is the usual projection. The restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant and defines a G-invariant Riemannian metric on $M = G/H$. A Riemannian metric defined in this way is called a *normal homogeneous metric*, and M together with a normal homogeneous metric is a *Riemannian normal homogeneous space*. Note the dependence on G. Here G is simple, so the normal homogeneous metric on G/H is unique up to scalar multiplications.

Any Cartan subalgebra of $\mathfrak{h}$ can be expanded to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $\mathfrak{t} = \mathfrak{t}\cap\mathfrak{h} + \mathfrak{t}\cap\mathfrak{m}$. As any two Cartan subalgebras of $\mathfrak{g}$ are conjugate, we can assume $\mathfrak{t}$ is the standard one. For example, when $\mathfrak{g} = \mathfrak{su}(n+1)$, $\mathfrak{t}$ is the subalgebra of all diagonal matrices. The standard $\mathfrak{u}(n) \hookrightarrow \mathfrak{so}(2n)$ comes from $a+b\sqrt{-1} \mapsto \left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right)$ for $a, b \in \mathbb{R}$. We view $\mathfrak{sp}(n)$ as the space of all skew–Hermitian skew $n \times n$ quaternion matrices where $q \mapsto \overline{q}$ is the usual conjugation of $\mathbb{H} = \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ over $\mathbb{R}$. Then $\mathfrak{u}(n) \subset \mathfrak{sp}(n)$ when we identify $\sqrt{-1}$ with $\mathbf{i}$. With these descriptions the space of diagonal matrices in $\mathfrak{u}(n)$ also provides the standard Cartan subalgebra $\mathfrak{t}$ for the other classical compact simple Lie algebras. The standard Cartan subalgebra of $\mathfrak{so}(2n)$ can also be viewed as that for $\mathfrak{so}(2n+1)$ with $\mathfrak{so}(2n)$ identified with the block at the lower right corner.

Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ be the root system of $\mathfrak{g}$, and Δ^+ be any positive root system in Δ. Because of the bi-invariant inner product on $\mathfrak{g}$, the roots of $\mathfrak{g}$ can be viewed as elements of $\mathfrak{t}$ instead of $\mathfrak{t}^*$. We have the standard decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{t} + \sum\nolimits_{\alpha\in\Delta^+} \mathfrak{g}_{\pm\alpha}, \tag{2.1}$$

in which each $\mathfrak{g}_{\pm\alpha}$ is the real two-dimensional root plane $(\mathfrak{g}^C_\alpha + \mathfrak{g}^C_{-\alpha}) \cap \mathfrak{g}$. Considering the subalgebra $\mathfrak{h}$, we have another decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} + \sum\nolimits_{\alpha'\in \mathrm{pr}_{\mathfrak{h}}(\Delta^+)} \widehat{\mathfrak{g}}_{\pm\alpha'} \text{ where } \widehat{\mathfrak{g}}_{\pm\alpha'} = \sum\nolimits_{\alpha\in\Delta^+, \mathrm{pr}_{\mathfrak{h}}(\pm\alpha)=\pm\alpha'} \mathfrak{g}_{\pm\alpha}. \tag{2.2}$$

Both (2.1) and (2.2) are orthogonal decompositions. More importantly, we have orthogonal decompositions

$$\mathfrak{t} = (\mathfrak{t}\cap\mathfrak{h}) + (\mathfrak{t}\cap\mathfrak{m}) \quad \text{and} \tag{2.3}$$

$$\widehat{\mathfrak{g}}_{\pm\alpha'} = (\widehat{\mathfrak{g}}_{\pm\alpha'}\cap\mathfrak{h}) + (\widehat{\mathfrak{g}}_{\pm\alpha'}\cap\mathfrak{m}), \tag{2.4}$$

in which the summands are equal to the images of the projection maps $\mathrm{pr}_{\mathfrak{h}}$ and $\mathrm{pr}_{\mathfrak{m}}$.

Let $\Delta' = \Delta(\mathfrak{h}, \mathfrak{t}\cap\mathfrak{h})$ denote the root system of $\mathfrak{h}$ and choose a positive subsystem $\Delta'^+ \subset \Delta'$. The restriction of the bi-invariant inner product of $\mathfrak{g}$ to $\mathfrak{h}$ is a bi-invariant inner product there, and Δ' can be viewed as a subset of $\mathfrak{t}\cap\mathfrak{h}$. For each root $\alpha' \in \Delta'^+$, the two-dimensional root plane $\widehat{\mathfrak{h}}_{\pm\alpha'}$ is just the factor $\widehat{\mathfrak{g}}_{\pm\alpha'}\cap\mathfrak{h}$ in (2.4).

For each simple Lie algebra $\mathfrak{g}$ we recall the Bourbaki description of the root system Δ^+, and the root planes in the classical cases.

(1) The case $\mathfrak{g} = \mathfrak{a}_n = \mathfrak{su}(n+1)$ for $n > 0$. Let $\{e_1, \ldots, e_{n+1}\}$ denote the standard orthonormal basis of $\mathbb{R}^{n+1}$. Then $\mathfrak{t}$ can be isometrically identified with the subspace $(e_1 + \cdots + e_{n+1})^\perp \subset \mathbb{R}^{n+1}$. The root system Δ is

$$\{\pm(e_i - e_j) \mid 1 \leqq i < j \leqq n+1\}. \tag{2.5}$$

Let $E_{i,j}$ be the matrix with all zeros except for a 1 in the (i,j) place. Then

$$e_i = \sqrt{-1}E_{i,i} \in \mathfrak{su}(n+1) \quad \text{and}$$
$$\mathfrak{g}_{\pm(e_i-e_j)} = \mathbb{R}(E_{i,j} - E_{j,i}) + \mathbb{R}\sqrt{-1}(E_{i,j} + E_{j,i}).$$

(2) The case $\mathfrak{g} = \mathfrak{b}_n = \mathfrak{so}(2n+1)$ for $n > 1$. The Cartan subalgebra $\mathfrak{t}$ can be isometrically identified with $\mathbb{R}^n$ with the standard orthonormal basis $\{e_1, \dots, e_n\}$. The root system Δ is

$$\{\pm e_i \mid 1 \leqq i \leqq n\} \cup \{\pm e_i \pm e_j \mid 1 \leqq i < j \leqq n\}. \tag{2.6}$$

Using matrices, we have

$$\begin{aligned} e_i &= E_{2i,2i+1} - E_{2i+1,2i}, \\ \mathfrak{g}_{\pm e_i} &= \mathbb{R}(E_{2i,1} - E_{1,2i}) + \mathbb{R}(E_{2i+1,1} - E_{1,2i+1}), \\ \mathfrak{g}_{\pm(e_i - e_j)} &= \mathbb{R}(E_{2i,2j} + E_{2i+1,2j+1} - E_{2j,2i} - E_{2j+1,2i+1}) \\ &\quad + \mathbb{R}(E_{2i,2j+1} - E_{2i+1,2j} + E_{2j,2i+1} - E_{2j+1,2i}), \quad \text{and} \\ \mathfrak{g}_{\pm(e_i + e_j)} &= \mathbb{R}(E_{2i,2j} - E_{2i+1,2j+1} - E_{2j,2i} + E_{2j+1,2i+1}) \\ &\quad + \mathbb{R}(E_{2i,2j+1} + E_{2i+1,2j} - E_{2j,2i+1} - E_{2j+1,2i}). \end{aligned}$$

(3) The case $\mathfrak{g} = \mathfrak{c}_n = \mathfrak{sp}(n)$ for $n > 2$. As before, $\mathfrak{t}$ is isometrically identified with $\mathbb{R}^n$ with the standard orthonormal basis $\{e_1, \dots, e_n\}$. The root system Δ is

$$\{\pm 2e_i \mid 1 \leqq i \leqq n\} \cup \{\pm e_i \pm e_j \mid 1 \leqq i < j \leqq n\}. \tag{2.7}$$

Using matrices, we have

$$\begin{aligned} e_i &= \mathbf{i}E_{i,i}, \\ \mathfrak{g}_{\pm 2e_i} &= \mathbb{R}\mathbf{j}E_{i,i} + \mathbb{R}\mathbf{k}E_{i,i}, \\ \mathfrak{g}_{\pm(e_i - e_j)} &= \mathbb{R}(E_{i,j} - E_{j,i}) + \mathbb{R}\mathbf{i}(E_{i,j} + E_{j,i}), \quad \text{and} \\ \mathfrak{g}_{\pm(e_i + e_j)} &= \mathbb{R}\mathbf{j}(E_{i,j} + E_{j,i}) + \mathbb{R}\mathbf{k}(E_{i,j} + E_{j,i}). \end{aligned}$$

(4) The case $\mathfrak{g} = \mathfrak{d}_n = \mathfrak{so}(2n)$ for $n > 3$. The Cartan subalgebra $\mathfrak{t}$ is identified with $\mathbb{R}^n$ with the standard orthonormal basis $\{e_1, \dots, e_n\}$. The root system Δ is

$$\{\pm e_i \pm e_j \mid 1 \leqq i < j \leqq n\}. \tag{2.8}$$

In matrices, we have formulas for the e_i and for the root planes for $e_i \pm e_j$ similar to those in the case of $\mathfrak{b}_n$, i.e.,

$$\begin{aligned} e_i &= E_{2i-1,2i} - E_{2i,2i-1}, \\ \mathfrak{g}_{\pm(e_i - e_j)} &= \mathbb{R}(E_{2i-1,2j-1} + E_{2i,2j} - E_{2j-1,2i-1} - E_{2j,2i}) \\ &\quad + \mathbb{R}(E_{2i-1,2j} - E_{2i,2j-1} + E_{2j-1,2i} - E_{2j,2i-1}), \quad \text{and} \\ \mathfrak{g}_{\pm(e_i + e_j)} &= \mathbb{R}(E_{2i-1,2j-1} - E_{2i,2j} - E_{2j-1,2i-1} + E_{2j,2i}) \\ &\quad + \mathbb{R}(E_{2i-1,2j} + E_{2i,2j-1} - E_{2j-1,2i} - E_{2j,2i-1}). \end{aligned}$$

(5) The case $\mathfrak{g} = \mathfrak{e}_6$. The Cartan subalgebra $\mathfrak{t}$ can be isometrically identified with $\mathbb{R}^6$ with the standard orthonormal basis $\{e_1, \dots, e_6\}$. The root system is

$$\begin{aligned} &\{\pm e_i \pm e_j \mid 1 \leqq i < j \leqq 5\} \\ &\qquad \cup \{\pm \tfrac{1}{2}e_1 \pm \cdots \pm \tfrac{1}{2}e_5 \pm \tfrac{\sqrt{3}}{2}e_6 \text{ with an odd number of +'s}\}. \end{aligned} \tag{2.9}$$

It contains a root system of type $\mathfrak{d}_5$.

(6) The case $\mathfrak{g} = \mathfrak{e}_7$. The Cartan subalgebra can be isometrically identified with $\mathbb{R}^7$ with the standard orthonormal basis $\{e_1, \ldots, e_7\}$. The root system is

$$\begin{aligned}&\{\pm e_i \pm e_j \mid 1 \leqq i < j < 7\} \cup \{\pm\sqrt{2}e_7; \tfrac{1}{2}(\pm e_1 \pm \cdots \pm e_6 \pm \sqrt{2}e_7)\\ &\qquad \text{with an odd number of plus signs among the first six coefficients}\}.\end{aligned} \tag{2.10}$$

It contains a root system of $\mathfrak{d}_6$.

(7) The case $\mathfrak{g} = \mathfrak{e}_8$. The Cartan subalgebra can be isometrically identified with $\mathbb{R}^8$ with the standard orthonormal basis $\{e_1, \ldots, e_8\}$. The root system Δ is

$$\begin{aligned}&\{\pm e_i \pm e_j \mid 1 \leqq i < j \leqq 8\}\\ &\qquad \cup \{\tfrac{1}{2}(\pm e_1 \pm \cdots \pm e_8) \text{ with an even number of +'s}\}.\end{aligned} \tag{2.11}$$

It contains a root system of $\mathfrak{d}_8$.

(8) The case $\mathfrak{g} = \mathfrak{f}_4$. The Cartan subalgebra is isometrically identified with $\mathbb{R}^4$ with the standard orthonormal basis $\{e_1, \ldots, e_4\}$. The root system is

$$\{\pm e_i \mid 1 \leqq i \leqq 4\} \cup \{\pm e_i \pm e_j \mid 1 \leqq i < j \leqq 4\} \cup \{\tfrac{1}{2}(\pm e_1 \pm \cdots \pm e_4)\}. \tag{2.12}$$

It contains the root system of $\mathfrak{b}_4$.

(9) The case $\mathfrak{g} = \mathfrak{g}_2$. The Cartan subalgebra is isometrically identified with $\mathbb{R}^2$ with the standard orthonormal basis $\{e_1, e_2\}$. The root system Δ is

$$\{(\pm\sqrt{3}, 0), (\pm\tfrac{\sqrt{3}}{2}, \pm\tfrac{3}{2}), (0, \pm 1), (\pm\tfrac{\sqrt{3}}{2}, \pm\tfrac{1}{2})\}. \tag{2.13}$$

There are many choices of the orthonormal basis $\{e_1, \ldots, e_n\}$ with respect to which the root systems have the same standard presentations as above, for example the ones obtained by applying elements of the Weyl group. In the classical cases this means any permutation of the e_i, with any number of sign changes $e_i \mapsto \pm e_i$ in cases $\mathfrak{b}$ and $\mathfrak{c}$, an even number of sign changes in case $\mathfrak{d}$. For type $\mathfrak{d}$ we can also use the outer automorphism and thus have $e_i \mapsto \pm e_i$ with any number of sign changes.

3. CK vector fields on compact normal homogeneous spaces

Assume $M = G/H$ is a Riemannian normal homogeneous space in which G is a compact connected simple Lie group, and H is a closed subgroup with $0 < \dim H < \dim G$. We keep all notation of the last section and further assume there is a nonzero vector $v \in \mathfrak{g}$ that defines a Clifford–Killing vector field on M. The value of v at $\pi(g)$, where $\pi\colon G \to G/H$ as usual, is $\pi_*|_g((L_g)_*(\mathrm{Ad}(g)v)) = g_*\pi_*|_e(\mathrm{Ad}(g)v)$. So the condition that v defines a nonzero CK vector field on M is that $\|\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(g)v)\|$ is a positive constant function of g. For the bi-invariant inner product,

$$\|v\|^2 = \|\mathrm{Ad}(g)v\|^2 = \|\mathrm{pr}_{\mathfrak{h}}(\mathrm{Ad}(g)v)\|^2 + \|\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(g)v)\|^2$$

is a constant function of $g \in G$, so the same is true for $\|\mathrm{pr}_{\mathfrak{h}}(\mathrm{Ad}(g)v)\|$. Suitably choosing v within its $\mathrm{Ad}(G)$-orbit, we can assume $v \in \mathfrak{t}$ (the standard special

Cartan subalgebra given in the last section). Now $\|\mathrm{pr}_{\mathfrak{h}}(\rho(v))\|$ and $\|\mathrm{pr}_{\mathfrak{m}}(\rho(v))\|$ are constant functions of ρ in the Weyl group. Because $\mathfrak{g}$ is simple and $v \neq 0$, both the functions $\|\mathrm{pr}_{\mathfrak{h}}(\mathrm{Ad}(g)(v))\|$ and $\|\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(g))(v)\|$ for $g \in G$, (or the functions $\|\mathrm{pr}_{\mathfrak{h}}(\rho(v))\|$ and $\|\mathrm{pr}_{\mathfrak{m}}(\rho(v))\|$ for ρ in the Weyl group) are positive constant functions. From the above observations, it is easy to prove two special cases of Theorem 1.1:

Proposition 3.1. *Let G be a compact connected simple Lie group and H a closed subgroup with $0 < \dim H < \dim G$. If $\mathfrak{g} = \mathfrak{a}_2$ or $\mathfrak{g}_2$ then there is no nonzero $v \in \mathfrak{g}$ that defines a CK vector field on the Riemannian normal homogeneous space G/H.*

Proof. Consider $\mathfrak{g} = \mathfrak{a}_2$ first. Assume conversely there is a nonzero CK vector field, defined by the nonzero vector $v \in \mathfrak{t}$. The subspaces $\mathfrak{t} \cap \mathfrak{h}$ and $\mathfrak{t} \cap \mathfrak{m}$ are a pair of orthogonal lines in $\mathfrak{t}$. Denote all different vectors in the Weyl group orbit of v as $v_1 = v, \ldots, v_k$, $k = 3$ or 6, then

$$\sum_{i=1}^{k} v_i = \sum_{i=1}^{k} \mathrm{pr}_{\mathfrak{h}}(v_i) = \sum_{i=1}^{k} \mathrm{pr}_{\mathfrak{m}}(v_i) = 0.$$

All the vectors $\mathrm{pr}_{\mathfrak{m}}(v_i)$ have the same nonzero length, which only have two possible choices in $\mathfrak{t} \cap \mathfrak{m}$. So we must have $k = 6$, and all v_i can be divided into two sets, such that, for example, $\mathrm{pr}_{\mathfrak{m}}(v_1) = \mathrm{pr}_{\mathfrak{m}}(v_2) = \mathrm{pr}_{\mathfrak{m}}(v_3)$ and $\mathrm{pr}_{\mathfrak{m}}(v_4) = \mathrm{pr}_{\mathfrak{m}}(v_5) = \mathrm{pr}_{\mathfrak{m}}(v_6)$ are opposite to each other. Obviously $v_1 + v_2 + v_3 \neq 0$, so there are two v_i among them, v_1 and v_2 for example, such that $v_1 = \rho(v_2)$, in which ρ is the reflection in some root of $\mathfrak{g}$. Thus $\mathfrak{t} \cap \mathfrak{h}$, containing $v_1 - v_2$, is linearly spanned by a root of $\mathfrak{g}$. Similar argument can also prove $\mathfrak{t} \cap \mathfrak{m}$ is spanned by a root of $\mathfrak{g}$. But for $\mathfrak{a}_2$, there do not exist a pair of orthogonal roots. This is a contradiction.

The Weyl group of $\mathfrak{g}_2$ contains that of $\mathfrak{a}_2$ as its subgroup, so the statement for $\mathfrak{g}_2$ also follows immediately the above argument. $\square$

To prove Theorem 1.1 in general we need some preparation. Suppose that $M = G/H$ is a Riemannian normal homogeneous space and $v \in \mathfrak{g}$ defines a CK vector field on M. If $\psi : G' \to G$ is a covering group and H' is an open subgroup of $\psi^{-1}(H)$, then $M' = G'/H'$ is a Riemannian normal homogeneous space and a Riemannian covering manifold of M, and the same $v \in \mathfrak{g}$ defines a CK vector field on M'. Thus we can always replace G by a covering group. Similarly we can go down to a certain class of subgroups:

Lemma 3.2. *Let $M = G/H$ be a Riemannian normal homogeneous space such that $v \in \mathfrak{g}$ defines a CK vector field on M. Let G' be a closed subgroup of G whose Lie algebra $\mathfrak{g}'$ satisfies $\mathfrak{g}' = \mathfrak{g}' \cap \mathfrak{h} + \mathfrak{g}' \cap \mathfrak{m}$. Let H' be a closed subgroup of G' with Lie algebra $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$. Then the restriction of the bi-invariant inner product of $\mathfrak{g}$ to $\mathfrak{g}'$ defines a Riemannian normal homogeneous metric on $M' = G'/H'$. If $v = v' + v''$ with $v' \in \mathfrak{g}'$, $\langle v'', \mathfrak{g}' \rangle = 0$, and $[v'', \mathfrak{g}'] = 0$, then v' defines a CK vector field on $M' = G'/H'$.*

Proof. Because of the decomposition $\mathfrak{g}' = (\mathfrak{g}' \cap \mathfrak{h}) + (\mathfrak{g}' \cap \mathfrak{m})$, we also have $\mathfrak{g}'^{\perp} = (\mathfrak{g}'^{\perp} \cap \mathfrak{h}) + (\mathfrak{g}'^{\perp} \cap \mathfrak{m})$. The condition that $v = v' + v''$ defines a CK vector field for the Riemannian normal homogeneous space $M = G/H$ implies that

$$\begin{aligned}\|\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(g')v)\|^2 &= \|\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(g')v') + \mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(g')v'')\|^2\\ &= \|\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(g')v') + \mathrm{pr}_{\mathfrak{m}}(v'')\|^2\\ &= \|\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(g')v')\|^2 + \|\mathrm{pr}_{\mathfrak{m}}(v'')\|^2\end{aligned}$$

is a constant function for $g' \in G'$. And so does $\|\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(g')v')\|^2$, i.e., v' defines a CK vector field for the Riemannian normal homogeneous space $M' = G'/H'$. □

We will frequently use Lemma 3.2 to reduce our considerations to smaller groups.

Lemma 3.3. *Suppose that $0 \neq v \in \mathfrak{t}$ defines a CK vector field on $M = G/H$. Then*

(1) *If the Weyl group orbit $W(v)$ contains an orthogonal basis of $\mathfrak{t}$ then*

$$\frac{\|\mathrm{pr}_{\mathfrak{h}}(v)\|^2}{\|v\|^2} = \frac{\dim(\mathfrak{t}\cap\mathfrak{h})}{\dim\mathfrak{t}}.$$

(2) *If $\mathrm{Ad}(G)v$ contains an orthogonal basis of $\mathfrak{g}$ then*

$$\frac{\|\mathrm{pr}_{\mathfrak{h}}(v)\|^2}{\|v\|^2} = \frac{\dim\mathfrak{h}}{\dim\mathfrak{g}}.$$

Proof. (1) For simplicity, we assume $\|v\| = 1$. Let $\{v_1, \ldots, v_n\} \subset W(v)$ be an orthogonal basis of $\mathfrak{t}$, and $\{u_1, \ldots, u_h\}$ an orthonormal basis of $\mathfrak{t}\cap\mathfrak{h}$. Expand $u_i = \sum_{j=1}^n a_{ij}v_j$; then $\mathrm{pr}_{\mathfrak{h}}(v_i) = \sum_{j=1}^h a_{ij}u_j$ and $\|\mathrm{pr}_{\mathfrak{h}}(v_i)\|^2 = \sum_{j=1}^h a_{ij}^2$. So

$$\frac{\|\mathrm{pr}_{\mathfrak{h}}(v)\|^2}{\|v\|^2} = \frac{1}{n}\left(\sum_{i=1}^n \|\mathrm{pr}_{\mathfrak{h}}(v_i)\|^2\right) = \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^h \|a_{ij}\|^2 = \frac{1}{n}\sum_{j=1}^h\sum_{i=1}^h \|a_{ij}\|^2 = \frac{h}{n}.$$

That proves the first assertion. The proof of the second is similar. □

Lemma 3.3 provides a useful tool when we deal with the cases $\mathfrak{g} = \mathfrak{b}_n$ and $\mathfrak{g} = \mathfrak{d}_n$.

Lemma 3.4. *Suppose that $0 \neq v \in \mathfrak{t}$ defines a CK vector field on $M = G/H$. Let $\alpha, \beta \in \Delta(\mathfrak{g}, \mathfrak{t})$ such that $\langle\alpha, \beta\rangle = 0$ and $\alpha(v) \neq 0 \neq \beta(v)$. Then*

$$\langle\mathrm{pr}_{\mathfrak{h}}(\alpha), \mathrm{pr}_{\mathfrak{h}}(\beta)\rangle = \langle\mathrm{pr}_{\mathfrak{m}}(\alpha), \mathrm{pr}_{\mathfrak{m}}(\beta)\rangle = 0.$$

Proof. Let the reflections for the roots α and β be denoted ρ_α and ρ_β respectively. Then the four points

$$\begin{aligned}v_1 &= \mathrm{pr}_{\mathfrak{h}}v,\\ v_2 &= \mathrm{pr}_{\mathfrak{h}}(\rho_\alpha(v)) = \mathrm{pr}_{\mathfrak{h}}(v) - \frac{2\langle v, \alpha\rangle}{\langle\alpha, \alpha\rangle}\mathrm{pr}_{\mathfrak{h}}(\alpha),\\ v_3 &= \mathrm{pr}_{\mathfrak{h}}(\rho_\beta(v)) = \mathrm{pr}_{\mathfrak{h}}(v) - \frac{2\langle v, \beta\rangle}{\langle\beta, \beta\rangle}\mathrm{pr}_{\mathfrak{h}}(\alpha), \quad \text{and}\\ v_4 &= \mathrm{pr}_{\mathfrak{h}}(\rho_\beta\rho_\alpha(v)) = \mathrm{pr}_{\mathfrak{h}}(v) - \frac{2\langle v, \alpha\rangle}{\langle\alpha, \alpha\rangle}\mathrm{pr}_{\mathfrak{h}}(\alpha) - \frac{2\langle v, \beta\rangle}{\langle\beta, \beta\rangle}\mathrm{pr}_{\mathfrak{h}}(\alpha)\end{aligned}$$

belong to a two-dimensional plane and have the same distance from 0. They are the vertices of a rectangle with adjacent edges parallel to $\mathrm{pr}_{\mathfrak{h}}(\alpha)$ and $\mathrm{pr}_{\mathfrak{h}}(\beta)$ respectively. Those edges are orthogonal, in other words $\langle \mathrm{pr}_{\mathfrak{h}}(\alpha), \mathrm{pr}_{\mathfrak{h}}(\beta)\rangle = 0$. The other statement, $\langle \mathrm{pr}_{\mathfrak{m}}(\alpha), \mathrm{pr}_{\mathfrak{m}}(\beta)\rangle = 0$, follows immediately. □

Lemma 3.4 is the key to our study of the CK vector fields on the Cartan subalgebra level. The next proposition implies, at least for classical $\mathfrak{g}$, that a nonzero vector v which defines a CK vector field must be very singular.

Proposition 3.5. *Suppose that $\mathfrak{g}$ is classical, i.e., $\mathfrak{g} = \mathfrak{a}_n$ for $n > 0$, $\mathfrak{b}_n$ for $n > 1$, $\mathfrak{c}_n$ for $n > 2$ or $\mathfrak{d}_n$ for $n > 3$. Suppose that $0 \neq v \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space $M = G/H$. Use the standard presentations for the Cartan subalgebra $\mathfrak{t}$ and root system Δ in* (2.5)–(2.8)*. Then, for a suitable choice of the e_i, v must be one of the following, up to multiplication by a nonzero scalar.*

(1) *Let $\mathfrak{g} = \mathfrak{a}_n$ with $n > 2$. Then $v = ne_1 - e_2 - \ldots - e_{n+1}$, or (if n is odd) $v = e_1 + \cdots + e_k - e_{k+1} - \cdots - e_{n+1}$ for $n = 2k - 1$.*
(2) *Let $\mathfrak{g} = \mathfrak{b}_n$ or $\mathfrak{c}_n$ with $n > 1$, or let $\mathfrak{g} = \mathfrak{d}_n$ with $n > 3$. Then $v = e_1$ or $v = e_1 + \cdots + e_n$.*

Proof. (1) Assume $\mathfrak{g} = \mathfrak{a}_n$ with $n > 2$. If the Weyl group orbit of v contains a multiple of $ne_1 - e_2 - \cdots - e_{n+1}$, then Assertion (1) is proved. Now suppose that the Weyl group orbit of v does not contain a multiple of $ne_1 - e_2 - \cdots - e_{n+1}$. Then for any orthogonal pair of roots, $\alpha = e_i - e_j$ and $\beta = e_k - e_l$ with i, j, k and l distinct, we can replace v by a Weyl group conjugate and assume $v = a_1e_1 + \cdots + a_{n+1}e_{n+1}$ where $a_i \neq a_j$ and $a_k \neq a_l$. Applying Lemma 3.4, we have

$$0 = \langle e_i - e_j, e_k - e_l\rangle = \langle \mathrm{pr}_{\mathfrak{h}}(e_i - e_j), \quad \mathrm{pr}_{\mathfrak{h}}(e_k - e_l)\rangle = \langle \mathrm{pr}_{\mathfrak{h}}(e_i - e_j), e_k - e_l\rangle \tag{3.6}$$

for any k and l such that i, j, k and l are distinct. Express $\mathrm{pr}_{\mathfrak{h}}(e_i - e_j) = \sum r_m e_m$. Hold i and j fixed, and let k and ℓ vary over $\{1, \ldots, n+1\} \setminus \{i, j\}$. Then (3.6) shows that all such $r_k = r_\ell$. Thus we have constants a and b such that

$$\mathrm{pr}_{\mathfrak{h}}(e_i - e_j) = ae_i + be_j + \frac{-a-b}{n-1}(e_1 + \cdots + e_{n+1} - e_i - e_j).$$

Similarly,

$$\mathrm{pr}_{\mathfrak{h}}(e_k - e_l) = ce_3 + de_4 + \frac{-c-d}{n-1}(e_1 + \cdots + e_{n+1} - e_k - e_l).$$

Now (3.6) tells us

$$\langle \mathrm{pr}_{\mathfrak{h}}(e_i - e_j), \mathrm{pr}_{\mathfrak{h}}(e_k - e_l)\rangle = \left(\frac{n-3}{(n-1)^2} - \frac{2}{n-1}\right)(a+b)(c+d) = 0, \tag{3.7}$$

i.e., either $a + b = 0$ or $c + d = 0$. If $a + b = 0$ then $\mathrm{pr}_{\mathfrak{h}}(e_i - e_j)$ is a multiple of $e_i - e_j$, and if $c + d = 0$ then $\mathrm{pr}_{\mathfrak{h}}(e_k - e_l)$ is a multiple of $e_k - e_l$. If $a + b = 0$, so $\mathrm{pr}_{\mathfrak{h}}(e_i - e_j) = r(e_i - e_j)$, then $\mathrm{pr}_{\mathfrak{h}}^2(e_i - e_j) = \mathrm{pr}_{\mathfrak{h}}(e_i - e_j)$ so $r^2 = r$; either

$r = 0$ and $e_i - e_j \in \mathfrak{m}$ or $r = 1$ and $e_i - e_j \in \mathfrak{h}$. Similarly if $c + d = 0$ then either $e_k - e_l \in \mathfrak{m}$ or $e_k - e_l \in \mathfrak{h}$. So if there is a root α contained neither in $\mathfrak{h}$ nor in $\mathfrak{m}$, then any roots orthogonal to it are contained either in $\mathfrak{h}$ or in $\mathfrak{m}$.

Suppose that there is a root α contained neither in $\mathfrak{h}$ nor in $\mathfrak{m}$. Applying a Weyl group element we may assume $\alpha = e_1 - e_2$. Then any root $e_i - e_j$, $2 < i < j \leqq n+1$, is contained in $\mathfrak{h}$ or $\mathfrak{m}$, and all such roots must be contained in the same subspace. Suppose they all belong to $\mathfrak{h}$; the argument will be the same if they all belong to $\mathfrak{m}$. Now $e_3 - e_4 \in \mathfrak{h}$ and $\langle e_1 - e_3, e_3 - e_4\rangle \neq 0$ shows $e_1 - e_3 \notin \mathfrak{m}$. If $e_1 - e_3 \notin \mathfrak{h}$, then by the above argument, $e_2 - e_4 \in \mathfrak{h}$. Suitably permuting the e_i, we see $e_i - e_j \in \mathfrak{h}$ for $1 < i < j \leqq n+1$, so $\mathfrak{m} = \mathbb{R}(ne_1 - e_2 - \cdots - e_{n+1})$. Recall $v = a_1e_1 + \cdots + a_{n+1}e_{n+1}$ with $\sum a_i = 0$. All $\mathrm{pr}_{\mathfrak{m}}(\rho(v))$ have the same length for any ρ in the Weyl group, in other words $|a_1n - a_2 - \cdots - a_{n+1}| = (n+1)|a_1|$ is constant under permutations of the a_i. Thus n is odd, say $n = 2k-1$, and after a suitable permutation of the e_i, v is a scalar multiple of $(e_1 + \cdots + e_k) - (e_{k+1} + \cdots + e_{n+1})$.

On the other hand, suppose that every root α is contained in either $\mathfrak{h}$ or $\mathfrak{m}$, say $\Delta = \Delta_{\mathfrak{h}} \cup \Delta_{\mathfrak{m}}$. If $\Delta_{\mathfrak{m}} = \emptyset$ then $\mathfrak{t} \subset \mathfrak{h}$ and the CK vector field v has a zero, forcing $v = 0$, which contradicts the hypothesis $v \neq 0$. If $\Delta_{\mathfrak{h}} = \emptyset$ then $\mathfrak{t} \subset \mathfrak{m}$, contradicting our construction of $\mathfrak{t}$, which starts with a Cartan subalgebra of $\mathfrak{h}$. Thus $\Delta_{\mathfrak{h}} \neq \emptyset$ and $\Delta_{\mathfrak{m}} \neq \emptyset$. Because $\langle \Delta_{\mathfrak{h}}, \Delta_{\mathfrak{m}}\rangle = 0$, this is a contradiction with the fact that $\mathfrak{g}$ is simple. That completes the proof of Assertion (1).

(2) Assume $\mathfrak{g} = \mathfrak{b}_n$ with $n > 1$. If the Weyl group orbit $W(v)$ of v contains a multiple of e_1 then Assertion (2) is proved for $\mathfrak{g} = \mathfrak{b}_n$. Now suppose that $W(v)$ does not contain a multiple of e_1. Express $v = a_1e_1 + \cdots + a_ne_n$; then at least two of the coefficients a_i are nonzero. Any two short roots, e_i and e_j with $i \neq j$, are orthogonal to each other, so we can suitably choose v from its Weyl group orbit such that $a_i \neq 0$ and $a_j \neq 0$. Applying Lemma 3.4, we see

$$\langle \mathrm{pr}_{\mathfrak{h}}(e_i), \mathrm{pr}_{\mathfrak{h}}(e_j)\rangle = \langle \mathrm{pr}_{\mathfrak{m}}(e_i), \mathrm{pr}_{\mathfrak{m}}(e_j)\rangle = 0 \quad \text{whenever } i \neq j.$$

These can only be true when each e_i is contained in either $\mathfrak{t} \cap \mathfrak{h}$ or $\mathfrak{t} \cap \mathfrak{m}$. Then, suitably permutting the e_i, we can assume that $\{e_1, \ldots, e_h\}$ spans $\mathfrak{t} \cap \mathfrak{h}$, and $\{e_{h+1}, \ldots, e_n\}$ spans $\mathfrak{t} \cap \mathfrak{m}$. All $\mathrm{pr}_{\mathfrak{h}}(\rho(v))$, ρ in the Weyl group, have the same length. In particular, $v = a_1e_1 + \cdots + a_ne_n$ must satisfy $|a_1| = \cdots = |a_n|$. By suitable scalar changes and Weyl group actions, we have $v = e_1 + \cdots + e_n$. That completes the proof of Assertion (2) for $\mathfrak{g} = \mathfrak{b}_n$. The proof for $\mathfrak{g} = \mathfrak{c}_n$ is similar.

Now assume $\mathfrak{g} = \mathfrak{d}_n$ with $n > 3$. The root system of $\mathfrak{d}_n$ contains two subsystems of type $\mathfrak{a}_{n-1}$ whose intersection is of type $\mathfrak{a}_{n-2}$. If the Weyl group orbit $W(v)$ contains a scalar multiple of e_1 or of e_1 or $e_1 + \cdots + e_n$ then Assertion (2) follows. If it contains a scalar multiple of $e_1 + \cdots + e_{n-1} - e_n$ we apply the outer automorphism that restricts to $e_1 + \cdots + e_{n-1} - e_n \mapsto e_1 + \cdots + e_{n-1} + e_n$, and Assertion (2) follows. Now suppose that neither of these holds: $W(v)$ contains neither a multiple of e_1 nor a multiple of $e_1 + \cdots + e_{n-1} \pm e_n$. Then $v = a_1e_1 + \cdots + a_ne_n$ has two nonzero coefficients and not all the $|a_i|$ are equal. If i, j, k and l are distinct we have $v' = a'_1e_1 + \cdots + a'_ne_n \in W(v)$ such that $a'_i \neq \pm a'_j$ and $a'_k \neq a'_l$, and $v'' = a''_1e_1 + \cdots + a''_ne_n \in W(v)$ such that $a''_i \neq \pm a''_j$ and $a''_k \neq -a''_l$. Applying Lemma 3.4 to $\alpha = e_i \pm e_j$ and $\beta = e_k \pm e_l$, or $\alpha = e_i + e_j$ and $\beta = e_i - e_j$, the result is

$$\langle \mathrm{pr}_{\mathfrak{h}}(e_i \pm e_j), \mathrm{pr}_{\mathfrak{h}}(e_k \pm e_l)\rangle = \langle \mathrm{pr}_{\mathfrak{m}}(e_i \pm e_j), \mathrm{pr}_{\mathfrak{m}}(e_k \pm e_l)\rangle = 0$$

and

$$\langle \mathrm{pr}_{\mathfrak{h}}(e_i + e_j), \mathrm{pr}_{\mathfrak{h}}(e_i - e_j)\rangle = \langle \mathrm{pr}_{\mathfrak{m}}(e_i + e_j), \mathrm{pr}_{\mathfrak{m}}(e_i - e_j)\rangle = 0,$$

whenever i, j, k and l are distinct. This is only possible when each one of $\pm e_i \pm e_j$ is contained in $\mathfrak{t} \cap \mathfrak{h}$ or $\mathfrak{t} \cap \mathfrak{m}$. By an argument similar to that used to prove (1), we have either $\mathfrak{t} \subset \mathfrak{h}$ and the Riemannian normal homogeneous space $M = G/H$ has no nonzero CK vector field, or $\mathfrak{t} \cap \mathfrak{h} = 0$ which contradicts our construction of $\mathfrak{t}$. $\square$

Proposition 3.5 is the key step in the proof of Theorem 1.1. It reduces our discussion for each classical $\mathfrak{g}$ to very few possibilities for the vector v. In the next section, we will apply this proposition to each exceptional $\mathfrak{g}$ and show there does not exist any nonzero CK vector field in those cases.

4. Proof of Theorem 1.1 for $\mathfrak{g}$ exceptional

In this section, we will apply Proposition 3.5 to prove Theorem 1.1 when $\mathfrak{g}$ is a compact exceptional simple Lie algebra. The proof is a case by case discussion.

(1) The case $\mathfrak{g} = \mathfrak{g}_2$ has already been proven in Proposition 3.1.

(2) Let $\mathfrak{g} = \mathfrak{f}_4$. We use the standard presentation (2.12) for its root system. Its root system has a subsystem of type $\mathfrak{b}_4$, which defines a subgroup W' of the Weyl group W. By the argument for the case of $\mathfrak{b}_n$ in Proposition 3.5, if $0 \neq v \in \mathfrak{g}$ defines a CK vector field on M, we can re-scale it and use the W' action and assume that either $v = e_1$ or $v = \frac{1}{2}(e_1 + \cdots + e_4)$. But those belong to the same orbit for W. Considering $v = \frac{1}{2}(e_1 + \cdots + e_4)$, it follows that each of $\mathfrak{t} \cap \mathfrak{h}$ and $\mathfrak{t} \cap \mathfrak{m}$ is linearly spanned by a non-empty subset of $\{e_1, \ldots, e_4\}$. Then use $v = e_1$ in the same Weyl group orbit; $\|\mathrm{pr}_{\mathfrak{m}}(\rho(v))\|$ varies with $\rho \in W'$, contradicting the CK property of v. We conclude that if $\mathfrak{g} = \mathfrak{f}_4$ then $M = G/H$ has no nonzero CK vector field.

(3) Let $\mathfrak{g} = \mathfrak{e}_6$. We use the standard presentation (2.9) for its root system. Its root system has a subsystem of type $\mathfrak{d}_5$, which defines subgroup W' of the Weyl group W of $\mathfrak{g}$. Suppose that $0 \neq v \in \mathfrak{g}$ defines a CK vector field on the Riemannian normal homogeneous space $M = G/H$. Using the reflections for roots of the form $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_5 \pm \sqrt{3} e_6)$, we can assume that $v = a_1 e_1 + \cdots + a_6 e_6$ has three nonzero coefficients among the first five a_i. By the argument in the $\mathfrak{d}_n$ case of Proposition 3.5, if $1 \leqq i < j \leqq 5$ then $\mathrm{pr}_{\mathfrak{h}}(e_i \pm e_j)$ and $\mathrm{pr}_{\mathfrak{m}}(e_i \pm e_j)$ are four orthogonal vectors in $\mathbb{R}e_i + \mathbb{R}e_j + \mathbb{R}e_6$. So if $1 \leqq i < j \leqq 5$ then either $e_i + e_j$ or $e_i - e_j$ is contained in $\mathfrak{t} \cap \mathfrak{h}$ or in $\mathfrak{t} \cap \mathfrak{m}$, and all those roots define the same subspace. We will argue the case where they are in $\mathfrak{t} \cap \mathfrak{h}$; with very minor modifications our argument also works when they are in $\mathfrak{t} \cap \mathfrak{m}$. In that case there are two possibilities: (1) there exist i and j with $1 \leqq i < j < 6$ and both $\pm e_i \pm e_j$ contained in $\mathfrak{t} \cap \mathfrak{h}$, or (2) whenever $1 \leqq i < j < 6$ either $e_i - e_j$ or $e_1 + e_j$ is not contained in $\mathfrak{t} \cap (\mathfrak{h} \cup \mathfrak{m})$.

Suppose that whenever $1 \leqq i < j < 6$ either $e_i - e_j$ or $e_i + e_j$ is not contained in $\mathfrak{t} \cap (\mathfrak{h} \cup \mathfrak{m})$. By suitably choosing the first five e_i, in the $\mathfrak{d}_5$ where W' acts, we can assume $e_i - e_j \in \mathfrak{h}$ and $e_i + e_j \notin \mathfrak{h} \cup \mathfrak{m}$ for $1 \leqq i < j < 5$. Let ρ be the reflection in the root $\frac{1}{2}(e_1 - e_2 + e_3 + e_4 + e_5 + \sqrt{3}e_6)$. Denote $e'_i = \rho(e_i)$. Apply the above argument to the new basis $\{e'_1, \cdots, e'_6\}$. Then for $1 \leqq i < j < 6$, either $e'_i + e'_j$ or $e'_i - e'_j$ is not contained in $\mathfrak{t} \cap (\mathfrak{h} \cup \mathfrak{m})$. Because $e'_1 + e'_2 = e_1 + e_2$ is not contained in $\mathfrak{t} \cap (\mathfrak{h} \cup \mathfrak{m})$, and because $e'_1 - e'_2 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - \sqrt{3}e_6)$ is not orthogonal to $e_1 - e_2 \in \mathfrak{t} \cap \mathfrak{h}$, we have

$$\tfrac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - \sqrt{3}e_6) \in \mathfrak{t} \cap \mathfrak{h}. \tag{4.1}$$

If we use the reflection in the root $\frac{1}{2}(e_1 - e_2 - e_3 - e_4 + e_5 + \sqrt{3}e_6)$, the above argument shows

$$\tfrac{1}{2}(e_1 - e_2 + e_3 + e_4 - e_5 - \sqrt{3}e_6) \in \mathfrak{h}. \tag{4.2}$$

By (4.1) and (4.2), $e_3 + e_4 \in \mathfrak{t} \cap \mathfrak{h}$, which contradicts our assumption. Thus there exist i and j such that $1 \leqq i < j < 6$ and the $\pm e_i \pm e_j$ are contained in $\mathfrak{t} \cap \mathfrak{h}$, Then $\mathfrak{h} = \mathbb{R}e_1 + \cdots + \mathbb{R}e_5$ and $\mathfrak{m} = \mathbb{R}e_6$.

Let ρ be the reflection in a root of the form $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm \sqrt{3}e_6)$, and denote $e'_i = \rho(e_i)$ for $1 \leqq i \leqq 6$. The e'_i are another orthonormal basis of $\mathfrak{t}$ for which the root system Δ is given by (2.9). In particular, the $\pm e'_i \pm e'_j$, $1 \leqq i < j < 6$ give a root system of $\mathfrak{d}_5$ in Δ. The above argument also implies $\mathfrak{m} = \mathbb{R}e'_6$. But $\mathbb{R}e'_6 \neq \mathbb{R}e_6$. This is a contradiction. We conclude that there is no nonzero vector v that defines a CK vector field for the Riemannian normal homogeneous space G/H.

(3) Use the standard presentation (2.10) for the root system of $\mathfrak{e}_7$, and apply an argument similar to the one above for $\mathfrak{e}_6$. Arguing mutatis mutandis we see that, when $\mathfrak{g} = \mathfrak{e}_7$, there is no nonzero vector v that defines a CK vector field for the Riemannian normal homogeneous space G/H.

(4) Let $\mathfrak{g} = \mathfrak{e}_8$. We use the standard presentation (2.11) for its root system. Its root system contains a root system of type $\mathfrak{d}_8$, and the Weyl group W' of that $\mathfrak{d}_8$ is of course a subgroup of the Weyl group W of $\mathfrak{g}$. Suppose that a nonzero vector $v \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space G/H. The argument for the case of $\mathfrak{d}_n$ for Proposition 3.5 can be applied here to show that, up to scalar multiplications and the action of W', either $v = e_1$ or $v = e_1 + \cdots + e_7 \pm e_8$. In either case, the reflection in the root $\frac{1}{2}(e_1 + e_2 - e_3 - \cdots - e_8)$ maps v to another $v' = a_1 e_1 + \cdots + a_8 e_8$ such that there are at least two nonzero a_is and not all $|a_i|$s are the same. We have shown $\|\mathrm{pr}_{\mathfrak{h}}(\rho(v))\|$ is not a constant function for all $\rho \in W'$ in the proof of Proposition 3.5. This contradicts our assumption on v. So there is no nonzero $v \in \mathfrak{g}$ defining a CK vector field on $M = G/H$.

In summary, we have proved

Proposition 4.3. *Let G be a compact connected exceptional simple Lie group, and H a closed subgroup with $0 < \dim H < \dim G$. Then there is no nonzero vector $v \in \mathfrak{g}$ that defines a CK vector field on the Riemannian normal homogeneous space $M = G/H$.*

5. Proof of Theorem 1.1 for $\mathfrak{g} = \mathfrak{a}_n$

In this section $\mathfrak{g} = \mathfrak{a}_n = \mathfrak{su}(n+1)$ and $0 \neq v \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space $M = G/H$. Proposition 3.5 says that, up to the action of the Weyl group, either v is a multiple of $ne_1 - e_2 - \cdots - e_{n+1}$ or $n+1 = 2k$ and v is a multiple of $(e_1 + \cdots + e_k) - (e_{k+1} + \cdots + e_{n+1})$. However, we must see whether those vectors v actually define CK vector fields. The case $n = 1$ is trivial. If there are nonzero CK vectors fields, then $\dim H = 0$. The case $n = 2$ has been proven in Proposition 3.1. So we assume $n > 2$.

5A. The case $n = 2k-1$ is odd and $v = (e_1+\cdots+e_k)-(e_{k+1}+\cdots+e_{n+1})$

From the argument in Proposition 3.5, either $\mathfrak{t} \cap \mathfrak{h} = \mathbb{R}(ne_1 - e_2 - \cdots - e_{n+1})$, or $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(ne_1 - e_2 - \cdots - e_{n+1})$, up to the action of the Weyl group.

Suppose $\mathfrak{t}\cap\mathfrak{h} = \mathbb{R}(ne_1-e_2-\cdots-e_{n+1})$. Then either $\mathfrak{h} = \mathbb{R}(ne_1-e_2-\cdots-e_{n+1})$ or $\mathfrak{h}$ is the $\mathfrak{a}_1$ with Cartan subalgebra $\mathfrak{h}\cap\mathfrak{t} = \mathbb{R}(ne_1-e_2-\cdots-e_{n+1})$. In the $\mathfrak{a}_1$ case the root plane $\mathfrak{j}$ of $\mathfrak{h}$ relative to $\mathfrak{t} \cap \mathfrak{h}$ is contained in $\mathfrak{g}_{\pm(e_1-e_2)} + \cdots + \mathfrak{g}_{\pm(e_1-e_{n+1})}$. Direct calculation shows $[\mathfrak{j},\mathfrak{j}] \not\subset \mathfrak{t}\cap\mathfrak{h}+\mathfrak{j}$ so $\mathfrak{t}\cap\mathfrak{h}+\mathfrak{j}$ is not a Lie algebra. This eliminates the $\mathfrak{a}_1$ case. Thus $\mathfrak{h} = \mathbb{R}(ne_1-e_2-\cdots-e_{n+1})$. Let $g \in G$ with $\mathrm{Ad}\,(g)v = \sqrt{-1}\,\mathrm{diag}(\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), 1, -1, \ldots, 1, -1)$. Then $\langle \mathrm{Ad}\,(g)v, ne_1 - e_2 - \cdots - e_{n+1}\rangle = 0$ while $\langle v, ne_1-e_2-\cdots-e_{n+1}\rangle = n+1$, so v cannot define a CK vector field on $M = G/H$. We have proved $\mathfrak{t}\cap\mathfrak{h} \neq \mathbb{R}(ne_1 - e_2 - \cdots - e_{n+1})$, so, up to the action of the Weyl group, we assume $\mathfrak{t}\cap\mathfrak{m} = \mathbb{R}(ne_1 - e_2 - \cdots - e_{n+1})$.

If $0 \neq \gamma \in \mathfrak{t}\cap\mathfrak{h}$ then $\gamma \perp e_1$ so $\dim \widehat{\mathfrak{g}}_{\pm\gamma} = 0$ or 2. The root planes of $\mathfrak{h}$ are root planes of $\mathfrak{g}$ for roots orthogonal to e_1. Consider one such, $\mathfrak{g}_{\pm(e_i-e_j)}$, where $1 < i < j \leqq n+1$, which is not a root plane of $\mathfrak{h}$. Then it is contained in $\mathfrak{m}$. Permuting the e_l we may assume $i = 3$ and $j = 4$. Then we have

$$v' = \sqrt{-1}\,\mathrm{diag}(1, -1, \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), 1, -1, \ldots, 1, -1) \in \mathrm{Ad}(G)v.$$

But $\|\mathrm{pr}_{\mathfrak{m}}(v')\| > \|\mathrm{pr}_{\mathfrak{m}}(v)\|$, which is a contradiction. This proves $\mathfrak{g}_{\pm(e_i-e_j)} \in \mathfrak{h}$ for $1 < i < j \leqq n+1$. The other root planes of $\mathfrak{g}$ involve e_1 in the root, so they are all contained in $\mathfrak{m}$. In conclusion, $\mathfrak{h}$ is a standard $\mathfrak{su}(n)$ in $\mathfrak{g} = \mathfrak{su}(n+1)$, and the universal cover of $M = G/H$ is $S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ with $n > 1$.

Remark. The vector $v = (e_1 + \cdots + e_k) - (e_{k+1} + \cdots + e_{n+1})$ defines a CK vector field on the sphere $S^{2n+1} = \mathrm{SO}(2n+2)/\mathrm{SO}(2n+1)$. However, the Riemannian normal homogeneous metric on $S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ is not Riemannian symmetric. The isotropy representation for $S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ decomposes $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$, in which $\dim \mathfrak{m}_0 = 1$ with trivial $\mathrm{Ad}(H)$-action. Let $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ be the inner product on $\mathfrak{m}$ which defines the Riemannian symmetric metric on S^{2n+1}. The decomposition $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$ is orthogonal for both $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$. By a suitable scalar change, $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ coincides with $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}_1$, and differs on $\mathfrak{m}_0$. If the same $v = (e_1 + \cdots + e_k) - (e_{k+1} + \cdots + e_{n+1})$ defines a CK vector field on $S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ for the Riemannian normal homogeneous metric, then by the general observations at the beginning of Section 3, $\mathrm{Ad}(G)v$ is contained in a hyperplane in $\mathfrak{g}$ which is parallel to $\mathfrak{h} + \mathfrak{m}_1$. That would contradict the fact $\mathfrak{g}$ is simple and v is nonzero.

5B. The case $v = ne_1 - e_2 - \cdots - e_{n+1}$

Suppose that some $\mathfrak{g}_{\pm(e_i-e_j)} \subset \mathfrak{m}$. We may assume $i = 1$ and $j = 2$. Then we have

$$v' = -e_1 + ne_2 - e_3 - \cdots - e_{n+1} \in \mathrm{Ad}(G)v$$

and

$$v'' = \sqrt{-1}\,\mathrm{diag}\left(\begin{pmatrix} (n-1)/2 & (n+1)/2 \\ (n+1)/2 & (n-1)/2 \end{pmatrix}, -1, \ldots, -1\right) \in \mathrm{Ad}(G)v.$$

Note $\|\mathrm{pr}_{\mathfrak{h}}(v)\| = \|\mathrm{pr}_{\mathfrak{h}}(v')\| = \|\mathrm{pr}_{\mathfrak{h}}(v'')\|$. By construction, $\mathrm{pr}_{\mathfrak{h}}(v) + \mathrm{pr}_{\mathfrak{h}}(v') = 2\mathrm{pr}_{\mathfrak{h}}(v'')$, so $\mathrm{pr}_{\mathfrak{h}}(v) = \mathrm{pr}_{\mathfrak{h}}(v')$, i.e., $e_1 - e_2 \in \mathfrak{t}\cap\mathfrak{m}$. From this argument $\mathfrak{g}_{\pm(e_i-e_j)} \subset \mathfrak{m}$ only when $e_i - e_j \in \mathfrak{t}\cap\mathfrak{m}$. In particular, for any root $e_i - e_j \notin \mathfrak{t}\cap\mathfrak{m}$, $\mathrm{pr}_{\mathfrak{h}}(e_i - e_j)$ is a root of $\mathfrak{h}$. Similarly, $\mathfrak{g}_{\pm(e_i-e_j)} \subset \mathfrak{h}$ only when $e_i - e_j \in \mathfrak{t}\cap\mathfrak{h}$.

It will be convenient to assume $\mathrm{pr}_{\mathfrak{h}}(e_1+\cdots+e_{n+1}) = 0$, so that $\mathrm{pr}_{\mathfrak{h}}$ is defined on the Euclidean space $\mathbb{R}^{n+1}$ that contains $\mathfrak{t}$. Denote $e'_i = \mathrm{pr}_{\mathfrak{h}}(e_i)$ for $1 \leqq i \leqq n+1$. Then $e'_i - e'_j$ is 0 or a root of $\mathfrak{h}$. The e'_i generate all the roots of $\mathfrak{h}$. It follows that $\mathfrak{h}$ is a simple Lie algebra. Summarizing the above argument, we have

Lemma 5.1. *Assume $\mathfrak{g} = \mathfrak{a}_n$ for $n > 2$. Suppose that $v = ne_1 - e_2 - \cdots - e_{n+1} \in \mathfrak{t}$ defines a CK vector field on the normal homogeneous space $M = G/H$. Then*

1. *If $\mathfrak{g}_{\pm(e_i-e_j)} \subset \mathfrak{m}$, then $e_i - e_j \in \mathfrak{t}\cap\mathfrak{m}$.*
2. *If $\mathfrak{g}_{\pm(e_i-e_j)} \subset \mathfrak{h}$, then $e_i - e_j \in \mathfrak{t}\cap\mathfrak{h}$.*
3. *Denote $e'_i = \mathrm{pr}_{\mathfrak{h}}(e_i)$ for $1 \leqq i \leqq n+1$. Then the root system of $\mathfrak{h}$ is the set of all nonzero vectors of the form $e'_i - e'_j$. In particular, $\mathfrak{h}$ is a compact simple Lie algebra.*

Now consider the case $e'_i - e'_j = e'_k - e'_l \neq 0$, in which $e'_i \neq e'_k$ or $e'_j \neq e'_l$ (both inequalities are satisfied). As we saw, $e'_i - e'_j$ is a root of $\mathfrak{h}$, so $e'_j \neq e'_k$, for otherwise $e'_i - e'_l = 2(e'_i - e'_j)$ is a root of $\mathfrak{h}$; similarly $e'_i \neq e'_l$. Then for the roots $\alpha' = e'_i - e'_j$ and $\beta' = e'_i - e'_k$ of $\mathfrak{h}$, both $\alpha' + \beta' = e'_i - e'_l$ and $\alpha' - \beta' = e'_k - e'_j$ are roots of $\mathfrak{h}$. There are only two possibilities for this:

1. The roots α' and β' of $\mathfrak{h}$ are short and have an angle $\pi/3$ or $2\pi/3$, and $\mathfrak{h} \cong \mathfrak{g}_2$.
2. The roots α' and β' of $\mathfrak{h}$ are short, $\langle\alpha', \beta'\rangle = 0$, and $\pm\alpha' \pm \beta'$ are long roots.

Based on this observation, we will prove the following lemma.

Lemma 5.2. *Let $\mathfrak{g} = \mathfrak{a}_n$ with $n > 2$ and suppose that $v = ne_1 - e_2 - \cdots - e_{n+1} \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space $M = G/H$. Denote $e'_i = \mathrm{pr}_{\mathfrak{h}}(e_i)$, so $e'_1 + \cdots + e'_{n+1} = 0$. Then we have the following.*

1. *No two e'_i are equal.*
2. *There exist distinct i, j, k and l such that $e'_i - e'_j = e'_k - e'_l$.*
3. *If i, j, k and l are distinct and satisfy $e'_i - e'_j = e'_k - e'_l$, then the roots α' and β' of $\mathfrak{h}$ are short, $\langle\alpha', \beta'\rangle = 0$, and $\pm\alpha' \pm \beta'$ are long roots.*

Proof. (1) For simplicity, we call e'_i *single* if its pre-image $\mathrm{pr}_{\mathfrak{h}}^{-1}(e'_i)$ consists of a single element e_i. If e'_1 is not single, and $\mathrm{pr}_{\mathfrak{h}}^{-1}(e'_1) = \{e_1, \ldots, e_k\}$, $k > 1$, then $e'_1 - e'_i$ is root of $\mathfrak{h}$ for $i > k$. Permuting e_i for $i > k$, we can assume $e'_1 - e'_{k+1}$ has the largest length among all the roots of the form $e'_1 - e'_i$, and $\mathrm{pr}_{\mathfrak{h}}^{-1}(e'_{k+1}) = e_{k+1}, \ldots, e_{k+l}$, $l > 0$. If $e'_1 - e'_{k+1} = e'_i - e'_j$ with $i > k$, then by the observation before this lemma, either there is a root $e'_1 - e'_j$ longer than $e'_1 - e'_{k+1}$, which is a contradiction, or $\mathfrak{h} = \mathfrak{g}_2$ and the angle between $e'_1 - e'_{k+1}$ and $e'_l - e'_i$ is $\pi/3$ or $2\pi/3$.

We first assume there do not exist i and j such that $i > k$ and $e'_1 - e'_{k+1} = e'_i - e'_j$. Then $\mathfrak{h}_{\pm(e'_1 - e'_{k+1})} \subset \widehat{\mathfrak{g}}_{\pm(e'_1 - e'_{k+1})} = \sum_{i=1}^{k}\sum_{j=1}^{l} \mathfrak{g}_{\pm(e_i - e_{k+j})}$, and direct calculation shows $e'_1 - e'_{k+1} \in \mathbb{R}(e_1 - e_2) + \cdots + \mathbb{R}(e_{k+l-1} - e_{k+l})$. Let G' be the closed subgroup of G with its algebra

$$\mathfrak{g}' = \sum_{1 \leq i < k+l} \mathbb{R}(e_i - e_{i+1}) + \sum_{1 \leq i < j \leq k+l} \mathfrak{g}_{\pm(e_i - e_j)}.$$

It is the standard $\mathfrak{su}(k+l)$ in $\mathfrak{g} = \mathfrak{su}(n+1)$ corresponding to the $(k+l) \times (k+l)$-block at the upper left corner. Let H' be the connected component of $G' \cap H$; its Lie algebra is $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$. The restriction of the bi-invariant inner product of $\mathfrak{g}$ to $\mathfrak{g}'$ defines a Riemannian normal homogeneous space G'/H'. Then we have the orthogonal decomposition $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$, which coincides with $\mathfrak{g}' = \mathfrak{g}' \cap \mathfrak{h} + \mathfrak{g}' \cap \mathfrak{m}$. Notice $v = ne_1 - e_2 - \cdots - e_{n+1}$ can be decomposed as a sum of

$$v' = \frac{n+1}{k+l}((k+l-1)e_1 - e_2 - \cdots - e_{k+l}) \in \mathfrak{g}'$$

and

$$v'' = \Big(\frac{n+1}{k+l} - 1\Big)(e_1 + \cdots + e_{k+l}) - (e_{k+l+1} + \cdots + e_{n+1}) \in \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}'),$$

with $\langle v'', \mathfrak{g}' \rangle = 0$, so by Lemma 3.2, $v' = (k+l-1)e_1 - e_2 - \cdots - e_{k+l}$ defines a CK vector field on the Riemannian normal homogeneous space G'/H'. The subalgebra $\mathfrak{h}'$ is isomorphic to $\mathfrak{a}_1$, which can be assumed to be linearly spanned by

$$u_1 = \sqrt{-1}\begin{pmatrix} aI_k & 0 \\ 0 & bI_l \end{pmatrix}, u_2 = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}, \text{ and } u_3 = \sqrt{-1}\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix},$$

in which $ak + bl = 0$, I_k and I_l are $k \times k$ and $l \times l$ identity matrices respectively, and A is a $k \times l$-complex matrix. Direct calculation for the condition $[u_2, u_3] \subset \mathbb{R}u_1$ indicates $a = -b$, $k = l$, and by suitable scalar changes and unitary conjugations, we can assume $A = I_k$. The orbit $\mathrm{Ad}(G')v'$ contains

$$v'_1 = \sqrt{-1}\Big((k-1)(E_{1,1} + E_{k+1,k+1}) + k(E_{1,k+1} + E_{k+1,1}) - \sum_{1 \neq i \neq k+1} E_{i,i}\Big)$$

and

$$v'_2 = \sqrt{-1}\Big((k-1)(E_{1,1} + E_{2k,2k}) + k(E_{1,2k} + E_{2k,1}) - \sum_{1 \neq i \neq 2k} E_{i,i}\Big).$$

Thus $\|\mathrm{pr}_{\mathfrak{h}'}(v_1')\| > \|\mathrm{pr}_{\mathfrak{h}'}(v_2')\| = 0$, which is a contradiction. We conclude that there do exist i and j such that $i > k$ and $e_1' - e_{k+1}' = e_i' - e_j'$. Note then that $i \neq j$.

Now there exist $i \neq j$ such that $i > k$ and $e_1' - e_{k+1}' = e_i' - e_j'$. In this case $\mathfrak{h} = \mathfrak{g}_2$. For simplicity, we can suitably permute the e_i(not assuming $e_1' = \cdots = e_k'$ any more), such that we have $e_1' - e_2' = e_3' - e_4'$, the angle between the short roots $e_1' - e_2' = e_3' - e_4'$ and $e_1' - e_3' = e_2' - e_4'$ of $\mathfrak{h}$ is $\pi/3$ or $2\pi/3$, and both $e_1' - e_4'$ and $e_2' - e_3'$ are roots of $\mathfrak{h}$ such that one is long and the other is short. Assume $e_2' - e_3'$ is the short root for example; assuming $e_1' - e_4'$ to be the short root introduces only very minor changes in the following argument.

If $e_2' - e_3' = e_p' - e_q'$, such that $e_p' \neq e_2'$ or $e_q' \neq e_3'$, then either e_p' or e_q' must be different from the e_r's with $1 \leqq r \leqq 4$. The short root $e_2' - e_p' = e_3' - e_q'$ must be one of $\pm(e_1' - e_2') = \pm(e_3' - e_4')$ or $\pm(e_1' - e_3') = \pm(e_2' - e_4')$. Then the same e_2' or e_3' appears in different presentations of a short root of $\mathfrak{h}$, and this contradicts our earlier observation. So there do not exist p and q such that $e_2' - e_3' = e_p' - e_q'$ with either $e_p' \neq e_2'$ or $e_q' \neq e_3'$. Thus

$$\widehat{\mathfrak{g}}_{\pm(e_2'-e_3')} = \sum\nolimits_{e_p'=e_2', e_q'=e_3'} \mathfrak{g}_{\pm(e_p-e_q)}.$$

Arguing as above, we see that both e_2' and e_3' are single. We can also get $e_2' - e_3' \neq e_2 - e_3$, otherwise $e_2' - e_3'$ reaches the maximal possible length of $\mathfrak{h}$, which must be a long root, but we have assumed it is a short root. By Lemma 5.1, $\mathfrak{h}_{\pm(e_2'-e_3')}$ is not a root plane of $\mathfrak{g}$, i.e., $\widehat{\mathfrak{g}}_{\pm(e_2'-e_3')} = \sum_{e_p'=e_2', e_q'=e_3'} \mathfrak{g}_{\pm(e_p-e_q)}$ has dimension bigger than 2. So e_2' and e_3' cannot both be single. This is a contradiction. Assertion (1) of Lemma 5.2 is proved.

(2) If there do not exist distinct indices i, j, k and l such that $e_i' - e_j' = e_k' - e_l'$, then by Lemma 5.2(1) and Lemma 5.1, each root $e_i - e_j$ is either contained in $\mathfrak{t} \cap \mathfrak{h}$ or $\mathfrak{t} \cap \mathfrak{m}$. That is only possible when $\mathfrak{t} \subset \mathfrak{m}$ or $\mathfrak{t} \subset \mathfrak{h}$, which we have seen is not the case.

(3) follows from the argument of Lemma 5.2(1) which shows that $\mathfrak{h} \not\cong \mathfrak{g}_2$. $\square$

Now we determine $\mathfrak{h}$ by the following lemma.

Lemma 5.3. *Let $\mathfrak{g} = \mathfrak{a}_n$ with $n > 2$ and suppose that $v = ne_1 - e_2 - \cdots - e_{n+1} \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space $M = G/H$. Keep all relevant notations. Then for any $1 \leqq i \leqq n+1$, there is a unique j such that $e_i - e_j = e_i' - e_j' \in \mathfrak{h}$ is a long root of $\mathfrak{h}$. Furthermore, n is an odd number.*

Proof. Since $\mathfrak{t}$ is not contained in $\mathfrak{h}$, for any e_i, there is another e_l such that $e_i - e_l \notin \mathfrak{h}$. Then the root $e_i' - e_l'$ of $\mathfrak{h}$ is not a root of $\mathfrak{g}$. By Lemma 5.1, $\mathfrak{h}_{\pm(e_i'-e_l')}$ is not a root plane of $\mathfrak{g}$ and then $\dim \widehat{\mathfrak{g}}_{\pm(e_i'-e_l')} > 2$. So we have $e_i' - e_l' = e_k' - e_j'$ with $k \neq i$ and $l \neq j$. By an earlier observation, and Lemma 5.2(3), $e_i' - e_j'$ is a long root of $\mathfrak{h}$. There do not exist p and q such that $p \neq i$, $q \neq j$ and $e_i' - e_j' = e_p' - e_q'$. So $\mathfrak{h}_{\pm(e_i'-e_j')} = \widehat{\mathfrak{g}}_{\pm(e_i'-e_j')}$ is a root plane of $\mathfrak{g}$. By Lemma 5.1(2), $e_i' - e_j' = e_i - e_j \in \mathfrak{t} \cap \mathfrak{h}$. There does not exist another index p such that $p \neq j$ and $e_i - e_p \in \mathfrak{h}$, because $e_i - e_p$ is not orthogonal to $e_i + e_j - e_l - e_k \in \mathfrak{m}$. Obviously the map from i to $j \neq i$ maps j back to i. It follows immediately that n must be odd. $\square$

After a suitable permutation of the e_i we can assume $\mathfrak{h}$ contains $e_1 - e_{k+1}$, $e_2 - e_{k+2}$, $\ldots$, $e_k - e_{n+1}$, where $k = (n+1)/2$, and at the same time $\mathfrak{m}$ contains $e_1 + e_{k+1} - e_2 - e_{k+2}$, $e_2 + e_{k+2} - e_3 - e_{k+3}$, $\ldots$, $e_{k-1} + e_{2k-1} - e_k - e_n$. Those are bases for the subspaces $\mathfrak{h}$ and $\mathfrak{m}$. The root system of $\mathfrak{h}$ is

$$\{\pm(e_i - e_{i+k}) \mid 1 \leqq i \leqq k\} \cup \{\pm(e'_i - e'_j) \mid 1 \leqq i < j \leqq k\} \\ \cup \{\pm(e'_i - e'_{j+k}) \mid 1 \leqq i < j \leqq k\}.$$

Thus $\mathfrak{h}$ is isomorphic to $\mathfrak{c}_k = \mathfrak{sp}(k)$. For the root planes, we have

$$\begin{aligned} \mathfrak{h}_{\pm(e_i - e_{i+k})} &= \widehat{\mathfrak{g}}_{\pm(e_i - e_{i+k})} = \mathfrak{g}_{\pm(e_i - e_{i+k})}, \\ \widehat{\mathfrak{g}}_{\pm(e'_i - e'_j)} &= \mathfrak{g}_{\pm(e_i - e_j)} + \mathfrak{g}_{\pm(e_{i+k} - e_{j+k})}, \text{ for } 1 \leqq i < j \leqq k, \quad \text{and} \\ \widehat{\mathfrak{g}}_{\pm(e'_i - e'_{j+k})} &= \mathfrak{g}_{\pm(e_i - e_{j+k})} + \mathfrak{g}_{\pm(e_j - e_{i+k})}, \text{ for } 1 \leqq i < j \leqq k. \end{aligned}$$

We will see $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, i.e., G/H is Riemannian symmetric. If α' is a root of $\mathfrak{h}$ such that $\widehat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{m} \neq 0$, and $u \in \mathfrak{t} \cap \mathfrak{m}$ then $\mathrm{ad}\,(u) : \widehat{\mathfrak{g}}_{\pm\alpha'} \to \widehat{\mathfrak{g}}_{\pm\alpha'}$ is the same $\left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right)$ on the root planes $\mathfrak{g}_{\pm\alpha}$ where α restricts to α'. Thus $\mathrm{ad}\,(u) : \widehat{\mathfrak{g}}_{\pm\alpha'} \to \widehat{\mathfrak{g}}_{\pm\alpha'}$ is a multiple of an isometry. For generic u, $ad(u)$ maps $\widehat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{h}$ onto $\widehat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{m}$, and thus $\widehat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{m}$ onto $\widehat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{h}$, because it maps orthocomplement to orthocomplement. There are no root planes of $\mathfrak{g}$ contained in $\mathfrak{m}$, so the above argument proves

$$[\mathfrak{m} \cap \mathfrak{t}, \mathfrak{m}] = [\mathfrak{m} \cap \mathfrak{t}, \mathfrak{m} \cap \mathfrak{t}^\perp] \subset \mathfrak{h}. \tag{5.4}$$

In particular, $[u, \mathfrak{h}] = \mathfrak{m} \cap \mathfrak{t}^\perp$ and $[u, \mathfrak{m}] \subset \mathfrak{h}$ for generic $u \in \mathfrak{t} \cap \mathfrak{m}$. If $w_1, w_2 \in \mathfrak{m} \cap \mathfrak{t}^\perp$ and $u \in \mathfrak{t} \cap \mathfrak{m}$ is generic, now $[u, w_1]$, $[u, w_2] \subset \mathfrak{h}$, and

$$[u, [w_1, w_2]] = [[u, w_1], w_2] + [w_1, [u, w_2]] \subset \mathfrak{m},$$

so

$$\langle [w_1, w_2], \mathfrak{m} \cap \mathfrak{t}^\perp \rangle = \langle [w_1, w_2], [u, \mathfrak{h}] \rangle = \langle [u, [w_1, w_2]], \mathfrak{h} \rangle = 0.$$

Also we have

$$\langle [w_1, w_2], u \rangle = \langle w_1, [w_2, u] \rangle = 0,$$

so $\langle [w_1, w_2], \mathfrak{t} \cap \mathfrak{m} \rangle = 0$. Thus

$$[\mathfrak{m} \cap \mathfrak{t}^\perp, \mathfrak{m} \cap \mathfrak{t}^\perp] \subset \mathfrak{h}. \tag{5.5}$$

By (5.4) and (5.5), $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$; in other words, G/H is a symmetric homogeneous space. It is locally Riemannian symmetric as well.

To summarize, we have proved the following proposition.

Proposition 5.6. *Let G be a compact connect simple Lie group of type $\mathfrak{a}_n$ and H a closed subgroup with $0 < \dim H < \dim G$. If there is a nonzero vector $v \in \mathfrak{g} = \mathrm{Lie}(G)$ that defines a CK vector field on the Riemannian normal homogeneous space $M = G/H$, then M is a local Riemannian symmetric space with universal Riemannian covering space* $\mathrm{SU}(2k)/\mathrm{Sp}(k)$.

6. Proof of Theorem 1.1 for $\mathfrak{g} = \mathfrak{b}_n$

In this section $\mathfrak{g} = \mathfrak{b}_n = \mathfrak{so}(2n+1)$ with $n > 1$, and $0 \neq v \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space $M = G/H$. Proposition 3.5 says that, up to the action of the Weyl group, either v is a multiple of e_1 or a multiple of $e_1 + \cdots + e_n$. We must see whether those vectors v define CK vector fields on M.

6A. The case $v = e_1 + \cdots + e_n$

Following the proof of Proposition 3.5, we may suitably permute the e_i and assume

$$\mathfrak{h} = \mathbb{R}e_1 + \mathbb{R}e_{n-h+2} + \cdots + \mathbb{R}e_n \quad \text{and} \quad \mathfrak{m} = \mathbb{R}e_2 + \cdots + \mathbb{R}e_{n-h+1}.$$

Let G' be the connected Lie subgroup of G whose Lie algebra $\mathfrak{g}'$ is the centralizer of $\mathbb{R}e_{n-h+2} + \cdots + \mathbb{R}e_n$ in $\mathfrak{g}$. Let H' be the connected component of $G' \cap H$; its Lie algebra $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$. We have a direct sum decomposition $\mathfrak{g}' = \mathfrak{g}'' \oplus \mathfrak{c}$ in which $\mathfrak{c}$ is the center and $\mathfrak{g}'' = \mathfrak{so}(2n-2h+3)$ corresponding to the e_i with $1 \leqq i \leqq n-h+1$. Here $\mathfrak{c} = \mathbb{R}e_{n-h+2} + \cdots + \mathbb{R}e_n$. We also have a direct sum decomposition $\mathfrak{h}' = \mathfrak{h}'' \oplus \mathfrak{c}$, in which either $\mathfrak{h}'' = \mathbb{R}e_1$ or $\mathfrak{h}''$ is of type $\mathfrak{a}_1$ with Cartan subalgebra $\mathfrak{t} \cap \mathfrak{h}'' = \mathbb{R}e_1$. Let G'' be the closed subgroup in the universal cover of G' with Lie algebra $\mathfrak{g}''$, and H'' the closed connected subgroup of G'' with Lie algebra $\mathfrak{h}''$. The restriction of the bi-invariant inner product of $\mathfrak{g}$ to $\mathfrak{g}'$ and $\mathfrak{g}''$ defines locally isometric Riemannian normal homogeneous metrics on G'/H' and G''/H'' respectively. Thus we have orthogonal decompositions

$$\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}' \quad \text{and} \quad \mathfrak{g}'' = \mathfrak{h}'' + \mathfrak{m}''. \tag{6.1}$$

Since $\mathfrak{g}'$ is the centralizer of a subalgebra of $\mathfrak{h}$, the first decomposition in (6.1) coincides with $\mathfrak{g}' = (\mathfrak{g}' \cap \mathfrak{h}) + (\mathfrak{g}' \cap \mathfrak{m})$. Since $\mathfrak{c} \subset \mathfrak{h}'$ is orthogonal to $\mathfrak{m}'$, i.e., $\mathfrak{m}' \subset \mathfrak{g}''$, the second decomposition in (6.1) coincides with

$$\mathfrak{g}'' = (\mathfrak{g}'' \cap \mathfrak{h}') + (\mathfrak{g}'' \cap \mathfrak{m}') = (\mathfrak{g}'' \cap \mathfrak{h}) + (\mathfrak{g}'' \cap \mathfrak{m}).$$

The vector v can be decomposed as the sum of $v'' = e_1 + \cdots + e_{n-h+1} \in \mathfrak{g}''$ and $v_{\mathfrak{c}} = e_{n-h+2} + \cdots + e_n \in \mathfrak{c}$, which is orthogonal to $\mathfrak{g}''$. So by Lemma 3.2, v'' defines a CK vector field on the Riemannian normal homogeneous space G''/H''.

If $\mathfrak{h}'' = \mathbb{R}e_1$ we can find

$$v_1'' = \mathrm{diag}\left(0, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \in \mathfrak{m}'' \cap \mathrm{Ad}\,(G)v''.$$

Then $\|\mathrm{pr}_{\mathfrak{h}''}(v'')\| > \|\mathrm{pr}_{\mathfrak{h}''}(v_1'')\| = 0$, which contradicts the CK property of v. If $\mathfrak{h}'' \cong \mathfrak{a}_1$, we use the Weyl group of G'' to change it to the standard $\mathfrak{so}(3) \subset \mathfrak{so}(2n-2h+3)$ corresponding the 3×3-block at the upper left corner. The argument used for the case $\mathfrak{h}'' = \mathbb{R}e_1$ also leads to a contradiction in this case.

6B. The case $v = e_1$

We now consider the case $v = e_1$ in Proposition 3.5. Denote $e_i' = \mathrm{pr}_{\mathfrak{h}}(e_i)$ for $1 \leqq i \leqq n$; they all have the same length. We have $E_{1,2i} - E_{2i,1}$ and $E_{1,2i+1} - E_{2i+1,1}$ in $\mathrm{Ad}\,(G)v \cap \mathfrak{g}_{\pm e_i}$. Thus, for $1 \leqq i \leqq n$, e_i' is a root of $\mathfrak{h}$, and $\dim \widehat{\mathfrak{g}}_{\pm e_i'} > 2$. $\mathrm{Ad}\,(G)v$ also contains $E_{2i,2j} - E_{2j,2i} \in \mathfrak{g}_{\pm(e_i+e_j)} + \mathfrak{g}_{\pm(e_i-e_j)}$ for $1 \leqq i < j \leqq n$, so either $e_i' + e_j'$ or $e_i' - e_j'$ is a root of $\mathfrak{h}$ whenever $i \neq j$. Any root of $\mathfrak{h}$ has form $\pm e_i'$ or $\pm e_i' \pm e_j'$, and it follows that $\mathfrak{h}$ is a compact simple Lie algebra. From the standard description of the roots of $\mathfrak{b}_n$, $\mathfrak{c}_n$, $\mathfrak{f}_4$ and $\mathfrak{g}_2$, the $\pm e_i'$ cannot be long roots, because all roots of $\mathfrak{h}$ are of the form $\pm e_i' \pm e_j'$ for $i \neq j$.

If $i \neq j$ then $e_i' \neq \pm e_j'$ because that would give a root $e_i' \pm e_j' = 0$ or $2e_j'$. Thus $\{\pm e_i'\}$ is a set of $2n$ distinct roots of $\mathfrak{h}$. Since $\dim \widehat{\mathfrak{g}}_{\pm e_i'} > 2$, and $\mathrm{pr}_{\mathfrak{h}}(\pm e_j) \neq e_i'$ for $i \neq j$, there must be a root α of the form $\pm e_j \pm e_k$, such that $\mathrm{pr}_{\mathfrak{h}}(\alpha) = e_i'$, i.e., $e_i' = \pm e_k' \pm e_l'$.

Summarizing the above argument, we have the following lemma.

Lemma 6.2. *Suppose that $\mathfrak{g} = \mathfrak{b}_n$ with $n > 1$ and that $v = e_1 \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space $M = G/H$. Denote $e_i' = \mathrm{pr}_{\mathfrak{h}}(e_i)$, then we have the following.*

(1) *$\{\pm e_1', \dots, \pm e_n'\}$ consists of $2n$ different roots of $\mathfrak{h}$. If $1 \leqq i < j \leqq n$ then either $e_i' + e_j'$ or $e_i' - e_j'$ is a root of $\mathfrak{h}$.*
(2) *The Lie algebra $\mathfrak{h}$ is compact simple. If it is isomorphic to $\mathfrak{b}_n$, $\mathfrak{c}_n$, $\mathfrak{f}_4$ or $\mathfrak{g}_2$ then the $\pm e_i'$ are short roots.*
(3) *Any e_i' can be expressed as $e_i' = \pm e_j' \pm e_k'$ in which i, j and k are different from each other.*

Because from the $\mathrm{Ad}(G)$-orbit of v, we can find an orthonormal basis for $\mathfrak{t}$ as well as an orthonormal basis for $\mathfrak{g}$, by Lemma 3.3,

$$2n + 1 = \frac{\dim \mathfrak{g}}{\dim \mathfrak{t}} = \frac{\dim \mathfrak{h}}{\dim \mathfrak{t} \cap \mathfrak{h}}. \tag{6.3}$$

Denote $h = \dim \mathfrak{t} \cap \mathfrak{h}$. Then the right side of (6.3) is $h + 2$, $2h + 1$, $2h + 1$, $2h - 1$, 13, 19, 31, 13 or 7, respectively, when $\mathfrak{h}$ is isomorphic to $\mathfrak{a}_h$, $\mathfrak{b}_h$, $\mathfrak{c}_h$, $\mathfrak{d}_h$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$ or $\mathfrak{g}_2$. Because $h < n$, $\mathfrak{h}$ can only be $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$ or $\mathfrak{g}_2$.

First consider the cases where $\mathfrak{h}$ is isomorphic to $\mathfrak{e}_7$, $\mathfrak{e}_8$ or $\mathfrak{f}_4$. If $e_i' = \pm e_j' \pm e_k'$, then e_j' and e_k' have an angle $\pi/3$ or $2\pi/3$, so the corresponding vector $\pm e_j' \pm (-e_k')$ is not a root of $\mathfrak{h}$. Based on this observation, we have the following lemma.

Lemma 6.4. *Assume $\mathfrak{g} = \mathfrak{b}_n$, $\mathfrak{h} = \mathfrak{e}_7$, $\mathfrak{e}_8$ or $\mathfrak{f}_4$, and $v = e_1 \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space G/H. Then*

$$\frac{\|v\|^2}{\|\mathrm{pr}_{\mathfrak{h}}(v)\|^2} = \frac{\dim \mathfrak{t}}{\dim(\mathfrak{t} \cap \mathfrak{h})} = \dim \widehat{\mathfrak{g}}_{\pm e_i'} - 1 \geq 3, \tag{6.5}$$

whenever $1 \leq i \leq n$.

Proof. Denote $\dim \widehat{\mathfrak{g}}_{\pm e_i'} = 2k + 2$ and $\widehat{\mathfrak{g}}_{\pm e_i'} = \sum_{j=1}^{k} \mathfrak{g}_{\pm \alpha_j}$. Consider any root α_j of $\mathfrak{g}$ in the above equality, and assume it has the form $\alpha_j = \pm e_p \pm e_q$ with $p < q$.

Denote $\bar{\alpha} = \pm e_p \mp e_q$. Because $\mathfrak{h}$ is not isomorphic to $\mathfrak{g}_2$, and all the e'_p, e'_q and $\mathrm{pr}_{\mathfrak{h}}(\alpha_j) = \pm e'_p \pm e'_q$ are short roots of $\mathfrak{h}$, now $\mathrm{pr}_{\mathfrak{h}}(\bar{\alpha}_j) = \pm e'_p \mp e'_q$ is not a root of $\mathfrak{h}$, i.e., $\mathfrak{g}_{\pm\bar{\alpha}_j} \subset \mathfrak{m}$.

Let $\mathfrak{v} = \mathfrak{g}_{\pm e_i} + \sum_{j=1}^k (\mathfrak{g}_{\pm\alpha_j} + \mathfrak{g}_{\pm\bar{\alpha}_j})$, then $\mathfrak{v} = (\mathfrak{v} \cap \mathfrak{h}) + (\mathfrak{v} \cap \mathfrak{m})$, and $\mathfrak{v} \cap \mathfrak{h} = \mathfrak{h}_{\pm e'_i}$ is real 2-dimensional. Because different roots α_j correspond to different pairs $\{p, q\}$ for $\alpha_j = \pm e_p \pm e_q$, all roots $\bar{\alpha}_j$ of $\mathfrak{g}$ are also different with each other. So $\mathfrak{v}$ is a real $4k+2$-dimensional linear space. Inside $\mathrm{Ad}(G)v = \mathrm{Ad}\,(G)e_1$ we have an orthonormal basis of $\mathfrak{v}$ consisting of $E_{1,2i} - E_{2i,1}$ and $E_{1,2i+1} - E_{2i+1,1}$ in $\mathfrak{g}_{\pm e_i}$ and $E_{2p,2q} - E_{2q,2p}$, $E_{2p,2q+1} - E_{2q+1,2q}$, $E_{2p+1,2q} - E_{2q,2p+1}$ and $E_{2p+1,2q+1} - E_{2q+1,2p+1}$ in each $\mathfrak{g}_{\pm\alpha_j} + \mathfrak{g}_{\pm\bar{\alpha}_j}$. By Lemma 6A, and using some arguments as before, we have

$$\frac{||v||^2}{||\mathrm{pr}_{\mathfrak{h}}||^2} = \frac{\dim \mathfrak{t}}{\dim(\mathfrak{t} \cap \mathfrak{h})} = \frac{\dim \mathfrak{v}}{\dim(\mathfrak{v} \cap \mathfrak{h})} = 2k+1 = \widehat{\mathfrak{g}}_{\pm e'_i} - 1. \qquad \square$$

If $\mathfrak{h}$ is $\mathfrak{e}_7$, $\mathfrak{e}_8$ or $\mathfrak{f}_4$, then, by (6.3), $n = \dim \mathfrak{t}$ is 9, 15 or 6, respectively, contradicting (6.5).

Finally we consider the case $\mathfrak{h} = \mathfrak{g}_2$ and show it is possible. By (6.3), we have $n = 3$ for this case, with $\pm e_1$, $\pm e_2$ and $\pm e_3$ corresponding to the three pairs of short roots. Suitably choosing e_i and applying sign changes $e_i \mapsto -e_i$ as appropriate, we may assume $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2 + e_3)$, and the root planes of $\mathfrak{h}$ are

$$\begin{aligned} \mathfrak{h}_{\pm(e_i - e_j)} &= \mathfrak{g}_{\pm(e_i - e_j)} \quad \text{for } 1 \leqq i < j \leqq 3, \quad \text{and} \\ \mathfrak{h}_{\pm\frac{1}{3}(e_i + e_j - 2e_k)} &\subset \widehat{\mathfrak{g}}_{\pm\frac{1}{3}(e_i + e_j - 2e_k)} = \mathfrak{g}_{\pm(e_i + e_j)} + \mathfrak{g}_{\pm e_k} \quad \text{for } \{i, j, k\} = \{1, 2, 3\}. \end{aligned}$$

The subalgebra $\mathfrak{h}$ is uniquely determined up to the action of $\mathrm{Ad}(G)$. Since the isotropy subgroup G_2 is transitive on directions in the tangent space of $S^7 = \mathrm{Spin}(7)/\mathrm{G}_2$, the $\mathrm{Spin}(7)$–invariant Riemannian metric on that space is the standard constant positive curvature metric. Now the vector $v = e_1 \in \mathfrak{t}$ defines a CK vector field on $S^7 = \mathrm{Spin}(7)/\mathrm{G}_2$.

It is well known that $v' = e_1 + \cdots + e_4$ defines a CK vector field on the symmetric space $S^7 = \mathrm{Spin}(8)/\mathrm{Spin}(7)$ for the standard imbedding $\mathfrak{so}(7) \hookrightarrow \mathfrak{so}(8)$. But if we change the setup by a suitable outer automorphism, using triality, v' can be changed to $v = e_1$, which belongs to the Cartan subalgebra $\mathfrak{t}$ of the standard $\mathfrak{so}(7) \subset \mathfrak{so}(8)$. Inside $\mathfrak{so}(8)$ the intersection of the standard $\mathfrak{so}(7)$ and the isotropic one is just the $\mathfrak{g}_2$. So $v = e_1$ also defines a CK vector field on the Riemannian symmetric $S^7 = \mathrm{Spin}(7)/\mathrm{G}_2$.

In summary, we have the following proposition.

Proposition 6.6. *Let G be a compact connected simple Lie group with $\mathfrak{g} = \mathfrak{b}_n$ where $n > 1$. Let H be closed subgroup with $0 < \dim H < \dim G$, such that G/H is a Riemannian normal homogeneous space. Assume there is a nonzero vector $v \in \mathfrak{g}$ that defines a CK vector field on $M = G/H$. Then $M = G/H$ is a locally symmetric Riemannian manifold whose universal Riemannian covering is $S^7 = \mathrm{Spin}(7)/\mathrm{G}_2$.*

7. The proof of Theorem 1.1 when $\mathfrak{g} = \mathfrak{c}_n$

In this section we assume that $\mathfrak{g} = \mathfrak{c}_n = \mathfrak{sp}(n)$ with $n > 2$, and that $0 \neq v \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space $M = G/H$. According to Proposition 3.5, we may assume that either $v = e_1 + \cdots + e_n$ or $v = e_1$.

7A. The case $v = e_1 + \cdots + e_n$

After a suitable permutation of the e_i we may assume $\mathfrak{h} = \mathbb{R}e_1 + \mathbb{R}e_{m+2} + \cdots + \mathbb{R}e_n$ and $\mathfrak{m} = \mathbb{R}e_2 + \cdots + \mathbb{R}e_{m+1}$. Consider the closed subgroup G' of G whose Lie algebra $\mathfrak{g}'$ is the centralizer of $\mathbb{R}e_{m+2} + \cdots + \mathbb{R}e_n$. Let H' be the identity component of $G' \cap H$. Then $\mathfrak{g}' = \mathfrak{g}'' \oplus \mathfrak{c}$ where $\mathfrak{g}'' = \mathfrak{sp}(m+1)$ corresponds to $\{e_1, \ldots, e_{m+1}\}$ and $\mathfrak{c} = \mathbb{R}e_{m+2} + \cdots + \mathbb{R}e_n$ is its center. The subalgebra $\mathfrak{h}' = \mathfrak{h}'' \oplus \mathfrak{c}$ where either $\mathfrak{h}'' = \mathfrak{h} \cap \mathfrak{g}'' = \mathbb{R}e_1$ or $\mathfrak{h}'' \cong \mathfrak{a}_1$ with Cartan subalgebra $\mathbb{R}e_1$ and root plane $\mathbb{R}\mathbf{j}E_{1,1} + \mathbb{R}\mathbf{k}E_{1,1}$. Let G'' be the analytic subgroup of G' for $\mathfrak{g}''$ and let H'' be the analytic subgroup of G'' for $\mathfrak{h}''$. They are closed subgroups. The restriction of the bi-invariant inner product of $\mathfrak{g}$ to $\mathfrak{g}'$ and $\mathfrak{g}''$ defines locally symmetric Riemannian normal homogeneous metrics on G'/H' and G''/H'' respectively. As argued before, the orthogonal decomposition $\mathfrak{g}'' = \mathfrak{h}'' + \mathfrak{m}''$ is the same as $\mathfrak{g}'' = (\mathfrak{g}'' \cap \mathfrak{h}) + (\mathfrak{g}'' \cap \mathfrak{m})$. We can also decompose v as the sum of $v'' = e_1 + \cdots + e_{m+1} \in \mathfrak{t} \cap \mathfrak{g}''$ and $v_\mathfrak{c} = e_{m+2} + \cdots + e_n \in \mathfrak{c}$ which is orthogonal to $\mathfrak{g}''$. By Lemma 3.2, v'' defines a CK vector field on the Riemannian normal homogeneous space G''/H''. From the $\mathrm{Ad}(G'')$-orbit of v'' we have

$$v_1'' = \mathbf{i}\left(\operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \ldots, 1\right)\right) \in \mathfrak{m}'',$$

i.e., $\|\mathrm{pr}_{\mathfrak{h}''}(v'')\| > \|\mathrm{pr}_{\mathfrak{h}''}(v_1'')\| = 0$, which is a contradiction.

Remark. As in the case 5A in Section 5, for $n > 2$, $v = e_1 + \cdots + e_n$ defines a CK vector field for the constant curvature metric on S^{4n-1}. That metric is normal homogeneous for $S^{4n-1} = \mathrm{SO}(4n)/\mathrm{SO}(4n-1)$, but it is not normal homogeneous for $S^{4n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$.

7B. The case $v = e_1$

Here $v = e_1$ in Proposition 3.5, and we denote $e_i' = \mathrm{pr}_\mathfrak{h}(e_i)$ for $1 \leqq i \leqq n$. The e_i' all have the same length.

The orbit $\mathrm{Ad}(G)v$ contains $v' = \mathbf{j}E_{i,i} \subset \mathfrak{g}_{\pm 2e_i}$, and $\mathrm{pr}_\mathfrak{h}(v') \neq 0 \neq \mathrm{pr}_\mathfrak{m}(v')$, for $1 \leqq i \leqq n$. So $2e_i'$ is a root of $\mathfrak{h}$ and $\dim \widehat{\mathfrak{g}}_{\pm 2e_i'} > 2$, for $1 \leqq i \leqq n$.

Suitably choosing e_i with any necessary sign changes $e_i \mapsto -e_i$, we may assume, for $1 \leqq i < j \leqq n$, that the roots $\pm 2e_i$ and $\pm 2e_j$ project to the same pair of roots of $\mathfrak{h}$ only when $e_i' = e_j'$. In other words $e_i' + e_j' \neq 0$, for $1 \leqq i < j \leqq n$. If we have a different presentation for the root $2e_i'$, e.g. $2e_i' = \pm e_j' \pm e_k'$, then the plus signs must be taken and $e_j' = e_k' = e_i'$.

If $e_i' = e_j'$ for some $i \neq j$, we can permute e_i so that $\mathrm{pr}_\mathfrak{h}^{-1}(e_1')$ contains e_i for $1 \leqq i \leqq k$, where $k > 1$, and it does not contain any other e_i. So

$$\widehat{\mathfrak{g}}_{\pm 2e_1'} = \sum_{1 \leqq i \leqq k} \mathfrak{g}_{\pm 2e_i} + \sum_{1 \leqq i < j \leqq k} \mathfrak{g}_{\pm(e_i + e_j)},$$

in such a way that the root plane $\mathfrak{h}_{\pm e'_1}$ is linearly generated by two matrices u and w in $\mathfrak{sp}(k) \subset \mathfrak{sp}(n)$ with nonzero elements only in the upper left $k \times k$ corner. So $[u, w]$, which is a nonzero multiple of e'_1, is represented by a matrix in $\mathbb{R}e_1 + \cdots + \mathbb{R}e_k \in \mathfrak{sp}(k)$. Since $e_i - e_j \in \mathfrak{m}$ for $1 \leqq i < j \leqq k$, we have $e'_1 = \frac{1}{k}(e_1 + \cdots + e_k)$. Any root plane $\mathfrak{g}_{\pm(e_i - e_j)}$, $1 \leqq i < j \leqq k$ is contained in $\mathfrak{m}$.

Let G' be the closed subgroup of G with Lie algebra $\mathfrak{g}' = \mathfrak{sp}(k)$ corresponding to $\{e_1, \ldots, e_k\}$, and H' the identity component of $G' \cap H$. The Lie algebra $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h} = \mathbb{R}e'_1 + \mathfrak{h}_{\pm e'_1}$. The restriction of the bi-invariant inner product of $\mathfrak{g}$ to $\mathfrak{g}'$ defines a Riemannian normal homogeneous space $M' = G'/H'$. The corresponding orthogonal decomposition $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$ coincides with $\mathfrak{g}' = (\mathfrak{g}' \cap \mathfrak{h}) + (\mathfrak{g}' \cap \mathfrak{m})$, $v = e_1 \in \mathfrak{g}'$ defines a CK vector field on the normal homogeneous space G'/H', and $\mathfrak{h}'$ is spanned by

$$w_1 = \mathbf{i}I_{k\times k}\,,\ w_2 = \mathbf{j}A + \mathbf{k}B,\ \text{and}\ w_3 = -\mathbf{j}B + \mathbf{k}A,$$

where A and B are real symmetric $k \times k$ matrices. From $w_1 \in \mathbb{R}[w_2, w_3]$, we have $AB = BA$ and $A^2 + B^2 = cI > 0$.

By a suitable $\mathrm{Ad}(G')$ conjugation (which is an $\mathrm{SO}(k)$ conjugation on $\mathfrak{sp}(k)$) we diagonalize A and B simultaneously. By a suitable $\mathrm{Ad}(G')$ conjugation and scalar multiplications, i.e., $\mathrm{Sp}(k)$-conjugation, we may assume $w_2 = \mathbf{j}I_{k\times k}$ and $w_3 = \mathbf{k}I_{k\times k}$. Notice

$$\begin{aligned}
v'_1 &= \mathrm{diag}\left(\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}, 1, \ldots, 1\right) \cdot v \cdot \mathrm{diag}\left(\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}, 1, \ldots, 1\right) \\
&= \mathrm{diag}\left(\begin{pmatrix} \mathbf{i}/2 & -\mathbf{i}/2 \\ -\mathbf{i}/2 & \mathbf{i}/2 \end{pmatrix}, 0, \ldots, 0\right) \quad \text{and} \\
v'_2 &= \mathrm{diag}(1, -\mathbf{j}, 1, \ldots, 1) \cdot v' \cdot \mathrm{diag}(1, \mathbf{j}, 1, \ldots, 1) \\
&= \mathrm{diag}\left(\begin{pmatrix} \mathbf{i}/2 & -\mathbf{k}/2 \\ -\mathbf{k}/2 & -\mathbf{i}/2 \end{pmatrix}, 0, \ldots, 0\right)
\end{aligned}$$

belong to $\mathrm{Ad}(G')v$, but $\|\mathrm{pr}_{\mathfrak{h}'}(v)\| > \|\mathrm{pr}_{\mathfrak{h}'}(v'')\| = 0$. This is a contradiction. So the e'_i must all be distinct and $\widehat{\mathfrak{g}}_{\pm 2e'_i} = \mathfrak{g}_{\pm 2e_i} = \mathfrak{h}_{\pm 2e'_i}$ for $1 \leqq i \leqq n$. This contradicts our earlier conclusion that $\dim \widehat{\mathfrak{g}}_{\pm 2e'_i} > 2$.

In summary, we have proved

Proposition 7.1. *Let G be a compact connected simple Lie group with $\mathfrak{g} = \mathfrak{c}_n$ and $n > 2$. Let H be a closed subgroup with $0 < \dim H < \dim G$ such that G/H is a Riemannian normal homogeneous space. Then there is no nonzero vector $v \in \mathfrak{g}$ that defines a CK vector field on G/H.*

8. The case $\mathfrak{g} = \mathfrak{d}_n$

In this section $\mathfrak{g} = \mathfrak{d}_n = \mathfrak{so}(2n)$ with $n > 3$, and $0 \neq v \in \mathfrak{g}$ defines a CK vector field on the Riemannian normal homogeneous space G/H.

8A. The case $v = e_1 + \cdots + e_n$

We consider the case $v = e_1 + \cdots e_n$ of Proposition 3.5 with $n > 4$. If $n = 4$, we can use the outer automorphism of $\mathfrak{g}$ that changes v to e_1, which will be discussed in the next case. By an argument similar to that of Proposition 3.5(2), we show

$$\langle \mathrm{pr}_{\mathfrak{h}}(\pm e_i \pm e_j), \pm e_k \pm e_l\rangle = \langle \mathrm{pr}_{\mathfrak{m}}(\pm e_i \pm e_j), \pm e_k \pm e_l\rangle = 0,$$

whenever i, j, k and l are different indices. Changing k and l arbitrarily, and taking linear combinations of these two equalities, $\mathrm{pr}_{\mathfrak{h}}(e_i)$ and $\mathrm{pr}_{\mathfrak{m}}(e_i)$ are contained in $\mathbb{R}e_i + \mathbb{R}e_j$. Change j as well; we get that $\mathrm{pr}_{\mathfrak{h}}(e_i)$ and $\mathrm{pr}_{\mathfrak{m}}(e_i)$ are contained by $\cap_{j\neq i}(\mathbb{R}e_i + \mathbb{R}e_j) = \mathbb{R}e_i$. So either $e_i \in \mathfrak{h}$ or $e_i \in \mathfrak{m}$ for each i.

Let $m = \dim \mathfrak{t}\cap\mathfrak{m}$. We will prove $m = 1$. For if $m > 1$ we can suitably permute e_i so that $\mathfrak{h}\cap\mathfrak{t} = \mathbb{R}e_1 + \mathbb{R}e_{m+2} + \cdots + \mathbb{R}e_n$ and $\mathfrak{m}\cap\mathfrak{t} = \mathbb{R}e_2 + \cdots + \mathbb{R}e_{m+1}$. Let $\mathfrak{g}'$ be the centralizer of $\mathbb{R}e_{m+2} + \cdots + \mathbb{R}e_n$ in $\mathfrak{g}$ and G' the analytic subgroup of G for $\mathfrak{g}'$. Similarly $\mathfrak{h}' = \mathfrak{g}'\cap\mathfrak{h}$ and H' is the corresponding analytic subgroup. Then $\mathfrak{g}' = \mathfrak{g}''\oplus\mathfrak{c}$ where $\mathfrak{g}'' \cong \mathfrak{so}(2m+2)$ corresponds to the $(2m+2)\times(2m+2)$-block in the upper left corner, and where $\mathfrak{c} = \mathbb{R}e_{m+1} + \cdots + \mathbb{R}e_n$ is the center of $\mathfrak{g}'$. Observe $\mathfrak{h}' = \mathfrak{h}''\oplus\mathfrak{c}$ where $\mathfrak{h}'' = \mathfrak{g}''\cap\mathfrak{h}'$ is either the abelian subalgebra $\mathbb{R}e_1$ or is isomorphic to $\mathfrak{a}_1$ with Cartan subalgebra $\mathbb{R}e_1$. Let G'' be the analytic subgroup of G' with Lie algebra $\mathfrak{g}''$ and H'' the analytic subgroup of G'' for $\mathfrak{h}''$. As we argued in Section 6A, the restriction of the bi-invariant inner product of $\mathfrak{g}$ to $\mathfrak{g}''$ defines a Riemannian normal homogeneous metric on G''/H'', such that the orthogonal decomposition $\mathfrak{g}'' = \mathfrak{h}'' + \mathfrak{m}''$ coincides with $\mathfrak{g}'' = (\mathfrak{g}''\cap\mathfrak{h}) + (\mathfrak{g}''\cap\mathfrak{m})$. For the corresponding decomposition $v = v'' + v_{\mathfrak{c}}$,

$$v'' = e_1 + \cdots + e_{m+1} \in \mathfrak{g}'' \quad \text{and} \quad v_{\mathfrak{c}} = e_{m+2} + \cdots + e_n \in \mathfrak{c} \subset \mathfrak{g}''^{\perp},$$

v'' defines a CK vector field on the normal homogeneous space $M'' = G''/H''$.

If $\mathfrak{h}''$ is abelian, then we can choose

$$v_1'' = \mathrm{diag}\left(\begin{pmatrix} 0_{2\times 2} & I_{2\times 2} \\ -I_{2\times 2} & 0_{2\times 2}\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right) \in \mathfrak{m}''\cap \mathrm{Ad}\,(G'')v''.$$

Then $\|\mathrm{pr}_{\mathfrak{h}''}(v'')\| > \|\mathrm{pr}_{\mathfrak{h}''}(v_1'')\| = 0$, which is a contradiction. If $\mathfrak{h}''$ is not abelian, then we can use a suitable $\mathrm{Ad}(G')$ action, i.e., $\mathrm{SO}(2m+2)$ conjugation, to move $\mathfrak{h}''$ to the subalgebra $\mathfrak{so}(3)$ for the 3×3-block in the upper left corner. We can choose

$$v_2'' = \mathrm{diag}\left(\begin{pmatrix} 0_{3\times 3} & I_{3\times 3} \\ -I_{3\times 3} & 0_{3\times 3}\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right) \in \mathfrak{m}''\cap \mathrm{Ad}\,(G'')v''.$$

Then we again have $\|\mathrm{pr}_{\mathfrak{h}''}(v'')\| > \|\mathrm{pr}_{\mathfrak{h}'}(v_2'')\| = 0$, which is a contradiction. This completes the proof that $m = 1$.

Now we suitably permute e_i so that $\mathfrak{t}\cap\mathfrak{h} = \mathbb{R}e_1 + \cdots + \mathbb{R}e_{n-1}$ and $\mathfrak{t}\cap\mathfrak{m} = \mathbb{R}e_n$. Whenever $1 \leqq i < j < n$, we have vectors

$$v' = \sum_{1\leqq k\leqq n-1, i\neq k\neq j} (E_{2k-1,2k} - E_{2k,2k-1}) + u' \in \mathrm{Ad}\,(G)v$$

in which the possibilities for u' are

$$E_{2i-1,2j-1} + E_{2i,2j} - E_{2j-1,2i-1} - E_{2j,2i} \quad \text{and} \quad E_{2i-1,2j} - E_{2i,2j-1} + E_{2j-1,2i} - E_{2j,2i-1}$$

in $\mathfrak{g}_{\pm(e_i-e_j)}$, and

$$E_{2i-1,2j-1}-E_{2i,2j}-E_{2j-1,2i-1}+E_{2j,2i} \text{ and } E_{2i-1,2j}+E_{2i,2j-1}+E_{2j-1,2i}+E_{2j,2i-1}$$

in $\mathfrak{g}_{\pm(e_i+e_j)}$. The condition that $\|\mathrm{pr}_{\mathfrak{h}}(v)\| = \|\mathrm{pr}_{\mathfrak{h}}(v')\|$ for each choice of u' indicates each $u' \in \mathfrak{h}$, i.e., $\mathfrak{g}_{\pm(e_i\pm e_j)} \subset \mathfrak{h}$ for $1 \leqq i < j < n$. A similar argument can also show each e_i is a root of $\mathfrak{h}$, and $\widehat{\mathfrak{g}}_{\pm e_i} = \mathfrak{g}_{\pm(e_i+e_n)} + \mathfrak{g}_{\pm(e_i-e_n)}$. Now, up to the action of $\mathrm{Ad}(G)$, $\mathfrak{h}$ is uniquely determined, and is the standard $\mathfrak{so}(2n-1)$ in $\mathfrak{so}(2n)$. We can also use a similar argument as for the case 5B in Section 5, to prove directly the homogeneous space is symmetric, i.e., $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$. Then G/H is Riemannian symmetric, covered by the sphere $S^{2n-1} = \mathrm{SO}(2n)/\mathrm{SO}(2n-1)$ of positive constant curvature. It is a well-known fact that $v = e_1 + \cdots + e_n$ defines a CK vector field on it, because its centralizer $\mathrm{U}(n)$ acts transitively on S^{2n-1}.

8B. The case $v = e_1$

Finally we consider the case $v = e_1$ in Proposition 3.5. Denote $e_i' = \mathrm{pr}_{\mathfrak{h}}(e_i)$ for $1 \leqq i \leqq n$. They all have the same length. Either $e_i' + e_j'$ or $e_i' - e_j'$ is a root of $\mathfrak{h}$ because $v' = E_{2i-1,2j-1} - E_{2j-1,2i-1} \in \mathfrak{g}_{\pm e_i+e_j} + \mathfrak{g}_{\pm e_i-e_j}$ belongs to the $\mathrm{Ad}(G)$-orbit of v, i.e., $\|\mathrm{pr}_{\mathfrak{h}}(v')\| = \|\mathrm{pr}_{\mathfrak{h}}(v)\| > 0$.

Distinct roots. We first prove that $\{\pm e_1', \ldots, \pm e_n'\}$ consists of $2n$ distinct roots of $\mathfrak{h}$. For if not, then there are indices $i \neq j$ such that one of $e_i' \pm e_j' = 0$. Then $2e_i' = e_i' \mp e_j'$ is a root of $\mathfrak{h}$. If $2e_i' = \pm e_k' \pm e_l'$, then $\|e_i'\| = \|e_k'\| = \|e_l'\|$ tells us that the pairs $\{\pm e_i'\}$, $\{\pm e_k'\}$ and $\{\pm e_l'\}$ are equal. If $e_i' \neq \pm e_k'$ and $e_i' + e_k'$ is a root of $\mathfrak{h}$ then, using $\langle 2e_i', e_i' + e_k'\rangle > 0$, $2e_i' - (e_i' + e_k') = e_i' - e_k'$ is a root of $\mathfrak{h}$. Similarly if $e_i' - e_k'$ is a root of $\mathfrak{h}$ then $e_i' + e_k'$ is a root of $\mathfrak{h}$. Notice that $2e_i'$ is a long root and $e_i' \pm e_k'$ are short roots orthogonal to each other. This can only happen in a subalgebra of type $\mathfrak{c}_2$. Thus $2(e_i' + e_k') - 2e_i' = 2e_k'$ is also a long root of $\mathfrak{h}$ with $\langle e_i', e_k'\rangle = 0$, and there must be an index l with $e_k' = \pm e_l'$. From this argument all $\pm 2e_i'$s are long roots of $\mathfrak{h}$. If there are exactly m different pairs $\{\pm e_i'\}$, then $\mathfrak{h}$ is isomorphic to $\mathfrak{c}_m$.

Suitably choosing e_i, then we can assume $e_i' + e_j' \neq 0$ for $i \neq j$. Without loss of generality, we may assume that $\mathrm{pr}_{\mathfrak{h}}^{-1}(e_1')$ contains $\{e_1, \ldots, e_k\}$ but no other e_l and does not contain any $-e_l$. Then

$$\mathfrak{g}_{\pm(e_i-e_j)} \subset \mathfrak{m} \text{ for } 1 \leqq i < j \leqq k \quad\text{and}\quad \mathfrak{h}_{\pm 2e_1'} \subset \widehat{\mathfrak{g}}_{\pm 2e_1'} = \sum\nolimits_{1\leqq i<j\leqq k} \mathfrak{g}_{\pm(e_i+e_j)}.$$

Now $e_1' \subset [\mathfrak{h}_{\pm 2e_1'}, \mathfrak{h}_{\pm 2e_1'}]$ is realized as a matrix in $\mathfrak{so}(2k)$ corresponding to the left upper $(2k\times 2k)$–block, i.e., $e_1' \in \mathbb{R}e_1 + \cdots + \mathbb{R}e_k$. By our assumption, $e_i - e_j \in \mathfrak{t}\cap\mathfrak{m}$ for $1 \leqq i < j \leqq k$, so $e_1' = \frac{1}{k}(e_1 + \cdots + e_k)$.

Let G' be the analytic subgroup of G with Lie algebra $\mathfrak{g}' = \mathfrak{so}(2k)$ corresponding to the upper left $2k \times 2k$ corner and H' the analytic subgroup for $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h} = \mathbb{R}e_1' + \mathfrak{h}_{\pm 2e_1'}$. Then $\mathfrak{h}' \cong \mathfrak{a}_1$. Also,

$$\mathfrak{g}' \cap \mathfrak{m} = \sum\nolimits_{1\leqq i<k} \mathbb{R}(e_{i+1} - e_i) + \sum\nolimits_{1\leqq i<j\leqq k} \mathfrak{g}_{\pm(e_i-e_j)}.$$

The restriction to $\mathfrak{g}'$ of the bi-invariant inner product on $\mathfrak{g}$ defines a Riemannian normal homogeneous metric on G'/H'. The orthogonal decomposition $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$

coincides with the decomposition $\mathfrak{g}' = (\mathfrak{g}' \cap \mathfrak{h}) + (\mathfrak{g}' \cap \mathfrak{m})$. So the vector $v = e_1 \in \mathfrak{g}'$ defines a CK vector field on the Riemannian normal homogeneous space G'/H'.

The orbit $\mathrm{Ad}(G')v$ contains an orthonormal basis of $\mathfrak{g}'$ which in turn contains an orthonormal basis for $\mathfrak{t} \cap \mathfrak{g}'$. By Lemma 3.3,

$$2k - 1 = \frac{\dim \mathfrak{g}'}{\dim(\mathfrak{t} \cap \mathfrak{g}')} = \frac{\dim \mathfrak{h}'}{\dim(\mathfrak{h}' \cap \mathfrak{t})} = 3, \quad \text{so} \quad k = 2.$$

Suitably permuting the e_i, we can assume the distinct e'_i are given by

$$\{e'_1 = e'_2, e'_3 = e'_4, \ldots, e'_{n-1} = e'_n\}.$$

Then $\dim \widehat{\mathfrak{g}}_{\pm(e'_1 \pm e'_3)} = 8$, in which 2 dimensions belong to $\mathfrak{h}$ and the other 6 dimensions belong to $\mathfrak{m}$. The $\mathrm{Ad}(G)$–orbit of $v = e_1$ contains an orthonormal basis of $\mathfrak{t}$ consisting of all the e_i. It also contains an orthonormal basis of $\widehat{\mathfrak{g}}_{\pm(e_1+e_3)} + \widehat{\mathfrak{g}}_{\pm(e_1-e_3)}$. As in Lemma 3.3, we have

$$\frac{\dim \mathfrak{t}}{\dim(\mathfrak{t} \cap \mathfrak{h})} = \frac{\dim(\widehat{\mathfrak{g}}_{\pm(e_1+e_3)} + \widehat{\mathfrak{g}}_{\pm(e_1-e_3)})}{\dim(\mathfrak{h} \cap (\widehat{\mathfrak{g}}_{\pm(e_1+e_3)} + \widehat{\mathfrak{g}}_{\pm(e_1-e_3)}))},$$

which is a contradiction because the left side is 2 and the right side is 4. In summary, we have proved

Lemma 8.1. *Suppose $\mathfrak{g} = \mathfrak{d}_n$ with $n > 3$. Suppose that $v = e_1 \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space G/H. Denote $e'_i = \mathrm{pr}_{\mathfrak{h}}(e_i)$ for $1 \leqq i \leqq n$, then $\{\pm e'_1, \ldots, \pm e'_n\}$ has cardinality $2n$.*

Second, we consider the case where one of $e'_i \pm e'_j$ is a root of $\mathfrak{h}$ and the other is not, for any $1 \leqq i < j \leqq n$. Given a root α' of $\mathfrak{h}$, we denote $\dim \widehat{\mathfrak{g}}_{\pm\alpha'} = 2k_{\alpha'}$ and $\dim(\mathfrak{t} \cap \mathfrak{h}) = h$, and we express $\widehat{\mathfrak{g}}_{\pm\alpha'} = \sum_{i=1}^{k_{\alpha'}} \mathfrak{g}_{\pm\alpha_i}$. By an argument similar to that of Lemma 6.4, we see

$$\frac{n}{h} = \frac{\dim \mathfrak{t}}{\dim(\mathfrak{t} \cap \mathfrak{h})} = \dim \widehat{\mathfrak{g}}_{\pm\alpha'} = 2k_{\alpha'}, \quad \text{i.e.,} \quad n = 2k_{\alpha'}h.$$

Thus $k = k_{\alpha'}$ is independent of the choice of α'. Denote the number of roots of $\mathfrak{g}$ and $\mathfrak{h}$ by $|\Delta|$ and $|\Delta'|$, respectively. Then $|\Delta| = 2k|\Delta'|$. But $|\Delta| = 2n(n-1)$, which implies $|\Delta'| = 2h(2kh-1) \geqq 4h^2 - 2h$. Calculate $|\Delta'|$ and $4h^2 - 2h$ for each simple Lie algebra:

	$\mathfrak{a}_q$	$\mathfrak{b}_q$	$\mathfrak{c}_q$	$\mathfrak{d}_q$	$\mathfrak{g}_2$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
$\lvert\Delta'\rvert$	$q(q+1)$	$2q^2$	$2q^2$	$2q(q-1)$	12	48	72	126	240
$4h^2 - 2h$	$4q^2 - 2q$	$4q^2 - 2q$	$4q^2 - 2q$	$4q^2 - 2q$	12	56	132	182	240

Now for any compact simple Lie algebra $\mathfrak{h}$ of rank h, the number $|\Delta'|$ of all roots satisfies $|\Delta'| \leqq 4h^2 - 2h$, with equality if and only if $k = 1$ and $\mathfrak{h}$ is isomorphic to $\mathfrak{a}_1$, $\mathfrak{g}_2$ or $\mathfrak{e}_8$.

If $\mathfrak{h}$ is not simple, say $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p \oplus \mathbb{R}^q$, let h_i denote the rank, and Δ'_i the root system, of $\mathfrak{h}_i$. Then

$$|\Delta'| = \sum_{i=1}^{p} |\Delta'_i| \leqq \sum_{i=1}^{p} (4h_i^2 - 2h_i) \leqq 4\left(\sum_{i=1}^{p} h_i\right)^2 - 2\sum_{i=1}^{p} h_i \leqq 4h^2 - 2h. \tag{8.2}$$

If $q > 0$ the last $\leqq$ in (8.2) is strict, and if $p > 1$ then the second to last $\leqq$ in (8.2) is strict. So in those cases $|\Delta'| < 4h^2 - 2h$. Now $\mathfrak{h}$ is simple.

Since $\mathfrak{h}$ is simple, it is isomorphic to $\mathfrak{a}_1$, $\mathfrak{g}_2$ or $\mathfrak{e}_8$, Here $\mathfrak{a}_1$ would imply $n = 2$, while we are assuming $n > 3$, so $\mathfrak{h}$ is $\mathfrak{g}_2$ or $\mathfrak{e}_8$. With some sign changes $e_i \mapsto -e_i$, now every $e'_i - e'_{i+1}$ is a root of $\mathfrak{h}$. As $\dim \widehat{\mathfrak{g}}_{\pm(e'_i - e'_{i+1})} = 2k_{e'_i - e'_{i+1}} = 2$, i.e., $\mathfrak{h}_{\pm(e'_i - e'_{i+1})} = \mathfrak{g}_{\pm(e_i - e_{i+1})}$,

$$e'_i - e'_{i+1} \in [\mathfrak{g}_{\pm(e_i - e_{i+1})}, \mathfrak{g}_{\pm(e_i - e_{i+1})}] \subset \mathfrak{t} \cap \mathfrak{h} \text{ for } 1 \leqq i < n.$$

Thus $\dim(\mathfrak{t} \cap \mathfrak{h}) \geqq n - 1 > h$. This is a contradiction.

Summarizing the above argument:

Lemma 8.3. *Assume $\mathfrak{g} = \mathfrak{d}_n$ with $n > 3$ and suppose that $v = e_1 \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space G/H. Denote $e'_i = \mathrm{pr}_{\mathfrak{h}}(e_i)$ for $1 \leqq i \leqq n$. Then there exist $i < j$ such that both $e'_i \pm e'_j$ are roots of $\mathfrak{h}$.*

$\mathfrak{h}$ *is simple.* Third, we will prove that $\mathfrak{h}$ is a compact simple Lie algebra. Suitably permuting e'_i we can assume both $e'_1 + e'_2$ and $e'_1 - e'_2$ are roots of $\mathfrak{h}$. They are orthogonal. For each $i > 2$, either $e'_1 + e'_i$ or $e'_1 - e'_i$ is a root of $\mathfrak{h}$. Suitably replace some e_i by its negative; then we can assume $e'_1 - e'_i$ is a root of $\mathfrak{h}$. Then we have $c_i = \pm 1$ for $i > 2$ such that $\langle c_i e'_2, e'_1 - e'_i \rangle \geq 0$. Then $\langle e'_1 + c_i e'_2, e'_1 - e'_i \rangle \geq \langle e'_1, e'_1 - e'_i \rangle > 0$, in other words $e'_1 - e'_i$ is a root for the same simple component of $\mathfrak{h}$ as $e'_1 + c_i e'_2$. The roots $\{e'_1 - e'_i \mid 1 < i \leqq n\} \cup \{e'_1 + e'_2\}$ of $\mathfrak{h}$ generate $\mathfrak{t} \cap \mathfrak{h}$, so $\mathfrak{h}$ is semi-simple. If $\mathfrak{h}$ is not simple then the above argument shows $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ where $e'_1 - e'_2$ is a root of $\mathfrak{h}_1$, $e'_1 + e'_2$ is a root of $\mathfrak{h}_2$, and $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are simple.

Suppose $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ as just described. Suppose that there are indices $i \neq j$, both > 2, $e'_1 - e'_i$ is a root of $\mathfrak{h}_1$, and $e'_1 - e'_j$ is a root of $\mathfrak{h}_2$. Because $e'_2 + e'_i = (e'_1 + e'_2) - (e'_1 - e'_i) \notin \mathfrak{t} \cap (\mathfrak{h}_1 \cup \mathfrak{h}_2)$ is not a root of $\mathfrak{h}$, $e'_2 - e'_i \in \mathfrak{t} \cap \mathfrak{h}_1$ is a root of $\mathfrak{h}_1$. Similarly, $e'_2 + e'_j \in \mathfrak{t} \cap \mathfrak{h}_2$ is a root of $\mathfrak{h}_2$. Then neither $e'_i + e'_j = (e'_i - e'_2) + (e'_2 + e'_j)$ nor $e'_i - e'_j = (e'_i - e'_1) + (e'_1 - e'_j)$ is contained in $\mathfrak{t} \cap (\mathfrak{h}_1 \cup \mathfrak{h}_2)$, so they are not roots of $\mathfrak{h}$. That is a contradiction. So all $e'_1 - e'_i$ for $i > 2$ are roots of the same $\mathfrak{h}_1$ or $\mathfrak{h}_2$.

Suitably choose e_2 from $\pm e_2$ so that $e'_1 - e'_i$ is a root of $\mathfrak{h}_1$ for $1 < i \leqq n$. It implies $\mathrm{rk}\mathfrak{h}_1 = \mathrm{rk}\mathfrak{h} - 1$, and then $\mathfrak{h}_2 \cong \mathfrak{a}_1$ with the only roots $\pm(e'_1 + e'_2)$. There does not exist any root $e'_i + e'_j$ of $\mathfrak{h}$ for $2 < i < j \leq n$, because otherwise it is a root of $\mathfrak{h}_1$, and it implies $\mathfrak{t} \cap \mathfrak{h}_1 = \mathfrak{t} \cap \mathfrak{h}$ which is a contradiction. As $e'_1 + e'_2 \perp e'_1 - e'_i$, $e'_1 + e'_2 \in \mathfrak{h}$ is orthogonal to $e_1 - e_i = (e'_1 - e'_i) + \mathrm{pr}_{\mathfrak{m}}(e_1 - e_i)$ as well, for $1 < i \leqq n$. That implies $e'_1 + e'_2 \in \mathbb{R}(e_1 + \cdots + e_n)$. Let G' be the analytic subgroup of G with Lie algebra

$$\mathfrak{g}' = \sum_{1 \leq i < n} \mathbb{R}(e_i - e_{i+1}) + \sum\nolimits_{1 \leqq i < j \leqq n} \mathfrak{g}_{\pm(e_i - e_j)}.$$

It is the standard $\mathfrak{su}(n-1)$ in $\mathfrak{so}(2n)$. The identity component H' of $G'\cap H$ has Lie algebra $\mathfrak{h}' = \mathfrak{h}_1$. The restriction of the bi-invariant inner product of $\mathfrak{g}$ to $\mathfrak{g}'$ defines a normal homogeneous metric on G'/H'. For $(\mathfrak{g}', \mathfrak{h}')$, we have the decomposition

$$\mathfrak{g}' = \mathfrak{t}\cap\mathfrak{g}' + \sum\nolimits_{\gamma\in\mathfrak{t}\,\cap\mathfrak{h}_1} \widehat{\mathfrak{g}}'_{\pm\gamma},$$

in which $\widehat{\mathfrak{g}}'_{\pm\gamma}$ coincides with $\widehat{\mathfrak{g}}_{\pm\gamma}$ because $\langle\gamma, e_1+\cdots+e_n\rangle = 0$ whenever γ is a root of $\mathfrak{h}_1$. The orthogonal decomposition $\mathfrak{g}' = \mathfrak{h}'+\mathfrak{m}'$ is given by $\mathfrak{g}' = (\mathfrak{g}'\cap\mathfrak{h})+(\mathfrak{g}'\cap\mathfrak{m})$. Decompose $v = e_1$ as $v' + v_{\mathfrak{c}}$ where $v' = \frac{1}{n}((n-1)e_1 - e_2 - \cdots - e_n) \in \mathfrak{t}\cap\mathfrak{g}'$ and $v_{\mathfrak{c}} = \frac{1}{n}(e_1+\cdots+e_n)$ with $[v_{\mathfrak{c}}, \mathfrak{g}'] = 0$ and $\langle v_{\mathfrak{c}}, \mathfrak{g}'\rangle = 0$. So v' defines a CK vector field on the Riemannian normal homogeneous space G'/H'.

By Proposition 5.6, n is even, say $n = 2k$ with $k \geq 2$, and conjugation by an element of $\mathrm{Ad}\,(G')$ carries $\mathfrak{h}'$ to the standard $\mathfrak{sp}(k)$ in $\mathfrak{su}(n)$. So $\mathfrak{h} \cong \mathfrak{c}_k \oplus \mathfrak{a}_1$. Since $\mathrm{Ad}(G)(v)$ contains an orthonormal basis for $\mathfrak{g}$ as well as an orthonormal basis for $\mathfrak{t}$, Lemma 3.3 implies

$$2n-1 = \frac{\dim\mathfrak{g}}{\dim\mathfrak{t}} = \frac{\dim\mathfrak{h}}{\dim(\mathfrak{t}\cap\mathfrak{h})} = \frac{2k^2+k+3}{k+1}, \text{ not an integer unless } k=3.$$

But if $k = 3$ then $2n-1 = 5$ while $(2k^2+k+3)/(k+1) = 6$. So in any case this is a contradiction. Summarizing the above arguments, we have proved

Lemma 8.4. *Assume $\mathfrak{g} = \mathfrak{d}_n$ with $n > 3$ and suppose that $v = e_1 \in \mathfrak{t}$ defines a CK vector field on the Riemannian normal homogeneous space G/H. Then the Lie algebra $\mathfrak{h}$ is simple.*

Since $\mathfrak{h}$ is simple, the ratio $r = \|e_1'+e_2'\|/\|e_1'-e_2'\|$ can only be $(\sqrt{3})^{\pm1}$, $(\sqrt{2})^{\pm1}$ or 1. The e_i' cannot all be mutually orthogonal because that would imply $\mathfrak{t}\subset\mathfrak{h}$, a contradiction. If $r \neq 1$, so $\mathfrak{h}$ has two root lengths and must be $\mathfrak{b}_h$ or $\mathfrak{c}_h$ with $h = \dim(\mathfrak{t}\cap\mathfrak{h})$, or $\mathfrak{f}_4$ with $h = 4$, or $\mathfrak{g}_2$ with $h = 2$. If $r = 1$ and we have e_i' and e_j', such that $i \neq j$ and $\langle e_i', e_j'\rangle \neq 0$, either $e_i'+e_j'$ or $e_i'-e_j'$ is a root whose length is different from that of $e_1' \pm e_2'$. So in that case also $\mathfrak{h}$ is isomorphic to $\mathfrak{b}_h$, $\mathfrak{c}_h$, $\mathfrak{f}_4$ or $\mathfrak{g}_2$.

As seen earlier, $\mathrm{Ad}(G)(v)$ contains orthonormal bases of $\mathfrak{t}$ and $\mathfrak{g}$. By Lemma 3.3,

$$2n-1 = \frac{\dim\mathfrak{g}}{\dim\mathfrak{t}} = \frac{\dim(\mathfrak{h})}{\dim(\mathfrak{t}\cap\mathfrak{h})}. \tag{8.5}$$

The $\mathfrak{g}_2$ case. When $\mathfrak{h}$ is isomorphic to $\mathfrak{g}_2$, the right side of (8.5) is 7, so $n = 4$. In this case e_i' and e_j' must have an angle $\pi/3$ or $2\pi/3$, when $i \neq j$. The only possible choices for all $\pm e_i'$ are $\pm e_1'$, $\pm e_2'$ and the shorter pair among $\pm e_1' \pm e_2'$. One cannot have four different pairs of $\pm e_i'$. This is a contradiction.

The $\mathfrak{b}_h$ and $\mathfrak{c}_h$ cases. When $\mathfrak{h}$ is isomorphic to $\mathfrak{b}_h$ or $\mathfrak{c}_h$, the right side of (8.5) is $2h+1$, so $n = h+1$. Note that $\mathfrak{h}\cap\mathfrak{t}$ is a hyperplane in $\mathfrak{t}$, so the complement has 2 components. With suitable sign changes $e_i \mapsto -e_i$ we may suppose that all e_i belong to the same component and they all have the same projection to $\mathfrak{m}$. So $e_i - e_j \in \mathfrak{h}$ for $i < j$ and $\mathfrak{m}\cap\mathfrak{t} = \mathbb{R}(e_1+\cdots+e_n)$.

Since $e_1'-e_2'=e_1-e_2$ is a long root of $\mathfrak{h}$, with length $\sqrt{2}$, $e_1'+e_2'$ can have length 1 only when $n=4$. In that case $\pm(e_1'+e_2')=\mp(e_3'+e_4')$, $\pm(e_1'+e_3)=\mp(e_2'+e_4')$ and $\pm(e_1'+e_4')=\mp(e_2'+e_3')$ are the only three possible pairs of short roots that are orthogonal to each other, so $\mathfrak{h}$ is isomorphic to $\mathfrak{b}_3$, where the root system contains all above short roots and all long roots of the form $\pm(e_i-e_j)$. Up to $\mathrm{Ad}(G)$ conjugacy now $\mathfrak{h}$ is uniquely determined, and it satisfies the symmetric space condition $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$. By a suitable outer automorphism of $\mathfrak{d}_4$, it can be changed back to the standard $\mathfrak{so}(7)$ inside $\mathfrak{g}=\mathfrak{so}(8)$, and so G/H is a locally Riemannian symmetric space, covered by the round $S^7=\mathrm{Spin}(8)/\mathrm{Spin}(7)=\mathrm{SO}(8)/\mathrm{SO}(7)$. At the same time, the automorphism changes $v=e_1$ to a vector of the form $\pm e_1\pm\cdots\pm e_4$, which is well known to define a CK vector field on S^7.

The $\mathfrak{f}_4$ *case.* For the rest of this section, $\mathfrak{h}$ is isomorphic to $\mathfrak{f}_4$. Then the right side of (8.5) is 13, so $n=7$. For simplicity, we rescale the inner product of $\mathfrak{g}$, so that $\|e_i'\|=1$ for $1\leqq i\leqq n$. Then $\langle e_1',e_2'\rangle$ must be 0 or $\pm1/3$.

Our next step is to prove that the case that $e_1'\pm e_2'$ are roots of $\mathfrak{h}$ with $\langle e_1',e_2'\rangle=\pm1/3$ is impossible. Assume conversely it happens, then one of $e_1'\pm e_2'$ is short with length $2/\sqrt{3}$ and the other is long with length $2\sqrt{2}/\sqrt{3}$. For any indices $i\neq j$, either $e_i'+e_j'$ or $e_i'-e_j'$ is a root of $\mathfrak{h}$, with length $2/\sqrt{3}$ or $2\sqrt{2}/\sqrt{3}$. So $\langle e_i,e_j\rangle=\pm1/3$ for $1\leqq i<j\leqq n$.

We make appropriate sign changes $e_i\mapsto -e_i$ so that $\langle e_1',e_i'\rangle=-1/3$ for all $i>1$. So if $e_1'-e_i'$ is a root of $\mathfrak{h}$, it is a long root, and if $e_1'+e_i'$ is a root of $\mathfrak{h}$, it is a short root.

For any $i>2$, if $e_1'+e_i'$ is a short root of $\mathfrak{h}$, then $e_1'+e_i'$ has an angle $\pi/4$ with the long root $e_1'-e_2'$, because $\langle e_1'+e_i',e_1'-e_2'\rangle=1-\langle e_i',e_2'\rangle>0$. So

$$\langle e_1'+e_i',e_1'-e_2'\rangle=|e_1'+e_i'|\cdot|e_1'-e_2'|\cdot\cos\frac{\pi}{4}=\frac{2}{\sqrt{2}}\cdot\frac{2\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{2}}{2}=\frac{4}{3}.$$

That implies $\langle e_2',e_i'\rangle=-1/3$. If $e_1'-e_i'$ is a long root of $\mathfrak{h}$, then $e_1'-e_i'$ has an angle $\frac{\pi}{4}$ with the short root $e_1'+e_2'$, because $\langle e_1'-e_i',e_1'+e_2'\rangle=1-\langle e_i',e_2'\rangle>0$. Arguing as above, $\langle e_2',e_i'\rangle=-1/3$.

Assume $i\neq j$ with $\{i,j\}\subset\{3,\ldots,n\}$. If $e_1'-e_i'$ and $e_1'-e_j'$ are both long roots of $\mathfrak{h}$, they must have an angle $\pi/3$ because $\langle e_1'-e_i',e_1'-e_j'\rangle=5/3+\langle e_i',e_j'\rangle>0$. So $\langle e_1'-e_i',e_1'-e_j'\rangle=4/3$, which implies $\langle e_i',e_j'\rangle=-1/3$. If $e_1'-e_i'$ is a long root and $e_1'+e_j'$ is short they must have an angle $\pi/4$ because $\langle e_1'-e_i',e_1'+e_j'\rangle=1-\langle e_i',e_j'\rangle>0$. So $\langle e_1'-e_i',e_1'+e_j'\rangle=\frac{4}{3}$, which implies $\langle e_i',e_j'\rangle=-1/3$.

Based on the above observations, we see if there is e_i', $i>2$, such that $e_1'-e_i'$ is a long root of $\mathfrak{h}$, we can suitably permute e_j, $j>2$ to make $i=3$. Then the matrix $(\langle e_p',e_q'\rangle)_{1\leqq p,q\leqq5}$ must be of the form

$$\begin{pmatrix}1&-1/3&-1/3&-1/3&-1/3\\-1/3&1&-1/3&-1/3&-1/3\\-1/3&-1/3&1&-1/3&-1/3\\-1/3&-1/3&-1/3&1&1/3\\-1/3&-1/3&-1/3&1/3&1\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}1&-1/3&-1/3&-1/3&-1/3\\-1/3&1&-1/3&-1/3&-1/3\\-1/3&-1/3&1&-1/3&-1/3\\-1/3&-1/3&-1/3&1&-1/3\\-1/3&-1/3&-1/3&-1/3&1\end{pmatrix},$$

which is non-singular in either case. That contradicts $\dim(\mathfrak{t}\cap\mathfrak{h})=4$. So $\mathfrak{h}$ has no long root of the form $e_1'-e_i'$, $i>2$. Furthermore, $\mathfrak{h}$ has no long root of the form $e_2' - e_i'$, $i > 2$; for if $e_2' - e_i'$ is a long root with $i > 2$, then it has an angle $\frac{2\pi}{3}$ with the long root $e_1' - e_2'$ of $\mathfrak{h}$, i.e., $e_1' - e_i' = (e_1' - e_2') + (e_2' - e_i')$ is a long root of $\mathfrak{h}$, which contradicts our previous statement. The number of long roots with the form $\pm e_i' \pm e_j'$ is at most 22, consisting of $\pm(e_1' - e_2')$, and at most a pair of long roots from each set $\{\pm e_i' \pm e_j'\}$ for $2 < i < j \leqq 7$. The number of long roots of $\mathfrak{h}$ cannot reach 24, which is a contradiction.

This completes the proof, in the $\mathfrak{f}_4$ case, that we do not have roots $e_1' \pm e_2'$ of $\mathfrak{h}$ with $\langle e_1', e_2' \rangle = \pm 1/3$.

Next, in the $\mathfrak{f}_4$ case, we consider the situation where $\langle e_1', e_2' \rangle = 0$ and $e_1' \pm e_2'$ are roots of $\mathfrak{h}$. Since $\mathfrak{h}$ has no root of length 2, any short root $\pm e_i' \pm e_j'$ has the length 1 with $\langle e_i', e_j' \rangle = \pm\frac{1}{2}$, and any long root $\pm e_i' \pm e_j'$ has the length $\sqrt{2}$ with $\langle e_i', e_j' \rangle = 0$. If i, j and k are distinct, $\langle e_i', e_j' \rangle = 0$, and $\langle e_i', e_k' \rangle = 0$, then for suitable $c_1 = \pm 1$ and $c_2 = \pm 1$ the roots $e_i' + c_1 e_j'$ and $e_i' + c_2 e_k'$ of $\mathfrak{h}$ are long. Because

$$\langle e_i' + c_1 e_j', e_i' + c_2 e_j' \rangle = 1 \pm \langle e_i', e_j' \rangle > 0,$$

the combination $c_1 e_j' - c_2 e_k' = (e_i' + c_1 e_j') - (e_i' + c_2 e_j')$ is a long root of $\mathfrak{h}$. That implies $\langle e_j', e_k' \rangle = 0$. Now $\{1, \ldots, 7\}$ is a disjoint union $\coprod_{a \in \mathcal{A}} S_a$ such that (i) if $i \neq j$ in the same S_a then $e_i' \perp e_j'$ and (ii) if $i \in S_a$ and $j \in S_b$ with $a \neq b$ then $\langle e_i, e_j \rangle = \pm 1/2$. If $i \neq j$ are in the same S_a , and $e_i \pm e_j$ are both long roots of $\mathfrak{h}$, then for whenever $k \in S_a$, $i \neq k \neq j$, there is a long root of the form $e_i \pm e_k$. It has angle $\pi/3$ with both $e_i + e_j$ and $e_i - e_j$, so both $e_k \pm e_j$ are long roots of $\mathfrak{h}$. Similarly, both $e_i \pm e_k$ are long roots of $\mathfrak{h}$. Extending this argument, whenever $k \neq l$ in the same S_a, both $e_k \pm e_l$ are long roots of $\mathfrak{h}$.

Each $|S_a| \leqq 4$. For if S_a contains five elements then $\dim \mathfrak{t} \cap \mathfrak{h} > 4$ which is impossible.

Suppose $|S_a| = 4$ with $\{1, 2\} \subset S_a$. We may permute the e_i so that $S_a = \{1, 2, 3, 4\}$. Then $\{e_1', \ldots, e_4'\}$ is an orthonormal basis of $\mathfrak{t} \cap \mathfrak{h}$. By our previous observation, $\pm e_i' \pm e_j'$ provide all long roots of $\mathfrak{h}$. From the standard presentation (2.12) of $\mathfrak{f}_4$, we can see, for any orthogonal pair of long roots α' and β' of $\mathfrak{h}$, $\frac{1}{2}(\alpha' \pm \beta')$ are short roots of $\mathfrak{h}$. Thus the $\pm e_i' = \pm\frac{1}{2}((e_i' + e_j') + (e_i' - e_j'))$, $1 \leqq i \leqq 4$ and $i \neq j$, are short roots of $\mathfrak{h}$. But $e_i' = \frac{1}{2}(\pm e_1' \pm \cdots \pm e_4')$ for $i = 5$, 6 and 7, so any short root $\pm e_i' \pm e_j'$ of $\mathfrak{h}$, for $1 \leqq i \leqq 4 < j \leqq 7$, is a vector of the form $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_4)$. And each set $\{\pm e_j' \pm e_k'\}$, $4 < j < k \leqq 7$, contains at most one pair of short roots, resulting in 3 pairs in total. That is not enough for the presentation just above for all the $\pm e_i'$, $1 \leqq i \leqq 4$. There is at least a short root e_i' of $\mathfrak{h}$, $1 \leq i \leq 4$, which cannot be given as any $\pm e_j' \pm e_k'$. This is a contradiction to our observation for the root system of $\mathfrak{h}$. So there is no S_a with $|S_a| = 4$ and $\{1, 2\} \subset S_a$.

Assume that one of the sets S_b satisfies $|S_b| = 4$ and it does not contain 1 and 2. Then we can permute the e_i so that $S_b = \{4, 5, 6, 7\}$. From the argument above, if $4 < i < j \leqq 7$, then either $e_i' + e_j'$ or $e_i' - e_j'$ is not a root of $\mathfrak{h}$. So S_b can at most provide 6 pairs of long roots of $\mathfrak{h}$. The only way that $\mathfrak{h}$ can have 12 pairs of long roots is that 1, 2 and 3 must belong to the same S_a , and $\pm e_i' \pm e_j'$ are long roots of $\mathfrak{h}$ for $1 \leqq i < j \leqq 3$. In that case we look at the short roots. Each set $\{\pm e_i' \pm e_j'\}$, $1 \leqq i \leqq 4 < j \leqq 7$, can only provide one pair of short roots, and in fact it must provide one pair to make the 12 pairs of short roots of $\mathfrak{h} = \mathfrak{f}_4$. Now, if α' is a root of $\mathfrak{h}$, then $\mathfrak{h}_{\pm\alpha'} = \widehat{\mathfrak{g}}_{\pm\alpha'}$ is a root plane of $\mathfrak{g}$, say $\mathfrak{h}_{\pm\alpha'} = \mathfrak{g}_{\pm\alpha}$. Then α' is also a root of $\mathfrak{g}$ because

$$\mathbb{R}\alpha' = [\mathfrak{h}_{\pm\alpha'}, \mathfrak{h}_{\pm\alpha'}] = [\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\alpha}] = \mathbb{R}\alpha.$$

Then the root system of $\mathfrak{h}$ is a subset of the root system of $\mathfrak{g}$. That is impossible because all the roots of $\mathfrak{g}$ have the same length. This completes the argument that none of the S_a can contain more than 3 elements.

Since it has at most 3 elements, each S_a can contribute no more than 12 long roots. For $\mathfrak{h}$ to have 24 long roots, $\{1, \ldots, 7\} = \coprod_{a\in\mathcal{A}} S_a$ is the union of three subsets, two with three elements each, one with one element. Suitably permuting the e_i we can assume $\mathcal{A} = \{a, b, c\}$, $S_a = \{1, 2, 3\}$, $S_b = \{4, 5, 6\}$, and $S_c = \{7\}$. All $\pm e_i \pm e_j$ must be long roots of $\mathfrak{h}$ for $1 \leqq i < j \leqq 3$ or $4 \leqq i < j \leqq 6$; they give all 24 long roots of $\mathfrak{h}$.

From the above argument, $\pm e'_i$, $1 \leqq i < 7$, are short roots of $\mathfrak{h}$, because they can be presented as $\frac{1}{2}(\pm\alpha' \pm \beta')$ for an orthogonal pair of long roots α' and β' of $\mathfrak{h}$. So for $1 \leqq i < 7$, we can find j and k, such that $e'_i = \pm e'_j \pm e'_k$. It is easy to see that, if $i < 4$, then $j > 3$ and $k > 3$; furthermore, j and k cannot both be chosen from $\{4, 5, 6\}$, so one of them is from $\{4, 5, 6\}$, which must be different for different e'_is, and the other is just 7. So suitably substitute some e'_is by $-e'_i$s, and we can have

$$e'_1 + e'_4 + e'_7 = 0, e'_2 + e'_5 + e'_7 = 0 \quad \text{and} \quad e'_3 + e'_6 + e'_7 = 0,$$

i.e., $\mathfrak{m}$ is linearly spanned by $e_1 + e_4 + e_7$, $e_2 + e_5 + e_7$ and $e_3 + e_6 + e_7$. Direct calculation shows, for $v = e_1$ and $v' = e_7$ in the $\mathrm{Ad}(G)$-orbit of v,

$$\|\mathrm{pr}_{\mathfrak{m}}(v)\| < \|\mathrm{pr}_{\mathfrak{m}}(v')\|,$$

which is a contradiction.

To summarize, we have the following proposition.

Proposition 8.6. *Suppose that G is a compact connected simple Lie group with $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{d}_n$ where $n > 1$, H is a closed subgroup with $0 < \dim H < \dim G$, and G/H is a Riemannian normal homogeneous space. Assume there is a nonzero vector $v \in \mathfrak{g}$ which defines a CK vector field on G/H. Then G/H is a locally Riemannian symmetric space which is covered by the sphere $S^{2n-1} = \mathrm{Spin}(2n)/\mathrm{Spin}(2n-1) = \mathrm{SO}(2n)/\mathrm{SO}(2n-1)$.*

Theorem 1.1 follows by combining Propositions 3.1, 4.3, 5.6, 6.6, 7.1 and 8.6.

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