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On the Analytic Structure of Commutative Nilmanifolds

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Abstract In the classification theorems of Vinberg and Yakimova for commutative nilmanifolds, the relevant nilpotent groups have a very surprising analytic property. The manifolds are of the form $G/K = N \rtimes K/K$ where, in all but three cases, the nilpotent group N has irreducible unitary representations whose coefficients are square integrable modulo the center Z of N. Here we show that, in those three "exceptional" cases, the group N is a semidirect product $N_1 \rtimes \mathbb{R}$ or $N_1 \rtimes \mathbb{C}$ where the normal subgroup N_1 contains the center Z of N and has irreducible unitary representations whose coefficients are square integrable modulo Z. This leads directly to explicit harmonic analysis and Fourier inversion formulae for commutative nilmanifolds.

Keywords Commutative nilmanifold · Weakly symmetric space · Square integrable representation · Fourier inversion formula

Mathematics Subject Classification Primary 22E27, 22E30, 22E47 · Secondary 53C35, 53C60

1 Introduction

A commutative space $X = G/K$, or equivalently a Gelfand pair (G, K), consists of a locally compact group G and a compact subgroup K such that the convolution algebra $L^1(K\backslash G/K)$ is commutative. When G is a connected Lie group it is equivalent to say that the algebra $\mathcal{D}(G, K)$ of G-invariant differential operators on G/K is commutative. We say that the commutative space G/K is a commutative nilmanifold if it is a nilmanifold in the sense that some nilpotent analytic subgroup N of G acts transitively.

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When G/K is connected and simply connected it follows that N is the nilradical of G, that N acts simply transitively on G/K, and that G is the semidirect product group $N \rtimes K$, so that $G/K = (N \rtimes K)/K$. In this paper we study commutative nilmanifolds $G/K = (N \rtimes K)/K$, examine the structure of N, and describe the consequences for harmonic analysis on G/K.

In Sect. 2 we review the relevant material on commutative spaces, Riemannian nilmanifolds, and commutative nilmanifolds. There we recall the Vinberg classification of irreducible commutative nilmanifolds and the Yakimova classification of those that satisfy certain technical conditions.

In Sect. 3 we review the theory of square integrable and stepwise square integrable representations of nilpotent Lie groups, and we indicate how it applies to commutative nilmanifolds. With three exceptions the commutative nilmanifolds $(N \rtimes K)/K$, described in the tables of Sect. 2, have the property that N has square integrable (modulo the center of N) representations. We then check each of these three "exceptional" cases and verify stepwise (in fact 2-step) square integrability for them.

In Sect. 4 we combine the results of Sect. 3 with principal orbit theory for the action of K on $\mathfrak{z}^*$ to obtain explicit Plancherel and Fourier Inversion theorems for our commutative nilmanifolds.

Finally, in Sect. 5 we specialize these results to weakly symmetric Riemannian nilmanifolds and extend that specialization to weakly symmetric Finsler nilmanifolds.

2 Commutative Nilmanifolds

A homogeneous space $X = G/K$ is called *commutative*, and the pair (G, K) is called a *Gelfand pair*, when G is a locally compact group, K is a compact subgroup, and the convolution algebra $L^1(K\backslash G/K)$ is commutative. Here $L^1(K\backslash G/K)$ denotes the space of L^1 functions on G that satisfy $f(kxk') = f(x)$ for $x \in G$ and $k, k' \in K$, and the composition in $L^1(K\backslash G/K)$ is the usual convolution $(f * h)(g) = \int_G f(x)h(x^{-1}g)d\mu_G(x)$ on G. In assembling the material we need on commutative spaces we will depend on the exposition and results from [13].

If G is a Lie group and K is a closed subgroup we write $\mathcal{D}(G, K)$ for the algebra of G-invariant differential operators on G/K. A theorem of Thomas [9] says: If G is a connected Lie group and K is a compact subgroup, then (G, K) is a Gelfand pair if and only if $\mathcal{D}(G, K)$ is commutative.

By *nilmanifold* we mean a differentiable manifold on which a nilpotent Lie group acts transitively. By *commutative nilmanifold* we mean a commutative space G/K such that G is a Lie group and a closed nilpotent subgroup N of G acts transitively. In that notation, if G/K is simply connected then ([12, Theorem 4.2], or see [13, Theorem 13.1.6]) N is the nilpotent radical of G, N acts simply transitively on G/K, and G is the semidirect product $N \rtimes K$. Further, there are several independent proofs that N is abelian or 2-step nilpotent; see [13, §13.1].

We look at the classification for reasons that will emerge in Sect. 3.

Let G/K be a connected simply connected commutative nilmanifold, $G = N \rtimes K$. We first consider the case where G/K and (G, K) are *irreducible* in the sense that $[\mathfrak{n}, \mathfrak{n}]$ (which must be central) is the center of $\mathfrak{n}$ and K acts irreducibly on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$.

Let Z_G^0 denote the identity component of the center of G. If Z is a closed connected $\mathrm{Ad}(K)$-invariant subgroup of Z_G^0, then $(G/Z, K/Z))$ is a Gelfand pair and is called a *central reduction* of (G, K). The pair (G, K) is called *maximal* if it is not a nontrivial central reduction. Here is a table of all the groups K and algebras $\mathfrak{n} = \mathfrak{z} + \mathfrak{v}$, $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$, for irreducible maximal Gelfand pairs $(N \rtimes K, K)$ where N is a connected simply connected nilpotent Lie group. Here $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, $\mathrm{Im}\,\mathbb{F}^{s\times s}$ is the space of skew hermitian $s \times s$ matrices over $\mathbb{F}$, $\mathrm{Re}\,\mathbb{F}^{s\times s}$ is the space of hermitian $s \times s$ matrices over $\mathbb{F}$; $\mathrm{Im}\,\mathbb{F}_0^{s\times s}$ and $\mathrm{Re}\,\mathbb{F}_0^{s\times s}$ are those of trace 0. The Lie algebra structure is given by $\mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ and should be clear, but is explained in detail in [13, §13.4B]. The result is due to E. B. Vinberg.

Maximal Irreducible Nilpotent Gelfand Pairs $(N \rtimes K, K)$ with $\mathfrak{n} \neq \mathfrak{z}$ ([10,11]) (2.1)

	Group K	$\mathfrak{v}$	$\mathfrak{z}$	$U(1)$	max
1	$SO(n)$	$\mathbb{R}^n$	$\Lambda^2\mathbb{R}^n = \mathfrak{so}(n)$		
2	$Spin(7)$	$\mathbb{R}^8 = \mathbb{O}$	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$		
3	G_2	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$		
4	$U(1) \cdot SO(n)$	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}$		$n \neq 4$
5	$(U(1)\cdot)SU(n)$	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n \oplus \mathrm{Im}\,\mathbb{C}$	n odd	
6	$SU(n)$, n odd	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n$		
7	$SU(n)$, n odd	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}$		
8	$U(n)$	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}^{n\times n} = \mathfrak{u}(n)$		
9	$(U(1)\cdot)Sp(n)$	$\mathbb{H}^n$	$\mathrm{Re}\,\mathbb{H}_0^{n\times n} \oplus \mathrm{Im}\,\mathbb{H}$		
10	$U(n)$	$S^2\mathbb{C}^n$	$\mathbb{R}$		
11	$(U(1)\cdot)SU(n)$, $n \geqq 3$	$\Lambda^2\mathbb{C}^n$	$\mathbb{R}$	n even	
12	$U(1) \cdot Spin(7)$	$\mathbb{C}^8$	$\mathbb{R}^7 \oplus \mathbb{R}$		
13	$U(1) \cdot Spin(9)$	$\mathbb{C}^{16}$	$\mathbb{R}$		
14	$(U(1)\cdot)Spin(10)$	$\mathbb{C}^{16}$	$\mathbb{R}$		
15	$U(1) \cdot G_2$	$\mathbb{C}^7$	$\mathbb{R}$		
16	$U(1) \cdot E_6$	$\mathbb{C}^{27}$	$\mathbb{R}$		
17	$Sp(1) \times Sp(n)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H} = \mathfrak{sp}(1)$		$n \geqq 2$
18	$Sp(2) \times Sp(n)$	$\mathbb{H}^{2\times n}$	$\mathrm{Im}\,\mathbb{H}^{2\times 2} = \mathfrak{sp}(2)$		
19	$(U(1)\cdot)SU(m) \times SU(n)$ $m, n \geqq 3$	$\mathbb{C}^m \otimes \mathbb{C}^n$	$\mathbb{R}$	$m = n$	
20	$(U(1)\cdot)SU(2) \times SU(n)$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$	$n = 2$	
21	$(U(1)\cdot)Sp(2) \times SU(n)$	$\mathbb{H}^2 \otimes \mathbb{C}^n$	$\mathbb{R}$	$n \leqq 4$	$n \geqq 3$
22	$U(2) \times Sp(n)$	$\mathbb{C}^2 \otimes \mathbb{H}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$		
23	$U(3) \times Sp(n)$	$\mathbb{C}^3 \otimes \mathbb{H}^n$	$\mathbb{R}$		$n \geqq 2$

Often one can replace K by a smaller group in such a way that (G, K) continues to be a Gelfand pair. For example, in (2.1), item 2, where N is an octonionic Heisenberg group, the pairs $(N \rtimes Spin(7), Spin(7))$, $(N \rtimes Spin(6), Spin(6))$ and $(N \rtimes Spin(5), Spin(5))$ all are Gelfand pairs; see [6, Proposition5.6].

Notation. All groups are real. If K is denoted $(U(1)\cdot)L$ it can be $U(1) \cdot L$ or L; where noted in the "$U(1)$" column it can only be $U(1) \cdot L$. Note $U(1) \cdot SU(n) = U(n)$. For $SO(n)$ it is understood that $n \geqq 3$, and for $U(n)$ and $SU(n)$ it is understood that $n \geqq 2$. If some pairs in the series are not maximal, the maximality condition is noted in the "maximal" column. ◇

The classification of commutative nilmanifolds is based on (2.1) and developed by O. Yakimova ([17,18]). For an exposition see §§13.4C and 13.4D, and Chap. 15, in [13]. In brief, Yakimova starts by defining technical conditions *principal* and $Sp(1)$-

saturated. Then she classifies Gelfand pairs $(N \rtimes K, K)$ that are indecomposable, principal, maximal and $Sp(1)$-saturated. See Table (2.2) below. Finally she introduces some combinatorial methods to complete the classification.

In Table (2.2), $\mathfrak{h}_{n;\mathbb{F}}$ denotes the Heisenberg algebra $\mathrm{Im}\,\mathbb{F} + \mathbb{F}^n$ of real dimension $(\dim_R \mathbb{F} - 1) + n \dim_{\mathbb{R}} \mathbb{F}$ with composition $[(z, u), (w, v)] = (\mathrm{Im}\,(\langle u, v\rangle), 0)$ where $\mathbb{F}$ is $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. Also in the table, $\mathfrak{v} = \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ and the summands in double parenthesis ((..)) are the subalgebras $[\mathfrak{w}, \mathfrak{w}] + \mathfrak{w}$ where $\mathfrak{w}$ is a K-irreducible subspace of $\mathfrak{v}$ with $[\mathfrak{w}, \mathfrak{w}] \neq 0$. The summands not in parentheses are K-invariant subspaces of $\mathfrak{w} \subset \mathfrak{v}$ with $[\mathfrak{w}, \mathfrak{w}] = 0$. Thus $\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] + \mathfrak{v}$, vector space direct sum, and its center $\mathfrak{z}$ is the sum of $[\mathfrak{n}, \mathfrak{n}]$ with those summands listed for $\mathfrak{v}$ that are *not* enclosed in double parenthesis ((..)).

Here is Yakimova's classification of indecomposable, principal, maximal and $Sp(1)$-saturated commutative pairs $(N \rtimes K, K)$ where the action of K on $\mathfrak{v}$ is reducible. We omit the case $[\mathfrak{n}, \mathfrak{n}] = 0$, where $N = \mathbb{R}^n$ and K is any closed subgroup of the orthogonal group $O(n)$.

Maximal Indecomposable Principal Saturated Nilpotent Gelfand Pairs $(N \rtimes K, K)$,
N Nonabelian, where the action of K on $\mathfrak{v} \cong \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is reducible [17,18]

	Group K	K-module $\mathfrak{v}$	K-module $[\mathfrak{n}, \mathfrak{n}]$	Algebra $\mathfrak{n}$
1	$U(n)$	$\mathbb{C}^n \oplus \mathrm{Im}\,\mathbb{C}_0^{n\times n}$	$\mathbb{R}$	$((\mathfrak{h}_{n;\mathbb{C}})) + \mathrm{Im}\,\mathbb{C}_0^{n\times n}$
2	$U(4)$	$\mathbb{C}^4 \oplus \mathbb{R}^6$	$\mathrm{Im}\,\mathbb{C} \oplus \Lambda^2\mathbb{C}^4$	$((\mathrm{Im}\,\mathbb{C} + \Lambda^2\mathbb{C}^4 + \mathbb{C}^4)) + \mathbb{R}^6$
3	$U(1) \times U(n)$	$\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{n;\mathbb{C}})) + ((\mathfrak{h}_{n(n-1)/2;\mathbb{C}}))$
4	$SU(4)$	$\mathbb{C}^4 \oplus \mathbb{R}^6$	$\mathrm{Im}\,\mathbb{C} \oplus \mathrm{Re}\,\mathbb{H}^{2\times 2}$	$((\mathrm{Im}\,\mathbb{C} + \mathrm{Re}\,\mathbb{H}^{2\times 2} + \mathbb{C}^4)) + \mathbb{R}^6$
5	$U(2) \times U(4)$	$\mathbb{C}^{2\times 4} \oplus \mathbb{R}^6$	$\mathrm{Im}\,\mathbb{C}^{2\times 2}$	$((\mathrm{Im}\,\mathbb{C}^{2\times 2} + \mathbb{C}^{2\times 4})) + \mathbb{R}^6$
6	$S(U(4) \times U(m))$	$\mathbb{C}^{4\times m} \oplus \mathbb{R}^6$	$\mathbb{R}$	$((\mathfrak{h}_{4m;\mathbb{C}})) + \mathbb{R}^6$
7	$U(m) \times U(n)$	$\mathbb{C}^{m\times n} \oplus \mathbb{C}^m$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{mn;\mathbb{C}})) + ((\mathfrak{h}_{m;\mathbb{C}}))$
8	$U(1) \times Sp(n) \times U(1)$	$\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{2n;\mathbb{C}})) + ((\mathfrak{h}_{2n;\mathbb{C}}))$
9	$Sp(1) \times Sp(n) \times U(1)$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H} \oplus \mathbb{R}$	$((\mathfrak{h}_{n;\mathbb{H}})) + ((\mathfrak{h}_{2n;\mathbb{C}}))$
10	$Sp(1) \times Sp(n) \times Sp(1)$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H} \oplus \mathrm{Im}\,\mathbb{H}$	$((\mathfrak{h}_{n;\mathbb{H}})) + ((\mathfrak{h}_{n;\mathbb{H}}))$
11	$Sp(n) \times \{Sp(1), U(1), \{1\}\}$ $\times Sp(m)$	$\mathbb{H}^n \oplus \mathbb{H}^{n\times m}$	$\mathrm{Im}\,\mathbb{H}$	$((\mathfrak{h}_{n;\mathbb{H}})) + \mathbb{H}^{n\times m}$
12	$Sp(n) \times \{Sp(1), U(1), \{1\}\}$	$\mathbb{H}^n \oplus \mathrm{Re}\,\mathbb{H}_0^{n\times n}$	$\mathrm{Im}\,\mathbb{H}$	$((\mathfrak{h}_{n;\mathbb{H}})) + \mathrm{Re}\,\mathbb{H}_0^{n\times n}$
13	$Spin(7) \times \{SO(2), \{1\}\}$	$(\mathbb{R}^8 = \mathbb{O}) \oplus \mathbb{R}^{7\times 2}$	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$	$((\mathfrak{h}_{1;\mathbb{O}})) + \mathbb{R}^{7\times 2}$
14	$U(1) \times Spin(7)$	$\mathbb{C}^7 \oplus \mathbb{R}^8$	$\mathbb{R}$	$((\mathfrak{h}_{7;\mathbb{C}})) + \mathbb{R}^8$
15	$U(1) \times Spin(7)$	$\mathbb{C}^8 \oplus \mathbb{R}^7$	$\mathbb{R}$	$((\mathfrak{h}_{8;\mathbb{C}})) + \mathbb{R}^7$
16	$U(1) \times U(1) \times Spin(8)$	$\mathbb{C}^8_+ \oplus \mathbb{C}^8_-$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{8;\mathbb{C}})) + ((\mathfrak{h}_{8;\mathbb{C}}))$
17	$U(1) \times Spin(10)$	$\mathbb{C}^{16} \oplus \mathbb{R}^{10}$	$\mathbb{R}$	$((\mathfrak{h}_{16;\mathbb{C}})) + \mathbb{R}^{10}$
18	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\}$ $\times SU(2)$	$\mathbb{C}^{n\times 2} \oplus \mathrm{Im}\,\mathbb{C}_0^{2\times 2}$	$\mathbb{R}$	$((\mathfrak{h}_{2n;\mathbb{C}})) + \mathrm{Im}\,\mathbb{C}_0^{2\times 2}$
19	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\}$ $\times U(2)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^2$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{2n;\mathbb{C}})) + ((\mathfrak{h}_{2;\mathbb{C}}))$
20	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\}$ $\times SU(2) \times$ $\{SU(m), U(m), U(1)Sp(\frac{m}{2})\}$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{2n;\mathbb{C}})) + ((\mathfrak{h}_{2m;\mathbb{C}}))$
21	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\}$ $\times SU(2) \times U(4)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times 4}$ $\oplus \mathbb{R}^6$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{2n;\mathbb{C}})) + ((\mathfrak{h}_{8;\mathbb{C}})) + \mathbb{R}^6$
22	$U(4) \times U(2)$	$\mathbb{R}^6 \oplus \mathbb{C}^{4\times 2}$ $\oplus \mathrm{Im}\,\mathbb{C}_0^{2\times 2}$	$\mathbb{R}$	$\mathbb{R}^6 + ((\mathfrak{h}_{8;\mathbb{C}})) + \mathrm{Im}\,\mathbb{C}_0^{2\times 2}$
23	$U(4) \times U(2) \times U(4)$	$\mathbb{R}^6 \oplus \mathbb{C}^{4\times 2}$ $\oplus \mathbb{C}^{2\times 4} \oplus \mathbb{R}^6$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}^6 + ((\mathfrak{h}_{8;\mathbb{C}}))$ $+ ((\mathfrak{h}_{8;\mathbb{C}})) + \mathbb{R}^6$
24	$U(1) \times U(1) \times SU(4)$	$\mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{R}^6$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{4;\mathbb{C}})) + ((\mathfrak{h}_{4;\mathbb{C}})) + \mathbb{R}^6$
25	$(U(1)\cdot)SU(4)(\cdot SO(2))$	$\mathbb{C}^4 \oplus \mathbb{R}^{6\times 2}$	$\mathbb{R}$	$((\mathfrak{h}_{4;\mathbb{C}})) + \mathbb{R}^{6\times 2}$

(2.2)

We go on to describe the representation theory relevant to these tables.

3 Square Integrable Representations

A connected simply connected Lie group N with center Z is called *square integrable* if it has unitary representations π whose coefficients $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ satisfy $|f_{u,v}| \in L^2(N/Z)$. C.C. Moore and the author worked out the structure and representation theory of these groups [7]. If N has one such square integrable representation then there is a certain polynomial function $\mathrm{Pf}\,(\lambda)$ on the linear dual space $\mathfrak{z}^*$ of the Lie algebra of Z that is key to harmonic analysis on N. Here $\mathrm{Pf}\,(\lambda)$ is the Pfaffian of the antisymmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$ given by $b_\lambda(x, y) = \lambda([x, y])$. The square integrable representations of N are the π_λ (corresponding to coadjoint orbits $\mathrm{Ad}^*(N)\lambda$) where $\lambda \in \mathfrak{z}^*$ with $\mathrm{Pf}\,(\lambda) \neq 0$, Plancherel almost irreducible unitary representations of N are square integrable, and up to an explicit constant $|\mathrm{Pf}\,(\lambda)|$ is the Plancherel density of the unitary dual $\widehat{N}$ at π_λ. Concretely,

Theorem 3.1 ([7]) *Let N be a connected simply connected nilpotent Lie group that has square integrable representations. Let Z be its center. If f is a Schwartz class function $N \to \mathbb{C}$ and $x \in N$ then*

$$f(x) = c \int_{\mathfrak{z}^*} \Theta_{\pi_\lambda}(r_x f)|\mathrm{Pf}\,(\lambda)|d\lambda \tag{3.1}$$

where $c = d!2^d$ with $2d = \dim \mathfrak{n}/\mathfrak{z}$, $r_x f$ is the right translate $(r_x f)(y) = f(yx)$, and Θ is the distribution character

$$\Theta_{\pi_\lambda}(f) = c^{-1}|\mathrm{Pf}\,(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi)d\nu_\lambda(\xi) \text{ for } f \text{ in the Schwartz space } \mathcal{S}(N). \tag{3.2}$$

Here f_1 is the lift $f_1(\xi) = f(\exp(\xi))$ of f from N to $\mathfrak{n}$, $\widehat{f_1}$ is its classical Fourier transform, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$, and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to $\mathrm{Ad}^*(N)\lambda$.

The connection with commutative nilmanifolds is

Theorem 3.2 ([13, Theorem 14.4.3]) *All of the nilpotent groups in* Table (2.1) *are square integrable except for those of table entries (1) and (6) with n odd, and those of table entry (3). All the groups N of* Table (2.2) *are square integrable.*

In [16] and [14] we extended the theory of square integrable nilpotent groups to "stepwise square integrable" nilpotent groups. See [13, Theorem 14.4.3]. Now we settle the three "exceptional" cases of Theorem 3.2 by checking that, in those cases, the nilpotent group is stepwise square integrable in a straightforward way. Case n refers to table entry (n) in Table (2.1).

Case 1: $\mathfrak{n} = \Lambda^2(\mathbb{R}^n) + \mathbb{R}^n$, $n = 2m + 1$ odd, with composition $[(z, u), (w, v)] = (u \wedge v, 0)$. Choose a basis $\{u_1, u_2, \ldots, u_{2m-1}, u_{2m}, u_{2m+1}\}$ of $\mathfrak{v} = \mathbb{R}^n$. Then $\{u_i \wedge u_j \mid i < j\}$ is a basis of $\mathfrak{z} = \Lambda^2(\mathbb{R}^n)$ and $\{(u_i \wedge u_j)^* \mid i < j\}$ is the dual basis of $\mathfrak{z}^*$. Define

$\mathfrak{l}_1 = \mathfrak{z} + \mathfrak{v}_1$ where $\mathfrak{v}_1 = \text{Span}\,\{u_1, \ldots, u_{2m}\}$ and $\mathfrak{l}_2 = u_{2m+1}\mathbb{R}$. Then $\mathfrak{l}_1$ is an ideal in $\mathfrak{n}$, $\mathfrak{n} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ semidirect sum, and $L_1 := \exp(\mathfrak{l}_1)$ has square integrable representations. For the latter define $\lambda_a = a_1(u_1 \wedge u_2)^* + a_2(u_3 \wedge u_4)^* + \cdots + a_m(u_{2m-1} \wedge u_{2m})^*$ where $a \in \mathbb{R}^m$. Then the bilinear form b_{λ_a} on $\mathfrak{l}_1/\mathfrak{z}$ has matrix, in the basis $\{u_1, u_2, \ldots, u_{2m-1}, u_{2m}\}$ of $\mathfrak{v}_1$, given by diag $\left\{\begin{pmatrix} 0 & -a_1 \\ a_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -a_m \\ a_m & 0 \end{pmatrix}\right\}$. So in this basis $|\text{Pf}\,(\lambda_a)| = |a_1 a_2 \ldots a_m|$. That proves the square integrability of L_1.

Now $N = L_1 \rtimes L_2$ where $L_2 = \exp(\mathfrak{l}_2) \cong \mathbb{R}$. This is a very simple case of the 2-step square integrability described in [16] and [14]. The square integrable representations π_λ of L_1 extend to representations π'_λ of N on the same Hilbert space, and unitary characters $\chi_\xi(exp(tu_n)) = e^{i\xi t}$ of L_2 can be viewed as unitary characters on N whose kernel contains L_1. Plancherel measure for N is concentrated on $\{\pi'_\lambda \boxtimes \chi_\xi \mid \text{Pf}\,(\lambda) \neq 0 \text{ and } \xi \in \mathfrak{l}_2^*\}$. If $x \in N$ denote $x = x_1 x_2$ with $x_i \in L_i$. Using Theorem 3.1 and the Mackey machine we have

Proposition 3.3 *If f is a Schwartz class function on N then*

$$f(x) = \tfrac{1}{\sqrt{2\pi}} m! 2^m \int_{\mathfrak{l}_2^*} \left(\int_{\mathfrak{z}^*} \Theta_{\pi_\lambda}(r_{x_1} f) |\text{Pf}\,(\lambda)| d\lambda \right) \chi_\xi(x_2) d\xi \tag{3.3}$$

where Θ_{π_λ} is the distribution character of $\pi_\lambda \in \widehat{L_1}$.

Case 6: $\mathfrak{n} = \Lambda^2(\mathbb{C}^n) + \mathbb{C}^n$, $n = 2m+1$ odd, with composition $[(z, u), (w, v)] = (u \wedge v, 0)$. Here $\mathfrak{n}$ is the underlying real Lie algebra of the complexification of the algebra of Case 1 just above. So $N = L_1 \rtimes L_2$ as before – complex instead of real. If $a \in \mathbb{C}^m$ and $\lambda_a = a_1(u_1 \wedge u_2)^* + a_2(u_3 \wedge u_4)^* + \cdots + a_m(u_{2m-1} \wedge u_{2m})^*$ then $|\text{Pf}\,(\lambda_a) = |a_1 a_2 \ldots a_m|^2$, so L_1 is square integrable and $N = L_1 \rtimes L_2$ where $L_2 = \exp(\mathfrak{l}_2) \cong \mathbb{C} \cong \mathbb{R}^2$. Now as in Case 1, Plancherel measure for N is concentrated on $\{\pi'_\lambda \boxtimes \chi_\xi \mid \text{Pf}\,(\lambda) \neq 0 \text{ and } \xi \in \mathbb{R}^2\}$. If $x \in N$ denote $x = x_1 x_2$ with $x_1 \in L_1$ and $x_2 \in L_2 = \mathbb{R}^2$.

Proposition 3.4 *If f is a Schwartz class function on N then*

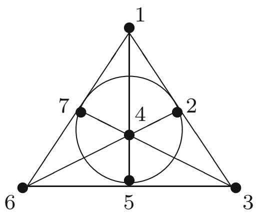
$$f(x) = \tfrac{1}{2\pi} (2m)! 2^{2m} \int_{\mathfrak{l}_2^*} \left(\int_{\mathfrak{z}^*} \Theta_{\pi_\lambda}(r_{x_1} f) |\text{Pf}\,(\lambda)| d\lambda \right) \chi_\xi(x_2) d\xi \tag{3.4}$$

where Θ_{π_λ} is the distribution character of $\pi_\lambda \in \widehat{L_1}$.

Case 3: $\mathfrak{n} = \text{Im}\,\mathbb{O} + \text{Im}\,\mathbb{O}$ with composition $[(z, u), (w, v)] = (\text{Im}\,(u\bar{v}), 0) = (-\text{Im}\,(uv), 0)$. Recall the multiplication table for the octonions: $\mathbb{O}$ is the algebra over $\mathbb{R}$ with basis $\{e_0, \ldots, e_7\}$ and multiplication defined[1] by (a) $e_0 e_j = e_j = e_j e_0$ for $1 \leqq j \leqq 7$, (b) $e_j^2 = -e_0$ for $1 \leqq j \leqq 7$, (c) $e_j e_k + e_k e_j = 0$ for $1 \leqq j, k \leqq 7$ with $j \neq k$, (d) $e_1 e_2 = e_3$, $e_3 e_5 = e_6$, $e_6 e_7 = e_1$, $e_1 e_4 = e_5$, $e_3 e_4 = -e_7$, $e_6 e_4 = e_2$ and

[1] This is one of several standards. The nice thing about standards is that there are so many of them.

$e_2e_5 = e_7$, and (e) each equation in (d) remains true when the subscripts involved in it are cyclically permuted. This multiplication table is summarized in the diagram



$$(1,2,3)\quad (3,5,6)\quad (6,7,1)\quad (1,4,5)\quad (3,4,7)\quad (6,4,2)\quad (2,5,7) \tag{3.5}$$

If $x = x_0e_0 + \cdots + x_7e_7 \in \mathbb{O}$ then Im x is the imaginary *component* $x_1e_1 + \cdots + x_7e_7$.

Denote $\mathfrak{l} = \mathfrak{z} + \mathfrak{v}$ as before where $\mathfrak{v}$ has basis $\{(0, e_1), \ldots, (0, e_7)\}$, the center $\mathfrak{z}$ has basis $\{(e_1, 0), \ldots, (e_7, 0)\}$, and $\{(e_1, 0)^*, \ldots, (e_7, 0)^*\}$ is the dual basis of $\mathfrak{z}^*$. Define $\mathfrak{v}_1 = \mathrm{Span}\,\{(0, e_1), \ldots, (0, e_6)\}$ and $\mathfrak{l}_1 = \mathfrak{z} + \mathfrak{v}_1$. Note $e_1e_2 = e_3$, $e_3e_5 = e_6$ and $e_6e_4 = e_2$. If $a \in \mathbb{R}^3$ define $\lambda_a = a_1(e_3, 0)^* + a_2(e_6, 0)^* + a_3(e_2, 0)^*$. In the ordered basis $\{(0, e_1), (0, e_2), (0, e_3), (0, e_5), (0, e_6), (0, e_4)\}$ of $\mathfrak{v}_1$ the bilinear form b_{λ_a} on $\mathfrak{l}_1/\mathfrak{z}$ has matrix diag $\left\{\begin{pmatrix} 0 & -a_1 \\ a_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -a_2 \\ a_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -a_3 \\ a_3 & 0 \end{pmatrix}\right\}$. In this basis $|\mathrm{Pf}\,(\lambda_a)| = |a_1a_2a_3|$. In particular $L_1 = \exp(\mathfrak{l}_1)$ has square integrable (modulo the center) representations. Recall the semidirect product structure $N = L_1 \rtimes L_2$ where $L_2 = \exp(e_7\mathbb{R}) \cong \mathbb{R}$. If $x \in N$ we write $x = x_1x_2$ with $x_i \in L_i$. Here $c = \frac{1}{\sqrt{2\pi}} 3!2^3 = \frac{48}{\sqrt{2\pi}}$. Now, as for Case 1,

Proposition 3.5 *If f is a Schwartz class function on N then*

$$f(x) = \frac{48}{\sqrt{2\pi}} \int_{\mathfrak{l}_2^*} \left(\int_{\mathfrak{z}^*} \Theta_{\pi_\lambda}(r_{x_1} f) |\mathrm{Pf}\,(\lambda)| d\lambda \right) \chi_\xi(x_2) d\xi \tag{3.6}$$

where Θ_{π_λ} is the distribution character of $\pi_\lambda \in \widehat{L_1}$.

The structural similarities between Propositions 3.3, 3.4 and 3.5 are clear. We summarize them as

Theorem 3.6 *Let $X = G/K$ be one of the three "exceptional" cases of* Table 2.1, *in other words entry (1) or (6) with n odd, or entry (3). Decompose $N = N_1 \rtimes N_2$ as in Sect. 3. Let $x \in X$, say $x = g_1g_2K$ where $g_i \in L_i$. If f is a Schwartz class function on X then*

$$f(x) = \frac{2^d d!}{(2\pi)^{\dim \mathfrak{l}_2/2}} \int_{\mathfrak{l}_2^*} \left(\int_{\mathfrak{z}^*} \Theta_{\pi_{\mathrm{Ad}^*(k)\lambda}}(r_{g_1} f) |\mathrm{Pf}\,(\lambda)| d\lambda \right) \chi_\xi(g_2) d\xi \tag{3.7}$$

where $d = \frac{1}{2} \dim \mathfrak{l}_1/\mathfrak{z}$ and Θ_{π_λ} is the distribution character of $\pi_\lambda \in \widehat{L_1}$.

We go on to carry these Fourier Inversion formulae over to the nilmanifolds themselves.

4 Analysis on the Nilmanifold

The commutative nilmanifolds $X = (N \rtimes K)/K$ evidently have the property that N is simply transitive. Thus one can interpret the inversion formulae, from Theorem 3.2 and Propositions 3.3, 3.4 and 3.5, as Fourier inversion formulae on X. However this ignores the role of the isotropy subgroup K, which already has a big impact in the case of ordinary Euclidean space.

Choose a K-invariant inner product (λ, ν) on $\mathfrak{z}^*$. Denote $\mathfrak{z}_t^* = \{\lambda \in \mathfrak{z}^* \mid (\lambda, \lambda) = t^2\}$, the sphere of radius t. Consider the action of K on $\mathfrak{z}_t^*$. Recall that two orbits $\mathrm{Ad}^*(K)\xi$ and $\mathrm{Ad}^*(M)\nu$ are of the *same orbit type* if the isotropy subgroups K_ξ and K_ν are conjugate, and an orbit is *principal* if all nearby orbits are of the same type. Since K and $\mathfrak{z}_t^*$ are compact, there are only finitely many orbit types of K on $\mathfrak{z}_t^*$, there is only one principal orbit type, and the union of the principal orbits forms a dense open subset of $\mathfrak{z}_t^*$ whose complement has codimension $\geqq 2$. There are many good expositions of this material, for example [1, Chap. 4,§3] for a complete treatment, [4, Part II, Chap. 3, §1] modulo references to [1], and [2, Cap. 5] for a more basic treatment but still with some references to [1]. Principal orbit isotropy subgroups of compact connected linear groups are studied in detail in the work [5] of W.-C. and W.-Y. Hsiang, so the possibilities for $K_\lambda(\mathrm{Ad}^*(K)\lambda$ principal) are essentially known. But it is not too difficult to work out the principal orbit isotropy subgroups K_λ directly in our cases.

Define the Pf -nonsingular principal orbit set for K on $\mathfrak{z}^*$ as follows:

$$\mathfrak{w}^* = \left\{\lambda \in \mathfrak{z}^* \mid \mathrm{Pf}\,(\lambda) \neq 0 \text{ and } \mathrm{Ad}^*(K)\lambda \text{ is a principal } K\text{-orbit on } \mathfrak{z}^*\right\}. \tag{4.1}$$

The principal orbit set $\mathfrak{w}^*$ is a dense open set whose complement has codimension $\geqq 2$ in $\mathfrak{z}^*$. If $\lambda \in \mathfrak{w}^*$ and $c \neq 0$ then $c\lambda \in \mathfrak{z}^*$ with isotropy $K_{c\lambda} = K_\lambda$. There is a Borel section σ to $\mathfrak{w}^* \to \mathfrak{w}^*/\mathrm{Ad}^*(K)$ which picks out an element in each K-orbit, so that K has the same isotropy subgroup at each of those elements. In other words in each K-orbit on $\mathfrak{w}^*$ we measurably choose an element $\lambda = \sigma(\mathrm{Ad}^*(K)\lambda)$ such that those isotropy subgroups K_λ are all the same. Let us denote

$$K_\diamond\text{: isotropy subgroup of } K \text{ at } \sigma(\mathrm{Ad}^*(K)\lambda) \text{ for every } \lambda \in \mathfrak{w}^* \tag{4.2}$$

Then we can replace K_λ by $K_\diamond$, independent of $\lambda \in \mathfrak{w}^*$, in our considerations.

Fix $\lambda = \sigma(\mathrm{Ad}^*(K)\lambda) \in \mathfrak{w}_t^* := \mathfrak{w} \cap \mathfrak{z}_t$. Consider the semidirect product group $N \rtimes K_\diamond$. We write $\mathcal{H}_\lambda$ for the representation space of π_λ. The next step is to extend the representation π_λ to a unitary representation π'_λ of $N \rtimes K_\diamond$ on the same representation space $\mathcal{H}_\lambda$. Following the Fock space argument of [15, Lemma 3.8] we see that the Mackey obstruction to this extension is trivial. Thus

Lemma 4.1 *The square integrable representation π_λ of N extends to an irreducible unitary representation π'_λ of $N \rtimes K_\diamond$ on the representation space of π_λ.*

We proceed *mutatis mutandis* along the lines of [15, §3]. Each $\lambda = \sigma(\mathrm{Ad}^*(K)\lambda) \in \mathfrak{w}^*$ defines classes

$$\mathcal{E}(\lambda) := \{\pi'_\lambda \otimes \gamma \mid \gamma \in \widehat{K_\diamond}\} \text{ and } \mathcal{F}(\lambda) := \left\{\mathrm{Ind}\,_{NK_\diamond}^{NK}(\pi'_\lambda \otimes \gamma) \mid \pi'_\lambda \otimes \gamma \in \mathcal{E}(\lambda)\right\} \tag{4.3}$$

of irreducible unitary representations of $N \rtimes K_\diamond$ and $N \rtimes K$. The Mackey little group method, plus the facts that the Plancherel density on $\widehat{N}$ is polynomial on $\mathfrak{z}^*$ and $\mathfrak{z}^* \setminus \mathfrak{w}^*$ has measure 0, give us

Lemma 4.2 *Plancherel measure for $N \rtimes K$ is concentrated on the set $\bigcup_{\lambda \in \mathfrak{w}^*} \mathcal{F}(\lambda)$ of (equivalence classes of) irreducible representations given by $\eta_{\lambda,\gamma} := \mathrm{Ind}\,_{NK_\diamond}^{NK}(\pi'_\lambda \otimes \gamma)$ with $\pi'_\lambda \otimes \gamma \in \mathcal{E}(\lambda)$ and $\lambda = \sigma(\mathrm{Ad}^*(K)\lambda) \in \mathfrak{w}^*$. Further*

$$\eta_{\lambda,\gamma}|_N = \left(\mathrm{Ind}\,_{NK_\diamond}^{NK}(\pi'_\lambda \otimes \gamma)\right)\Big|_N = \int_{K/K_\diamond} (\dim \gamma)\, \pi_{\mathrm{Ad}^*(k)\lambda}\, d(kK_\diamond). \tag{4.4}$$

The open subset $\mathfrak{w}^*$ of $\mathfrak{z}^*$ fibers over $\mathfrak{w}^*/\mathrm{Ad}^*(K)$ with compact fiber $K/K_\diamond$. So the Euclidean measure on $\mathfrak{z}^*$ pushes down to a measure $\overline{\mu}$ on $\mathfrak{w}^*/\mathrm{Ad}^*(K)$. Taking into account Lemma 4.2 and the Peter–Weyl Theorem for $K_\diamond$ this gives us

$$L^2(N \rtimes K) = \sum_{\gamma \in \widehat{K_\diamond}} \deg(\gamma) \int_{\mathfrak{w}^*/\mathrm{Ad}^*(K)} (\mathcal{H}_{\eta_{\lambda,\gamma}} \widehat{\otimes} \mathcal{H}^*_{\eta_{\lambda,\gamma}}) |\mathrm{Pf}\,(\lambda)| d\overline{\mu}(\mathrm{Ad}^*(K)\lambda). \tag{4.5}$$

In order to push (4.5) down to the commutative space $(N \rtimes K)/K$ we need

Lemma 4.3 *The representation $\eta_{\lambda,\gamma}$ of (4.4) has a K-fixed unit vector if and only if γ is the trivial representation of $K_\diamond$.*

Proof We take the representation space of π_λ to be the space of Hermite polynomials $p(z)$ on $\mathfrak{v} = \mathbb{C}^m$. There K_λ acts as a subgroup of $U(m)$. So $p(z) = 1$ is a fixed unit vector for $\pi'_\lambda(K_\diamond)$. Let $1_\diamond$ denote the trivial representation of $K_\diamond$. Now $\eta_{\lambda,1_\diamond}$ has a K-fixed unit vector. Conversely if $\eta_{\lambda,\gamma}$ has a K-fixed unit vector, then $\mathrm{Ind}\,_{K_\diamond}^K(\gamma)$ has a K-fixed unit vector. Then, by Frobenius Reciprocity for K, γ is a subrepresentation of the $K_\diamond$-restriction of the trivial representation of K. Thus $\gamma = 1_\diamond$. □

Since K is compact we can understand the Schwartz space of $G = N \rtimes K$ as

$$\mathcal{C}(G) = \left\{f \in C^\infty(G) \mid f(\cdot\, k) \in \mathcal{C}(N) \text{ for each } k \in K\right\}.$$

Combine Theorems 3.1 and 3.2 with (4.5) and Lemma 4.3, and average over K, to see

Theorem 4.4 *Let $X = (N \rtimes K)/K$ be a commutative nilmanifold from* Table 2.1*—except (1) or (6) with n odd or (3)—or from* Table 2.2. *Then $L^2(X) = \int_K \int_{\mathfrak{w}^*} \mathcal{H}_{\eta_{\mathrm{Ad}^*(k)\lambda,1_\diamond}} d\lambda\, dk$. If $f \in \mathcal{S}(G)$ and $x \in X$, say $x = gK$ where $g \in N$, then*

$$f(x) = d!\, 2^d \int_{K/K_\diamond} \left(\int_{\mathfrak{w}^*/\mathrm{Ad}^*(K)} \Theta_{\pi_{\mathrm{Ad}^*(k)\lambda}}(r_{gf})\, |\mathrm{Pf}\,(\lambda)| d\overline{\mu}(\mathrm{Ad}^*(K)\lambda)\right) d(kK_\diamond)$$

where $d = \frac{1}{2}\dim \mathfrak{n}/\mathfrak{z}$, r_{gf} *is the right translate* $(r_{gf})(hK) = f(ghK)$ *for* $h \in N$*, and* Θ *denotes the distribution character as in* Theorem 3.1.

The argument of Lemma 4.3 exhibits the spherical function in each of the $\mathcal{H}_{\eta_{\lambda,1_\diamond}}$ of Theorem 3.2.

Now we examine the three exceptional cases of Sect. 3. Retain the notation of Sect. 4. These cases are a bit more delicate because $\mathrm{Ad}^*(K)$ moves the factors in the semidirect product decomposition $N = L_1 \rtimes L_2$.

Case (1). $K = SO(2m+1)$ so $\mathfrak{w}^*$ consists of the $SO(2m+1)$-images of the $\lambda_a, a \in \mathbb{R}^m$ with $a_1 a_2 \dots a_m \neq 0$, given by $a_1(u_1 \wedge u_2)^* + a_2(u_3 \wedge u_4)^* + \cdots + a_m(u_{2m-1} \wedge u_{2,})^*$. This follows from the Darboux normal form of antisymmetric bilinear forms. So $\mathfrak{m}^*/\mathrm{Ad}^*(K) \subset \Lambda^2(\mathbb{R}^{2m+1})/SO(2m+1)$ has representatives λ_a as above with $a_1 \leqq a_2 \leqq \cdots \leqq a_m$, or, up to measure 0, $a_1 < a_2 < \cdots < a_m$.

Case (6). $K = SU(2m+1)$ so $\mathfrak{w}^*$ consists of the $SU(2m+1)$-images of the $\lambda_a, a \in \mathbb{C}^m$ with $a_1 a_2 \dots a_m \neq 0$, given by $a_1(u_1 \wedge u_2)^* + a_2(u_3 \wedge u_4)^* + \cdots + a_m(u_{2m-1} \wedge u_{2m})^*$, and one can choose representatives with $a_1 \leqq a_2 \leqq \cdots \leqq a_{m-1}$ real and $a_{m-1} \leqq |a_m|$.

Case (3). K is the exceptional simple group G_2 so $\mathfrak{w}^*$ consists of all G_2-images of the $\lambda_a = a_1(e_3, 0)^* + a_2(e_6, 0)^* + a_3(e_2, 0)^*$, $a \in \mathbb{R}^3$ and $a_1 a_2 a_3 \neq 0$. The point is that, inside $\mathrm{Im}\, \mathbb{O}$, e_2 and e_3 are orthogonal unit vectors that generate a subalgebra $\mathbb{H} \subset \mathbb{O}$, and e_6 is a unit vector orthogonal to that subalgebra.

We now proceed as in the transition from Theorem 3.2 to Theorem 4.4, by averaging over K. We write $\mathcal{O}_{K,\lambda}$ for the orbit $\mathrm{Ad}^*(K)\lambda$ in $\mathfrak{w}^*$. In view of Propositions 3.3, 3.4 and 3.5 we arrive at

Theorem 4.5 *Let* $X = G/K$ *be one of the three "exceptional" cases of* Table 2.1*, in other words entry (1) or (6) with n odd, or entry (3). Decompose* $N = N_1 \rtimes N_2$ *as in* Sect. *3. Let* $x \in X$*, say* $x = gK$ *where* $g \in N$*. If* f *is a Schwartz class function on* X *then*

$$f(x) = \frac{2^d d!}{(\sqrt{2\pi})^{d_2}} \int_{K/K_\diamond} \left(\int_{\mathfrak{l}_2^*} \left(\int_{\mathfrak{w}^*/\mathrm{Ad}^*(K)} \Theta_{\pi_{\mathrm{Ad}^*(k)\lambda}}(r_{g_{1,k}} f) |\mathrm{Pf}\,(\lambda)| d\overline{\mu}(\mathcal{O}_{K,\lambda}) \right) \chi_{\mathrm{Ad}^*(k)\xi}(g_{2,k}) d\xi \right) d(kK_\diamond) \tag{4.6}$$

where $x = g_{1,k} g_{2,k} K$ *with* $g_{i,k} \in \mathrm{Ad}(k^{-1}) L_i$, $d = \frac{1}{2}\dim \mathfrak{l}_1/\mathfrak{z}$, $d_2 = \dim \mathfrak{l}_2$*, and* $\Theta_{\pi_{\mathrm{Ad}^*(k)\lambda}}$ *is the distribution character of the representation* $\pi_{\mathrm{Ad}^*(k)\lambda}$ *of* $\mathrm{Ad}(k^{-1}) L_1$.

5 Weakly Symmetric Spaces

Let (X, ds^2) be a connected Riemannian manifold and $I(X, ds^2)$ its isometry group. Recall that (X, ds^2) is *symmetric* if, given $x \in X$, there is an isometry $s_x \in I(X, ds^2)$ such that $s_x(x) = x$ and $ds_x(\xi) = -\xi$ for every tangent vector $\xi \in T_x(X)$ at x. In other words, (X, ds^2) is symmetric just when, for every $x \in X$, there is an isometry that fixes x and reverses every geodesic through x. Then s_x is the *symmetry* at x. We are going to discuss an extension of this notion, called "weak symmetry", and then make some comments about weakly symmetric Finsler manifolds.

Again let (X, ds^2) be a connected Riemannian manifold and $I(X, ds^2)$ its isometry group. Suppose that, given $x \in X$ and $\xi \in T_x(X)$, that there is an isometry $s_{x,\xi} \in I(X, ds^2)$ such that $s_{x,\xi}(x) = x$ and $ds_{x,\xi}(\xi) = -\xi$. Then (X, ds^2) is *weakly symmetric*. Here note that $s_{x,\xi}$ depends on ξ as well as x. Obviously, symmetric Riemannian manifolds are weakly symmetric. As in the symmetric case, one composes symmetries along a geodesic to see that weakly symmetric Riemannian manifolds are complete, in fact homogeneous. In terms of the Lie group structure $X = G/K$ where $G = I(X, ds^2)$ (or the identity component $I(X, ds^2)^0$), this is equivalent to the existence of an automorphism $\sigma : G \to G$ such that $\sigma(g) \in Kg^{-1}K$ for every $g \in G$. Then (G, K) is a *weakly symmetric pair* with *weak symmetry* σ. More generally one can make these definitions for any pair (G, K) where G is a Lie group, K is a compact subgroup and G/K is connected. The connection with this note is

Theorem 5.1 [8]. *Every weakly symmetric pair is a Gelfand pair. In other words every weakly symmetric Riemannian manifold is a commutative space.*

The converse doesn't quite hold. If (G, K) is a Gelfand pair with G reductive then (G, K) has a weak symmetry. On the other hand there are commutative nilmanifolds that are not weakly symmetric, for example [6] the pair $(\{(\mathrm{Re}\,\mathbb{H}_0^{n\times n} \oplus \mathrm{Im}\,\mathbb{H}) + \mathbb{H}^n\} \rtimes Sp(n), Sp(n))$ in Table 2.1. Nevertheless,

Proposition 5.2 *All the commutative pairs of* Table 2.1 *are weakly symmetric except for item (9).*

Thus Theorems 4.4 and 4.5 apply to weakly symmetric nilmanifolds $G/K = (N \rtimes K)/K$ where K acts irreducibly on the tangent space. See [17] and [18], or the exposition in [13, Chap. 15], for a complete (modulo some combinatorics) classification of the weakly symmetric nilmanifolds.

All this holds for Finsler manifolds. If (X, F) is a weakly symmetric (using the same definition as in the Riemannian case) Finsler manifold, then it is geodesically complete and homogeneous, say $X = G/K$ where $G = I(X, ds^2)$ (or the identity component $I(X, ds^2)^0$). Also, X has a G-invariant weakly symmetric Riemannian metric [3, Theorem 2.1]. Thus the classification of weakly symmetric pairs (G, K) is the same for Finsler manifolds as for Riemannian manifolds, and Theorem 5.1, Proposition 5.2 and the remarks after Proposition 5.2 hold for weakly symmetric Finsler nilmanifolds.

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