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STEPWISE SQUARE INTEGRABLE REPRESENTATIONS FOR LOCALLY NILPOTENT LIE GROUPS

JOSEPH A. WOLF*

Department of Mathematics
University of California
Berkeley, CA 94720–3840, USA
jawolf@math.berkeley.edu

Abstract. In a recent paper we found conditions for a nilpotent Lie group N to have a filtration by normal subgroups whose successive quotients have square integrable representations, and such that these square integrable representations fit together nicely to give an explicit construction of Plancherel for almost all representations of N. The prototype for this sort of group is the group of upper triangular real matrices with 1's down the diagonal. More generally, this class of groups contains the nilradicals of minimal parabolic subgroups of all (finite-dimensional) reductive real or complex Lie groups, in other words, all groups N in Iwasawa decompositions of reductive real or complex Lie groups.

The construction of stepwise square integrable representations resulted in explicit character formulae, Plancherel formulae and multiplicity formulae. Here we extend those results to direct limits of stepwise square integrable nilpotent Lie groups. There are two keys to this extension. The first is to set up the corresponding direct system so that it respects the construction at every finite level. In the case of simple (or more generally reductive) groups this means that the restricted root Dynkin diagrams increase in a particular manner. The second is to follow Schwartz space theory through the direct limit process, develop a Schwartz space theory for certain direct limit nilpotent groups, and use it to study stepwise square integrability for coefficients of direct limits of stepwise square integrable nilpotent Lie groups. This leads to the main result, an explicit Fourier inversion formula for that class of infinite-dimensional Lie groups. One important consequence is the Fourier inversion formula for nilradicals of classical minimal parabolic subgroups of finitary real reductive Lie groups such as $\mathrm{GL}(\infty;\mathbb{R})$, $\mathrm{Sp}(\infty;\mathbb{C})$ and $\mathrm{SO}(\infty,\infty)$.

1. Introduction

A connected simply connected Lie group N with center Z is called *square integrable* if it has unitary representations π whose coefficients $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ satisfy $|f_{u,v}| \in L^2(N/Z)$. C. C. Moore and the author worked out the structure and representation theory of these groups [1]. If N has one such square integrable representation then there is a certain polynomial function $\mathrm{Pf}\,(\lambda)$ on the linear

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Corresponding Author: J. A. Wolf, e-mail: jawolf@math.berkeley.edu.

dual space $\mathfrak{z}^*$ of the Lie algebra of Z that is key to harmonic analysis on N. Here $\mathrm{Pf}\,(\lambda)$ is the Pfaffian of the antisymmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$ given by $b_\lambda(x,y) = \lambda([x,y])$. The square integrable representations of N are the π_λ (corresponding to coadjoint orbits $\mathrm{Ad}^*(N)\lambda$) where $\lambda \in \mathfrak{z}^*$ with $\mathrm{Pf}\,(\lambda) \neq 0$. Plancherel almost all irreducible unitary representations of N are square integrable. Up to an explicit constant $|\mathrm{Pf}\,(\lambda)|$ is the Plancherel density on the unitary dual $\widehat{N}$ at π_λ. This theory has proved to have serious analytic consequences. For example, for most commutative nilmanifolds G/K, i.e., Gelfand pairs (G,K) where a nilpotent subgroup N of G acts transitively on G/K, the group N has square integrable representations [5]. And it is known just which maximal parabolic subgroups of semisimple Lie groups have square integrable nilradical [4].

In [9] and [10] the theory of square integrable nilpotent groups was extended to "stepwise square integrable" nilpotent groups. They are the connected simply connected nilpotent Lie groups of (1.1) just below. We use L and $\mathfrak{l}$ to avoid conflict of notation with the M and $\mathfrak{m}$ of minimal parabolic subgroups.

$$\left.\begin{aligned}
&N = L_1L_2\dots L_{m-1}L_m \text{ where}\\
&\quad \text{(a) each factor } L_r \text{ has unitary representations with coefficients}\\
&\qquad \text{in } L^2(L_r/Z_r),\\
&\quad \text{(b) each } N_r := L_1L_2\dots L_r \text{ is a normal subgroup of } N \text{ with}\\
&\qquad N_r = N_{r-1} \rtimes L_r,\\
&\quad \text{(c) decompose } \mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r \text{ and } \mathfrak{n} = \mathfrak{s} + \mathfrak{v} \text{ as vector direct sums}\\
&\qquad \text{where } \mathfrak{s} = \textstyle\bigoplus \mathfrak{z}_r \text{ and } \mathfrak{v} = \bigoplus \mathfrak{v}_r;\ \text{then } [\mathfrak{l}_r, \mathfrak{z}_s] = 0 \text{ and } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v}\\
&\qquad \text{for } r > s.
\end{aligned}\right\} \tag{1.1}$$

Denote

$$\left.\begin{aligned}
&\text{(a) } d_r = \tfrac{1}{2}\dim(\mathfrak{l}_r/\mathfrak{z}_r) \text{ so } \tfrac{1}{2}\dim(\mathfrak{n}/\mathfrak{s}) = d_1 + \dots + d_m, \text{ and}\\
&\qquad c = 2^{d_1+\dots+d_m} d_1!d_2!\cdots d_m!,\\
&\text{(b) } b_{\lambda_r}\colon (x,y) \mapsto \lambda([x,y]) \text{ viewed as a bilinear form on } \mathfrak{l}_r/\mathfrak{z}_r,\\
&\text{(c) } S = Z_1Z_2\cdots Z_m = Z_1 \times \dots \times Z_m \text{ where } Z_r \text{ is the center of } L_r,\\
&\text{(d) Pf}:\ \text{polynomial } \mathrm{Pf}\,(\lambda) = \mathrm{Pf}_{\,\mathfrak{l}_1}(b_{\lambda_1})\mathrm{Pf}_{\,\mathfrak{l}_2}(b_{\lambda_2})\cdots\mathrm{Pf}_{\,\mathfrak{l}_m}(b_{\lambda_m}) \text{ on } \mathfrak{s}^*,\\
&\text{(e) } \mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid \mathrm{Pf}\,(\lambda) \neq 0\},\\
&\text{(f) } \pi_\lambda \in \widehat{N} \text{ where } \lambda \in \mathfrak{t}^*:\ \text{irreducible unitary rep. of } N = L_1L_2\cdots L_m.
\end{aligned}\right\} \tag{1.2}$$

The basic result for these groups is

Theorem 1.3. [10, Thm. 6.16] *Let N be a connected simply connected nilpotent Lie group that satisfies* (1.1). *Then the Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ, then the coefficient $f_{u,v}(x) = \langle u, \pi_\nu(x)v\rangle$ satisfies*

$$\|f_{u,v}\|^2_{L^2(N/S)} = \frac{\|u\|^2\|v\|^2}{|\mathrm{Pf}\,(\lambda)|}. \tag{1.4}$$

The distribution character Θ_{π_λ} of π_λ satisfies

$$\Theta_{\pi_\lambda}(f) = c^{-1}|\mathrm{Pf}\,(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi) d\nu_\lambda(\xi) \textit{ for } f \in \mathcal{C}(N) \tag{1.5}$$

where $\mathcal{C}(N)$ is the Schwartz space, f_1 is the lift $f_1(\xi) = f(\exp(\xi))$ of f from N to $\mathfrak{n}$, $\widehat{f_1}$ is its classical Fourier transform, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$, $c = 2^{d_1+\cdots+d_m} d_1! d_2! \cdots d_m!$ as in (1.2a), and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to $\mathrm{Ad}^*(N)\lambda$. The Fourier inversion formula on N is

$$f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f) |\mathrm{Pf}\,(\lambda)| d\lambda \textit{ for } f \in \mathcal{C}(N). \tag{1.6}$$

Definition 1.7. The representations π_λ of (1.2(f)) are the *stepwise square integrable* representations of N relative to the decomposition (1.1).

One of the main results of [9] and [10] is that nilradicals of minimal parabolic subgroups of finite-dimensional real reductive Lie groups are stepwise square integrable. Even the simplest case, the case of minimal parabolic subgroups in $\mathrm{SL}(n;\mathbb{R})$, was a definite improvement over earlier results on the group of strictly upper triangular real matrices. Here we extend the construction of stepwise square integrable representations to a class of locally nilpotent groups that are direct limits in a manner that respects the basic setup (1.1) of the finite-dimensional case, and we show how this applies to the nilradicals of direct limit minimal parabolic subgroups of the real and complex finitary reductive Lie groups, including $\mathrm{GL}(\infty;\mathbb{F})$, $\mathrm{SL}(\infty;\mathbb{F})$, $\mathrm{U}(p,q;\mathbb{F})$ and $\mathrm{SU}(p,q;\mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $p+q=\infty$), $\mathrm{Sp}(\infty;\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), and $\mathrm{SO}^*(2\infty)$.

At present, the main application of this theory is to the nilradicals of the direct limit minimal parabolic subgroups just mentioned. However, even for $\mathrm{SL}(\infty;\mathbb{R}) = \varinjlim \mathrm{SL}(n;\mathbb{R})$, where the nilpotent group consists of the real finitary upper triangular matrices with 1's on the diagonal, the arguments are no less delicate than in the general case treated here. So we would not save much effort by restricting considerations to the case of nilradicals of direct limit minimal parabolic subgroups of the real and complex finitary reductive Lie groups.

In Section 2 we examine strict direct systems $\{N_n, \varphi_{m,n}\}$ of finite-dimensional connected and simply connected nilpotent Lie groups that satisfy (1.1) in a manner that respects the maps $\varphi_{m,n} : N_n \to N_m$ $(m \geqq n)$. We show how this leads to sequences $\{\pi_{\gamma_n}\}$ of closely related stepwise square integrable representations of the groups N_n, and then to their unitary representation limits $\pi_\gamma = \varprojlim \pi_{\gamma_n}$.

In Section 3 we prove stepwise Frobenius-Schur orthogonality relations and restriction theorems for the coefficients of the representations π_{γ_n}. This will make it possible in Section 4 to apply a variation on the renormalization method of [6], [7] and [8] for coefficients of the limits $\pi_\gamma = \varprojlim \pi_{\gamma_n}$ of stepwise square integrable representations.

In Section 4 we apply the tools of Section 3 to obtain inverse systems, by restriction, of the spaces $\mathcal{A}(\pi_{\gamma_n})$ of coefficients of the representations π_{γ_n}. Then we combine density of $\mathcal{A}(\pi_{\gamma_n})$ in $\mathcal{H}_{\pi_{\gamma_n}} \widehat{\otimes} \mathcal{H}^*_{\pi_{\gamma_n}}$ with the renormalization method of

[8] to construct inverse systems, in the Hilbert space category, of the $\mathcal{H}_{\pi_{\gamma_n}} \widehat{\otimes} \mathcal{H}^*_{\pi_{\gamma_n}}$. These mirror the inverse systems of the $\mathcal{A}(\pi_{\gamma_n})$, resulting in an interpretation of the function space $\mathcal{A}(\pi_\gamma) = \varprojlim \mathcal{A}(\pi_{\gamma_n})$ as a dense subspace of the Hilbert space $\mathcal{H}_{\pi_\gamma} \widehat{\otimes} \mathcal{H}^*_{\pi_\gamma} = \varprojlim \mathcal{H}_{\pi_{\gamma_n}} \widehat{\otimes} \mathcal{H}^*_{\pi_{\gamma_n}}$. This is somewhat analogous to the infinite-dimensional Peter–Weyl Theorem of [7, Sect. 4].

In Section 5 we set up the Schwartz space machinery that will allow us to carry over the somewhat abstract $\mathcal{H}_{\pi_\gamma} \widehat{\otimes} \mathcal{H}^*_{\pi_\gamma} = \varprojlim \mathcal{H}_{\pi_{\gamma_n}} \widehat{\otimes} \mathcal{H}^*_{\pi_{\gamma_n}}$ to an explicit Fourier inversion formula. This, incidentally, strengthens the stepwise L^2 property for coefficients involving C^∞ vectors from L^2 to L^1.

In Section 6 we work out the explicit Fourier inversion formula for the direct limit group $N = \varinjlim N_n$. See Theorem 6.1.

In Section 7 we discuss the classical direct systems $\{G_n, \varphi_{m,n}\}$ of finite-dimensional real reductive Lie groups. We study conditions on their restricted root systems $\Delta(\mathfrak{g}_n, \mathfrak{a}_n)$, that lead to an appropriate limit restricted root system $\Delta(\mathfrak{g}, \mathfrak{a}) = \varprojlim \Delta(\mathfrak{g}_n, \mathfrak{a}_n)$ of the Lie algebra of $G = \varinjlim \{G_n, \varphi_{m,n}\}$. This describes the stepwise square integrable structure of the nilradicals of minimal parabolic subgroups.

Finally, in Section 8, we arrive at the goal of this paper, Theorem 8.4, an explicit Fourier inversion formula for the classical direct limit of the nilradicals of those minimal parabolics. This is done by combining the tools of Section 7 with Theorem 6.1

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2. Alignment and construction

For our direct limit considerations it will be necessary to adjust the decompositions (1.1) of the connected simply connected nilpotent Lie groups N_n. This is so that the adjusted decompositions will fit together as n increases. We do that by reversing the indices and keeping the L_r constant as n goes to infinity. First, we suppose that

$$\begin{gathered}\{N_n\} \text{ is a strict direct system of connected} \\ \text{simply connected nilpotent Lie groups,}\end{gathered} \tag{2.1}$$

in other words, the connected simply connected nilpotent Lie groups N_n have the property that N_n is a closed analytic subgroup of N_ℓ for all $\ell \geqq n$. As usual, Z_r denotes the center of L_r. For each n, we require that

$$\left.\begin{array}{l}
N_n = L_1 L_2 \cdots L_{m_n} \text{ where} \\
\quad \text{(a) } L_r \text{ is a closed analytic subgroup of } N_n \text{ for } 1 \leqq r \leqq m_n \text{ and} \\
\quad \text{(b) each } L_r \text{ has unitary representations with coefficients in } L^2(L_r/Z_r). \\
\text{Let } L_{p,q} = L_{p+1}L_{p+2}\cdots L_q \text{ for } p < q \text{ and } N_{\ell,n} = L_{m_\ell+1}L_{m_\ell+2}\cdots L_{m_n} = \\
\qquad L_{m_\ell,m_n} \text{ for } \ell < n. \text{ Then} \\
\quad \text{(c) } N_{\ell,n} \text{ is normal in } N_n \text{ and } N_n = N_r \ltimes N_{r,n} \text{ semidirect product,} \\
\quad \text{(d) decompose } \mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r \text{ and } \mathfrak{n}_n = \mathfrak{s}_n + \mathfrak{u}_n \text{ as vector space} \\
\qquad \text{direct sums where } \mathfrak{s}_n = \bigoplus_{r \leqq m_n} \mathfrak{z}_r \text{ and } \mathfrak{u}_n = \bigoplus_{r \leqq m_n} \mathfrak{v}_r; \\
\qquad \text{then } [\mathfrak{l}_r, \mathfrak{z}_s] = 0 \text{ and } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v} \text{ for } r < s.
\end{array}\right\} \tag{2.2}$$

With this setup we can follow the lines of the constructions in [10, Sect. 5]. We have the Pfaffian polynomials on the $\mathfrak{z}_r^*$ and on $\mathfrak{s}_n^*$ as follows. Given $\lambda_r \in \mathfrak{z}_r^*$, extended to an element of $\mathfrak{l}_r^*$ by $\lambda_r(\mathfrak{v}_r) = 0$, we have the antisymmetric bilinear form b_{λ_r} on $\mathfrak{l}_r/\mathfrak{z}_r$ defined as usual by $b_{\lambda_r}(x, y) = \lambda_r([x, y])$, and $\mathrm{Pf}_r(\lambda_r)$ denotes its Pfaffian. If $\gamma_n = \lambda_1 + \cdots + \lambda_{m_n} \in \mathfrak{s}_n^*$ with each $\lambda_r \in \mathfrak{z}_r^*$, then we have the product

$$P_n(\gamma_n) = \mathrm{Pf}_1(\lambda_1)\mathrm{Pf}_2(\lambda_2)\cdots\mathrm{Pf}_{m_n}(\lambda_{m_n}) \tag{2.3}$$

and the nonsingular set

$$\mathfrak{t}_n^* = \{\gamma_n \in \mathfrak{s}_n^* \mid P_n(\gamma_n) \neq 0\}. \tag{2.4}$$

Recall the construction ([10]) of stepwise square integrable representations π_{γ_n} of N_n, where $\gamma_n \in \mathfrak{t}_n^*$, and where we adjust the indices to our situation. If $m_n = 1$ then π_{γ_n} is just the square integrable representation π_{λ_1} of L_1 defined by $\gamma_n = \lambda_1$. Let $m_n > 1$ and use $N_n = (L_1L_2\cdots L_{m_n-1}) \ltimes L_{m_n} = L_{0,m_n-1} \ltimes L_{m_n}$. By induction on m_n we have the stepwise square integrable representation $\pi_{\lambda_1+\cdots+\lambda_{m_n-1}}$ of L_{0,m_n-1}, and we view it as a representation of N_n whose kernel contains L_{m_n}. We also have the square integrable representation $\pi_{\lambda_{m_n}}$ of L_{m_n}. Write $\pi'_{\lambda_{m_n}}$ for the extension of $\pi_{\lambda_{m_n}}$ to a unitary representation of N_n on the same Hilbert space $\mathcal{H}_{\pi_{\lambda_{m_n}}}$ (the Mackey obstruction vanishes). Now

$$\pi_{\gamma_n} = \pi_{\lambda_1+\cdots+\lambda_{m_n-1}} \widehat{\otimes} \pi'_{\lambda_{m_n}}. \tag{2.5}$$

The parameter space for our representations of the direct limit Lie group $N = \varinjlim N_n$ will be

$$\mathfrak{t}^* = \bigcup_{n>0} \left\{ \gamma = \sum \lambda_r \in \mathfrak{s}^* \middle| \gamma_\ell \in \mathfrak{t}_\ell^* \text{ for } \ell \leqq n \text{ and } \lambda_r = 0 \in \mathfrak{z}_r^* \text{ for } r > m_n \right\} \tag{2.6}$$

where $\mathfrak{s}^* := \bigcup_{\ell>0} \mathfrak{s}_\ell^* = \sum_{r>0} \mathfrak{z}_r^*$. The representations π_γ of N are defined in a manner similar to that of (2.5). Given $\gamma = \sum \lambda_r \in \mathfrak{t}^*$ we have the index $n = n(\gamma)$ defined by $\gamma_\ell \in \mathfrak{t}_\ell^*$ for $\ell \leqq n(\gamma)$ and $\lambda_r = 0 \in \mathfrak{z}_r^*$ for $\ell > m_{n(\gamma)}$. Express

$$N = N_{n(\gamma)} \ltimes N_{n(\gamma),\infty} \text{ semidirect product, where } N_{n(\gamma),\infty} = \prod\nolimits_{r>m_{n(\gamma)}} L_r. \tag{2.7}$$

In particular, the closed normal subgroup $N_{n(\gamma),\infty}$ satisfies $N_{n(\gamma)} \cong N/N_{n(\gamma),\infty}$, and we denote

$$\pi_\gamma\text{: lift to } N \text{ of the stepwise square integrable } \pi_{\lambda_1+\cdots+\lambda_{m_{n(\gamma)}}} \in \widehat{N_{n(\gamma)}}. \tag{2.8}$$

The representation space of π_γ is the projective (jointly continuous) tensor product

$$\mathcal{H}_{\pi_\gamma} = \mathcal{H}_{\pi_{\lambda_1}} \widehat{\otimes} \mathcal{H}_{\pi_{\lambda_2}} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{H}_{\pi_{\lambda_{n(\gamma)}}} \tag{2.9}$$

These representations π_γ are the *limit stepwise square integrable* representations of N. We go on to see the extent to which their coefficients and characters imitate the properties of Theorem 1.3.

3. Coefficient functions

Let $\mathcal{H}_{\pi_\gamma}$ denote the representation space of π_γ and $\langle\,\cdot\,,\cdot\,\rangle_{\pi_\gamma}$ the hermitian inner product on $\mathcal{H}_{\pi_\gamma}$. Given $u, v \in \mathcal{H}_{\pi_\gamma}$ we have the coefficient function on N given by

$$f_{\pi_\gamma,u,v}(g) = \langle u, \pi_\gamma(g)v\rangle_{\pi_\gamma}. \tag{3.1}$$

We use the standard $(r(x)f)(g) = f(gx)$ and $(\ell(y)f)(g) = f(y^{-1}g)$. These right and left translations commute with each other. They are well defined on the $f_{\pi_\gamma,u,v}$ and satisfy

$$\ell(x)r(y) : f_{\pi_\gamma,u,v} \mapsto f_{\pi_\gamma,\pi_\gamma(x)u,\pi_\gamma(y)v}. \tag{3.2}$$

By our construction (2.8), the value $f_{\pi_\gamma,u,v}(g)$ depends only on the coset $gN''_{n(\gamma)}$. In other words, it really is a function on $N_{n(\gamma)} \cong N/N''_{n(\gamma)}$. Further, $|f_{\pi_\gamma,u,v}(g)|$ depends only on the coset $gS_{n(\gamma)}N''_{n(\gamma)}$ where $S_{n(\gamma)}$ is the quasicenter $Z_1Z_2\cdots Z_{m_{n(\gamma)}}$ of $N_{n(\gamma)} = L_1L_2\cdots L_{m_{n(\gamma)}}$. Building on (1.4), we have the following variation on the Frobenius-Schur orthogonality relations for finite groups:

Proposition 3.3. *If $\gamma \in \mathfrak{t}^*$ and $n = n(\gamma)$ then*

$$\|f_{\pi_\gamma,u,v}\|^2_{L^2(N/S_nN''_n)} = \frac{\|u\|^2_{\pi_\gamma}\|v\|^2_{\pi_\gamma}}{|P_n(\gamma)|}.$$

Proof. This is an induction on n. The case $n = 1$ is (1.4). Now go from n to $n+1$. Express $N_{n+1} = N_n \ltimes N_{n,n+1}$ where

$$N_n = L_1L_2\cdots L_{m_n} \text{ and } N_{n,q} = L_{m_n+1}L_{m_n+2}\cdots L_{m_q} \text{ for } q > n.$$

Then $S_{n+1} = S_n \times S_{n,n+1}$ where the quasi-centers

$$S_n = Z_1Z_2\cdots Z_{m_n} \text{ and } S_{n,q} = Z_{m_n+1}Z_{m_n+2}\cdots Z_{m_q} \text{ for } q > n.$$

Now let $\gamma_n \in \mathfrak{t}^*_n$ and $\gamma_{n,n+1} \in \mathfrak{t}^*_{n,n+1}$ where, as before, $\mathfrak{t}^*$ is the nonzero set of the Pfaffian in $\mathfrak{s}^*$. Note that $\pi_{\gamma_n} \in \widehat{N_n}$ and $\pi_{\gamma_{n,n+1}} \in \widehat{N_{n,n+1}}$ are stepwise square integrable. Write $\pi'_{\gamma_{n,n+1}}$ for the extension of $\pi_{\gamma_{n,n+1}}$ from $N_{n,n+1}$ to N_{n+1}. Let $u, v \in \mathcal{H}_{\pi_{\gamma_n}}$ and $x, y \in \mathcal{H}_{\pi_{\gamma_{n,n+1}}}$ so $u\otimes x, v\otimes y \in \mathcal{H}_{\pi_{\gamma_{n+1}}}$. Let a run over N_n and let b run over $N_{n,n+1}$. Compute

$$\begin{aligned}
&\|f_{\pi_{\gamma_{n+1}},u\otimes x,v\otimes y}\|^2_{L^2(N_{n+1}/S_{n+1})}\\
&\quad= \int_{N_{n+1}/S_{n+1}} |\langle u\otimes x, (\pi_{\gamma_n}\widehat{\otimes}\pi'_{\gamma_{n,n+1}})(ab)(v\otimes y)\rangle|^2 da\,db\\
&\quad= \int_{N_{n+1}/S_{n+1}} |\langle u\otimes x, \pi_{\gamma_n}(a)\pi'_{\gamma_{n,n+1}}(b)(v)\otimes \pi_{\gamma_{n,n+1}}(b)(y)\rangle|^2 da\,db\\
&\quad= \int_{N_{n,n+1}/S_{n,n+1}} \left(\int_{N_n/S_n} |\langle u, \pi_{\gamma_n}(a)\pi'_{\gamma_{n,n+1}}(b)(v)\rangle|^2 da\right) |\langle x, \pi_{\gamma_{n,n+1}}(b)(y)\rangle|^2 db\\
&\quad= \int_{N_{n,n+1}/S_{n,n+1}} \frac{\|u\|^2\|\pi'_{\gamma_{n,n+1}}(b)(v)\|^2}{|\mathrm{Pf}_{\,n}(\gamma_n)|} |\langle x, \pi_{\gamma_{n,n+1}}(b)(y)\rangle|^2 db\\
&\quad= \int_{N_{n,n+1}/S_{n,n+1}} \frac{\|u\|^2\|v\|^2}{|\mathrm{Pf}_{\,n}(\gamma_n)|} |\langle x, \pi_{\gamma_{n,n+1}}(b)(y)\rangle|^2 db\\
&\quad= \frac{\|u\|^2\|v\|^2}{|\mathrm{Pf}_{\,n}(\gamma_n)|}\cdot\frac{\|x\|^2\|y\|^2}{|\mathrm{Pf}_{\,n,n+1}(\gamma_{n,n+1})|} = \frac{\|u\otimes x\|^2\|v\otimes y\|^2}{|\mathrm{Pf}_{\,n+1}(\gamma_{n+1})|}.
\end{aligned}$$

The proposition follows. □

In the notation of the proof of Proposition 3.3,

$$f_{\pi_{\gamma_{n+1}},u\otimes x,v\otimes y}(a) = \langle u, \pi_{\gamma_n}(a)v\rangle \cdot \langle x, y\rangle = \langle x, y\rangle f_{\pi_{\gamma_n},u,v}(a) \text{ for } a \in N_n. \tag{3.4}$$

In other words, $f_{\pi_{\gamma_{n+1}},u\otimes x,v\otimes y}|_{N_n} = \langle x, y\rangle f_{\pi_{\gamma_n},u,v}$. In particular, the case where $x = e = y$, where e is a unit vector, is

$$f_{\pi_{\gamma_{n+1}},u\otimes e,v\otimes e}|_{N_n} = f_{\pi_{\gamma_n},u,v} \tag{3.5}$$

Iterating this and combining it with Proposition 3.3 we arrive at

Proposition 3.6. *Let $\gamma \in \mathfrak{t}^*$ and $n = n(\gamma)$. Let $\gamma' \in \mathfrak{t}^*$ and $n' = n(\gamma')$ with $n' > n$ and $\gamma'|_{\mathfrak{s}_n} = \gamma$. Then $\pi_{\gamma'}|_{N_n}$ is an infinite multiple of π_γ. Split $\mathcal{H}_{\pi_{\gamma'}} = \mathcal{H}_{\pi_\gamma}\widehat{\otimes}\mathcal{H}''$ where $\mathcal{H}'' = \mathcal{H}_{\pi_{\gamma'_{n+1}}}\widehat{\otimes}\cdots\widehat{\otimes}\mathcal{H}_{\pi_{\gamma'_{n'}}}$ in the notation of (2.9). Choose a unit vector $e \in \mathcal{H}''$. Then*

$$\mathcal{H}_{\pi_\gamma} \hookrightarrow \mathcal{H}_{\pi_{\gamma'}} \text{ by } v \mapsto v \otimes e \tag{3.7}$$

is a well-defined N_n–equivariant isometric injection. If $u, v \in \mathcal{H}_{\pi_\gamma}$ then

$$\|f_{\pi_{\gamma'},u\otimes e,v\otimes e}\|^2_{L^2(N/S_{n'}N''_{n'})} = \frac{|P_n(\gamma)|}{|P_{n'}(\gamma')|}\|f_{\pi_\gamma,u,v}\|^2_{L^2(N/S_nN''_n)}. \tag{3.8}$$

Proposition 3.6 will lead to construction of a Hilbert space $L^2(N)$. Corollary 5.17 will use coefficients and Schwartz class functions to show that $L^2(N)$ is independent of choice of the vectors e in (3.7).

4. Hilbert space limits

Now we combine the restriction maps of Section 3. Let $\gamma \in \mathfrak{t}^*$ and $n = n(\gamma)$. Then γ defines a unitary character $\zeta_\gamma = \exp(2\pi i\gamma)$ by

$$\zeta_\gamma(\exp(\xi)y) = e^{2\pi i\gamma(\xi)} \text{ where } \xi \in \mathfrak{s}_n \text{ and } y \in N''_n. \tag{4.1}$$

That defines the Hilbert space

$$\begin{aligned} L^2(N/S_nN''_n, \zeta_\gamma)\text{: } &\text{functions } f : N \to \mathbb{C} \text{ such that } f(gx) = \zeta_\gamma(x)^{-1}f(g) \\ &\text{and } |f| \in L^2(N/S_nN''_n) \text{ for } g \in N \text{ and } x \in S_nN''_n. \end{aligned} \tag{4.2}$$

The finite linear combinations of the coefficients $f_{\pi_\gamma,u,v}$ (where $u, v \in \mathcal{H}_{\pi_\gamma}$) form a dense subspace of $L^2(N/S_nN''_n, \zeta_\gamma)$, and that gives an $N \times N$ equivariant Hilbert space isomorphism

$$L^2(N/S_nN''_n, \zeta_\gamma) \cong \mathcal{H}_{\pi_\gamma}\widehat{\otimes}\mathcal{H}^*_{\pi_\gamma}. \tag{4.3}$$

We know that the stepwise square integrable group $N_n = N/N''_n$ satisfies

$$L^2(N_n) = L^2(N/N''_n) = \int_{\gamma\in\mathfrak{t}^* \text{ and } n(\gamma)=n} (\mathcal{H}_{\pi_\gamma}\widehat{\otimes}\mathcal{H}^*_{\pi_\gamma})|P_n(\gamma)|d\gamma. \tag{4.4}$$

In brief, that expands the functions on N that depend only on the first $m(n)$ factors in $N = N_1N_2N_3\cdots$. To expand the functions that depend on more factors, say the first $m(n')$ factors in the notation of Proposition 3.6, we would like to inject

$$L^2(N/N_n'') = \int_{\gamma\in\mathfrak{t}_n^*} L^2(N/S_nN_n'', \zeta_\gamma)|P_n(\gamma)|d\gamma$$

into

$$L^2(N/N_{n'}'') = \int_{\gamma'\in\mathfrak{t}_{n'}^*} L^2(N/S_{n'}N_{n'}'', \zeta_{\gamma'})|P_{n'}(\gamma')|d\gamma'$$

using the renormalizations of (3.8). However, γ has many extensions γ' with the given $n(\gamma') = n'$, so this will not work directly. But we can take the orthogonal projections dual to the injections of (3.8) and form an inverse system of Hilbert spaces.

To start, if $u, v \in \mathcal{H}_{\pi_\gamma}$ and $x, y \in \mathcal{H}''$, using (3.4) and Proposition 3.6,

$$p_{\gamma',\gamma} : f_{\pi_{\gamma'}, u\otimes x, v\otimes y} \mapsto \langle x, y\rangle \left|\frac{P_n(\gamma)}{P_{n'}(\gamma')}\right|^{1/2} f_{\pi_\gamma,u,v} \tag{4.5}$$

is the orthogonal projection dual to the isometric inclusion (3.7). Since γ is the restriction of γ' from $\mathfrak{s}_{n(\gamma')}$ to $\mathfrak{s}_{n(\gamma)}$ we can reformulate (4.5) as

$$p_{\gamma',n} : f_{\pi_{\gamma'}, u\otimes x, v\otimes y} \mapsto \langle x, y\rangle \left|\frac{P_n(\gamma'|_{\mathfrak{s}_n})}{P_{n'}(\gamma')}\right|^{1/2} f_{\pi_{\gamma'|_{\mathfrak{s}_n}},u,v} \text{ where } n = n(\gamma). \tag{4.6}$$

The maps $p_{\gamma,n}$ of (4.6) sum to a Hilbert space projection, essentially restriction of coefficients,

$$p_{n',n} : L^2(N_{n'}) \to L^2(N_n) \text{ for } n = n(\gamma'|_{\mathfrak{s}_n}) \text{ and } n' = n(\gamma') \geqq n \tag{4.7}$$

where $p_{n',n} = \left(\int_{\gamma'\in\mathfrak{s}_{n'}^*} p_{\gamma',n}\, d\gamma'\right)$. The maps $p_{n',n}$ of (4.7) define an inverse system in the category of Hilbert spaces and partial isometries:

$$L^2(N_1) \xleftarrow{p_{2,1}} L^2(N_2) \xleftarrow{p_{3,2}} L^2(N_3) \xleftarrow{p_{4,3}} \cdots \leftarrow L^2(N) \tag{4.8}$$

where the projective limit $L^2(N) := \varprojlim\{L^2(N_n), p_{n',n}\}$ is taken in the category of Hilbert spaces and partial isometries. We now have the Hilbert space

$$L^2(N) := \varprojlim\{L^2(N_n), p_{n',n}\}. \tag{4.9}$$

5. The Schwartz spaces

In order to refine (4.9) to a Fourier inversion formula we must first make it more explicit. The span $\mathcal{A}(\pi_{\gamma_n})$ of the coefficients of the representation π_{γ_n} is dense in the space of functions on N_n given by $\mathcal{H}_{\pi_{\gamma_n}}\widehat{\otimes}\mathcal{H}_{\pi_{\gamma_n}}$. The idea in the background here is to realize Schwartz class functions as wave packets $f(a) =$

$\int_{\mathfrak{s}_n^*} \varphi(\gamma_n) f_{\pi_{\gamma_n},u(\gamma_n),v(\gamma_n)}(a) d\gamma_n$ where φ is a Schwartz class function on $\mathfrak{s}_n$ and where $u(\gamma_n)$ and $v(\gamma_n)$ are fields of C^∞ unit vectors in the $\mathcal{H}_{\pi_{\gamma_n}}$. More concretely, we show that the coefficient $f_{\pi_{\gamma_n},u,v}$ belongs to an appropriate Schwartz space (and thus an appropriate L^1 space) when u and v are C^∞ vectors for π_{γ_n}.

We first collect some standard facts from Kirillov theory concerning the analog of the Schrödinger representation of the Heisenberg group. Let L be a connected simply connected nilpotent Lie group that has square integrable representations. Z is the center of L, and $\lambda \in \mathfrak{z}^*$ with $\mathrm{Pf}_{\mathfrak{l}}(\lambda) \neq 0$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be totally real polarizations for λ, $\mathfrak{p} = \mathfrak{z} + \mathfrak{a}$ and $\mathfrak{q} = \mathfrak{z} + \mathfrak{b}$, and suppose that we chose them so that $b_\lambda(x,y) = \lambda([x,y])$ gives a nondegenerate pairing of $\mathfrak{a}$ with $\mathfrak{b}$. In this setting, the square integrable representation π_λ of L is $\mathrm{Ind}_P^N(\exp(2\pi i \lambda)$, and it represents L on $L^2(N/P) = L^2(B)$. Further, here π_λ maps the universal enveloping algebra $\mathcal{U}(\mathfrak{l})$ onto the set of all polynomial (in linear coordinates from $\exp\colon \mathfrak{b} \to B$) differential operators on B. In particular,

Lemma 5.1. *The C^∞ vectors for the representation π_λ are the Schwartz class functions on B. In other words, if p and q are polynomials on B, then D is a constant coefficient differential operator on B. Further, if $u\colon B \to \mathbb{C}$ is a C^∞ vector for π_λ, then $|q(x)p(D)u|$ is bounded.*

In order to extend this to stepwise square integrable representations we must take into account the problem that S_n need not be central in N_n. We do this by decomposing

$$N_n \simeq L_1 \times \cdots \times L_{m(n)} \tag{5.2}$$

where $\simeq$ is the measure preserving C^ω diffeomorphism given by the polynomial map $\exp'\colon \mathfrak{n}_n \to N_n$, defined by

$$\exp'(\xi_1 + \cdots + \xi_{m(n)}) = \exp(\xi_1)\exp(\xi_2)\cdots\exp(\xi_{m(n)}) \text{ where each } \xi_r \in \mathfrak{l}_r. \tag{5.3}$$

Using the part of (2.2d) that says $[\mathfrak{l}_r, \mathfrak{z}_s] = 0$ for $r < s$ the decomposition (5.2) gives us

$$\begin{aligned} N_n/S_n &= \{x_{m(n)} \cdots x_2 x_1 Z_{m(n)} \cdots Z_2 Z_1 \mid x_r \in L_r\} \\ &= \{x_{m(n)} Z_{m(n)} \cdots x_2 Z_2 x_1 Z_1 \mid x_r \in L_r\} \\ &= (L_{m(n)}/Z_{m(n)}) \times \cdots \times (L_1/Z_1) \\ &\simeq (L_1/Z_1) \times \cdots \times (L_{m(n)}/Z_{m(n)}). \end{aligned} \tag{5.4}$$

Now let $\gamma_n = \lambda_1 + \cdots + \lambda_{m(n)} \in \mathfrak{t}_n^*$. Let $\mathfrak{p}_r$ and $\mathfrak{q}_r$ be totally real polarizations on $\mathfrak{l}_r$ for λ_r, paired as above by b_{λ_r}. We do not claim that $\mathfrak{p} = \sum \mathfrak{p}_r$ and $\mathfrak{q} = \sum \mathfrak{q}_r$ are polarizations on $\mathfrak{n}_n$ for γ_n (we do not know that they are algebras), but still $\mathfrak{p}_r = \mathfrak{z}_r + \mathfrak{a}_r$ and $\mathfrak{q}_r = \mathfrak{z}_r + \mathfrak{b}_r$ where b_{λ_r} pairs $\mathfrak{a}_r$ with $\mathfrak{b}_r$, so b_{γ_n} is a nondegenerate pairing of $\mathfrak{a} = \sum \mathfrak{a}_r$ with $\mathfrak{b} = \sum \mathfrak{b}_r$. Now the stepwise square integrable representation π_{γ_n} of N_n is realized on $L^2(B)$ where $B = \exp'(\mathfrak{b})$ in the notation of (5.3). Again, in this setting, π_{γ_n} maps the universal enveloping algebra of $\mathfrak{n}_n$ onto the set of all polynomial (in linear coordinates from $\exp'\colon \mathfrak{b} \to B$) differential operators on B. This extends Lemma 5.1 to

Lemma 5.5. *Identify $B = \exp'(\mathfrak{b})$ with the real vector space $\mathfrak{b}$. The C^∞ vectors for the representation π_{γ_n} are the Schwartz class functions on B. In other words, if p and q are polynomials on B, if D is a constant coefficient differential operator on B, and if $u\colon B \to \mathbb{C}$ is a C^∞ vector for π_{γ_n}, then $|q(x)p(D)u|$ is bounded.*

Now consider the Schwartz space analog of the definition (4.2). We define the *relative Schwartz space* $\mathcal{C}(N/S_nN''_n, \zeta_\gamma) = \mathcal{C}(N_n/S_n, \zeta_{\gamma_n})$ to be all functions $f \in C^\infty(N)$ such that

$$\begin{aligned}&f(xs) = \zeta_\gamma(s)^{-1} f(x) \text{ for all } x \in N_n \text{ and } s \in S_n, \text{ and } |q(x)p(D)f| \\ &\text{is bounded for all polynomials } p, q \text{ on } N_n/S_n \text{ and all } D \in \mathcal{U}(\mathfrak{n}_n).\end{aligned} \tag{5.6}$$

It is a nuclear Fréchet space and is dense in $L^2(N/S_nN''_n, \zeta_\gamma) = L^2(N_n/S_n, \zeta_{\gamma_n})$.

We define $C_c^\infty(N/S_nN''_n, \zeta_\gamma) = C_c^\infty(N_n/S_n, \zeta_{\gamma_n})$ to be the space of all functions $f \in C^\infty(N_n)$ such that $f(xs) = \zeta_\gamma(s)^{-1}f(x)$ for $x \in N_n$ and $s \in S_n$, and where $|f| \in C_c^\infty(N_n/S_n) = C_c^\infty(N/S_nN''_n)$. It is dense in the corresponding Schwartz space. Thus we have the expected continuous inclusions $C_c^\infty \hookrightarrow \mathcal{C} \hookrightarrow L^2$ with dense images.

Theorem 5.7. *Let u and v be C^∞ vectors for the stepwise square integrable representation π_{γ_n} of N_n. Define ζ_γ and ζ_{γ_n} as in* (4.1)*, and $A = \exp'(\mathfrak{a})$ and $B = \exp'(\mathfrak{b})$ as in the discussion following* (5.4)*. Then the coefficient function $f_{\pi_{\gamma_n},u,v}$ belongs to the relative Schwartz space $\mathcal{C}(N/S_nN''_n, \zeta_\gamma) = \mathcal{C}(N_n/S_n, \zeta_{\gamma_n})$.*

Proof. Write $f_{u,v}$ for $f_{\pi_{\gamma_n},u,v}$ and π for π_{γ_n}. So $f_{u,v}(x) = \langle u, \pi(x)v\rangle$. The left/right action of the enveloping algebra is $Df_{u,v}E = f_{\pi(D)u,\pi(E)v}$. View $u \in \mathcal{C}(A)$ and $v \in \mathcal{C}(B)$. Here $\pi(D)u$ is the image of u under the (arbitrary) polynomial differential operator $\pi(D)$ on A and $\pi(E)v$ is the image of v under the (arbitrary) polynomial differential operator $\pi(D)$ on B. Together they give the image of $f_{u,v}$ under the polynomial differential operator $\pi(D)\otimes\pi(E)$ on $A\times B = N_n/S_n$. Every polynomial differential operator on $A \times B$ is a finite sum of such operators $\pi(D)\otimes\pi(E)$. Since coefficients are bounded, here $|f_{\pi(D)u,\pi(E)v}(x)| \leqq \|\pi(D)u\| \cdot \|\pi(E)v\|$, and since $f_{\pi(D)u,\pi(E)v}(xs) = \zeta(s)^{-1} f_{\pi(D)u,\pi(E)v}(x)$, the coefficient $f_{u,v} \in \mathcal{C}(N_n/S_n, \zeta_{\gamma_n})$. $\square$

Corollary 5.8. *Let u and v be C^∞ vectors for the stepwise square integrable representation π_{γ_n} of N_n. Then the coefficient function $f_{\pi_{\gamma_n},u,v} \in L^1(N_n/S_n, \zeta_{\gamma_n})$.*

Corollary 5.9. *Let L be a connected simply connected nilpotent Lie group, Z its center, and π a square integrable representation of L. Let $\zeta \in \widehat{Z}$ such that $\pi|_Z$ is a multiple of ζ. Let u and v be C^∞ vectors for π. Then $f_{\pi,u,v} \in L^1(L/Z, \zeta)$.*

Any norm $|\xi|$ on $\mathfrak{n}_n$ carries over to a norm $|\exp(\xi)| := |\xi|$ on N_n. We have the standard Schwartz space $\mathcal{C}(N_n)$, given by the seminorms

$$\nu_{k,D,E}(f) = \sup_{x\in N_n} |(1 + |x|^2)^k (DfE)(x)|$$

where k is a positive integer and $D, E \in \mathcal{U}(\mathfrak{n}_n)$ acting on the left and right. Since $\exp\colon \mathfrak{n}_n \to N_n$ is a polynomial diffeomorphism it gives a topological isomorphism of $\mathcal{C}(N_n)$ onto the classical Schwartz space $\mathcal{C}(\mathfrak{n}_n)$. Fourier transform and inverse Fourier transform of Schwartz class functions preserve $\mathcal{C}(N_n)$.

Remark 5.10. If $\gamma_n \in \mathfrak{s}_n^*$ and $f \in \mathcal{C}(N_n)$ define $f_{\gamma_n}(x) = \int_{S_n} f(xs)\zeta_{\gamma_n}(s)ds$. Then $f_{\gamma_n} \in \mathcal{C}(N_n/S_n, \zeta_{\gamma_n})$. Let $z \in S_n$. Since S_n is commutative,

$$\begin{aligned} f_{\gamma_n}(xz) = \int_{S_n} f(xzs)\zeta_{\gamma_n}(s)ds &= \int_{S_n} f(xsz)\zeta_{\gamma_n}(s)ds \\ &= \int_{S_n} f(xs)\zeta_{\gamma_n}(z^{-1}s)ds = \zeta_{\gamma_n}(z)^{-1} f_{\gamma_n}(x). \end{aligned}$$

Given $x \in N_n$ we view $f_{\gamma_n}(x)$ as a function on $\mathfrak{s}_n^*$ by $\varphi_x(\gamma_n) := f_{\gamma_n}(x)$. Note that φ_x is (a multiple of) the Fourier transform of the left translate $(\ell(x^{-1})f)|_{S_n}$, say $\mathcal{F}_{S_n}(\ell(x^{-1})f)|_{S_n}$. The inverse Fourier $\mathcal{F}_{S_n}^{-1}(\varphi_x)$ transform reconstructs f from the f_{γ_n}. Each of the f_{γ_n} is a limit (in $\mathcal{C}(N_n/S_n, \zeta_{\gamma_n})$) of finite linear combinations of coefficient functions $f_{\pi_{\gamma_n},u,v}$. Thus every $f \in \mathcal{C}(N_n)$ is approximated (in $\mathcal{C}(N_n)$) by Schwartz class function packets of coefficient functions of stepwise square integrable representations.

Proceeding as in Section 4, let $n' \geqq n$ and consider $\gamma_{n'} \in \mathfrak{t}_{n'}^*$ with $\gamma_{n'}|_{\mathfrak{s}_n} = \gamma_n$. For brevity write $\gamma = \gamma_n$ and $\gamma' = \gamma_{n'}$. We reformulate (4.7) through (4.9) for the Schwartz spaces:

$$q_{n',n} \colon \mathcal{C}(N_{n'}) \to \mathcal{C}(N_n) \text{ by } f \mapsto f|_{N_n}. \tag{5.11}$$

The maps $q_{n',n}$ of (5.11) define an inverse system in the category of complete locally convex topological vector spaces

$$\mathcal{C}(N_1) \xleftarrow{q_{2,1}} \mathcal{C}(N_2) \xleftarrow{q_{3,2}} \mathcal{C}(N_3) \xleftarrow{q_{4,3}} \cdots . \tag{5.12}$$

We define the projective limit

$$\mathcal{C}(N) := \varprojlim \{\mathcal{C}(N_n), q_{n',n}\} \tag{5.13}$$

to be the Schwartz space of $N = \varinjlim N_n$. This is dual to our earlier construction in [8, (2.20)]. Now we relate it to (4.9). We scale the natural injections to maps

$$r_{n,\gamma} \colon \mathcal{C}(N_n/S_n, \zeta_\gamma) \to L^2(N_n/S_n, \zeta_\gamma) \text{ by } f \mapsto |\mathrm{Pf}_{\,\mathfrak{n}_n}(\gamma)|^{1/2} f. \tag{5.14}$$

They sum to maps

$$r_n = \left(\int_{\mathfrak{s}_n^*} r_{n,\gamma}\, d\gamma \right) : \mathcal{C}(N_n) \to L^2(N_n) \tag{5.15}$$

that are equivariant for the maps $p_{n',n}$ and $q_{n',n}$. The arguments leading to [8, Prop. 2.22] can be dualized from direct limits to projective limits. Thus, dual to [8, Prop. 2.22]:

Proposition 5.16. *The maps r_n of* (5.15) *satisfy $p_{n',n} \cdot r_{n'} = r_n \cdot q_{n',n}$ for $n' \geqq n$ and send the inverse system $\{\mathcal{C}(N_n), q_{n',n}\}$ into the inverse system $\{L^2(N_n), p_{n',n}\}$. That defines a continuous N–equivariant injection*

$$r \colon \mathcal{C}(N) \to L^2(N)$$

with dense image. In particular r defines a pre-Hilbert space structure on $\mathcal{C}(N)$ with completion isometric to $L^2(N)$.

Since $\mathcal{C}(N)$ is independent of the choices involved in the construction of $L^2(N)$ we have

Corollary 5.17. *The limit Hilbert space* $L^2(N) = \varprojlim\{L^2(N_n), p_{n',n}\}\}$ *of* (4.9), *and the left/right regular representation of* $N \times N$ *on* $L^2(N)$, *are independent of the choice of vectors* e *in* (3.7).

6. Fourier inversion for the limit group

In this section we apply the material of Section 5 to extend the Fourier inversion portion of Theorem 1.3 from the N_n to the limit group $N = \varinjlim N_n$. To set this up recall that

- $\mathfrak{t}^* = \varprojlim \mathfrak{t}_n^*$ consists of all collections $\gamma = (\gamma_n)$ where each $\gamma_n \in \mathfrak{t}_n^*$ and if $n' \geqq n$ then $\gamma_{n'}|_{\mathfrak{s}_n} = \gamma_n$.
- Given $\gamma = (\gamma_n) \in \mathfrak{t}^*$ the limit representation $\pi_\gamma = \varprojlim \pi_{\gamma_n}$ is constructed as in Section 2.
- The distribution character $\Theta_{\pi_{\gamma_n}}$ are given by (1.5).
- $\mathcal{C}(N) = \varprojlim \mathcal{C}(N_n)$ consists of all sets $f = (f_n)$ where each $f_n \in \mathcal{C}(N_n)$, and where if $n' \geqq n$ then $f_{n'}|_{N_n} = f_n$.

Then the limit Fourier inversion formula is

Theorem 6.1. *Suppose that* $N = \varinjlim N_n$ *where* $\{N_n\}$ *satisfies* (2.2). *Then the Plancherel measure for* N *is concentrated on* $\mathfrak{t}^*$. *Let* $f = (f_n) \in \mathcal{C}(N)$ *and* $x \in N$. *Then* $x \in N_n$ *for some* n *and*

$$f(x) = c_n \int_{\mathfrak{t}_n^*} \Theta_{\pi_{\gamma_n}}(r_x f)|\mathrm{Pf}_{\,\mathfrak{n}_n}(\gamma_n)|d\gamma_n \tag{6.2}$$

where $c_n = 2^{d_1+\cdots+d_m} d_1! d_2! \cdots d_m!$ *as in* (1.2a) *and* m *is the number of factors* L_r *in* N_n.

Proof. Apply Theorem 1.3 to N_n: $f(x) = f_n(x) = c_n \int_{\mathfrak{t}_n^*} \Theta_{\pi_{\gamma_n}}(r_x f)|\mathrm{Pf}_{\,\mathfrak{n}_n}(\gamma_n)|\, d\gamma_n$.
□

Remark 6.3. A Plancherel Formula of the sort $\|f\|^2_{L^2} = \int \|\pi(f)\|^2_{HS}\, d\pi$ usually is somewhat easier than a Fourier inversion formula. This in part is because it usually is easier to prove that operators $\pi(f)$ are Hilbert-Schmidt than to prove that (for appropriate functions f) they are of trace class. Thus one might expect that a formula $\|f\|^2_{L^2(N)} = \lim c_n' \int_{\mathfrak{t}_n^*} \|\pi_{\gamma_n}(f|_{N_n})\|^2_{HS}|\mathrm{Pf}_{\,\mathfrak{n}_n}(\gamma_n)|\, d\gamma_n$ would be easier to prove than (6.2). But it is not clear how to relate the Hilbert–Schmidt norms to the limit process, because we have not yet found an appropriate form of the Frobenius–Schur orthogonality relations. Thus the "less delicate" Plancherel Formula remains problematical.

7. Nilradicals of parabolics in finite-dimensional groups

In Section 8 we will specialize our results to nilradicals of minimal parabolic subgroups of finitary real reductive Lie groups such as the infinite special and general linear groups and the infinite real, complex and quaternionic unitary groups. In order to do that, in this section we review the relevant restricted root structure that gives the finite-dimensional case, reversing some of the enumerations used in [10] to be appropriate for our direct limit systems.

Let G be a finite-dimensional connected real reductive Lie group. We recall some structural results on its minimal parabolic subgroups, some standard and some from [10].

Fix an Iwasawa decomposition $G = KAN$. Write $\mathfrak{k}$ for the Lie algebra of K, $\mathfrak{a}$ for the Lie algebra of A, and $\mathfrak{n}$ for the Lie algebra of N. Complete $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Now we have root systems

- $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$: roots of $\mathfrak{g}_\mathbb{C}$ relative to $\mathfrak{h}_\mathbb{C}$ (ordinary roots), and
- $\Delta(\mathfrak{g}, \mathfrak{a})$: roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ (restricted roots).
- $\Delta_0(\mathfrak{g}, \mathfrak{a}) = \{\gamma \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid 2\gamma \notin \Delta(\mathfrak{g}, \mathfrak{a})\}$ (nonmultipliable restricted roots).

Sometimes we will identify a restricted root $\gamma = \alpha|_\mathfrak{a}$, $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and $\alpha|_\mathfrak{a} \neq 0$, with the set

$$[\gamma] := \{\alpha' \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \mid \alpha'|_\mathfrak{a} = \gamma\} \tag{7.1}$$

of all roots that restrict to it. Further, $\Delta(\mathfrak{g}, \mathfrak{a})$ and $\Delta_0(\mathfrak{g}, \mathfrak{a})$ are root systems in the usual sense. Any positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \subset \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ defines positive systems

- $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha|_\mathfrak{a} \mid \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \text{ and } \alpha|_\mathfrak{a} \neq 0\}$ and
- $\Delta_0^+(\mathfrak{g}, \mathfrak{a}) = \Delta_0(\mathfrak{g}, \mathfrak{a}) \cap \Delta^+(\mathfrak{g}, \mathfrak{a})$.

We can (and do) choose $\Delta^+(\mathfrak{g}, \mathfrak{h})$ so that

- $\mathfrak{n}$ is the sum of the positive restricted root spaces and
- if $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and $\alpha|_\mathfrak{a} \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ then $\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.

Recall that two roots are *strongly orthogonal* if their sum and their difference are not roots. Then they are orthogonal. We define

$$\begin{aligned} \beta_1' \in \Delta^+(\mathfrak{g}, \mathfrak{a}) &\text{ is a maximal positive restricted root and} \\ \beta_{r+1}' \in \Delta^+(\mathfrak{g}, \mathfrak{a}) &\text{ is a maximum among the roots of } \Delta^+(\mathfrak{g}, \mathfrak{a}) \\ &\text{ that are orthogonal to all } \beta_i' \text{ with } i \leqq r. \end{aligned} \tag{7.2}$$

Then the β_r' are mutually strongly orthogonal. Note that each $\beta_r' \in \Delta_0^+(\mathfrak{g}, \mathfrak{a})$. This is the Kostant cascade coming down from the maximal root. Denote

$$\{\beta_1', \dots, \beta_m'\}: \text{ the set of strongly orthogonal roots constructed in (7.2).} \tag{7.3}$$

The enumeration (7.3) is not appropriate for the direct limit process, but we need it for some of the lemmas below. For direct limit considerations we will use the reversed ordering

$$\beta_r = \beta_{m-r+1}', \text{ so the ordered sets } \{\beta_1, \dots, \beta_m\} = \{\beta_m', \dots, \beta_1'\}. \tag{7.4}$$

For $1 \leqq r \leqq m$ define

$$\begin{aligned} \Delta_m^+ &= \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \beta_m - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\} \text{ and} \\ \Delta_{m-r-1}^+ &= \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \setminus (\Delta_m^+ \cup \dots \cup \Delta_{m-r}^+) \mid \beta_{m-r-1} - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\}. \end{aligned} \tag{7.5}$$

Lemma 7.6 ([10, Lemma 6.3]). *If $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ then either $\alpha \in \{\beta_1, \dots, \beta_m\}$ or α belongs to exactly one of the sets Δ_r^+. In particular the Lie algebra $\mathfrak{n}$ of N is the vector space direct sum of its subspaces*

$$\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum\nolimits_{\Delta_r^+} \mathfrak{g}_\alpha \text{ for } 1 \leqq r \leqq m. \tag{7.7}$$

Lemma 7.8 ([10, Lemma 6.4]). *The set Δ_r^+ of* (7.5) *satisfies*

$$\Delta_r^+ \cup \{\beta_r\} = \{\alpha \in \Delta^+ \mid \alpha \perp \beta_i \text{ for } i > r \text{ and } \langle \alpha, \beta_r \rangle > 0\}.$$

In particular, $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}_t$ where $t = \max\{r, s\}$. Thus $\mathfrak{n}$ has an increasing foliation based on the ideals

$$\mathfrak{l}_{r,m} = \mathfrak{l}_{r+1} + \cdots + \mathfrak{l}_m \text{ for } 0 \leqq r < m \tag{7.9}$$

with a corresponding group level decomposition by normal subgroups $L_{r,m}$ where

$$N = L_{0,m} = L_1 L_2 \cdots L_m \text{ and } L_{r,m} = L_{r+1} \ltimes N_{r+1,m} \text{ for } 0 \leqq r < m. \tag{7.10}$$

The structure of Δ_r^+, and later of $\mathfrak{l}_r$, is exhibited by a particular Weyl group element of $\Delta(\mathfrak{g}, \mathfrak{a})$ and the negative of that Weyl group element. Denote

$$s_{\beta_r}\colon \text{ Weyl reflection in } \beta_r \text{ and } \sigma_r\colon \Delta(\mathfrak{g}, \mathfrak{a}) \to \Delta(\mathfrak{g}, \mathfrak{a}) \text{ by } \sigma_r(\alpha) = -s_{\beta_r}(\alpha). \tag{7.11}$$

Here $\sigma_r(\beta_s) = -\beta_s$ for $s \neq r$, $+\beta_s$ if $s = r$. If $\alpha \in \Delta_r^+$ we still have $\sigma_r(\alpha) \perp \beta_i$ for $i > r$ and $\langle \sigma_r(\alpha), \beta_r \rangle > 0$. If $\sigma_r(\alpha)$ is negative then $\beta_r - \sigma_r(\alpha) > \beta_r$, contradicting the maximality property of β_{m-r+1}. Thus, using Lemma 7.8, $\sigma_r(\Delta_r^+) = \Delta_r^+$. This divides each Δ_r^+ into pairs:

Lemma 7.12 ([10, Lemma 6.8]). *If $\alpha \in \Delta_r^+$ then $\alpha + \sigma_r(\alpha) = \beta_r$.* (*Of course it is possible that $\alpha = \sigma_r(\alpha) = (1/2)\beta_r$ when $(1/2)\beta_r$ is a root.*) *If $\alpha, \alpha' \in \Delta_r^+$ and $\alpha + \alpha' \in \Delta(\mathfrak{g}, \mathfrak{a})$ then $\alpha + \alpha' = \beta_r$.*

It comes out of Lemmas 7.6 and 7.8 that the decompositions of (7.5), (7.7) and (7.9) satisfy (2.2), so Theorem 1.3 applies to nilradicals of minimal parabolic subgroups. In other words, as in Theorem 1.3,

Theorem 7.13 ([10, Thm. 6.16]). *Let G be a real reductive Lie group, $G = KAN$ an Iwasawa decomposition, $\mathfrak{l}_r$ and $\mathfrak{n}_r$ the subalgebras of $\mathfrak{n}$ defined in* (7.7) *and* (7.9), *and L_r and N_r the corresponding analytic subgroups of N. Then the L_r and N_r satisfy* (2.2). *In particular, the Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ, then the coefficient $f_{u,v}(x) = \langle u, \pi_\lambda(x) v \rangle$ satisfies*

$$\|f_{u,v}\|^2_{L^2(N/S)} = \frac{\|u\|^2 \|v\|^2}{|\mathrm{Pf}\,(\lambda)|}. \tag{7.14}$$

The distribution character Θ_{π_λ} of π_λ satisfies

$$\Theta_{\pi_\lambda}(f) = c^{-1} |\mathrm{Pf}\,(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi) d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N) \tag{7.15}$$

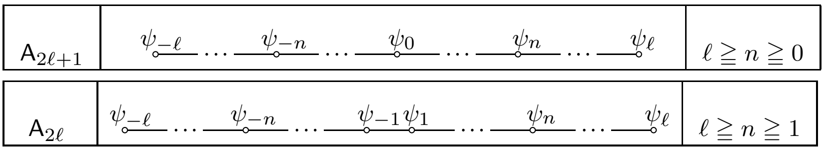
where $\mathcal{C}(N)$ is the Schwartz space, f_1 is the lift $f_1(\xi) = f(\exp(\xi))$, $\widehat{f_1}$ is its classical Fourier transform, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$, *$c = 2^{d_1+\cdots+d_m} d_1! d_2! \cdots d_m!$ as in* (1.2a), *and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to* $\mathrm{Ad}^*(N)\lambda$. *The Fourier inversion formula on N is*

$$f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f) |\mathrm{Pf}\,(\lambda)| d\lambda \text{ for } f \in \mathcal{C}(N). \tag{7.16}$$

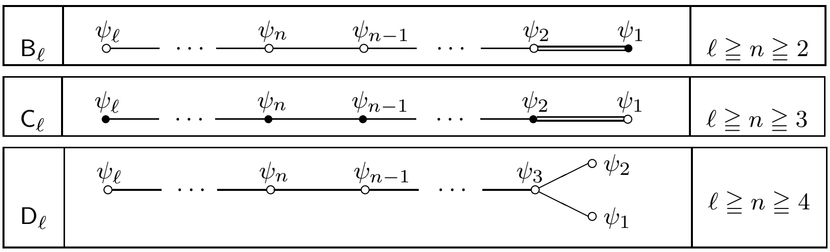
8. Nilradicals of parabolics in infinite-dimensional groups

We now look at the classical real forms of the three classical simple locally finite countable–dimensional Lie algebras $\mathfrak{g}_\mathbb{C} = \varinjlim \mathfrak{g}_{n,\mathbb{C}}$, and their real forms $\mathfrak{g}_\mathbb{R}$. The Lie algebras $\mathfrak{g}_\mathbb{C}$ are the classical direct limits, $\mathfrak{sl}(\infty, \mathbb{C}) = \varinjlim \mathfrak{sl}(n; \mathbb{C})$, $\mathfrak{so}(\infty, \mathbb{C}) = \varinjlim \mathfrak{so}(2n; \mathbb{C}) = \varinjlim \mathfrak{so}(2n + 1; \mathbb{C})$, and $\mathfrak{sp}(\infty, \mathbb{C}) = \varinjlim \mathfrak{sp}(n; \mathbb{C})$, where the direct systems are given by the inclusions of the form $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ or $A \mapsto \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. We often consider the locally reductive algebra $\mathfrak{gl}(\infty; \mathbb{C}) = \varinjlim \mathfrak{gl}(n; \mathbb{C})$ along with $\mathfrak{sl}(\infty; \mathbb{C})$.

Let G_n be a real (this includes complex) simple Lie group of classical type and real rank n. We have just described it as sitting in a direct system $\{G_n\}$ of Lie algebras in the same series. Set $G = \varinjlim G_n$ as above. Then we have coherent Iwasawa decompositions $G_n = K_n A_n N_n$ with $K_n \subset K_\ell$, $A_n \subset A_\ell$ and $N_n \subset N_\ell$ for $\ell \geqq n$. We need to do this so that the direct limit respects the restricted root structures, in particular the strongly orthogonal root structures, of the N_n. To do that we enumerate the set $\Psi_n = \Psi(\mathfrak{g}_n, \mathfrak{h}_n)$ of nonmultipliable simple restricted roots so that, in the Dynkin diagram, for type A we spread from the center of the diagram. For types B, C and D ψ_1 is the *right* endpoint; in other words, for $\ell \geqq n$ Ψ_ℓ is constructed from Ψ_n adding simple roots to the *left* end of their Dynkin diagrams. Thus



(8.1)



(8.2)

We describe this by saying that *G_ℓ propagates G_n*. For types B, C and D this is the same as the notion of propagation in [2] and [3], but for type A it is a bit different. With the simple root enumeration of (8.1) and (8.2) the set $\{\beta_1, \ldots, \beta_m\}$

of strongly orthogonal positive restricted roots of (7.4) is

$$
\begin{aligned}
\mathsf{A}_{2n+1}\colon & \beta_1 = \psi_0; \beta_2 = \psi_{-1} + \psi_0 + \psi_1; \dots; \beta_r = \psi_{-r+1} + \beta_{r-1} + \psi_{r-1}; \dots \\
\mathsf{A}_{2n}\colon & \beta_1 = \psi_{-1} + \psi_1; \beta_2 = \psi_{-2} + \psi_{-1} + \psi_1 + \psi_2; \dots; \beta_r = \psi_{-r} + \beta_{r-1} + \psi_r; \dots \\
\mathsf{B}_{2n+1}\colon & \beta_1 = \psi_1; \beta_2 = \psi_3 \text{ and } \beta_3 = 2(\psi_1 + \psi_2) + \psi_3; \dots; \\
& \beta_{2r} = \psi_{2r+1} \text{ and } \beta_{2r+1} = 2(\psi_1 + \cdots \psi_{2r}) + \psi_{2r+1}; \dots \\
\mathsf{B}_{2n}\colon & \beta_1 = \psi_2 \text{ and } \beta_2 = 2\psi_1 + \psi_2; \beta_3 = \psi_4 \text{ and } \beta_4 = 2(\psi_1 + \psi_2 + \psi_3) + \psi_4; \dots; \\
& \beta_{2r+1} = \psi_{2r-1} \text{ and } \beta_{2r} = 2(\psi_1 + \dots \psi_{2r-1}) + \psi_{2r}; \dots \\
\mathsf{C}_n\colon & \beta_1 = \psi_1; \beta_2 = \psi_1 + 2\psi_2; \dots; \beta_r = \psi_1 + 2(\psi_2 + \cdots + \psi_r); \dots \\
\mathsf{D}_{2n+1}\colon & \beta_1 = \psi_3; \beta_2 = \psi_1 + \psi_2 + \psi_3; \beta_3 = \psi_5; \beta_4 = \psi_1 + \psi_2 + 2(\psi_3 + \psi_4) + \psi_5; \\
& \beta_{2r-1} = \psi_{2r+1} \text{ and } \beta_{2r} = \psi_1 + \psi_2 + 2(\psi_3 + \cdots + \psi_{2r}) + \psi_{2r+1}; \dots \\
\mathsf{D}_{2n}\colon & \beta_1 = \psi_1; \beta_2 = \psi_2; \beta_3 = \psi_4 \text{ and } \beta_4 = \psi_1 + \psi_2 + 2\psi_3 + \psi_4; \beta_5 = \psi_6; \text{ and} \\
& \beta_6 = \psi_1 + \psi_2 + 2(\psi_3 + \psi_4 + \psi_5) + \psi_6; \dots; \beta_{2r-1} = \psi_{2r}; \text{ and} \\
& \beta_{2r} = \psi_1 + \psi_2 + 2(\psi_3 + \cdots + \psi_{2r-1}) + \psi_{2r}; \dots
\end{aligned}
$$

In order to simplify use of these constructions we denote

Definition 8.3. Let $G = \varinjlim G_n$ be a classical simple locally finite countable-dimensional Lie group. Possibly passing to a cofinal subsequence, suppose that we have coherent Iwasawa decompositions $G_n = K_n A_n N_n$ such that G_ℓ propagates G_n for $\ell \geqq n$. Then, passing to a cofinal subsequence if necessary, we can assume that all of the nonmultipliable restricted root systems $\Delta_0(\mathfrak{g}_n, \mathfrak{a}_n)$ are of the same type A_{2n+1}, A_{2n}, B_{2n+1}, B_{2n}, C_n, D_{2n+1} or D_{2n}. Then we will say that the direct system $\{G_n\}$ is *well aligned.*

The condition that $\{G_n\}$ be well aligned is exactly what we need for $\{N_n\}$ to satisfy (2.2), and given G we have a realization $G = \varinjlim G_n$ for which $\{G_n\}$ is well aligned. In summary,

Theorem 8.4. *Let G be a classical connected countable-dimensional real reductive Lie group. Express $G = \varinjlim G_n$ with $\{G_n\}$ well aligned. Then $\{N_n\}$ satisfies* (2.2). *In particular, Theorem* 7.13 *holds for the maximal locally unipotent subgroup $N = \varinjlim N_n$ of G.*

Remark 8.5. In Theorem 8.4 the possibilities for G are the finite-dimensional simple Lie groups and the infinite-dimensional $\mathrm{SL}(\infty;\mathbb{C})$, $\mathrm{SO}(\infty;\mathbb{C})$, $\mathrm{Sp}(\infty;\mathbb{C})$, $\mathrm{SL}(\infty;\mathbb{R})$, $\mathrm{SL}(\infty;\mathbb{H})$, $\mathrm{SU}(\infty,q)$ with $q \leqq \infty$, $\mathrm{SO}(\infty,q)$ with $q \leqq \infty$, $\mathrm{Sp}(\infty,q)$ with $q \leqq \infty$, $\mathrm{Sp}(\infty;\mathbb{R})$ and $\mathrm{SO}^*(2\infty)$. Further, the normalizer $P = MAN$ of N in G is a classical minimal parabolic subgroup $\varinjlim (P_n = M_n A_n N_n)$ where P_n is the minimal parabolic in G_n that is the normalizer of N_n.

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