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The Plancherel Formula for Minimal Parabolic Subgroups

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Abstract. In a recent paper we found conditions for a nilpotent Lie group to be foliated into subgroups that have square integrable unitary representations that fit together to form a filtration by normal subgroups. That resulted in explicit character formulae, Plancherel Formulae and multiplicity formulae. We also showed that nilradicals N of minimal parabolic subgroups $P = MAN$ enjoy that "stepwise square integrable" property. Here we extend those results from N to P. The Pfaffian polynomials, which give orthogonality relations and Plancherel density for N, also give a semi-invariant differential operator that compensates lack of unimodularity for P. The result is a completely explicit Plancherel Formula for P.

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1. Introduction

A connected simply connected Lie group N with center Z is called *square integrable* if it has unitary representations π whose coefficients $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ satisfy $|f_{u,v}| \in \mathcal{L}^2(N/Z)$. C.C. Moore and the author worked out the structure and representation theory of these groups [11]. If N has one such square integrable representation then there is a certain polynomial function $\mathrm{Pf}\,(\lambda)$ on the linear dual space $\mathfrak{z}^*$ of the Lie algebra of Z that is key to harmonic analysis on N. Here $\mathrm{Pf}\,(\lambda)$ is the Pfaffian of the antisymmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$ given by $b_\lambda(x, y) = \lambda([x, y])$. The square integrable representations of N are the π_λ where $\lambda \in \mathfrak{z}^*$ with $\mathrm{Pf}\,(\lambda) \neq 0$, Plancherel almost all irreducible unitary representations of N are square integrable, and up to an explicit constant $|\mathrm{Pf}\,(\lambda)|$ is the Plancherel density of the unitary dual $\widehat{N}$ at π_λ. This theory has proved to have serious analytic consequences. For example, for most commutative nilmanifolds G/K, i.e. Gelfand pairs (G, K) where a nilpotent subgroup N of G acts transitively on G/K, the group N has square integrable representations [15]. And it is known just which maximal parabolic subgroups of semisimple Lie groups have square integrable nilradical [14].

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In [17] and [18] the theory of square integrable nilpotent groups was extended to "stepwise square integrable" nilpotent groups. By definition they are the connected simply connected nilpotent Lie groups that satisfy (1.1) just below. We use L and $\mathfrak{l}$ to avoid conflict of notation with the M and $\mathfrak{m}$ of minimal parabolic subgroups. Z_r denotes the center of L_r and $\mathfrak{v}_r$ is a vector space complement to $\mathfrak{z}_r$ in $\mathfrak{l}_r$.

$$
\begin{aligned}
&N = L_1 L_2 \dots L_{m-1} L_m \text{ where}\\
&\quad \text{(a) each } L_r \text{ has unitary representations with coefficients in } \mathcal{L}^2(L_r/Z_r),\\
&\quad \text{(b) each } N_r := L_1 L_2 \dots L_r \text{ is normal in } N \text{ with } N_r = N_{r-1} \rtimes L_r \text{ semidirect},\\
&\quad \text{(c) decompose } \mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r \text{ and } \mathfrak{n} = \mathfrak{s} + \mathfrak{v} \text{ as vector direct sums where}\\
&\qquad\qquad \mathfrak{s} = \oplus\, \mathfrak{z}_r \text{ and } \mathfrak{v} = \oplus\, \mathfrak{v}_r; \text{ then } [\mathfrak{l}_r, \mathfrak{z}_s] = 0 \text{ and } [\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v} \text{ for } r > s\,.
\end{aligned}
\tag{1.1}
$$

The choice of the $\mathfrak{v}_r$ is not important in (1.1), as long as $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v}$ for $r > s$, because integration and Lie brackets in $\mathfrak{l}_r$ are really over $\mathfrak{l}_r/\mathfrak{z}_r$ rather than $\mathfrak{v}_r$. Denote

$$
\begin{aligned}
&\text{(a) } d_r = \tfrac{1}{2}\dim(\mathfrak{l}_r/\mathfrak{z}_r) \text{ so } \tfrac{1}{2}\dim(\mathfrak{n}/\mathfrak{s}) = d_1 + \dots + d_m\,,\\
&\qquad\qquad\qquad\qquad \text{and } c = 2^{d_1+\dots+d_m} d_1! d_2! \dots d_m!\\
&\text{(b) } b_{\lambda_r} : (x,y) \mapsto \lambda([x,y]) \text{ viewed as a bilinear form on } \mathfrak{l}_r/\mathfrak{z}_r\\
&\text{(c) } S = Z_1 Z_2 \dots Z_m = Z_1 \times \dots \times Z_m \text{ where } Z_r \text{ is the center of } L_r\\
&\text{(d) Pf : polynomial } \mathrm{Pf}\,(\lambda) = \mathrm{Pf}_{\,\mathfrak{l}_1}(b_{\lambda_1}) \mathrm{Pf}_{\,\mathfrak{l}_2}(b_{\lambda_2}) \dots \mathrm{Pf}_{\,\mathfrak{l}_m}(b_{\lambda_m}) \text{ on } \mathfrak{s}^*\\
&\text{(e) } \mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid \mathrm{Pf}\,(\lambda) \neq 0\}\\
&\text{(f) } \pi_\lambda \in \widehat{N} \text{ where } \lambda \in \mathfrak{t}^* : \text{ irreducible unitary rep. of } N = L_1 L_2 \dots L_m
\end{aligned}
\tag{1.2}
$$

We recall the Schwartz space $\mathcal{C}(N)$, following the lines of the exposition in [2, Section 1]. Start with a norm on N. For example the operator norm $||\alpha(x)||$, where α is a faithful finite dimensional representation of N by unipotent linear transformations of a Hilbert space, defines a norm $|x| = \sup(||\alpha(x)||, ||\alpha(x^{-1})||)$. Or one can use $|x| = (1 + \mathit{distance}(x,1)^2)$ with a left invariant Riemannian metric on N. The properties we need are that the norm be continuous and satisfy (i) $|1| = 1$, (ii) $|x| \geqq 1$, (iii) $|x^{-1}| = |x|$ and (iv) $|x| \cdot |y|^{-1} \leqq |xy| \leqq |x| \cdot |y|$. Write ℓ for the left action of the universal enveloping algebra $\mathcal{U}(\mathfrak{n})$ on $C^\infty(N)$ and r for the right action. The *Schwartz space* $\mathcal{C}(N)$, also called the space of rapidly decreasing smooth functions on N, consists of all $f \in C^\infty(N)$ such that

$$\nu_{a,k}(f) := \sup_{x \in N} |x|^k |\ell(a)(f)(x)| < \infty \text{ for every } a \in \mathcal{U}(\mathfrak{n}) \text{ and every integer } k \geqq 0.$$

The seminorms $\nu_{a,k}$ define a nuclear Fréchet space topology on $\mathcal{C}(N)$ and we have continuous inclusions $C^\infty(N) \hookrightarrow \mathcal{C}(N) \hookrightarrow \mathcal{L}^2(N)$ with dense images. Two continuous norms that satisfy our conditions (i) through (iv) are equivalent (each bounded by a multiple of the other), so they give the same Schwartz space. If $f \in \mathcal{C}(N)$ then $\ell(a)\, r(b)\, (f) \in \mathcal{C}(N) \subset \mathcal{L}^2(N)$ for all $a, b \in \mathcal{U}(\mathfrak{n})$. Since N is connected, simply connected and nilpotent, the exponential map $\exp : \mathfrak{n} \to N$ is polynomial, and $f \in \mathcal{C}(N)$ if and only if its lift $f_1(\xi) = f(\exp(\xi))$ belongs to the classical Schwartz space of the real vector space $\mathfrak{n}$.

If $\pi \in \widehat{N}$ and $f \in \mathcal{C}(N)$ then $\pi(f) := \int_N f(x)\pi(x)dx$ is trace class and $\Theta_\pi : f \mapsto \operatorname{trace} \pi(f)$ is a tempered distribution (distribution that extends by continuity from C_c^∞ to $\mathcal{C}$) on N called the *distribution character* of π. The point, now, is that Plancherel measure on $\widehat{N}$ is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$, and

Theorem 1.3. *Let N be a connected simply connected nilpotent Lie group that satisfies* (1.1). *Then Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ, then the coefficient $f_{u,v}(x) = \langle u, \pi_\nu(x)v\rangle$ satisfies*

$$||f_{u,v}||^2_{\mathcal{L}^2(N/S)} = \frac{||u||^2||v||^2}{|\mathrm{Pf}\,(\lambda)|}. \tag{1.4}$$

Recall $c = 2^{d_1+\cdots+d_m} d_1!d_2!\dots d_m!$ from $(1.2(a))$. *Then the distribution character Θ_{π_λ} of π_λ satisfies*

$$\Theta_{\pi_\lambda}(f) = c^{-1}|\mathrm{Pf}\,(\lambda)|^{-1}\int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi)d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N) \tag{1.5}$$

where $\mathcal{C}(N)$ is the Schwartz space, f_1 is the lift $f_1(\xi) = f(\exp(\xi))$, $\widehat{f_1}$ is its classical Fourier transform, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$, *and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to* $\mathrm{Ad}^*(N)\lambda$. *The Plancherel Formula on N is*

$$f(x) = c\int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|\mathrm{Pf}\,(\lambda)|d\lambda \text{ for } f \in \mathcal{C}(N). \tag{1.6}$$

Definition 1.7. The representations π_λ of (1.2(f)) are the *stepwise square integrable* representations of N relative to the decomposition (1.1). $\diamondsuit$

One of the main results of [17] and [18] is that nilradicals of minimal parabolic subgroups are stepwise square integrable. Even the simplest case, the case of a minimal parabolic in $SL(n;\mathbb{R})$, was a big improvement over earlier results on the group of strictly upper triangular real matrices. Here we extend the results of [17] and [18] to obtain explicit Plancherel Formulae for the minimal parabolic P itself. This is done by construction of a Dixmier–Pukánszky operator on $\mathcal{L}^2(P)$, i.e. a pseudo–differential operator that compensates lack of unimodularity on P. The Dixmier–Pukánszky operator is explicit; it is constructed from the Pfaffian polynomials of (1.2(d)). The construction gives a beautiful relation between the Dixmier–Pukánszky operator of P and the Plancherel density of its nilradical.

In Section 2 we review the restricted root structure, stepwise square integrable representations, character formulae and the Plancherel (or Fourier Inversion) Formula for nilradicals of minimal parabolic subgroups. Some of the restricted root results are discussed further in Section 7, a sort of appendix, where we placed them because they add to, but are not needed for, the main results.

Is Section 3 we discuss the structure and action of the group M in a minimal parabolic $P = MAN$. The notion of principal orbit gives a uniform description of the stabilizers of stepwise square integrable representations of N. We also show triviality of a certain Mackey obstruction, leading to an explicit Plancherel Formula for MN.

In Section 4 we work out the Dixmier–Pukánszky operator of P in terms of the Pfaffian (which gives Plancherel density on N) and a certain explicit "quasi–central determinant" polynomial.

In Section 5 we apply the Mackey machine to give an explicit description of subsets of $\widehat{P}$ and $\widehat{AN}$ that carry Plancherel measure. The point here is that the description is explicit.

Finally in Section 6 we give explicit Plancherel Formulae for the minimal parabolic subgroups $P = MAN$ and their exponential solvable subgroups AN.

2. Minimal Parabolics: Structure of the Nilradical

Let G be a real reductive Lie group. We recall some structural results on its minimal parabolic subgroups, some standard and some from [18].

Fix an Iwasawa decomposition $G = KAN$. As usual, write $\mathfrak{k}$ for the Lie algebra of K, $\mathfrak{a}$ for the Lie algebra of A, and $\mathfrak{n}$ for the Lie algebra of N. Complete $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Now we have root systems

- $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$: roots of $\mathfrak{g}_\mathbb{C}$ relative to $\mathfrak{h}_\mathbb{C}$ (ordinary roots), and
- $\Delta(\mathfrak{g}, \mathfrak{a})$: roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ (restricted roots).
- $\Delta_0(\mathfrak{g}, \mathfrak{a}) = \{\gamma \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid 2\gamma \notin \Delta(\mathfrak{g}, \mathfrak{a})\}$ (nonmultipliable restricted roots).

Sometimes we will identify a restricted root $\gamma = \alpha|_\mathfrak{a}$, $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and $\alpha|_\mathfrak{a} \neq 0$, with the set

$$[\gamma] := \{\alpha' \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \mid \alpha'|_\mathfrak{a} = \alpha|_\mathfrak{a}\} \tag{2.1}$$

of all roots that restrict to it. Further, $\Delta(\mathfrak{g}, \mathfrak{a})$ and $\Delta_0(\mathfrak{g}, \mathfrak{a})$ are root systems in the usual sense. Any positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \subset \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ defines positive systems

- $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha|_\mathfrak{a} \mid \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \text{ and } \alpha|_\mathfrak{a} \neq 0\}$ and
- $\Delta_0^+(\mathfrak{g}, \mathfrak{a}) = \Delta_0(\mathfrak{g}, \mathfrak{a}) \cap \Delta^+(\mathfrak{g}, \mathfrak{a})$.

We can (and do) choose $\Delta^+(\mathfrak{g}, \mathfrak{h})$ so that

- $\mathfrak{n}$ is the sum of the positive restricted root spaces and
- if $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and $\alpha|_\mathfrak{a} \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ then $\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.

Two roots are called *strongly orthogonal* if their sum and their difference are not roots. Then they are orthogonal. We define

$$\begin{aligned} &\beta_1 \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximal positive restricted root and} \\ &\beta_{r+1} \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximum} \\ &\qquad \text{among the roots of } \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ orthogonal to all } \beta_i \text{ with } i \leqq r \end{aligned} \tag{2.2}$$

Then the β_r are mutually strongly orthogonal. This is Kostant's cascade construction. Note that each $\beta_r \in \Delta_0^+(\mathfrak{g},\mathfrak{a})$. Also note that β_1 is unique if and only if $\Delta(\mathfrak{g},\mathfrak{a})$ is irreducible. For $1 \leqq r \leqq m$ define

$$\begin{aligned}&\Delta_1^+ = \{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a}) \mid \beta_1 - \alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})\} \text{ and}\\ &\Delta_{r+1}^+ = \{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a}) \setminus (\Delta_1^+ \cup \cdots \cup \Delta_r^+) \mid \beta_{r+1} - \alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})\}.\end{aligned} \tag{2.3}$$

Lemma 2.4. [18, Lemma 6.3] *If $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})$ then either $\alpha \in \{\beta_1,\dots,\beta_m\}$ or α belongs to exactly one of the sets Δ_r^+. In particular the Lie algebra $\mathfrak{n}$ of N is the vector space direct sum of its subspaces*

$$\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum\nolimits_{\Delta_r^+} \mathfrak{g}_\alpha \text{ for } 1 \leqq r \leqq m \tag{2.5}$$

Lemma 2.6. [18, Lemma 6.4] *The set $\Delta_r^+ \cup \{\beta_r\} = \{\alpha \in \Delta^+ \mid \alpha \perp \beta_i \text{ for } i < r \text{ and } \langle \alpha, \beta_r \rangle > 0\}$. In particular, $[\mathfrak{l}_r,\mathfrak{l}_s] \subset \mathfrak{l}_t$ where $t = \min\{r,s\}$. Thus $\mathfrak{n}$ has an increasing filtration by ideals*

$$\mathfrak{n}_r = \mathfrak{l}_1 + \mathfrak{l}_2 + \cdots + \mathfrak{l}_r \text{ for } 1 \leqq r \leqq m \tag{2.7}$$

with a corresponding group level decomposition by normal subgroups N_r where

$$N = L_1 L_2 \dots L_m \text{ with } N_r = N_{r-1} \rtimes L_r \text{ for } 1 \leqq r \leqq m. \tag{2.8}$$

The structure of Δ_r^+, and later of $\mathfrak{l}_r$, is exhibited by a particular Weyl group element $s_{\beta_r} \in W(\mathfrak{g},\mathfrak{a})$ and its negative. Specifically,

$$\begin{aligned}&s_{\beta_r} \text{ is the Weyl group reflection in } \beta_r \text{ and}\\ &\sigma_r : \Delta(\mathfrak{g},\mathfrak{a}) \to \Delta(\mathfrak{g},\mathfrak{a}) \text{ by } \sigma_r(\alpha) = -s_{\beta_r}(\alpha).\end{aligned} \tag{2.9}$$

Here $\sigma_r(\beta_s) = -\beta_s$ for $s \neq r$, $+\beta_s$ if $s = r$. If $\alpha \in \Delta_r^+$ we still have $\sigma_r(\alpha) \perp \beta_i$ for $i < r$ and $\langle \sigma_r(\alpha), \beta_r \rangle > 0$. If $\sigma_r(\alpha)$ is negative then $\beta_r - \sigma_r(\alpha) > \beta_r$ contradicting the maximality property of β_r. Thus, using Lemma 2.6, $\sigma_r(\Delta_r^+) = \Delta_r^+$. This divides each Δ_r^+ into pairs:

Lemma 2.10. [18, Lemma 6.8] *If $\alpha \in \Delta_r^+$ then $\alpha + \sigma_r(\alpha) = \beta_r$. (Of course it is possible that $\alpha = \sigma_r(\alpha) = \frac{1}{2}\beta_r$ when $\frac{1}{2}\beta_r$ is a root.). If $\alpha, \alpha' \in \Delta_r^+$ and $\alpha + \alpha' \in \Delta(\mathfrak{g},\mathfrak{a})$ then $\alpha + \alpha' = \beta_r$.*

It comes out of Lemmas 2.4 and 2.6 that the decompositions of (2.3), (2.5) and (2.7) satisfy (1.1), so Theorem 1.3 applies to nilradicals of minimal parabolic subgroups. In other words,

Theorem 2.11. [18, Theorem 6.16] *Let G be a real reductive Lie group, $G = KAN$ an Iwasawa decomposition, $\mathfrak{l}_r$ and $\mathfrak{n}_r$ the subalgebras of $\mathfrak{n}$ defined in (2.5) and (2.7), and L_r and N_r the corresponding analytic subgroups of N. Then the L_r and N_r satisfy (1.1). In particular, Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the* *representation space* $\mathcal{H}_{\pi_\lambda}$ *of* π_λ*, then the coefficient* $f_{u,v}(x) = \langle u, \pi_\lambda(x)v\rangle$ *satisfies*

$$||f_{u,v}||^2_{\mathcal{L}^2(N/S)} = \frac{||u||^2||v||^2}{|\mathrm{Pf}\,(\lambda)|}\,. \tag{2.12}$$

The distribution character Θ_{π_λ} *of* π_λ *satisfies*

$$\Theta_{\pi_\lambda}(f) = c^{-1}|\mathrm{Pf}\,(\lambda)|^{-1}\int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi)d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N) \tag{2.13}$$

where $\mathcal{C}(N)$ *is the Schwartz space,* f_1 *is the lift* $f_1(\xi) = f(\exp(\xi))$*,* $\widehat{f_1}$ *is its classical Fourier transform,* $\mathcal{O}(\lambda)$ *is the coadjoint orbit* $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$*, and* $d\nu_\lambda$ *is the translate of normalized Lebesgue measure from* $\mathfrak{v}^*$ *to* $\mathrm{Ad}^*(N)\lambda$*. The Plancherel Formula on* N *is*

$$f(x) = c\int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|\mathrm{Pf}\,(\lambda)|d\lambda \text{ for } f \in \mathcal{C}(N). \tag{2.14}$$

3. Minimal Parabolics: M-Orbit Structure

Recall the Iwasawa decomposition $G = KAN$ and the corresponding minimal parabolic subgroup $P = MAN$ where M is the centralizer of A in K. We write ^{0} for identity component, so $P^0 = M^0AN$.

Lemma 3.1. *Recall the* Pf *–nonsingular set* $\mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid \mathrm{Pf}\,(\lambda) \neq 0\}$ *of* (1.2e)*. Then* $\mathrm{Ad}^*(M)\mathfrak{t}^* = \mathfrak{t}^*$*. Further, if* $\lambda \in \mathfrak{t}^*$ *and* $c \neq 0$ *then* $c\lambda \in \mathfrak{t}^*$*, in fact* $\mathrm{Pf}\,(c\lambda) = c^{\dim(\mathfrak{n}/\mathfrak{s})/2}\mathrm{Pf}\,(\lambda)$*.*

Proof. All the ingredients in the formula for $\lambda \mapsto \mathrm{Pf}\,(\lambda)$ are $\mathrm{Ad}^*(M)$–equivariant, so $\mathrm{Ad}^*(M)\mathfrak{t}^* = \mathfrak{t}^*$. By definition the bilinear form b_λ on $\mathfrak{n}/\mathfrak{s}$ satisfies $b_{c\lambda} = cb_\lambda$, so $\mathrm{Pf}\,(c\lambda) = c^{\dim(\mathfrak{n}/\mathfrak{s})/2}\mathrm{Pf}\,(\lambda)$. ■

Choose an M–invariant inner product (μ,ν) on $\mathfrak{s}^*$. Denote $\mathfrak{s}^*_t = \{\lambda \in \mathfrak{s}^* \mid (\lambda,\lambda) = t^2\}$, the sphere of radius t. Consider the action of M on $\mathfrak{s}^*_t$. Recall that two orbits $\mathrm{Ad}^*(M)\mu$ and $\mathrm{Ad}^*(M)\nu$ are of the *same orbit type* if the isotropy subgroups M_μ and M_ν are conjugate, and an orbit is *principal* if all nearby orbits are of the same type. Since M and $\mathfrak{s}^*_t$ are compact, there are only finitely many orbit types of M on $\mathfrak{s}^*_t$, there is only one principal orbit type, and the union of the principal orbits forms a dense open subset of $\mathfrak{s}^*_t$ whose complement has codimension $\geqq 2$. There are many good expositions of this material, for example [1, Chapter 4, Section 3] for a complete treatment, [4, Part II, Chapter 3, Section 1] modulo references to [1], and [12, Cap. 5] for a more basic treatment but still with some references to [1].

Since the action of M on $\mathfrak{s}^*$ commutes with dilation, the above mentioned structural results on the $\mathfrak{s}_t$ also hold on $\mathfrak{s}^* = \bigcup_{t\geq 0}\mathfrak{s}^*_t$. Define the Pf -nonsingular principal orbit set as follows:

$$\mathfrak{u}^* = \{\lambda \in \mathfrak{t}^* \mid \mathrm{Ad}^*(M)\lambda \text{ is a principal } M\text{-orbit on } \mathfrak{s}^*\}. \tag{3.2}$$

Summarizing the short discussion,

Lemma 3.3. *The principal orbit set* $\mathfrak{u}^*$ *is a dense open set with complement of codimension* $\geqq 2$ *in* $\mathfrak{s}^*$. *If* $\lambda \in \mathfrak{u}^*$ *and* $c \neq 0$ *then* $c\lambda \in \mathfrak{u}^*$ *with isotropy* $M_{c\lambda} = M_\lambda$.

Fix $\lambda \in \mathfrak{u}_t^* := \mathfrak{u}^* \cap \mathfrak{s}_t^*$, so $\mathrm{Ad}^*(M)\lambda$ is a Pf -nonsingular principal orbit of M on the sphere $\mathfrak{s}_t^*$. Then $\mathrm{Ad}^*(M^0)\lambda$ is a principal orbit of M^0 on $\mathfrak{s}_t^*$. Principal orbit isotropy subgroups of compact connected linear groups are studied in detail in [5] so the possibilities for $(M^0)_\lambda$ are essentially known.

Lemma 3.4. *Let* G *be connected and linear. Then* $M = (\exp(i\mathfrak{a}) \cap K)Z_G M^0$ *where* Z_G *is the center of* G, *and its action on a restricted root space* $\mathfrak{g}_\alpha$ *has form* $\exp(i\alpha(\xi))|_{\mathfrak{g}_\alpha} = \pm 1$. *In particular* $(\exp(i\mathfrak{a}) \cap K)$ *is an elementary abelian* 2*-subgroup of* M *that meets each of its topological components.*

Proof. A Cartan subgroup $B \subset M$ meets every component of M. The complex Cartan $(BA)_\mathbb{C} = \exp(\mathfrak{b}_\mathbb{C})\exp(\mathfrak{a}_\mathbb{C}) \subset G_\mathbb{C}$ is connected, and $\exp(\mathfrak{b})$ and $\exp(\mathfrak{a})$ are connected as well, so the components of $(BA) \cap G$ are given by $\exp(i\mathfrak{b})\exp(i\mathfrak{a}) \cap G$. As $\exp(i\mathfrak{b})$ is split over $\mathbb{R}$ the components of $(BA) \cap G$ are given by $\exp(i\mathfrak{a}) \cap G = \exp(i\mathfrak{a}) \cap K$. The Cartan involution θ of G with fixed point set K fixes every element of K and sends every element of $\exp(i\mathfrak{a})$ to its inverse, so $\exp(i\mathfrak{a}) \cap K$ is an elementary abelian 2-group that meets every component of M. The restricted root spaces $\mathfrak{g}_\alpha$ are joint eigenspaces of $\mathfrak{a}$, so every element of $\exp(i\mathfrak{a}) \cap K$ acts on each $\mathfrak{g}_\alpha$ by a scalar multiplication ± 1. ■

Define F to be the elementary abelian 2-subgroup $\exp(i\mathfrak{a}) \cap K$ of M considered in Lemma 3.4. In order to see exactly how F acts on $\mathfrak{s}^*$ we use a result of Kostant applied to the centralizer of $Z_M(M^0)A$:

Lemma 3.5. [16, Theorem 8.13.3] *Suppose that* G *is connected. Then the adjoint representation of* M *on* $\mathfrak{g}$ *preserves each restricted root space, say acting by* η_α *on* $\mathfrak{g}_\alpha$, *and each* $\eta_\alpha|_{M^0}$ *is irreducible.*

Now we have the action of F on $\mathfrak{s}^*$, as follows.

Proposition 3.6. *The group* $\mathrm{Ad}^*(F)$ *acts trivially on* $\mathfrak{s}^*$.

Proof. Each of the strongly orthogonal roots gives us a θ-stable subalgebra $\mathfrak{g}[\beta_r] \cong \mathfrak{sl}(2;\mathbb{R})$ of $\mathfrak{g}$. It has standard basis $\{x_r, y_r, h_r\}$ where $h_r \in \mathfrak{a}$ and each $x_r \in \mathfrak{z}_r \subset \mathfrak{s}$. Now $\mathfrak{a} = \mathfrak{a}_\diamondsuit \oplus \bigoplus \sum \mathbb{R}x_r$ where $\mathfrak{a}_\diamondsuit$ (notation to be justified by (5.1)) is the intersection of the kernels of the β_r. As defined, $\mathrm{ad}^*(\mathfrak{a}_\diamondsuit)$ vanishes on $\sum \mathbb{R}x_r$. By strong orthogonality of $\{\beta_r\}$, each $\mathrm{ad}^*(h_s\mathbb{C})$ is trivial on $\mathbb{R}x_r$ for $s \neq r$. Further $\mathrm{ad}(\exp(\mathbb{C}h_r) \cap K)$ is trivial on $\mathbb{R}x_r$ by a glance at $\mathfrak{sl}(2;\mathbb{R})$. We have shown that $\mathrm{Ad}(F)x_r = x_r$ for each r. Since M^0 is irreducible on each $\mathfrak{z}_r = \mathfrak{g}_{\beta_r}$ by Lemma 3.5, and M centralizes A, now $\mathrm{Ad}(F)x = x$ for all $x \in \mathfrak{z}_r$ and all r. ■

Combining Lemma 3.4 and Proposition 3.6, the action of M_λ is given by the action of the identity component of M:

Lemma 3.7. *If $\lambda \in \mathfrak{t}^*$ then its M-stabilizer M_λ is given by $M_\lambda = F \cdot (M^0)_\lambda$.*

In view of Lemma 3.7, the group M_λ is specified by the work of W.–C. and W.–Y. Hsiang [5] on the structure and classification of principal orbits of compact connected linear groups.

Fix $\lambda \in \mathfrak{t}^*$, so $\pi_\lambda \in \widehat{N}$ is stepwise square integrable (Definition 1.7). Consider the semidirect product group $N \rtimes M_\lambda$. We write $\mathcal{H}_\lambda$ for the representation space of π_λ . The next step is to extend the representation π_λ to a unitary representation $\pi_\lambda^\dagger$ of $N \rtimes M_\lambda$ on the same representation space $\mathcal{H}_\lambda$. By [3, Théorème 6.1] the Mackey obstruction $\varepsilon \in H^2(M_\lambda; U(1))$ to this extension, where $U(1) = \{|z| = 1\}$, has order 1 or 2. But here the Mackey obstruction is trivial so we can be more precise:

Lemma 3.8. *The stepwise square integrable π_λ extends to a representation $\pi_\lambda^\dagger$ of $N \rtimes M_\lambda$ on the representation space of π_λ .*

Proof. The group M preserves each $\mathfrak{z}_r^*$, so $M_\lambda = \bigcap_{\lambda_r} M_{\lambda_r}$ where $\lambda = \sum \lambda_r$ with $\lambda_r \in \mathfrak{z}_r^*$. Recall the construction of π_λ from the decomposition $N = L_1 \dots L_m$ of (1.1) and the square integrable representations π_{λ_r} of the Heisenberg (or abelian) groups L_r from [18] . The point is that π_{λ_1} extends to $\widetilde{\pi_{\lambda_1}} \in \widehat{L_1 L_2}$ and then we have $\pi_{\lambda_1+\lambda_2} := \widetilde{\pi_{\lambda_1}} \hat{\otimes} \pi_{\lambda_2}$, $\pi_{\lambda_1+\lambda_2}$ extends to $\widetilde{\pi_{\lambda_1+\lambda_2}} \in \widehat{L_1 L_2 L_3}$ giving $\pi_{\lambda_1+\lambda_2+\lambda_3} := \widetilde{\pi_{\lambda_1+\lambda_2}} \hat{\otimes} \pi_{\lambda_3}$, etc. Note that we use tilde to denote extension to the next step in the decomposition (1.1) of N.

The Fock representation of the $2n+1$ dimensional Heisenberg group H extends to the semidirect product $H \rtimes U(n)$ [13]; so each π_{λ_r} extends to $L_r \rtimes M_{\lambda_r}$. We use this to modify the construction of π_λ just described. We will use dagger to denote extension from N_* to $N_* \rtimes M_*$, prime to denote dagger together with tilde, and double prime to denote an appropriate restriction of dagger or prime.

Let $\pi_{\lambda_1}^\dagger$ denote the extension of π_{λ_1} from L_1 to $L_1 \rtimes M_{\lambda_1}$. Now extend $\pi_{\lambda_1}^\dagger$ (instead of π_{λ_1}), obtaining an extension π'_{λ_1} of π_{λ_1} from $L_1 \rtimes M_{\lambda_1}$ to $(L_1 L_2) \rtimes M_{\lambda_1}$. It restricts to a representation π''_{λ_1} of $(L_1 L_2) \rtimes (M_{\lambda_1} \cap M_{\lambda_2})$. We have the extension $\pi_{\lambda_2}^\dagger$ of π_{λ_2} from L_2 to $L_2 \rtimes M_{\lambda_2}$; let π''_{λ_2} denote its restriction to $L_2 \rtimes (M_{\lambda_1} \cap M_{\lambda_2})$. That gives us an extension $\pi_{\lambda_1+\lambda_2}^\dagger := \pi''_{\lambda_1} \hat{\otimes} \pi''_{\lambda_2}$ of $\pi_{\lambda_1+\lambda_2}$ from $L_1 L_2$ to $(L_1 L_2) \rtimes (M_{\lambda_1} \cap M_{\lambda_2})$. Continuing this way, we construct the extension of π_λ from N to $N \rtimes M_\lambda$. ■

Remark 3.9. One can also prove Lemma 3.8 by combining the Mackey obstructions $[\gamma_r] \in H^2(M_{\lambda_r}; U(1))$ to extension of π_{λ_r} from N_r to $N_r \rtimes M_{\lambda_r}$. In effect the cocycle $\gamma \in Z^2(M_\lambda; U(1))$ whose cohomology class is the Mackey obstruction to extension of π_λ from N to $N \rtimes M_\lambda$, is cohomologous to the pointwise product of the $(\gamma_r)|_{M_\lambda \times M_\lambda}$, and each $[(\gamma_r)|_{M_\lambda \times M_\lambda}] \in H^2(M_\lambda; U(1))$ is trivial because each $[\gamma_r] \in H^2(M_{\lambda_r}; U(1))$ is trivial.

Each $\lambda \in \mathfrak{t}^*$ now defines classes

$$\begin{aligned}\mathcal{E}(\lambda) &:= \left\{\pi_\lambda^\dagger \otimes \gamma \mid \gamma \in \widehat{M_\lambda}\right\} \text{ and} \\ \mathcal{F}(\lambda) &:= \left\{\operatorname{Ind}_{NM_\lambda}^{NM}(\pi_\lambda^\dagger \otimes \gamma) \mid \pi_\lambda^\dagger \otimes \gamma \in \mathcal{E}(\lambda)\right\}\end{aligned} \tag{3.10}$$

of irreducible unitary representations of $N \rtimes M_\lambda$ and NM. The Mackey little group method, plus the fact that the Plancherel density on $\widehat{N}$ is polynomial on $\mathfrak{s}^*$, and $\mathfrak{s}^* \setminus \mathfrak{u}^*$ has measure 0 in $\mathfrak{t}^*$, gives us

Proposition 3.11. *Plancherel measure for NM is concentrated on the set $\bigcup_{\lambda\in\mathfrak{u}^*} \mathcal{F}(\lambda)$ of (equivalence classes of) irreducible representations given by $\eta_{\lambda,\gamma} := \operatorname{Ind}_{NM_\lambda}^{NM}(\pi_\lambda^\dagger \otimes \gamma)$ with $\pi_\lambda^\dagger \otimes \gamma \in \mathcal{E}(\lambda)$ and $\lambda \in \mathfrak{u}^*$. Further*

$$\eta_{\lambda,\gamma}|_N = \left(\operatorname{Ind}_{NM_\lambda}^{NM}(\pi_\lambda^\dagger \otimes \gamma)\right)\Big|_N = \int_{M/M_\lambda} (\dim \gamma)\, \pi_{\operatorname{Ad}^*(m)\lambda}\, d(mM_\lambda).$$

In view of Lemma 3.3 there is a Borel section σ to $\mathfrak{u}^* \to \mathfrak{u}^*/\operatorname{Ad}^*(M)$ which picks out an element in each M-orbit so that M has the same isotropy subgroup at each of those elements. In other words in each M-orbit on $\mathfrak{u}^*$ we measurably choose an element $\lambda = \sigma(\operatorname{Ad}^*(M)\lambda)$ such that those isotropy subgroups M_λ are all the same. Let us denote

$$M_\diamondsuit\text{: isotropy subgroup of } M \text{ at } \sigma(\operatorname{Ad}^*(M)\lambda) \text{ for every } \lambda \in \mathfrak{u}^* \tag{3.12}$$

Then we can replace M_λ by $M_\diamondsuit$, independent of $\lambda \in \mathfrak{u}^*$, in Proposition 3.11. That lets us assemble to representations of Proposition 3.11 for a Plancherel Formula, as follows. Since M is compact, we have the Schwartz space $\mathcal{C}(NM)$ just as in the discussion of $\mathcal{C}(N)$ between (1.2) and Theorem 1.3, except that the pullback $\exp^* \mathcal{C}(NM) \neq \mathcal{C}(\mathfrak{n} + \mathfrak{m})$. The same applies to $\mathcal{C}(NA)$ and $\mathcal{C}(NAM)$

Proposition 3.13. *Let $f \in \mathcal{C}(NM)$ and write $(f_m)(n) = f(nm) = ({}_nf)(m)$ for $n \in N$ and $m \in M$. The Plancherel density at $\operatorname{Ind}_{NM_\diamondsuit}^{NM}(\pi_\lambda^\dagger \otimes \gamma)$ is $(\dim \gamma)|\operatorname{Pf}(\lambda)|$ and the Plancherel Formula for NM is*

$$f(nm) = c \int_{\mathfrak{u}^*/\operatorname{Ad}^*(M)} \sum_{\mathcal{F}(\lambda)} \operatorname{trace} \eta_{\lambda,\gamma}({}_nf_m) \cdot \dim(\gamma) \cdot |\operatorname{Pf}(\lambda)| d\lambda$$

where $c = 2^{d_1+\cdots+d_m} d_1! d_2! \dots d_m!$, from (1.2), *as in* Theorem 1.3.

4. The Pfaffian and the Dixmier–Pukánszky Operator

Let Q be a separable locally compact group of type I. Then [9, §1] the Plancherel Formula for Q has form

$$f(x) = \int_{\widehat{Q}} \operatorname{trace} \pi(D(r(x)f)) d\mu_Q(\pi) \tag{4.1}$$

where D is an invertible positive self adjoint operator on $\mathcal{L}^2(Q)$, conjugation–semi-invariant of weight equal to the modular function δ_Q, and μ is a positive

Borel measure on the unitary dual $\widehat{Q}$. The operator D is needed for the following reason. If Q were unimodular its Plancherel Formula would be of the form $f(1) = \int_{\widehat{Q}} \mathrm{trace}\, \pi(f))d\mu_Q(\pi)$ with both sides invariant under conjugation by elements of Q. In general, however, the left hand side $f(1)$ is conjugation–invariant while conjugation transforms $\pi(f) = \int_Q f(x)\pi(x)dx$, and thus the the right hand side $\int_{\widehat{Q}} \mathrm{trace}\, \pi(f))d\mu_Q(\pi)$, by the modular function. Thus the modular function has to be somehow compensated, and that is the role of D. If Q is unimodular then D is the identity and (4.1) reduces to the usual Plancherel Formula. The point is that semi-invariance of D compensates any lack of unimodularity. See [9, §1] for a detailed discussion, including a discussion of the domain of D and $D^{1/2}$.

Uniqueness of the pair (D, μ) remains unsettled, though of course $D \otimes \mu$ is unique (up to normalization of Haar measures), so one tries to find a "best" choice of D. Given any such pair (D, μ) we refer to D as a *Dixmier–Pukánszky Operator* on Q and to μ as the associated *Plancherel measure* on $\widehat{Q}$.

In this section we exhibit an explicit Dixmier–Pukánszky Operator for the minimal parabolic $P = MAN$ and its solvable subgroup AN. Those groups are never unimodular. Our Dixmier–Pukánszky Operator is constructed from the Pfaffian polynomial $\mathrm{Pf}\,(\lambda)$ and a certain "quasi-central determinant" function on $\mathfrak{s}^*$.

Let δ denote the modular function on $P = MAN$. As M is compact and $\mathrm{Ad}_P(N)$ is unipotent on $\mathfrak{p}$, MN is in the kernel of δ. So δ is determined by its values on A, where it is given by $\delta(\exp(\xi)) = \exp(\mathrm{trace}\,(\mathrm{ad}\,(\xi)))$. There $\xi = \log a \in \mathfrak{a}$.

Lemma 4.2. *Let $\xi \in \mathfrak{a}$. Then $\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r) \in \mathbb{Z}$ for $1 \leqq r \leqq m$ and*
(i) *the trace of* $\mathrm{ad}\,(\xi)$ *on* $\mathfrak{l}_r$ *is* $\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)\beta_r(\xi)$,
(ii) *the trace of* $\mathrm{ad}\,(\xi)$ *on* $\mathfrak{n}$ *and on* $\mathfrak{p}$ *is* $\frac{1}{2}\sum_r(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)\beta_r(\xi)$, *and*
(iii) *the determinant of* $\mathrm{Ad}\,(\exp(\xi))$ *on* $\mathfrak{n}$ *and on* $\mathfrak{p}$ *is* $\prod_r \exp(\beta_r(\xi))^{\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)}$.

Proof. Decompose $\mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r$ where $\mathfrak{z}_r = \mathfrak{g}_{\beta_r}$ is its center and $\mathfrak{v}_r = \sum_{\alpha \in \Delta_r^+} \mathfrak{g}_\alpha$. The set Δ_r^+ is the disjoint union of sets $\{\alpha, \beta_r - \alpha\}$ and (if $\frac{1}{2}\beta_r$ is a root) $\{\frac{1}{2}\beta_r\}$. That proves the integrality assertion. From (2.9) and Lemma 2.10 we have $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{\beta_r - \alpha}$. So the trace of $\mathrm{ad}\,(\xi)$ on $\mathfrak{v}_r$ adds up to $\frac{1}{2}(\dim \mathfrak{v}_r)\beta_r(\xi)$. On $\mathfrak{z}_r = \mathfrak{g}_{\beta_r}$ it is of course $(\dim \mathfrak{z}_r)\beta_r(\xi)$. That proves (i). For (ii) we take the sum over $\{\beta_1, \dots, \beta_m\}$ and then for (iii) we exponentiate. ■

Since $\delta = \det \mathrm{Ad}$, Lemma 4.2(iii) can be formulated as

Lemma 4.3. *The modular function $\delta = \delta_P$ of $P = MAN$ is $\delta(man) = \prod_r \exp(\beta_r(\log a))^{\frac{1}{2}(\dim \mathfrak{l}_r + \dim \mathfrak{z}_r)}$. The modular function δ_{AN} of AN is $\delta_P|_{AN}$.*

We consider semi-invariance of the Pfaffian. Let $\xi \in \mathfrak{a}$ and consider a basis $\{x_i\}$ of $\mathfrak{v}_r$, each element in some $\mathfrak{g}_\alpha$ with $\alpha \in \Delta_r^+$, in which b_λ has matrix consisting of 2×2 blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ down the diagonal. But $-\mathrm{ad}^*(\xi)(\lambda)[x_i, x_j] = \lambda(\mathrm{ad}\,(\xi)[x_i, x_j]) = \lambda[\mathrm{ad}\,(\xi)x_i, x_j] + \lambda([x_i, \mathrm{ad}\,(\xi)x_j] = \beta_r(\xi)\lambda([x_i, x_j])$ as in the proof of Lemma 4.2. Now $(\mathrm{ad}\,(\xi)\mathrm{Pf}\,)|_{\mathfrak{v}_r}(\lambda) = \mathrm{Pf}\,|_{\mathfrak{v}_r}(-\mathrm{ad}\,^*(\xi)(\lambda)) = \frac{1}{2}\dim \mathfrak{v}_r\, \beta_r(\xi)\mathrm{Pf}\,|_{\mathfrak{v}_r}$. Sum over r:

Lemma 4.4. *Let $\xi \in \mathfrak{a}$ and $a = \exp(\xi) \in A$. Then*

$$\mathrm{ad}\,(\xi)\mathrm{Pf}\; = \left(\frac{1}{2}\sum_r \dim(\mathfrak{l}_r/\mathfrak{z}_r)\beta_r(\xi)\right)\mathrm{Pf}$$

and

$$\mathrm{Ad}\,(a)\mathrm{Pf}\; = \left(\prod_r \exp(\beta_r(\xi))^{\frac{1}{2}\dim(\mathfrak{l}_r/\dim\mathfrak{z}_r)}\right)\mathrm{Pf}\,.$$

At this point it is convenient to introduce some notation and definitions.

Definition 4.5. The algebra $\mathfrak{s}$ is the *quasi-center* of $\mathfrak{n}$. Then $\mathrm{Det}_{\mathfrak{s}^*}(\lambda) := \prod_{\mathrm{r}}(\beta_{\mathrm{r}}(\lambda))^{\dim \mathfrak{g}_{\beta_{\mathrm{r}}}}$ is a polynomial function on $\mathfrak{s}^*$, the *quasi-center determinant.*

If $\xi \in \mathfrak{a}$ and $a = \exp(\xi) \in A$ we compute

$$\begin{aligned}(\mathrm{Ad}\,(a)\mathrm{Det}_{\mathfrak{s}^*})(\lambda) &= \mathrm{Det}_{\mathfrak{s}^*}(\mathrm{Ad}\,^*(\mathrm{a}^{-1})(\lambda)) \\ &= \prod\nolimits_r (\beta_r(\mathrm{Ad}\,(a^{-1})^*\lambda))^{\dim \mathfrak{g}_{\beta_r}} = \prod\nolimits_r (\beta_r(\exp(\beta_r(\xi))\lambda))^{\dim\mathfrak{g}_{\beta_r}}.\end{aligned} \tag{4.6}$$

Combining Lemmas 4.2 and 4.4 with (4.6) we have

Proposition 4.7. *The product* $\mathrm{Pf}\,\cdot \mathrm{Det}_{\mathfrak{s}^*}$ *is an* $\mathrm{Ad}\,(MAN)$*-semi-invariant polynomial on* $\mathfrak{s}^*$ *of degree* $\frac{1}{2}(\dim\mathfrak{n} + \dim\mathfrak{s})$ *and of weight equal to the modular function* δ_{MAN} *.*

Our fixed decomposition $\mathfrak{n} = \mathfrak{v} + \mathfrak{s}$ gives $N = VS$ where $V = \exp(\mathfrak{v})$ and $S = \exp(\mathfrak{s})$. Now define

$$D:\ \text{Fourier transform of } \mathrm{Pf}\,\cdot \mathrm{Det}_{\mathfrak{s}^*}, \text{ acting on } MAN = MAVS \text{ by acting on } S. \tag{4.8}$$

We use the fact that the definition of $\mathcal{C}(N)$ between (1.2) and Theorem 1.3 applies to $\mathcal{C}(MAN)$:

Theorem 4.9. *The operator D of* (4.8) *is an invertible self-adjoint differential operator of degree* $\frac{1}{2}(\dim\mathfrak{n} + \dim\mathfrak{s})$ *on* $\mathcal{L}^2(MAN)$ *with dense domain* $\mathcal{C}(MAN)$*, and it is* $\mathrm{Ad}\,(MAN)$*-semi-invariant of weight equal to the modular function* δ_{MAN} *. In other words $|D|$ is a Dixmier–Pukánszky Operator on MAN with domain equal to the space of rapidly decreasing C^∞ functions.*

Proof. Since it is the Fourier transform of a real polynomial, D is a differential operator which is invertible and self-adjoint on $\mathcal{L}^2(MAN)$. Its degree as a differential operator is the same as that of the polynomial. Further, it has dense domain $\mathcal{C}(MAN)$. Proposition 4.7 ensures that the degree is $\frac{1}{2}(\dim\mathfrak{n}+\dim\mathfrak{s})$ and that D is $\mathrm{Ad}\,(MAN)$–semi–invariant as asserted. ∎

5. Generic Representations

In this section we complete the description of a dense open subset of the unitary dual of $\widehat{P} = \widehat{MAN}$ that carries Plancherel measure. In the next section we will combine this with Theorem 4.9, using the framework of (4.1), to obtain explicit Plancherel Formulae for MAN and AN.

There are two paths here. We can obtain the generic representations of P by inducing the representations $\mathrm{Ind}_{NM_\lambda}^{NM} \eta_{\lambda,\gamma}$ discussed in Proposition 3.11. But one has a cleaner final statement if he avoids that induction by stages and induces directly from $N \rtimes (MA)_\lambda$ to P.

Since $\lambda \in \mathfrak{t}^*$ has nonzero projection on each summand $\mathfrak{z}_r^*$ of $\mathfrak{s}^*$, and $a \in A$ acts by the positive real scalar $\exp(\beta_r(\log(a)))$ on $\mathfrak{z}_r$,

$$A_\lambda = \exp(\{\xi \in \mathfrak{a} \mid \text{each } \beta_r(\xi) = 0\}), \text{ independent of } \lambda \in \mathfrak{t}^*. \tag{5.1}$$

Because of this independence, and in view of our earlier definition of

$$\mathfrak{a}_\diamondsuit = \{\xi \in \mathfrak{a} \mid \text{ each } \beta_r(\xi) = 0\},$$

we define

$$A_\diamondsuit = A_\lambda \text{ for any (and thus for all) } \lambda \in \mathfrak{t}^*. \tag{5.2}$$

Lemma 5.3. *In the notation of* (3.12) *and* (5.2), *if* $\lambda \in \sigma(\mathfrak{u}^*)$ *then the stabilizer* $(MA)_\lambda = M_\diamondsuit A_\diamondsuit$.

Proof. As $\lambda \in \mathfrak{t}^*$ it has expression $\lambda = \sum \lambda_r$ with $0 \neq \lambda_r \in \mathfrak{z}^* = \mathfrak{g}_{\beta_r}$. Let $\xi \in \mathfrak{a}$ and $m \in M$ with $\mathrm{Ad}^*(\exp(\xi)m)\lambda = \lambda$. Then each $\mathrm{Ad}^*(\exp(\xi)m)\lambda_r = \lambda_r$. In an $\mathrm{Ad}^*(M)$-invariant inner product, $||\mathrm{Ad}^*(\exp(\xi)m)\lambda_r|| = \exp(\beta_r(\xi))||\lambda_r||$ so each $\beta_r(\xi) = 0$, i.e. $\xi \in \mathfrak{a}_\diamondsuit$ and $\mathrm{Ad}^*(\exp(\xi)m)\lambda = \mathrm{Ad}^*(m)\lambda$. Thus $m \in M_\diamondsuit$ and $\exp(\xi) \in A_\diamondsuit$, as asserted. ∎

Now we are ready to use the Mackey little group method. First, there is no problem with obstructions:

Lemma 5.4. *Let* $\lambda \in \sigma(\mathfrak{u}^*)$ *and note the extension* $\pi_\lambda^\dagger$ *of* π_λ *from* N *to* $NM_\diamondsuit$ *defined by* Lemma 3.8. *Then* $\pi_\lambda^\dagger$ *extends further to a unitary representation* $\widetilde{\pi_\lambda}$ *of* $NM_\diamondsuit A_\diamondsuit$ *on the representation space of* π_λ .

Proof. Since $A_\diamondsuit$ is a vector group, it retracts to a point, so $H^2(A_\diamondsuit; U(1)) = H^2(\text{point}; U(1)) = \{1\}$. Thus the Mackey obstruction vanishes. ∎

Let $\lambda \in \sigma(\mathfrak{u}^*)$. Note that $\widehat{A_\diamondsuit}$ consists of the unitary characters $\exp(i\phi) : a \mapsto e^{i\phi(\log a)}$ with $\phi \in \mathfrak{a}_\diamondsuit^*$. With that notation, the representations of P corresponding to λ are the

$$\pi_{\lambda,\gamma,\phi} := \mathrm{Ind}_{NM_\diamondsuit A_\diamondsuit}^{NMA}(\widetilde{\pi_\lambda} \otimes \gamma \otimes \exp(i\phi)) \text{ where } \gamma \in \widehat{M_\diamondsuit} \text{ and } \phi \in \mathfrak{a}_\diamondsuit^* . \tag{5.5}$$

Here the action of A fixes γ because A centralizes M, and it fixes ϕ because A is commutative, so

$$\pi_{\lambda,\gamma,\phi} \cdot \mathrm{Ad}\,((ma)^{-1}) = \pi_{\mathrm{Ad}^*(ma)\lambda,\gamma,\phi} \tag{5.6}$$

Proposition 5.7. *Plancherel measure for MAN is concentrated on the set of unitary equivalence classes of representations $\pi_{\lambda,\gamma,\phi}$ for $\lambda \in \sigma(\mathfrak{u}^*)$, $\gamma \in \widehat{M_\diamondsuit}$ and $\phi \in \mathfrak{a}_\diamondsuit^*$. The equivalence class of $\pi_{\lambda,\gamma,\phi}$ depends only on* $(\mathrm{Ad}^*(MA)\lambda, \gamma, \phi)$.

Representations of AN are the case $\gamma = 1$. In effect, let π'_λ denote the obvious extension $\widetilde{\pi_\lambda}|_{AN}$ of the stepwise square integrable representation π_λ from N to $NA_\diamondsuit$ where $\widetilde{\pi_\lambda}$ is given by Lemma 5.4. Denote

$$\pi_{\lambda,\phi} = \mathrm{Ind}_{NA_\diamondsuit}^{NA}(\pi'_\lambda \otimes \exp(i\phi)) \text{ where } \lambda \in \mathfrak{u}^* \text{ and } \phi \in \mathfrak{a}_\diamondsuit^*. \tag{5.8}$$

Then $\pi_{\lambda,\phi}$ and $\pi_{\lambda',\phi}$ are equivalent if and only if $\lambda' \in \mathrm{Ad}^*(A)\lambda$. We have proved

Corollary 5.9. *Plancherel measure for AN is concentrated on the set $\{\pi_{\lambda,\phi} \mid \lambda \in \mathfrak{u}^* \text{ and } \phi \in \mathfrak{a}_\diamondsuit^*\}$ of (equivalence classes of) irreducible representations of $AN = NA$ described in* (5.8).

Finally we describe the set $\mathrm{Ad}^*(MA)\lambda$ of Proposition 5.7. A result of C.C. Moore says that $\mathrm{Ad}(P_\mathbb{C})$ has a Zariski open orbit on $\mathfrak{n}_\mathbb{C}^*$, so there is a finite set of open $\mathrm{Ad}(P)$-orbits on $\widehat{N}$ such that Plancherel measure is concentrated on the union of those open orbits. Moore presented this and a number of related results in a January 1972 seminar at Berkeley but he didn't publish it. Carmona circulated a variation on this but he also seems to have left it unpublished. Using Lemma 5.3, Moore's result leads directly to

Lemma 5.10. *The* Pf *-nonsingular principal orbit set $\mathfrak{u}^*$ is a finite union of open $\mathrm{Ad}^*(MA)$-orbits.*

Let $\{\mathcal{O}_1, \dots \mathcal{O}_v\}$ denote the (open) $\mathrm{Ad}^*(MA)$-orbits on $\mathfrak{u}^*$. Denote $\lambda_i = \sigma(\mathcal{O}_i)$ so

$$\mathcal{O}_i = \mathrm{Ad}^*(MA)\lambda_i \quad \text{and} \quad (MA)_{\lambda_i} = M_\diamondsuit A_\diamondsuit \quad \text{for} \quad 1 \leqq i \leqq v. \tag{5.11}$$

Then Proposition 5.7 becomes

Theorem 5.12. *Plancherel measure for MAN is concentrated on the set (of equivalence classes of) unitary representations $\pi_{\lambda_i,\gamma,\phi}$ for $1 \leqq i \leqq v$, $\gamma \in \widehat{M_\diamondsuit}$ and $\phi \in \mathfrak{a}_\diamondsuit^*$.*

6. Non–Unimodular Plancherel Formulae

Recall the Dixmier–Pukánsky operator D from (4.8) and Theorem 4.9. The Plancherel Formula (or Fourier inversion formula) for MAN is

Theorem 6.1. *Let $P = MAN$ be a minimal parabolic subgroup of the real reductive Lie group G. Given $\pi_{\lambda,\gamma,\phi} \in \widehat{MAN}$ as described in statement* (5.5), *let $\Theta_{\pi_{\lambda,\gamma,\phi}} : h \mapsto \mathrm{trace}\, \pi_{\lambda,\gamma,\phi}(h)$ denote its distribution character. Then $\Theta_{\pi_{\lambda,\gamma,\phi}}$ is* *a tempered distribution. If $f \in \mathcal{C}(MAN)$ then*

$$f(x) = c\sum_{i=1}^{v} \sum_{\gamma \in \widehat{M_\diamondsuit}} \int_{\mathfrak{a}_\diamondsuit^*} \Theta_{\pi_{\lambda_i,\gamma,\phi}}(D(r(x)f))|\mathrm{Pf}\,(\lambda_i)| \dim\gamma\ d\phi$$

where $c > 0$ depends on normalizations of Haar measures.

Proof. We compute along the lines of the argument of [10, Theorem 2.7], ignoring multiplicative constants that depend of normalizations of Haar measures. From [6, Theorem 3.2], $\mathrm{trace}\,\pi_{\lambda_i,\gamma,\phi}(Dh)\mathrm{trace}\,\pi_{\lambda_i,\gamma,\phi}(Dh)$

$$\begin{aligned}
&= \int_{x\in MA/M_\diamondsuit A_\diamondsuit} \delta(x)^{-1}\mathrm{trace} \int_{NM_\diamondsuit A_\diamondsuit} (Dh)(x^{-1}nmax)\cdot(\pi_{\lambda_i}\otimes\gamma\otimes\exp(i\phi))(nma)\,dn\,dm\,da\,dx\\
&= \int_{x\in MA/M_\diamondsuit A_\diamondsuit} \mathrm{trace} \int_{NM_\diamondsuit A_\diamondsuit} (Dh)(nx^{-1}max)\cdot(\pi_{\lambda_i}\otimes\gamma\otimes\exp(i\phi))(xnx^{-1}ma)\,dn\,dm\,da\,dx.
\end{aligned}$$

Now $\int_{\widehat{M_\diamondsuit A_\diamondsuit}} \mathrm{trace}\,\pi_{\lambda_i,\gamma,\phi}(Dh)\dim\gamma\ d\phi$

$$\begin{aligned}
&= \int_{\widehat{M_\diamondsuit A_\diamondsuit}} \mathrm{trace}\,\pi_{\lambda_i,\gamma,\phi}(Dh)\dim\gamma\ d\phi\\
&= \int_{\widehat{M_\diamondsuit A_\diamondsuit}} \int_{x\in MA/M_\diamondsuit A_\diamondsuit} \mathrm{trace} \int_{NM_\diamondsuit A_\diamondsuit} (Dh)(nx^{-1}max)\times\\
&\qquad\qquad \times(\pi_{\lambda_i}\otimes\gamma\otimes\exp(i\phi))(xnx^{-1}ma)\,dn\,dm\,da\,dx\ \dim\gamma\ d\phi\\
&= \int_{x\in MA/M_\diamondsuit A_\diamondsuit} \int_{\widehat{M_\diamondsuit A_\diamondsuit}} \mathrm{trace} \int_{NM_\diamondsuit A_\diamondsuit} (Dh)(nx^{-1}max)\times\\
&\qquad\qquad \times(\pi_{\lambda_i}\otimes\gamma\otimes\exp(i\phi)(xnx^{-1}ma)\,dn\,dm\,da\ \dim\gamma\ d\phi\ dx\\
&= \int_{x\in MA/M_\diamondsuit A_\diamondsuit} \mathrm{trace} \int_N (Dh)(n)\pi_{\lambda_i}(xnx^{-1})dn\,dx\\
&= \int_{x\in MA/M_\diamondsuit A_\diamondsuit} \mathrm{trace} \int_N (Dh)(n)(x^{-1}\cdot\pi_{\lambda_i})(n)dn\,dx \qquad (6.2)\\
&= \int_{x\in MA/M_\diamondsuit A_\diamondsuit} \mathrm{trace}\,((x^{-1}\cdot\pi_{\lambda_i})(Dh))\,dx\\
&= \int_{x\in MA/M_\diamondsuit A_\diamondsuit} (x^{-1}\cdot\pi_{\lambda_i})_*(D)\ \mathrm{trace}\,(x^{-1}\cdot\pi_{\lambda_i})(h)dx\\
&= \int_{x\in MA/M_\diamondsuit A_\diamondsuit} (\pi_{\lambda_i})_*(x\cdot D)\,\mathrm{trace}\,(x^{-1}\cdot\pi_{\lambda_i})(h)\,dx\\
&= \int_{x\in MA/M_\diamondsuit A_\diamondsuit} \delta_{MAN}(x)\,\mathrm{trace}\,(x^{-1}\cdot\pi_{\lambda_i})(h)\,dx = \int_{\mathrm{Ad}^*(MA)\lambda_i} \mathrm{trace}\,\pi_\lambda(h)|\mathrm{Pf}\,(\lambda)|d\lambda.
\end{aligned}$$

Summing over the orbits $\mathcal{O}_i$ of $\mathrm{Ad}^*(MA)$ on $\mathfrak{u}^*$ we now have

$$\begin{aligned}\sum_{i=1}^{v}\sum_{\gamma\in\widehat{M_\diamondsuit}}\int_{\mathfrak{a}_\diamondsuit^*}\operatorname{trace}\pi_{\lambda_i,\gamma,\phi}(Dh)\dim\gamma\; d\phi &= \sum_{i=1}^{v}\int_{\widehat{M_\diamondsuit A_\diamondsuit}}\operatorname{trace}\pi_{\lambda_i,\gamma,\phi}(Dh)\dim\gamma\; d\phi\\ = \sum_{i=1}^{v}\int_{\mathcal{O}_i}\operatorname{trace}\pi_\lambda(h)|\mathrm{Pf}\,(\lambda)|d\lambda &= \int_{\mathfrak{u}^*}\operatorname{trace}\pi_\lambda(h)|\mathrm{Pf}\,(\lambda)|d\lambda = h(1_N) = h(1_P)\,.\end{aligned} \tag{6.3}$$

Let h denote any right translate of f. The theorem follows. ∎

The Plancherel Theorem for NA follows similar lines. For the main computation (6.2) in Theorem 6.1 we omit M and γ. That gives

$$\int_{\mathfrak{a}_\diamondsuit^*}\operatorname{trace}\pi_{\lambda_0,\phi}(Dh)\,d\phi = \int_{\mathrm{Ad}^*(A)\lambda_0}\operatorname{trace}\pi_\lambda(h)|\mathrm{Pf}\,(\lambda)|d\lambda \tag{6.4}$$

In order to go from an $\mathrm{Ad}^*(A)\lambda_0$ in (6.4) to an integral over $\mathfrak{u}^*$ we use M to parameterize the space of $\mathrm{Ad}^*(A)$-orbits on $\mathfrak{u}^*$. We first note that

$$\text{If } \lambda\in\mathfrak{u}^* \text{ then } \mathrm{Ad}^*(A)\lambda\cap\mathrm{Ad}^*(M)\lambda = \{\lambda\} \tag{6.5}$$

because $\mathrm{Ad}^*(A)$ acts on each $\mathfrak{z}_r^*$ be positive scalars and $\mathrm{Ad}^*(M)$ preserves the norm on each $\mathfrak{z}_r^*$. Thus the space of $\mathrm{Ad}^*(A)$-orbits on $\mathfrak{u}^*$ is partitioned by the space of $\mathrm{Ad}^*(M)$-orbits on $\mathfrak{u}^*/\mathrm{Ad}^*(A)$. Each such $\mathrm{Ad}^*(M)$-orbit is in fact an $\mathrm{Ad}^*(MA)$-orbit on $\mathfrak{u}^*$. Recall the decomposition $\mathfrak{u}^* = \bigcup \mathcal{O}_i$ where $\mathcal{O}_i = \mathrm{Ad}^*(MA)\lambda_i$ with $\lambda_i = \sigma(\mathrm{Ad}^*(M)\lambda_i)$. Define $S_i = \mathrm{Ad}^*(M)\lambda_i$, so $\mathfrak{u}^* = \bigcup_i \mathrm{Ad}^*(A)S_i$. Now

Proposition 6.6. *Plancherel measure for NA is concentrated on the equivalence classes of representations $\pi_{\lambda,\phi} = \mathrm{Ind}_{NA_\diamondsuit}^{NA}(\pi'_\lambda\otimes\exp(i\phi))$ where $\lambda\in S_i := \mathrm{Ad}^*(M)\lambda_i$ $(1\leqq i\leqq v)$, π'_λ is the extension of π_λ from N to $NA_\diamond$ and $\phi\in\mathfrak{a}_\diamond^*$. Representations $\pi_{\lambda,\phi}$ and $\pi_{\lambda',\phi'}$ are equivalent if and only if $\lambda'\in\mathrm{Ad}^*(A)\lambda$ and $\phi'=\phi$. Further, $\pi_{\lambda,\phi}|_N = \int_{a\in A/A_\diamondsuit}\pi_{\mathrm{Ad}^*(a)\lambda}da$.*

Now we sum both sides of (6.4) as follows.

$$\begin{aligned}\sum_i\int_{\lambda'\in S_i}\int_{\mathfrak{a}_\diamondsuit^*}\operatorname{trace}\pi_{\lambda',\phi}(Dh)\,d\phi\,d\lambda' &= \sum_i\int_{\mathcal{O}_i}\operatorname{trace}\pi_\lambda(h)|\mathrm{Pf}\,(\lambda)|d\lambda\\ &= \int_{\mathfrak{u}^*}\operatorname{trace}\pi_\lambda(h)|\mathrm{Pf}\,(\lambda)|d\lambda = h(1_N) = h(1_{AN}).\end{aligned} \tag{6.7}$$

Again taking $h = r(x)f$ we have

Theorem 6.8. *Let $P = MAN$ be a minimal parabolic subgroup of the real reductive Lie group G. Given $\pi_{\lambda,\phi}\in\widehat{AN}$ as described in* Proposition 6.6 *let $\Theta_{\pi_{\lambda,\phi}} : h\mapsto \operatorname{trace}\pi_{\lambda,\phi}(h)$ denote its distribution character. Then $\Theta_{\pi_{\lambda,\phi}}$ is a tempered distribution. If $f\in\mathcal{C}(AN)$ then*

$$f(x) = c\sum_{i=1}^{v}\int_{\lambda\in S_i}\int_{\mathfrak{a}_\diamondsuit^*}\operatorname{trace}\pi_{\lambda,\phi}(D(r(x)f))|\mathrm{Pf}\,(\lambda)|d\lambda d\phi.$$

where $c = 2^{d_1+\cdots+d_m} d_1!d_2!\dots d_m!$, *from* (1.2), *as in* Theorem 1.3 *and* Proposition 3.13.

7. Remark on Strongly Orthogonal Restricted Roots

The goal of this paper was to extend our earlier result, Theorem 2.11, from nilradicals of minimal parabolic subgroups to the minimal parabolics themselves. In part we needed to extend some results of Kostant ([7], [8]) on strongly orthogonal roots from Borel subalgebras of complex semisimple Lie algebras to minimal parabolic subalgebras of real semisimple algebras. But some of the technical results in ([7], [8]), which we didn't use but are of strong independent interest, also extend. We use the notation of Section 2.

Lemma 7.1. $\Delta_r^+ = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid \alpha \perp \beta_i \text{ for } i < r \text{ and } \langle \alpha, \beta_r \rangle > 0\}$.

Proof. In view of (2.3) we need only show that if $\alpha \in -\Delta^+(\mathfrak{g}, \mathfrak{a})$ and $\alpha \perp \beta_i$ for $i < r$ then $\langle \alpha, \beta_r \rangle \leqq 0$. But if that fails, so $\langle \alpha, \beta_r \rangle > 0$, then $\beta_r - \alpha$ is a root greater than β_r and $\perp \beta_i$ for $i < r$, which contradicts the construction (2.2) of the cascade of strongly orthogonal roots β_j . ■

Proposition 7.2. *The composition* $s_{\beta_1} s_{\beta_2} \dots s_{\beta_r}$ *sends* $(\Delta_1^+ \cup \cdots \cup \Delta_r^+)$ *to* $-(\Delta_1^+ \cup \cdots \cup \Delta_r^+)$. *In particular, the longest element of the restricted Weyl group* $W = W(\mathfrak{g}, \mathfrak{a}, \Delta^+)$, *defined by* $w_0(\Delta^+(\mathfrak{g}, \mathfrak{a})) = -\Delta^+(\mathfrak{g}, \mathfrak{a})$, *is given by* $w_0 = s_{\beta_1} s_{\beta_2} \dots s_{\beta_m}$.

Proof. This is an induction on r. For $r = 1$ the statement is in the discussion immediately preceding Lemma 2.10. Now suppose that $s_{\beta_1} s_{\beta_2} \dots s_{\beta_{r-1}}$ sends $(\Delta_1^+ \cup \cdots \cup \Delta_{r-1}^+)$ to its negative. Since $s_{\beta_r}(\beta_i) = \beta_i$ for $i < r$, Lemma 7.1 shows that s_{β_r} preserves $(\Delta_1^+ \cup \cdots \cup \Delta_{r-1}^+)$, so $s_{\beta_1} s_{\beta_2} \dots s_{\beta_r}$ sends $(\Delta_1^+ \cup \cdots \cup \Delta_{r-1}^+)$ to its negative. But Lemma 7.1 also shows that $s_{\beta_1} s_{\beta_2} \dots s_{\beta_{r-1}}$ preserves Δ_r^+, and the discussion just before Lemma 2.10 shows that s_{β_r} sends Δ_r^+ to its negative. This completes the induction. In view of Lemma 2.4, the case $r = m$ says that $s_{\beta_1} s_{\beta_2} \dots s_{\beta_m} \Delta^+(\mathfrak{g}, \mathfrak{a}) = -\Delta^+(\mathfrak{g}, \mathfrak{a})$. ■

Corollary 7.3. *Let* $\nu \in \mathfrak{a}^*$ *be the highest weight of an irreducible finite dimensional representation* τ_ν *of* $\mathfrak{g}$, *so the dual representation* τ_ν^* *has highest weight* $\nu^* := -w_0(\nu)$. *Then* $\nu + \nu^* = \sum \frac{2\langle \nu, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} \beta_i$, *integral linear combination of* $\beta_1, \dots, \beta_m$.

Proof. Write $(\alpha, \gamma) = \frac{2\langle \alpha, \gamma \rangle}{\langle \gamma, \gamma \rangle}$. Compute $s_{\beta_1}(\nu) = \nu - (\nu, \beta_1)\beta_1$, then $s_{\beta_2} s_{\beta_1}(\nu) = \nu - (\nu, \beta_1)\beta_1 - (\nu, \beta_2)\beta_2$, continuing on to $s_{\beta_m} s_{\beta_{m-1}} \dots s_{\beta_1}(\nu) = \nu - \sum (\nu, \beta_i)\beta_i$. Using the last statement of Proposition 7.2 now $\nu + \nu^* = \nu - w_0(\nu) = \sum (\nu, \beta_i)\beta_i$ as asserted. ■

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