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Stepwise square integrable representations of nilpotent Lie groups

Joseph A. Wolf

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Abstract We study the conditions for a nilpotent Lie group to be foliated into subgroups that have square integrable (relative discrete series) unitary representations, that fit together to form a filtration by normal subgroups. Then we use that filtration to construct a class of "stepwise square integrable" representations on which Plancherel measure is concentrated. Further, we work out the character formulae for those stepwise square integrable representations, and we give an explicit Plancherel formula. Next, we use some structure theory to check that all these constructions and results apply to nilradicals of minimal parabolic subgroups of real reductive Lie groups. Finally, we develop multiplicity formulae for compact quotients N/Γ where Γ respects the filtration.

1 Introduction

There is a well developed theory of square integrable representations of nilpotent Lie groups [8]. It is, of course, based on the general representation theory [6] for nilpotent Lie groups. A connected simply connected Lie group N with center Z is called *square integrable* if it has unitary representations π whose coefficients $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ satisfy $|f_{u,v}| \in L^2(N/Z)$. If N has one such square integrable representation then there is a certain polynomial function $P(\lambda)$ on the linear dual space $\mathfrak{z}^*$ of the Lie algebra of Z that is key to harmonic analysis on N. Here $P(\lambda)$ is the Pfaffian of the antisymmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$ given by $b_\lambda(x, y) = \lambda([x, y])$. The square integrable representations of N are certain easily–constructed representations π_λ where

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J. A. Wolf (✉)
Department of Mathematics, University of California, Berkeley, CA 94720, USA
e-mail: jawolf@math.berkeley.edu

$\lambda \in \mathfrak{z}^*$ with $P(\lambda) \neq 0$, Plancherel almost all irreducible unitary representations of N are square integrable, and up to an explicit constant $|P(\lambda)|$ is the Plancherel density of the unitary dual $\widehat{N}$ at π_λ.

This theory has proved to have serious analytic consequences [12].

In this paper we present an extension of that theory. Under certain conditions, the nilpotent Lie group N has a particular decomposition into subgroups that have square integrable representations, and the Plancherel formula then is synthesized explicitly in terms of the Plancherel formulae of those subgroups. Many of our calculations are on the Lie algebra level, reflecting the setting of the square integrability conditions mentioned above.

The guiding example for these decompositions is that of the upper triangular real matrices. We go through it separately in Sect. 2 for the decompositions, Sect. 3 for the square integrability, and Sect. 4 for the Plancherel formula. We separate out the upper triangular groups for two reasons. First, they give a clear illustration of the technical conditions that we need more generally to decompose the group and to reconstitute the Plancherel formula. Second, and equally important, they appear in many situations and are of independent interest. The precise conditions are given in (5.1).

The decompositions for upper triangular matrices were suggested by the work of Barberis and Dotti on abelian complex structures using Aroldo Kaplan's idea of groups of type H. See, for example, [2,3] and [5]. I thank them for discussions of their work in progress [4] on those structures.

The general formulation and the resulting Plancherel formulae are the content of Sect. 5. There we extend the material of Sects. 2, 3 and 4 to a much more general setting.

In Sect. 6 we verify the conditions of Sect. 5 for the unipotent radicals of minimal parabolic subgroups of real semisimple Lie groups. This has many potential applications in differential geometry, in hypoelliptic differential equations, and in harmonic analysis. For example, in the case of cuspidal parabolics, it has the potential of simplifying some of the integrations in Harish–Chandra's theory of the constant term. These unipotent radical examples also appear in many other geometric and analytic settings.

Finally, in Sect. 7 we consider the case where our connected simply connected nilpotent Lie group N has a discrete co-compact subgroup Γ that fits into the pattern of Sect. 5. We show that the compact nilmanifold N/Γ has a corresponding foliation and derive analytic results analogous to those of Theorem 5.1. These results include multiplicity formulae for stepwise square integrable representations as summands of the regular representation $\mathrm{Ind}_\Gamma^N(1_\Gamma)$ of N on $L^2(N/\Gamma)$. They apply in particular to the nilradicals of minimal parabolic subgroups, as studied in Sect. 6.

2 Decomposition of upper triangular matrices

Let $\mathfrak{n}$ denote the real Lie algebra of $\ell \times \ell$ matrices with zeroes on and below the diagonal, and let N the corresponding unipotent Lie group of $\ell \times \ell$ matrices with zeroes below the diagonal and ones on the diagonal.

As usual $e_{i,j} \in \mathfrak{n}$ denotes the matrix with 1 in row i column j and zeroes elsewhere, so $\mathfrak{n}$ is the span of $\{e_{i,j} \mid 1 \leqq i < j \leqq \ell\}$. Here $[e_{i,j}, e_{m,n}]$ is $e_{i,n}$ if $i < j = m < n$,

$-e_{m,j}$ if $m < n = i < j$, 0 in all other cases. Thus, for $1 \leqq r \leqq [\frac{\ell}{2}]$, we define

$$\begin{aligned} \mathfrak{m}_r &:= \text{Span}\,\{e_{r,s} \mid r+1 \leqq s \leqq \ell-r\} \cup \text{Span}\,\{e_{q,\ell-r+1} \mid r+1 \leqq q \leqq \ell-r\} \cup e_{r,\ell-r+1}\mathbb{R} \\ \text{and } \mathfrak{n}_r &:= \mathfrak{m}_1+\mathfrak{m}_2+\cdots+\mathfrak{m}_r = \text{Span}\,\{e_{i,j} \mid i \leqq r \leqq \ell-r+1 < j\}. \end{aligned} \tag{2.1}$$

Then $\mathfrak{m}_r$ is a subalgebra of $\mathfrak{n}$ that is isomorphic to the Heisenberg algebra $\mathfrak{h}_{\ell-2r}$ of dimension $2(\ell-2r)+1$; it has center $\mathfrak{z}_r := e_{r,\ell-r+1}\mathbb{R}$. Note that

$$[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{n}_{min(i,j)}\,, \text{ so each } \mathfrak{n}_r \text{ is an ideal in } \mathfrak{n}. \tag{2.2}$$

In particular we have semidirect sum decompositions

$$\mathfrak{n}_r = \mathfrak{n}_{r-1} \inplus \mathfrak{m}_r \tag{2.3}$$

and a filtration

$$\mathfrak{n}_1 \subset \mathfrak{n}_2 \subset \cdots \subset \mathfrak{n}_{[\ell/2]} = \mathfrak{n} \tag{2.4}$$

by ideals.

Fix the positive definite inner product on $\mathfrak{n}$ in which the $e_{i,j}$ are orthonormal. Define $J_r : \mathfrak{n} \to \mathfrak{n}$ by $\langle J_r(x), y\rangle = \langle e_{r,\ell-r+1}, [x, y]\rangle$. Its image is the non-central part

$$\text{Span}\,\{e_{r,s} \mid r+1 \leqq s \leqq \ell-r\} \cup \text{Span}\,\{e_{q,\ell-r+1} \mid r+1 \leqq q \leqq \ell-r\} = \mathfrak{m}_r \cap \mathfrak{v}_r \tag{2.5}$$

of $\mathfrak{m}_r$ and its kernel is the central part $\mathfrak{z}_r = e_{r,\ell-r+1}\mathbb{R}$. That is the connection with abelian complex structures mentioned in the Introduction.

Group structure here follows algebra structure immediately, as all the groups are unipotent and thus equal to the exponential image of their Lie algebras. Thus we have closed connected subgroups $M_r \cong H_{\ell-2r}$ and normal closed connected subgroups $N_r = M_1 M_2 \dots M_r$ in N, and semidirect product decompositions $N_r = N_{r-1} \rtimes M_r$.

3 Square integrability for upper triangular matrices

Now we cascade down antidiagonally from the upper right hand corner. For convenience let $m := [\frac{\ell}{2}]$. If $r < m$ them M_r is the Heisenberg group of dimension $2(\ell-2r)+1$. If ℓ is odd then M_m is the 3-dimensional Heisenberg group, and if ℓ is even then $M_m \cong \mathbb{R}$ is the 1-dimensional vector group. The point is that Plancherel-almost-every irreducible unitary representation of M_r is a representation π_{λ_r} specified by a nonzero linear functional λ_r on the center $\mathfrak{z}_r$ of $\mathfrak{m}_r$, and that representation has matrix coefficients in $L^2(M_r/Z_r)$. Write $\mathcal{H}_{\pi_{\lambda_r}}$ for the representation space, or just $\mathcal{H}_r$ if there is no chance of confusion.

Let $\lambda = \lambda_1 + \cdots + \lambda_m$ where $0 \neq \lambda_r \in \mathfrak{z}_r$ for $1 \leqq r \leqq m$. We are going to put together the square integrable representations $\pi_{\lambda_r} \in \widehat{M_r}$ to form a representation

$\pi_r \in \widehat{N}$. This will be a recursion on r and we will need the

$$S_r = Z_1 Z_2 \dots Z_r = S_{r-1} \times Z_r$$

for that recursive construction.

Lemma 3.1 *M_r centralizes S_{r-1}.*

Proof The Lie algebra $\mathfrak{s}_{r-1}$ is spanned by the $e_{j,\ell+1-j}$ for $j = 1, \dots, r-1$. Let $e_{u,v} \in \mathfrak{m}_r$. Then $[e_{u,v}, e_{j,\ell+1-j}] = 0$ because $v > j$ and $u < \ell + 1 - j$.

Express N_2 as the semidirect product $N_1 \rtimes M_2$. Plancherel-almost-every irreducible unitary representation of $N_1 = M_1$ is a representation π_{λ_1} specified by a nonzero linear functional $\lambda_1 \in \mathfrak{z}_1^*$. View λ_1 as an element of $\mathfrak{n}_1^*$ that vanishes on the non-central matrices $e_{i,j}$ in $\mathfrak{n}_1$. Choose an invariant polarization $\mathfrak{p}_1' \subset \mathfrak{n}_2$ for the linear functional $\lambda_1' \in \mathfrak{n}_2^*$ that agrees with λ_1 on $\mathfrak{n}_1$ and vanishes on $\mathfrak{m}_2$. Lemma 3.1 implies $\mathrm{ad}^*(\mathfrak{m}_2)(\lambda_1')|_{\mathfrak{z}_1+\mathfrak{m}_2} = 0$, so $\mathfrak{p}_1' = \mathfrak{p}_1 + \mathfrak{m}_2$ where $\mathfrak{p}_1$ is an invariant polarization for the linear functional $\lambda_1 \in \mathfrak{n}_1^*$. The associated representations are $\pi_{\lambda_1'} \in \widehat{N_2}$ and $\pi_{\lambda_1} \in \widehat{N_1}$. Note that $N_2/P_1' = N_1/P_1$, so the representation spaces $\mathcal{H}_{\pi_{\lambda_1'}} = L^2(N_2/P_1') = L^2(N_1/P_1) = \mathcal{H}_{\pi_{\lambda_1}}$. In other words, $\pi_{\lambda_1'}$ extends π_{λ_1} to a unitary representation of N_2 on the same Hilbert space $\mathcal{H}_{\pi_{\lambda_1}}$, and $d\pi_{\lambda_1'}(\mathfrak{z}_2) = 0$. Now the Mackey Little Group method gives us

Lemma 3.2 *The irreducible unitary representations of N_2, whose restrictions to N_1 are multiples of π_{λ_1}, are the $\pi_{\lambda_1'} \widehat{\otimes} \gamma$ where $\gamma \in \widehat{M_2} = \widehat{N_2/N_1}$.*

Given nonzero $\lambda_1 \in \mathfrak{z}_1^*$ and $\lambda_2 \in \mathfrak{z}_2^*$ we have representations $\pi_{\lambda_1} \in \widehat{M_1}$ and $\pi_{\lambda_2} \in \widehat{M_2}$ with coefficients in $L^2(M_1/Z_1)$ and $L^2(M_2/Z_2)$ respectively. Using the notation of Lemma 3.2 we define

$$\pi_{\lambda_1+\lambda_2} \in \widehat{N_2} \quad \text{by } \pi_{\lambda_1+\lambda_2} = \pi_{\lambda_1}' \widehat{\otimes} \pi_{\lambda_2}. \tag{3.1}$$

We now use the square integrability of π_{λ_1} and π_{λ_2} for some square integrability of $\pi_{\lambda_1+\lambda_2}$.

Proposition 3.3 *The coefficients $f_{z,w}(xy) = \langle z, \pi_{\lambda_1+\lambda_2}(xy)w\rangle$ of $\pi_{\lambda_1+\lambda_2}$ are in $L^2(N_2/S_2)$, in fact satisfy $||f_{z,w}||^2_{L^2(N_r/S_r)} = \frac{||z||^2||w||^2}{\deg(\pi_{\lambda_1})\dots\deg(\pi_{\lambda_r})}$.*

Proof We write $\mathcal{H}_r$ for the representation space of π_{λ_r}. $\mathcal{H}_1$ also is the representation space for $\pi_{\lambda_1'}$, so $\pi_{\lambda_1+\lambda_2}$ has representation space $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$. Choose nonzero vectors $u, v \in \mathcal{H}_1$ and $u', v' \in \mathcal{H}_2$. We need only prove that the function $f(x, y) = \langle u, \pi_{\lambda_1}'(xy)v\rangle\langle u', \pi_{\lambda_2}(y)v'\rangle$, $x \in M_1$ and $y \in M_2$, satisfies $||f||^2_{L^2(N_2/S_2)} = \left(\frac{||u||^2||v||^2}{\deg(\pi_{\lambda_1})}\right)\left(\frac{||u'||^2||v'||^2}{\deg(\pi_{\lambda_2})}\right)$, so that the coefficients $xy \mapsto \langle u \otimes u', \pi_{\lambda_1+\lambda_2}(xy)(v \otimes v')\rangle$ of decomposable vectors are in $L^2(N_2/S_2)$. For that, let $\{z_i\}$ and $\{w_j\}$ be complete orthonormal sets in $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$. Suppose that both $\sum |a_{i,j}|^2$ and $\sum |b_{i,j}|^2$ are finite, so $z = \sum a_{i,j} z_i \otimes w_j$ and $w = \sum b_{i,j} z_i \otimes w_j$ are general elements of $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$.

Then the coefficient $\langle x, \pi_{\lambda_1+\lambda_2}(xy)w\rangle = \sum a_{i,j}\overline{b_{i',j'}}\langle z_i, \pi_{\lambda_1+\lambda_2}(xy)w_j\rangle$ has square $L^2(N_2/S_2)$–norm given by

$$\frac{1}{\deg(\pi_{\lambda_1})}\frac{1}{\deg(\pi_{\lambda_2})}\sum |a_{i,j}|^2 \sum |b_{i',j'}|^2 = \frac{||z||^2||w||^2}{\deg(\pi_{\lambda_1})\deg(\pi_{\lambda_2})} < \infty$$

.

In order to integrate $|f|^2$ over $N_2 = M_1M_2$ modulo $S_2 = Z_1Z_2$ we use the fact that the action of M_2 on $\mathfrak{m}_1$ is unipotent, so there is a measure preserving decomposition

$$N_2/S_2 = (M_1/Z_1) \times (N_2/Z_2). \tag{3.2}$$

Using the extension of Schur Orthogonality to representations with coefficients that are square integrable modulo the center of the group, and writing v_y for $\pi_{\lambda_1'}(y)v$, we compute

$$\begin{aligned}
||f||^2_{L^2(N_2/S_2)} &= \int_{N_2/S_2} |\langle u, \pi'_{\lambda_1}(xy)v\rangle|^2 |\langle u', \pi_{\lambda_2}(y)v'\rangle|^2 d(xyZ_1Z_2)\\
&= \int_{M_2/Z_2} |\langle u', \pi_{\lambda_2}(y)v'\rangle|^2 \left(\int_{M_1/Z_1} |\langle u, \pi_{\lambda_1'}(xy)v\rangle|^2 d(xZ_1)\right) d(yZ_2)\\
&= \int_{M_2/Z_2} |\langle u', \pi_{\lambda_2}(y)v'\rangle|^2 \left(\int_{M_1/Z_1} |\langle u, \pi_{\lambda_1'}(x)v_y\rangle|^2 d(xZ_1)\right) d(yZ_2)\\
&= \int_{M_2/Z_2} |\langle u', \pi_{\lambda_2}(y)v'\rangle|^2 \left(\int_{M_1/Z_1} |\langle u, \pi_{\lambda_1}(x)v_y\rangle|^2 d(xZ_1)\right) d(yZ_2)\\
&= \int_{M_2/Z_2} |\langle u', \pi_{\lambda_2}(y)v'\rangle|^2 \tfrac{||u||^2||v_y||^2}{\deg(\pi_{\lambda_1})} d(yZ_2)\\
&= \left(\frac{||u||^2||v_y||^2}{\deg(\pi_{\lambda_1})}\right)\left(\frac{||u'||^2||v'||^2}{\deg(\pi_{\lambda_2})}\right) = \left(\frac{||u||^2||v||^2}{\deg(\pi_{\lambda_1})}\right)\left(\frac{||u'||^2||v'||^2}{\deg(\pi_{\lambda_2})}\right)\\
&= \frac{||u\otimes u'||^2||v\otimes v'||^2}{\deg(\pi_{\lambda_1})\deg(\pi_{\lambda_2})} < \infty.
\end{aligned}$$

That completes the proof of Proposition 3.3.

Proposition 3.3 starts our recursive construction. More generally, N_{r+1} is the semi-direct product $N_r \rtimes M_{r+1}$. We fix nonzero $\lambda_i \in \mathfrak{z}_i^*$ for $1 \leqq i \leqq r+1$, and we start with the representation $\pi_{\lambda_1+\dots+\lambda_r}$ constructed step by step from the square integrable representations $\pi_{\lambda_i} \in \widehat{M_i}$ for $1 \leqq i \leqq r$. The representation space $\mathcal{H}_{\pi_{\lambda_1+\dots+\lambda_r}} = \mathcal{H}_{\pi_{\lambda_1}}\widehat{\otimes}\cdots\widehat{\otimes}\mathcal{H}_{\pi_{\lambda_r}}$. The coefficients of $\pi_{\lambda_1+\dots+\lambda_r}$ have absolute value in $L^2(N_r/S_r)$. In fact they satisfy

$$||f_{z,w}||^2_{L^2(N_r/S_r)} = \frac{||z||^2||w||^2}{\deg(\pi_{\lambda_1})\ldots\deg(\pi_{\lambda_r})}.$$

Then $\pi_{\lambda_1+\cdots+\lambda_r}$ extends to a representation $\pi_{\lambda'_1+\cdots+\lambda_r}$ of L_{r+1} on the same Hilbert space $\mathcal{H}_{\pi_{\lambda_1+\cdots+\lambda_r}}$, and it satisfies $d\pi'_{\lambda_1+\cdots+\lambda_r}(\mathfrak{z}_{r+1}) = 0$. Thus, as in Lemma 3.2,

Lemma 3.4 *The irreducible unitary representations of N_{r+1}, whose restrictions to N_r are multiples of $\pi_{\lambda_1+\cdots+\lambda_r}$, are the $\pi'_{\lambda_1+\cdots+\lambda_r}\widehat{\otimes}\gamma$ where $\gamma \in \widehat{M_{r+1}} = \widehat{N_{r+1}/N_r}$.*

Recall $0 \neq \lambda_{r+1} \in \mathfrak{z}^*_{r+1}$ and the square integrable representation $\pi_{\lambda_{r+1}}$ of $M_{r+1} = L_{r+1}/L_r$. Computing exactly as in Proposition 3.3, we define $\pi_{\lambda_1+\cdots+\lambda_{r+1}} = \pi_{\lambda'_1+\cdots+\lambda_r}\widehat{\otimes}\pi_{\lambda_{r+1}}$ and conclude that

Proposition 3.5 *The coefficients $f_{z,w}(x_1\ldots x_{r+1}) = \langle z, \pi_{\lambda_1+\cdots+\lambda_{r+1}}(x_1x_2\cdots x_{r+1})w\rangle$ of $\pi_{\lambda_1+\cdots+\lambda_{r+1}}$ are in $L^2(N_{r+1}/S_{r+1})$, in fact satisfy $||f_{z,w}||^2_{L^2(N_{r+1}/S_{r+1})} = \frac{||z||^2||w||^2}{\deg(\pi_{\lambda_1})\ldots\deg(\pi_{\lambda_{r+1}})}$.*

Since the M_r are Heisenberg groups, except that M_m , the last one, is 1-dimensional abelian in case the size ℓ of the matrices is even, we have $\deg \pi_{\lambda_r} = |\lambda_r|^{d_r}$ where $\dim M_r = 2d_r + 1$ and $d_r = \ell - 2r$. Proposition 3.5 is the recursion step for our construction, and the end case $r + 1 = m$ is

Theorem 3.6 *Let $0 \neq \lambda_r \in \mathfrak{z}^*_r$ for $1 \leqq r \leqq m$ and set $\lambda = \lambda_1 + \cdots + \lambda_m$. Denote $\deg(\pi_\lambda) = \deg(\pi_{\lambda_1})\cdots\deg(\pi_{\lambda_m})$. Then the coefficients $f_{z,w}(x) = \langle z, \pi_\lambda(z)w\rangle$ of the irreducible unitary representation π_λ on N are in $L^2(N/S)$ and satisfy $||f_{z,w}||^2_{L^2(N/S)} = \frac{||z||^2||w||^2}{\deg(\pi_\lambda)} = ||z||^2||w||^2/\prod|\lambda_r|^{\ell-2r}$.*

Definition 3.7 The representations π_λ , constructed as just above, are the *stepwise square integrable* representations of N relative to the decompositions (2.1), (2.3) and (2.4).

4 Plancherel formula for upper triangular matrices

The Plancherel measure for the group M_r is $2^{d_r}d_r!|\lambda_r|^{d_r}d\lambda_r$ where $\dim M_r = 2d_r + 1$ and $d\lambda_r$ is Lebesgue measure on $\mathfrak{z}^*_r$. If $f \in L^1(M_r)$ we have $\dot{\pi}_{\lambda_r}(f) = \int_{M_r} f(x_r)\pi_{\lambda_r}(x_r)dx_r$. One version of the Plancherel formula for M_r is

$$||f||^2_{L^2(M_r)} = 2^{d_r}d_r!\int_{\mathfrak{z}^*_r} ||\dot{\pi}_{\lambda_r}(f)||^2_{HS}|\lambda_r|^{d_r}d\lambda_r \quad (d_r = \ell - 2r) \tag{4.1}$$

where $f \in L^1(M_r)\cap L^2(M_r)$ and $||\cdot||_{HS}$ is Hilbert–Schmidt norm, and another is

$$f(x) = c_r\int_{\mathfrak{z}^*_r} \Theta_{\pi_{\lambda_r}}(r_x f)|\lambda_r|^{d_r}d\lambda_r \tag{4.2}$$

where $\Theta_{\pi_{\lambda_r}}$ is the distribution character of π_{λ_r}, given by $\Theta_{\pi_{\lambda_r}}(h) = \operatorname{trace}\pi_{\lambda_r}(h)$, $f \in C_c^\infty(M_r)$, $c_r = 2^{d_r}d_r!$, and r_x is right translation of functions, $(r_x f)(g) = f(gx)$.

As we will see in a moment from the formula, the distribution $\Theta_{\pi_{\lambda_r}}$ is tempered, i.e. extends by continuity to the Schwartz space $\mathcal{C}(M_r)$.

To make this explicit one needs the character formula for π_{λ_r}, i.e. the formula for the tempered distribution $\Theta_{\pi_{\lambda_r}}$. That is given as follows. Define $h_1 \in C_c^\infty(\mathfrak{m}_r)$ by $h_1(\xi) = h(\exp(\xi))$. The geometric tangent space of $\mathrm{Ad}^*(M_r)\lambda_r$ is the coadjoint orbit $\mathrm{Ad}^*(M_r)\lambda_r$ itself, the affine hyperplane $\lambda_r + \mathfrak{z}_r^\perp$ in $\mathfrak{m}_r^* \cong (\mathfrak{m}_r/\mathfrak{z}_r)^*$. We use Lebesgue measure $d\nu_r$ on $(\mathfrak{m}_r/\mathfrak{z}_r)^*$ normalized so that Fourier transform is an isometry from $L^2(\mathfrak{m}_r/\mathfrak{z}_r)$ onto $L^2(\mathfrak{m}_r/\mathfrak{z}_r)^*$, and we translate $d\nu_r$ to a measure $d\nu_{\lambda_r}$ on the orbit. Then, from [9] and [8],

$$\Theta_{\pi_{\lambda_r}}(h) = c_r^{-1}|\lambda_r|^{-d_r} \int\limits_{\mathrm{Ad}^*(M_r)\lambda_r} \widehat{h_1}(\xi)d\nu_{\lambda_r}(\xi) \quad (d_r = \ell - 2r) \tag{4.3}$$

where $\widehat{h_1}$ is the Fourier transform of h_1. For all this see [8, Theorem 6 and its proof].

In order to extend these results from one group M_r to the upper triangular group N we need

Proposition 4.1 *Plancherel measure on $\widehat{N}$ is concentrated on the set*

$$\{\pi_\lambda \mid \lambda = \lambda_1 + \cdots + \lambda_m\,,\ 0 \neq \lambda_r \in \mathfrak{z}_r^* \quad \forall r\}.$$

Proof If $\zeta \in \mathfrak{s}^*$ then $e^{2\pi\sqrt{-1}\zeta} : \xi \mapsto e^{2\pi\sqrt{-1}\zeta(\log\xi)}$ on S is a unitary character on S. Denote the induced representation $\widetilde{\zeta} = \mathrm{Ind}_S^N(e^{2\pi\sqrt{-1}\zeta})$. Induction by stages says that the left regular representation of N is $\mathrm{Ind}_{\{1\}}^N(1) = \int_{\mathfrak{s}^*}\widetilde{\zeta}d\zeta$ where $d\zeta$ is Lebesgue measure on $\mathfrak{s}^*$. Let

$$\mathfrak{t}^* = \{\lambda = \lambda_1 + \cdots + \lambda_m\,,\ 0 \neq \lambda_r \in \mathfrak{z}_r^* \quad \forall r\} \text{ and } P(\lambda) = \lambda_1^{d_1}\lambda_2^{d_2}\dots\lambda_m^{d_m}. \tag{4.4}$$

Since P is not identically zero we can ignore its zero set in the direct integral, so left regular representation of N is $\mathrm{Ind}_{\{1\}}^N(1) = \int_{\mathfrak{t}^*}\widetilde{\zeta}d\zeta$.

Next, we break up the bilinear form b_λ.

Lemma 4.2 *Decompose each $\mathfrak{m}_r = \mathfrak{z}_r + \mathfrak{v}_r$ where $\mathfrak{v}_r$ is the span of the $e_{i,j}$ in $\mathfrak{m}_r$ but not in $\mathfrak{z}_r$, and similarly $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$. If $\lambda \in \mathfrak{t}^*$ then the antisymmetric bilinear form b_λ on $\mathfrak{v}$ is the direct sum $b_{\lambda_1} \oplus \cdots \oplus b_{\lambda_m}$ of nondegenerate bilinear forms on the $\mathfrak{v}_r$. Equivalently, if $r \neq t$ then $[\mathfrak{m}_r, \mathfrak{m}_t] \subset \mathfrak{v}$.*

Proof The equivalence is clear from the definition of b_λ. Let ρ denote reflection on the antidiagonal. Suppose that $e_{i,j} \in \mathfrak{m}_r$ and $e_{a,b} \in \mathfrak{m}_t$ with $r \neq t$. Then $e_{i,j}e_{a,b}$ is on the antidiagonal if and only if $\rho(e_{i,j}) = e_{a,b}$, and this happens if and only if $e_{a,b}e_{i,j}$ is on the antidiagonal. As $\rho(\mathfrak{m}_r) = \mathfrak{m}_r$ and $\rho(\mathfrak{m}_t) = \mathfrak{m}_t$, it follows that $[\mathfrak{m}_r, \mathfrak{m}_t] \subset \mathfrak{v}$.

Given $\lambda \in \mathfrak{t}^*$, the coadjoint orbit $\mathcal{O}(\lambda) := \mathrm{Ad}^*(N)\lambda$ is just $\mathrm{Ad}(M_1)\lambda_1 \times \cdots \times \mathrm{Ad}(M_m)\lambda_m$, and we have the measure $d\nu_\lambda = d\nu_{\lambda_1} \times \cdots \times d\nu_{\lambda_m}$ on it. Denote $c = c_1c_2\dots c_m = 2^{d_1+\cdots+d_m}d_1!d_2!\dots d_m!$ where we recall $d_r = \frac{1}{2}(\dim\mathfrak{m}_r - 1) = \ell - 2r$. Let $f \in C_c^\infty(N)$ (or more generally $f \in \mathcal{C}(N)$) and $\widehat{f_1}$ the classical Fourier transform of the lift $f_1(\xi) = f(\exp(\xi))$ of f to $\mathfrak{n}$. We use Lebesgue measure $d\nu$ on $(\mathfrak{m}/\mathfrak{s})^*$ normalized so that Fourier transform is an isometry of $L^2(\mathfrak{m}/\mathfrak{s})$ onto $L^2(\mathfrak{m}/\mathfrak{s})^*$.

Now, exactly as in (4.2) and (4.3) we combine the result [9, Theorem, p. 17] of Pukánszky with the method of [8, proof of Theorem 6] to obtain.

Theorem 4.3 *Let N be the group of real strictly triangular $\ell \times \ell$ matrices, $\mathfrak{m}_r$ and $\mathfrak{n}_r$ the algebras of Sect. 2, and M_r and N_r the corresponding analytic subgroups of N. Let $\lambda = \lambda_1 + \cdots + \lambda_m \in \mathfrak{t}^*$, and $P(\lambda) = \lambda_1^{\ell-2}\lambda_2^{\ell-4}\ldots\lambda_m^{\ell-2m}$, as in (4.4). Then $\pi_\lambda \in \widehat{N}$ has distribution character*

$$c_{\pi_\lambda}(f) = \operatorname{trace} \dot{\pi}_\lambda(f) = \frac{1}{c}\frac{1}{|P(\lambda)|}\int\limits_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi)d\nu(\xi)$$

and N has Plancherel formula

$$f(x) = c\int\limits_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|P(\lambda)|d\lambda.$$

5 General theory

Here's what we need to extend our considerations beyond the group of upper triangular matrices. The connected simply connected nilpotent Lie group should decompose as

$N = M_1 M_2 \ldots M_{m-1} M_m$ where

(a) each factor M_r has unitary representations with coefficients in $L^2(M_r/Z_r)$,

(b) each $N_r := M_1 M_2 \ldots M_r$ is a normal subgroup of N with

$N_r = N_{r-1} \rtimes M_r$ semidirect,

(c) decompose $\mathfrak{m}_r = \mathfrak{z}_r + \mathfrak{v}_r$ and $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$ as vector direct sums where

$\mathfrak{s} = \oplus\, \mathfrak{z}_r$ and $\quad \mathfrak{v} = \oplus\, \mathfrak{v}_r$; then $[\mathfrak{m}_r, \mathfrak{z}_s] = 0$ and $[\mathfrak{m}_r, \mathfrak{m}_s] \subset \mathfrak{v}$ for $r > s$. (5.1)

In order to follow the arguments leading to Theorem 4.3, we denote

(a) $d_r = \frac{1}{2}\dim(\mathfrak{m}_r/\mathfrak{z}_r)$ so $\frac{1}{2}\dim(\mathfrak{n}/\mathfrak{s}) = d_1 + \cdots + d_m$, and $c = 2^{d_1+\cdots+d_m} d_1! d_2! \ldots d_m!$

(b) $b_{\lambda_r} : (x, y) \mapsto \lambda([x, y])$ viewed as a bilinear form on $\mathfrak{m}_r/\mathfrak{z}_r$

(c) $S = Z_1 Z_2 \ldots Z_m = Z_1 \times \cdots \times Z_m$ where Z_r is the center of M_r

(d) P : polynomial $P(\lambda) = \mathrm{Pf}\,(b_{\lambda_1})\mathrm{Pf}\,(b_{\lambda_2}) \ldots \mathrm{Pf}\,(b_{\lambda_m})$ on $\mathfrak{s}^*$ (5.2)

(e) $\mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid P(\lambda) \neq 0\}$

(f) $\pi_\lambda \in \widehat{N}$ where $\lambda \in \mathfrak{t}^*$: irreducible unitary rep. of $N = M_1 M_2 \ldots M_m$ as in Section 4

Proposition 4.1 extends immediately to this setting: Plancherel measure is concentrated on the set $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. It is slightly more delicate to extend Lemma 4.2, but (5.1)(c) does the job.

Theorem 5.1 *Let N be a connected simply connected nilpotent Lie group that satisfies* (5.1). *Then Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ, then the coefficient $f_{u,v}(x) = \langle u, \pi_\lambda(x)v\rangle$ satisfies*

$$||f_{u,v}||^2_{L^2(N/S)} = \frac{||u||^2||v||^2}{|P(\lambda)|}. \tag{5.3}$$

The distribution character Θ_{π_λ} of π_λ satisfies

$$\Theta_{\pi_\lambda}(f) = c^{-1}|P(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi)d\nu_\lambda(\xi) \quad \text{for } f \in \mathcal{C}(N) \tag{5.4}$$

where $\mathcal{C}(N)$ is the Schwartz space, f_1 is the lift $f_1(\xi) = f(\exp(\xi))$, $\widehat{f_1}$ is its classical Fourier transform, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$, and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to $\mathrm{Ad}^*(N)\lambda$. The Plancherel formula on N is

$$f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|P(\lambda)|d\lambda \quad \text{for } f \in \mathcal{C}(N). \tag{5.5}$$

Definition 5.2 The representations π_λ of [5.2(f)] are the *stepwise square integrable* representations of N relative to (5.1).

6 Iwasawa decompositions

Let G be a real reductive Lie group. We now carry out the program of Sect. 5 for the groups N of Iwasawa decompositions $G = KAN$. Let $m = \mathrm{rank}_{\mathbb{R}} G = \dim_{\mathbb{R}} A$ and notice that we've done the case $G = SL(m+1;\mathbb{R})$. The idea is to use the Kostant cascade construction of strongly orthogonal roots: β_1 is the maximal root, β_{r+1} is a maximum among the positive roots orthogonal to $\{\beta_1, \ldots, \beta_r\}$, etc.

We fix an Iwasawa decomposition $G = KAN$. As usual, write $\mathfrak{k}$ for the Lie algebra of K, $\mathfrak{a}$ for the Lie algebra of A, and $\mathfrak{n}$ for the Lie algebra of N. Complete $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Now we have root systems

- $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$: roots of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}_{\mathbb{C}}$ (ordinary roots),
- $\Delta(\mathfrak{g}, \mathfrak{a})$: roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ (restricted roots),
- $\Delta_0(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid 2\alpha \notin \Delta(\mathfrak{g}, \mathfrak{a})\}$ (nonmultipliable restricted roots).

Here $\Delta(\mathfrak{g}, \mathfrak{a}) = \{\gamma|_{\mathfrak{a}} \mid \gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ and } \gamma|_{\mathfrak{a}} \neq 0\}$. Further, $\Delta(\mathfrak{g}, \mathfrak{a})$ and $\Delta_0(\mathfrak{g}, \mathfrak{a})$ are root systems in the usual sense. Any positive root system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \subset \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ defines positive systems

- $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\gamma|_{\mathfrak{a}} \mid \gamma \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ and } \gamma|_{\mathfrak{a}} \neq 0\}$
  and $\Delta_0^+(\mathfrak{g}, \mathfrak{a}) = \Delta_0(\mathfrak{g}, \mathfrak{a}) \cap \Delta^+(\mathfrak{g}, \mathfrak{a})$.

We can (and do) choose $\Delta^+(\mathfrak{g}, \mathfrak{h})$ so that

- $\mathfrak{n}$ is the sum of the positive restricted root spaces and
- if $\gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and $\gamma|_{\mathfrak{a}} \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ then $\gamma \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

Two roots are called *strongly orthogonal* if their sum and their difference are not roots. Then they are orthogonal. We define

$$\begin{aligned}&\beta_1 \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximal positive restricted root and}\\&\beta_{r+1} \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximum among the roots of } \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ orthogonal}\\&\text{to all } \beta_i \text{ with } i \leqq r\end{aligned} \tag{6.1}$$

Then the β_r are mutually strongly orthogonal. This is Kostant's cascade construction. Note that each $\beta_r \in \Delta_0^+(\mathfrak{g}, \mathfrak{a})$. Also note that β_1 is unique if and only if $\Delta(\mathfrak{g}, \mathfrak{a})$ is irreducible.

For $1 \leqq r \leqq m$ define

$$\begin{aligned}&\Delta_1^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \beta_1 - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\} \text{ and}\\&\Delta_{r+1}^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \setminus (\Delta_1^+ \cup \cdots \cup \Delta_r^+) \mid \beta_{r+1} - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\}.\end{aligned} \tag{6.2}$$

Lemma 6.1 *If $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ then either $\alpha \in \{\beta_1, \ldots, \beta_m\}$ or α belongs to exactly one of the sets Δ_r^+.*

Proof Suppose that $\alpha \notin \{\beta_1, \ldots, \beta_m\}$ and that $\alpha \notin \Delta_r^+$ for any r. As $\alpha \notin \Delta_1^+$ it is strongly orthogonal to β_1. Then as $\alpha \notin \Delta_2^+$ it is strongly orthogonal to β_2 as well. Continuing, α is strongly orthogonal to each of the β_r, contradicting maximality of $\{\beta_1, \ldots, \beta_m\}$.

Lemma 6.2 *The set $\Delta_r^+ \cup \{\beta_r\} = \{\alpha \in \Delta^+ \mid \alpha \perp \beta_i$ for $i < r$ and $\langle \alpha, \beta_r \rangle > 0\}$. In particular, $[\mathfrak{m}_r, \mathfrak{m}_s] \subset \mathfrak{m}_t$ where $t = \min\{r, s\}$.*

Proof Let $\alpha \in \Delta^+$ such that (i) $\alpha \perp \beta_i$ for $i < r$ and (ii) $\langle \alpha, \beta_r \rangle > 0$. Here (ii) shows that $\beta_r - \alpha$ is a root. If it's negative then $\alpha > \beta_r$, contradicting maximality of β_r for the property of being orthogonal to β_i for every $i < r$. So $\beta_r - \alpha \in \Delta^+$. Let s be the smallest integer such that $\alpha \in \Delta_s^+$. This relies on Lemma 6.1. As argued a moment ago, $\beta_s + \alpha$ is not a root. If $s < r$ then (i) says that α is strongly orthogonal to β_s, contradicting $\alpha \in \Delta_s^+$. Thus $r = s$ and $\alpha \in \Delta_r^+$.

Conversely we want to show that $\alpha \in \Delta_r^+$ implies $\alpha \perp \beta_i$ for $i < r$ and $\langle \alpha, \beta_r \rangle > 0$. This is clear for $r = 1$. We assume it for $r < t$, for a fixed $t \leqq m$, and prove it for $r = t$. Let $\alpha \in \Delta_t^+$. If $\alpha \not\perp \beta_r$ where $r < t$, and $\alpha + \beta_r$ is a root, then $\alpha + \beta_r \in \Delta_s^+$ where $s < r$, and $\langle \alpha + \beta_r, \beta_s \rangle > 0$. That is impossible because $\alpha \perp \beta_s \perp \beta_r$. If $\alpha \not\perp \beta_r$ now $\beta_r - \alpha$ is a root. It is positive by the maximality property of β_r, so $\alpha \in \Delta_r^+$, contradicting $\alpha \in \Delta_t^+$ with $r < t$. Thus $\alpha \perp \beta_r$ for all $r < t$. As argued before, $\alpha + \beta_t$ is not a root. Since $\beta_t - \alpha \in \Delta^+$ now $\langle \alpha, \beta_t \rangle > 0$. That completes the induction.

Finally, let $\alpha \in \Delta_r^+ \cup \{\beta_r\}$, $\gamma \in \Delta_s^+ \cup \{\beta_s\}$, and $t = \min\{r, s\}$. Suppose that $\alpha + \gamma$ is a root. If $i < t$ then $\langle \alpha + \gamma, \beta_i \rangle = \langle \alpha, \beta_i \rangle + \langle \gamma, \beta_i \rangle = 0$, and $\langle \alpha + \gamma, \beta_t \rangle > 0$ because at least one of $\langle \alpha, \beta_t \rangle$ and $\langle \gamma, \beta_t \rangle$ is positive.

Lemma 6.1 shows that the Lie algebra $\mathfrak{n}$ of N is the vector space direct sum of its subspaces

$$\mathfrak{m}_r = \mathfrak{g}_{\beta_r} + \sum\nolimits_{\Delta_r^+} \mathfrak{g}_\alpha \quad \text{for } 1 \leqq r \leqq m \tag{6.3}$$

and Lemma 6.2 shows that $\mathfrak{n}$ has an increasing foliation by ideals

$$\mathfrak{n}_r = \mathfrak{m}_1 + \mathfrak{m}_2 + \cdots + \mathfrak{m}_r \quad \text{for } 1 \leqq r \leqq m. \tag{6.4}$$

Now we will see that the corresponding group level decomposition $N = M_1 M_2 \ldots M_m$ and the semidirect product decompositions $N_r = N_{r-1} \rtimes M_r$ satisfy all the requirements of (5.1).

The structure of Δ_r^+, and later of $\mathfrak{m}_r$, is exhibited by a particular Weyl group element of $\Delta(\mathfrak{g}, \mathfrak{a})$. Denote

$$s_{\beta_r} \text{ is the Weyl group reflection in } \beta_r \text{ and } \sigma_r : \Delta(\mathfrak{g}, \mathfrak{a}) \to \Delta(\mathfrak{g}, \mathfrak{a}) \text{ by } \sigma_r(\alpha) = -s_{\beta_r}(\alpha). \tag{6.5}$$

Note that $\sigma_r(\beta_s) = -\beta_s$ for $s \neq r$, $+\beta_s$ if $s = r$. If $\alpha \in \Delta_r^+$ we still have $\sigma_r(\alpha) \perp \beta_i$ for $i < r$ and $\langle \sigma_r(\alpha), \beta_r \rangle > 0$. If $\sigma_r(\alpha)$ is negative then $\beta_r - \sigma_r(\alpha) > \beta_r$ contradicting the maximality property of β_r. Thus, using Lemma 6.2, $\sigma_r(\Delta_r^+) = \Delta_r^+$.

Lemma 6.3 *If $\alpha \in \Delta_r^+$ then $\alpha + \sigma_r(\alpha) = \beta_r$. (Of course it is possible that $\alpha = \sigma_r(\alpha) = \frac{1}{2}\beta_r$ when $\frac{1}{2}\beta_r$ is a root.). If $\alpha, \alpha' \in \Delta_r^+$ and $\alpha + \alpha' \in \Delta(\mathfrak{g}, \mathfrak{a})$ then $\alpha + \alpha' = \beta_r$.*

Proof If $\alpha \in \Delta_r^+$ with $\sigma_r(\alpha) = \alpha$ then $s_{\beta_r}(\alpha) = -\alpha$ so α is proportional to β_r. As β_r is nonmultipliable and $\langle \alpha, \beta_r \rangle > 0$ that forces $\alpha = \frac{1}{2}\beta_r$. In particular $\alpha + \sigma_r(\alpha) = \beta_r$.

Now suppose $\alpha \in \Delta_r^+$ with $\sigma_r(\alpha) \neq \alpha$. Then $\alpha + \sigma_r(\alpha) = \alpha - s_{\beta_r}(\alpha) = \alpha - (\alpha - \frac{2\langle \alpha, \beta_r \rangle}{\langle \beta_r, \beta_r \rangle}\beta_r) = \frac{2\langle \alpha, \beta_r \rangle}{\langle \beta_r, \beta_r \rangle}\beta_r$. As $\langle \alpha, \beta_r \rangle > 0$ and β_r is nonmultipliable this forces $\alpha + \sigma_r(\alpha) = \beta_r$.

Suppose that there exist $\alpha, \alpha' \in \Delta_r^+$ such that $\alpha + \alpha' = \alpha'' \in \Delta(\mathfrak{g}, \mathfrak{a})$ but $\alpha'' \neq \beta_r$. Fix such a pair $\{\alpha, \alpha'\}$ with α maximal for that property. Then α'' lacks that property. So $\beta_r = \alpha'' + \sigma_r(\alpha'') = (\alpha + \alpha') + \sigma_r(\alpha + \alpha') = (\alpha + \sigma_r(\alpha)) + (\alpha' + \sigma_r(\alpha')) = 2\beta_r$. Thus the specified α cannot exist.

Now we are in a position to start the proof of the main technical result of this section—that the M_r have square integrable representations. For that it suffices to consider the case where $\mathfrak{g}$ is simple as a real Lie algebra and run through some possibilities:

Lemma 6.4 *Let $\mathfrak{n}$ be a nilpotent Lie algebra, $\mathfrak{z}$ its center, and $\mathfrak{v}$ a vector space complement to $\mathfrak{z}$ in $\mathfrak{n}$. Suppose that we have vector space direct sum decompositions $\mathfrak{v} = \mathfrak{u} + \mathfrak{u}'$, $\mathfrak{u} = \sum \mathfrak{u}_a$ and $\mathfrak{u}' = \sum \mathfrak{u}'_a$, and $\mathfrak{z} = \sum \mathfrak{z}_b$ with $\dim \mathfrak{z}_b = 1$. Suppose further that* (i) *each $[\mathfrak{u}_a, \mathfrak{u}_a] = 0 = [\mathfrak{u}'_a, \mathfrak{u}'_a]$,* (ii) *if $a_1 \neq a_2$ then $[\mathfrak{u}_{a_1}, \mathfrak{u}'_{a_2}] = 0$ and* (iii) *for each index a there is a unique b_a such that $\mathfrak{u}_a \otimes \mathfrak{u}'_a \to \mathfrak{z}_{b_a}$, by $u \otimes u' \mapsto [u, u']$, is a nondegenerate pairing. Then $\mathfrak{n}$ is a direct sum of Heisenberg algebras $\mathfrak{z}_{b_a} + \mathfrak{u}_a + \mathfrak{u}'_a$ and the commutative algebra that is the sum of the remaining $\mathfrak{z}_b$.*

Lemma 6.5 *If $\mathfrak{g}$ is the split real form of $\mathfrak{g}_{\mathbb{C}}$ then each M_r has square integrable representations.*

Lemma 6.6 *If $\mathfrak{g}$ is simple but not absolutely simple then each M_r has square integrable representations.*

Lemma 6.7 *If G is the quaternion special linear group $SL(n;\mathbb{H})$ then M_1 has square integrable representations.*

Lemma 6.8 *If G is the group E_{6,F_4} of collineations of the Cayley projective plane then M_1 has square integrable representations.*

Lemma 6.9 *The group M_1 has square integrable representations.*

Lemma 6.10 *If $\mathfrak{g}$ is absolutely simple then each M_r has square integrable representations.*

Proof (Lemma 6.4.) The assertion is obvious.

Proof (Lemma 6.5.) This is the case where $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ consists of the $\mathbb{C}$-linear extensions of the roots in $\Delta(\mathfrak{g}, \mathfrak{a})$. All roots are indivisible, so Lemma 6.3 divides Δ_r^+ into two disjoint subsets, thus divides $\sum_{\alpha\in\Delta_r^+} \mathfrak{g}_\alpha$ as a direct sum $\mathfrak{u} \oplus \mathfrak{u}'$ of two subspaces, such that those subspaces satisfy the conditions of Lemma 6.4. As $\mathfrak{m}_r$ has 1-dimensional center $\mathfrak{g}_{\beta_r}$ it follows that $\mathfrak{m}_r$ is a Heisenberg algebra. Now M_r is a Heisenberg group and thus has square integrable representations.

Proof (Lemma 6.6.) This is the case where $\mathfrak{g}$ is the underlying real structure of a complex simple Lie algebra $\mathfrak{s}$. The Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ of $\mathfrak{s}$ is given by $\mathfrak{t} = \sqrt{-1}\mathfrak{a}$, and $\mathfrak{a}$ is the (real) subspace on which the roots take real values. As a real Lie algebra, $\mathfrak{n} \cong \sum_{\alpha\in\Delta^+(\mathfrak{s},\mathfrak{a})} \mathfrak{s}_\alpha$.

Let $\mathfrak{g}'$ denote the split real form of $\mathfrak{s}$. In the Iwasawa decomposition $G' = K'A'N'$ now $\mathfrak{a}' = \mathfrak{a}$, $\mathfrak{n}'$ is a real form of $\mathfrak{n}$, and for each r the algebra $\mathfrak{m}_r' := \mathfrak{m}_r \cap \mathfrak{n}'$ is a real form of $\mathfrak{m}_r$. From the latter, [11, Theorem 2.1] says that the corresponding group M_r' has square integrable representations if and only if its complexification M_r has square integrable representations. However, Lemma 6.5 says that M_r' has square integrable representations. Our assertion follows.

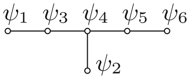
Proof (Lemma 6.7.) This is the case where $\mathfrak{g} = \mathfrak{sl}(n;\mathbb{H})$. In the usual root ordering, $\widetilde{\beta_1} = \psi_1 + \cdots + \psi_{2n-1}$ and $\psi_i|_\mathfrak{a} = 0$ just when i is odd. Thus the ordinary roots that restrict to β_1 are $\psi_1 + \cdots + \psi_{2n-1}$, $\psi_2 + \cdots + \psi_{2n-1}$, $\psi_1 + \cdots + \psi_{2n-2}$ and $\psi_2 + \cdots + \psi_{2n-2}$; their root spaces span the center $\mathfrak{z}_1$ of $\mathfrak{m}_1$. Further Δ_1^+ consists of the restrictions of pairs of roots that

- sum to $\psi_1+\cdots+\psi_{2n-1}$: $\{\psi_1+\cdots+\psi_j\,,\,\psi_{j+1}+\cdots+\psi_{2n-1}\}$, $1 \leqq j < 2n-1$,
- sum to $\psi_2+\cdots+\psi_{2n-1}$: $\{\psi_2+\cdots+\psi_j\,,\,\psi_{j+1}+\cdots+\psi_{2n-1}\}$, $2 \leqq j < 2n-1$,
- sum to $\psi_1+\cdots+\psi_{2n-2}$: $\{\psi_1+\cdots+\psi_j\,,\,\psi_{j+1}+\cdots+\psi_{2n-2}\}$, $1 \leqq j < 2n-2$,
- sum to $\psi_2+\cdots+\psi_{2n-2}$: $\{\psi_2+\cdots+\psi_j\,,\,\psi_{j+1}+\cdots+\psi_{2n-2}\}$. $2 \leqq j < 2n-2$.

Their root spaces span a complement $\mathfrak{v}_1$ to $\mathfrak{z}_1$ in $\mathfrak{m}_1$. Eliminating duplicates, the set of ordinary roots that restrict to elements of Δ_1^+ is $\{\psi_1 + \cdots + \psi_j\,;\,\psi_{j+1} + \cdots +$

ψ_{2n-1} ; $\psi_2 + \cdots + \psi_j$; $\psi_{j+1} + \cdots + \psi_{2n-2}\}$. Now let $\lambda \in \mathfrak{z}_1^*$ be zero on the root spaces for $\psi_2 + \cdots + \psi_{2n-1}$ and $\psi_1 + \cdots + \psi_{2n-2}$, nonzero on the root spaces for $\psi_1 + \cdots + \psi_{2n-1}$ and $\psi_2 + \cdots + \psi_{2n-2}$. Then the corresponding antisymmetric bilinear form b_λ on $\mathfrak{v}_1$ is nonsingular. Thus M_1 has square integrable (modulo its center) representations.

Proof (Lemma 6.8.) This is the case where $\mathfrak{g} = \mathfrak{e}_{6,F_4}$. Then $\text{rank}_{\mathbb{R}}\mathfrak{g} = 2$. In the Bourbaki order for the simple roots



$\widetilde{\beta}_1 = \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$, and the roots that restrict to 0 on $\mathfrak{a}$ are ψ_2, ψ_3, ψ_4 and ψ_5. So Δ_1^+ consists of the restrictions of pairs of roots that

– sum to $\psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$:

$$\begin{aligned}
&\{\{\psi_2,\ \psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6\},\\
&\ \{\psi_2 + \psi_4,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\},\\
&\ \{\psi_2 + \psi_3 + \psi_4,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\},\\
&\ \{\psi_2 + \psi_4 + \psi_5,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\ \{\psi_1 + \psi_2 + \psi_3 + \psi_4,\ \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\},\\
&\ \{\psi_2 + \psi_3 + \psi_4 + \psi_5,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\ \{\psi_2 + \psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5\},\\
&\ \{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5,\ \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\ \{\psi_2 + \psi_3 + 2\psi_4 + \psi_5,\ \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\ \{\psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5\}\}
\end{aligned}$$

– sum to $\psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$:

$$\begin{aligned}
&\{\{\psi_4,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\},\\
&\ \{\psi_3 + \psi_4,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\},\\
&\ \{\psi_4 + \psi_5,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\ \{\psi_1 + \psi_3 + \psi_4,\ \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\},\\
&\ \{\psi_3 + \psi_4 + \psi_5,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\ \{\psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5\},\\
&\ \{\psi_1 + \psi_3 + \psi_4 + \psi_5,\ \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\ \{\psi_2 + \psi_3 + 2\psi_4 + \psi_5,\ \psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\ \{\psi_3 + \psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5\}\}
\end{aligned}$$

– sum to $\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6$:

$$\begin{aligned}
&\{\{\psi_3,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\},\\
&\{\psi_5,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_1 + \psi_3,\ \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\},\\
&\{\psi_3 + \psi_4 + \psi_5,\ \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_5 + \psi_6,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5\},\\
&\{\psi_1 + \psi_3 + \psi_4 + \psi_5,\ \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_2 + \psi_3 + \psi_4 + \psi_5,\ \psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_3 + \psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5\}\}
\end{aligned}$$

– sum to $\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6$:

$$\begin{aligned}
&\{\{\psi_5,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_1,\ \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6\},\\
&\{\psi_4 + \psi_5,\ \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_5 + \psi_6,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5\},\\
&\{\psi_1 + \psi_3 + \psi_4 + \psi_5,\ \psi_2 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_2 + \psi_4 + \psi_5,\ \psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5\}\}
\end{aligned}$$

– sum to $\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6$:

$$\begin{aligned}
&\{\{\psi_3,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_1 + \psi_3,\ \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_3 + \psi_4,\ \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_6,\ \psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5\},\\
&\{\psi_1 + \psi_3 + \psi_4,\ \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_2 + \psi_3 + \psi_4,\ \psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_3 + \psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + \psi_3 + \psi_4\}\}
\end{aligned}$$

– sum to $\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6$:

$$\begin{aligned}
&\{\{\psi_1,\ \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_4,\ \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_6,\ \psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5\},\\
&\{\psi_1 + \psi_3 + \psi_4,\ \psi_2 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_2 + \psi_4,\ \psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\
&\{\psi_4 + \psi_5 + \psi_6,\ \psi_1 + \psi_2 + \psi_3 + \psi_4\}\}
\end{aligned}$$

– sum to $\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6$:

$$\begin{gathered}\{\{\psi_1,\ \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\ \{\psi_6,\ \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5\},\\ \{\psi_1 + \psi_3,\ \psi_2 + \psi_4 + \psi_5 + \psi_6\},\\ \{\psi_2,\ \psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\ \{\psi_5 + \psi_6,\ \psi_1 + \psi_2 + \psi_3 + \psi_4\}\}\end{gathered}$$

– sum to $\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6$:

$$\begin{gathered}\{\{\psi_1,\ \psi_3 + \psi_4 + \psi_5 + \psi_6\},\\ \{\psi_1 + \psi_3,\ \psi_4 + \psi_5 + \psi_6\},\\ \{\psi_1 + \psi_3 + \psi_4,\ \psi_5 + \psi_6\},\\ \{\psi_1 + \psi_3 + \psi_4 + \psi_5,\ \psi_6\}\}\end{gathered}$$

Eliminating duplicates, the set of ordinary roots that restrict to elements of Δ_1^+ consists of

- The 20 positive roots listed above in the first group, summing to $\psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$. These are the roots $\sum a_i\psi_i$ for which $a_2 = 1$ and (a_1, a_6) is either $(1, 0)$ or $(0, 1)$. We denote the sum of their root spaces by $\mathfrak{v}_1'$.
- The 8 positive roots listed above in the last group, summing to $\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6$. These are the roots $\sum a_i\psi_i$ for which $a_2 = 0$ and (a_1, a_6) is either $(1, 0)$ or $(0, 1)$. We denote the sum of their root spaces by $\mathfrak{v}_1''$.

Now the space $\mathfrak{v}_1 := \mathfrak{v}_1' + \mathfrak{v}_1''$ is a vector space complement to $\mathfrak{z}_1$ in $\mathfrak{m}_1$. Let $\lambda \in \mathfrak{z}_1^*$ be zero on the root spaces for $\psi_1 + \psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$, $\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + 2\psi_5 + \psi_6$, $\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + \psi_6$, $\psi_1 + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5 + \psi_6$, $\psi_1 + \psi_2 + \psi_3 + 2\psi_4 + \psi_5 + \psi_6$ and $\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6$, and nonzero on the root spaces for $\psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6$ and $\psi_1 + \psi_3 + \psi_4 + \psi_5 + \psi_6$. Then the corresponding antisymmetric bilinear form b_λ on $\mathfrak{v}_1$ is nonsingular, so M_1 has square integrable (modulo its center) representations.

Proof (Lemma 6.9.) It suffices to consider the case where $\mathfrak{g}_\mathbb{C}$ is a simple complex Lie algebra, but $\mathfrak{g}$ need not be its split real form. We do, however, assume that it is not the compact real form, for in that case $N = \{1\}$.

Suppose first that $\dim \mathfrak{g}_{\beta_1} = 1$. In other words the highest root in $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, call it $\widetilde{\beta}_1$, is the only root of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ that restricts to β_1. Applying Lemma 6.3 as in the proof of Lemma 6.5 it follows that M_1 has square integrable representations.

Now suppose that $\dim \mathfrak{g}_{\beta_1} > 1$. Note that the roots in $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ that restrict to β_1 are just the roots of the form $\widetilde{\beta}_1 - \sum t_i\gamma_i$ where every one of the $\gamma_i|_\mathfrak{a} = 0$. In particular the root(s) of the extended Dynkin diagram of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, to which $-\widetilde{\beta}_1$ attaches, have restriction 0 on $\mathfrak{a}$. We have already dealt with the cases $\mathfrak{g} = \mathfrak{sl}(n; \mathbb{H})$ and $\mathfrak{g} = \mathfrak{e}_{6,F_4}$, so there remain only a few easy cases:

Case $\mathfrak{g} = \mathfrak{so}(1, n)$. Then $\text{rank}_\mathbb{R}\mathfrak{g} = 1$, $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\widetilde{\beta}_1\}$ and $M_1 = N$ is abelian. In particular M_1 has square integrable (modulo its center) representations.

Case $\mathfrak{g} = \mathfrak{su}(1, n)$. Then $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} = 1$, $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\widetilde{\beta}_1, \frac{1}{2}\widetilde{\beta}_1\}$, and $M_1 = N$ is a Heisenberg group $\operatorname{Im}\mathbb{C} + \mathbb{C}^{n-1}$. In particular M_1 has square integrable representations.

Case $\mathfrak{g} = \mathfrak{sp}(p, q)$, $p \leqq q$. Then $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} = p$, $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\widetilde{\beta}_1, \frac{1}{2}\widetilde{\beta}_1\}$, and $M_1 = N$ is a quaternionic Heisenberg group $\operatorname{Im}\mathbb{H} + \mathbb{H}^s$. In particular M_1 has square integrable (modulo its center) representations.

Case $\mathfrak{g} = \mathfrak{f}_{4,B_4}$. Then $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} = 1$, $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\widetilde{\beta}_1, \frac{1}{2}\widetilde{\beta}_1\}$, and $M_1 = N$ is an octonionic Heisenberg group $\operatorname{Im}\mathbb{O} + \mathbb{O}$. In particular M_1 has square integrable (modulo its center) representations.

Proof (Lemma 6.10.) This is the case where $\mathfrak{g}_{\mathbb{C}}$ is a simple complex Lie algebra, but $\mathfrak{g}$ need not be its split real form. We do, however, assume that it is not the compact real form, for in that case $N = \{1\}$. Then $\beta_1(\mathfrak{t}) = 0$. Note that β_1 is the restriction to $\mathfrak{a}$ of the highest root $\widetilde{\beta}_1$ in $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and that $\widetilde{\beta}_1$ is a long root. Thus $\widetilde{\beta}_1$ is the only root in $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ that restricts of β_1. Applying Lemma 6.3 as in the proof of Lemma 6.5 it follows that M_1 has square integrable representations. This starts the induction.

Suppose we know that $M_1, \ldots, M_r$ have square integrable representations and that $r < m$. Let $\mathfrak{g}_r$ be the semisimple subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and all (restricted) root spaces for simple roots that are orthogonal to $\beta_1, \beta_2, \ldots, \beta_r$. Then β_{r+1} is a maximum among the positive restricted roots of $\mathfrak{g}_r$ and $\mathfrak{m}_{r+1}$ is the subalgebra of $\mathfrak{g}_r$ that is the counterpart of $\mathfrak{m}_1$ for $\mathfrak{g}$. Thus by the argument just above for M_1, and by Lemmas 6.5 and 6.6 as needed for the simple summands of $\mathfrak{g}_r$, we conclude that M_{r+1} has square integrable representations.

We now apply Lemmas 6.6 and 6.10 to the list (5.1) of conditions for setting up the character formula and Plancherel formula as in Theorem 5.1. Those Lemmas supply the key condition, that each M_r has unitary representations with coefficients in $L^2(M_r/Z_r)$. Lemma 6.2 ensures that each $N_r := M_1 M_2 \ldots M_r$ is a normal subgroup of N with $N_r = N_{r-1} \rtimes M_r$ semidirect product, and then Lemma 6.2 says that $N = M_1 M_2 \ldots M_{m-1} M_m$ as needed. The decompositions $\mathfrak{m}_r = \mathfrak{z}_r + \mathfrak{v}_r$ and $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$ now are immediate from the construction of the $\mathfrak{m}_r$. It remains only to verify that $[\mathfrak{m}_r, \mathfrak{z}_s] = 0$ and $[\mathfrak{m}_r, \mathfrak{m}_s] \subset \mathfrak{v}$ for $r > s$.

Let $\alpha \in \Delta_r^+$ and $s < r$. Then $\alpha \notin (\Delta_1^+ \cup \cdots \cup \Delta_s^+)$ and $\alpha \perp \beta_i$ for $i \leqq s$. Now $\beta_s + \alpha \in \Delta^+$ would imply $\beta_s - \alpha \in \Delta^+$, contradicting $\alpha \notin \Delta_s^+$. It follows that $[\mathfrak{m}_r, \mathfrak{z}_s] = 0$.

Let $\alpha \in \Delta_r^+$ and $\alpha' \in \Delta_s^+$, $s < r$, with $\alpha + \alpha' = \beta_t$. Lemma 6.2 says $s = t$ so $\beta_s - \alpha' = \alpha$. But then $\beta_s - \alpha' \in \Delta_s^+$ contradicting $\alpha \notin (\Delta_1^+ \cup \cdots \cup \Delta_s^+)$. We conclude that $[\mathfrak{m}_r, \mathfrak{m}_s] \subset \mathfrak{v}$ for $r > s$.

Summarizing, we have just shown that Theorem 5.1 applies to milradicals of minimal parabolic subgroups. In other words,

Theorem 6.11 *Let G be a real reductive Lie group, $G = KAN$ an Iwasawa decomposition, $\mathfrak{m}_r$ and $\mathfrak{n}_r$ the subalgebras of $\mathfrak{n}$ defined in* (6.3) *and* (6.4)*, and M_r and N_r the corresponding analytic subgroups of N. Then the M_r and N_r satisfy* (5.1)*. In particular, Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ, then the coefficient $f_{u,v}(x) = \langle u, \pi_\lambda(x)v\rangle$ satisfies*

$$||f_{u,v}||^2_{L^2(N/S)} = \frac{||u||^2||v||^2}{|P(\lambda)|}. \tag{6.6}$$

The distribution character Θ_{π_λ} of π_λ satisfies

$$\Theta_{\pi_\lambda}(f) = c^{-1}|P(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi) d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N) \tag{6.7}$$

where $\mathcal{C}(N)$ is the Schwartz space, f_1 is the lift $f_1(\xi) = f(\exp(\xi))$, $\widehat{f_1}$ is its classical Fourier transform, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$, and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to $\mathrm{Ad}^*(N)\lambda$. The Plancherel formula on N is

$$f(x) = c \int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|P(\lambda)| d\lambda \text{ for } f \in \mathcal{C}(N). \tag{6.8}$$

7 Arithmetic quotients

In this section we consider the case where our connected simply connected nilpotent Lie group N has a discrete co-compact subgroup Γ that fits into a decomposition of the form (5.1). We show that the compact nilmanifold N/Γ has a corresponding foliation and derive analytic results analogous to those of Theorem 5.1. These results include multiplicity formulae for the regular representation of N on $L^2(N/\Gamma)$. They apply in particular to the nilradicals of minimal parabolic subgroups, as studied in Sect. 6.

Here are some basic facts about discrete uniform (i.e. co-compact) subgroups of connected simply connected nilpotent Lie groups, mostly due to Malčev. See [10, Chapter 2] for an exposition.

Proposition 7.1 *Let N be a connected simply connected nilpotent Lie group. Then the following are equivalent.*

- *N has a discrete subgroup Γ with N/Γ compact.*
- *$N \cong N_{\mathbb{R}}$ where $N_{\mathbb{R}}$ is the group of real points in a unipotent linear algebraic group defined over the rational number field $\mathbb{Q}$*
- *$\mathfrak{n}$ has a basis $\{\xi_j\}$ for which the coefficients $c^k_{i,j}$ in $[\xi_i, \xi_j] = \sum c^k_{i,j}\xi_k$ are rational numbers.*

Under those conditions let $\mathfrak{n}_{\mathbb{Q}}$ denote the rational span of $\{\xi_j\}$ and let $\mathfrak{n}_{\mathbb{Z}}$ be the integral span. Then $\exp(\mathfrak{n}_{\mathbb{Z}})$ generates a discrete subgroup $N_{\mathbb{Z}}$ of $N = N_{\mathbb{R}}$ and $N_{\mathbb{R}}/N_{\mathbb{Z}}$ is compact. Conversely, if Γ is a discrete co-compact subgroup of N then the $\mathbb{Z}$–span of $\exp^{-1}(\Gamma)$ is a lattice in $\mathfrak{n}$ for which any generating set $\{\xi_j\}$ is a basis of $\mathfrak{n}$ such that the coefficients $c^k_{i,j}$ in $[\xi_i, \xi_j] = \sum c^k_{i,j}\xi_k$ are rational numbers.

Note that the conditions of Proposition 7.1 hold for the nilpotent groups studied in Sect. 6, where in fact one can choose the basis $\{\xi_j\}$ of $\mathfrak{n}$ so that the $c^k_{i,j}$ are integers.

Here are the basic facts on square integrable representations in this setting, from [8, Theorem 7]:

Proposition 7.2 *Let N be a connected simply connected nilpotent Lie group that has square integrable representations, and let Γ a discrete co-compact subgroup. Let Z be the center of N and normalize the volume form on $\mathfrak{n}/\mathfrak{z}$ by normalizing Haar measure on N so that $N/Z\Gamma$ has volume* 1. *Let P be the corresponding Pfaffian polynomial on $\mathfrak{z}^*$. Note that $\Gamma \cap Z$ is a lattice in Z and* $\exp^{-1}(\Gamma \cap Z)$ *is a lattice (denote it Λ) in $\mathfrak{z}$. That defines the dual lattice Λ^* in $\mathfrak{z}^*$. Then a square integrable representation π_λ occurs in $L^2(N/\Gamma)$ if and only if $\lambda \in \Lambda^*$, and in that case π_λ occurs with multiplicity $|P(\lambda)|$.*

Definition 7.3 Let $N = N_{\mathbb{R}}$ be defined over $\mathbb{Q}$ as in Proposition 7.1, so we have a fixed rational form $N_{\mathbb{Q}}$. We say that a connected Lie subgroup $M \subset N$ is *rational* if $M \cap N_{\mathbb{Q}}$ is a rational form of M, in other words if $\mathfrak{m} \cap \mathfrak{n}_{\mathbb{Q}}$ contains a basis of $\mathfrak{m}$. We say that a decomposition (5.1) is *rational* if the subgroups M_r and N_r are rational. $\diamond$

The following is immediate from this definition.

Lemma 7.4 *Let N be defined over $\mathbb{Q}$ as in* Proposition 7.1 *with rational structure defined by a discrete co-compact subgroup Γ. If the decomposition* (5.1) *is rational then each $\Gamma \cap Z_r$ in Z_r, each $\Gamma \cap M_r$ in M_r, each $\Gamma \cap S_r$ in S_r, and each $\Gamma \cap N_r$ in N_r, is a discrete co-compact subgroup defining the same rational structure as the one defined by its intersection with $N_{\mathbb{Q}}$.*

For the rest of this section we will assume that N and Γ satisfy the rationality conditions of Lemma 7.4, in particular that (5.1) is rational. Then for each r, $Z_r \cap \Gamma$ is a lattice in the center Z_r of M_r, and $\Lambda_r := \log(Z_r \cap \Gamma)$ is a lattice in its Lie algebra $\mathfrak{z}_r$. That defines the dual lattice Λ_r^* in $\mathfrak{z}_r^*$. We normalize the Pfaffian polynomials on the $\mathfrak{z}_r^*$, and thus the polynomial P on $\mathfrak{s}^*$, by requiring that the $N_r/(S_r \cdot (N_r \cap \Gamma))$ have volume 1.

Theorem 7.5 *Let $\lambda \in \mathfrak{t}^*$, in other words $\lambda = \sum \lambda_r$ where $\lambda_r \in \mathfrak{z}_r^*$ with* $\mathrm{Pf}\,(b_{\lambda_r}) \neq 0$. *Then a stepwise square integrable representation π_λ of N occurs in $L^2(N/\Gamma)$ if and only if each $\lambda_r \in \Lambda_r^*$, and in that case the multiplicity of π_λ on $L^2(N/\Gamma)$ is $|P(\lambda)|$.*

Proof Recall $N_r = M_1M_2\ldots M_r = N_{r-1} \rtimes M_r$ semidirect product, where $N = M_1M_2\ldots M_m$ and the center Z_r of M_r is central in N_r. Fix $r \leqq m$. By induction on dimension we assume that Theorem 7.5 holds for N_{r-1} and $\Gamma \cap N_{r-1}$. We may also assume that $\dim Z_r = 1$, following the argument of the first paragraph of the proof of [8, Theorem 7].

Now we proceed as in [8], adapted to our situation. Choose nonzero rational $x \in \mathfrak{m}_r \setminus \mathfrak{z}_r$ and $z \in \mathfrak{z}_r$ in such a way that (i) $\exp(z)$ generates the infinite cyclic group $\Gamma \cap Z_r$, (ii) $[x, \mathfrak{m}_r] \subset \mathfrak{z}_r$, and (iii) $\exp(x)$ and $\exp(z)$ generate $\Gamma \cap P_r$ where $P_r = \exp(\mathfrak{p}_r)$ where $\mathfrak{p}_r$ is the span of x and z. The centralizer $Z_{M_r}(x)$ of of x in M_r is a rational normal subgroup of codimension 1 in M_r,, so $Q_r := N_{r-1} \rtimes Z_{M_r}(x)$ is a rational normal subgroup of codimension 1 in N_r. The group $\Gamma Q_r/Q_r$ is infinite cyclic. Parameterize $\mathfrak{z}_r^*$ by $a = a(\nu_r) = \nu_r(z)$. The Pfaffian polynomial on $\mathfrak{z}_r^*$, normalized by the condition that $M_r/\Gamma Z_r$ has volume 1, satisfies $P_r(a) = \mathrm{Pf}\,(b_{\nu_r}) = c_r a^{d_r}$ where $\nu_r(z) = a$, $\dim M_r = 2d_r + 1$, and c_r is a nonzero constant.

Choose $\gamma \in \Gamma$ whose image in $\Gamma Q_r/Q_r$ is a generator and let $y = \log(\gamma)$. Then $[x, y]$ is a rational multiple of x, say $[x, y] = uz$. Since $\exp(x)$, $\exp(y)$ and $\exp(z)$

span a rational 3-dimensional Heisenberg algebra which we denote $\mathfrak{h}_r$; H_r denotes the corresponding group. It follows [1] that u is an integer.

Let $\pi_\nu \in \widehat{N_r}$ occur in $L^2(N_r/(\Gamma \cap N_r))$ where $\nu = \nu_1 + \cdots + \nu_r$ with $\nu_i \in \mathfrak{z}_i$ and $\mathrm{Pf}\,(b_{\nu_i}) \neq 0$. By induction, $\nu_i \in \Lambda^*$ for $i < r$, and the argument immediately above shows that $\nu_r \in \Lambda^*$. In other words, by induction on dimension and on r, if π_λ occurs on $L^2(N/\Gamma)$ then for each index i we have $\lambda_i \in \Lambda_i^*$.

By construction, Q_r satisfies (5.1) with $Z_{M_r}(x)$ in place of M_r. If $\nu_r \in \mathfrak{z}_r^*$ defines a square integrable (mod Z_r) representation of M_r, then it also defines a square integrable (mod P_r) representation of $Z_{M_r}(x)$. Let $\xi_\nu \in \widehat{Q_r}$ correspond to $\nu = \nu_1 + \cdots + \nu_r$ with each $\nu_i \in \Lambda^* \cap \mathfrak{z}_i^*$ and $\mathrm{Pf}\,(b_{\nu_i}) \neq 0$. By induction on dimension we may assume that ξ_ν has multiplicity $|\mathrm{Pf}\,(b_{\nu_1}) \ldots \mathrm{Pf}\,(b_{\nu_{r-1}})\mathrm{Pf}\,'(b_{\nu_r})|$ on $L^2(Q_r/(\Gamma \cap Q_r))$, where $\mathrm{Pf}\,'(b_{\nu_r})$ is the Pfaffian computed on the Lie algebra of $Z_{M_r}(x)$ (modulo its center $\mathfrak{p}_r$).

The square integrable representations of $Z_{M_r}(x)$ are parameterized by the linear functionals μ_r on $\mathfrak{p}_r = \mathfrak{z}_r + x\mathbb{R}$ with $\mathrm{Pf}\,' \neq 0$. We parameterize μ_r by $a = \mu_r(z)$ and $b = \mu_r(x)$ so $\mathrm{Pf}\,'(b_{\mu_r})$ is a polynomial in a and b. By construction it is independent of x, so $\mathrm{Pf}\,'(b_{\mu_r}) = c_r' a^{d_r-1}$ where c_r' is a constant and $\dim \mathfrak{m}_r/\mathfrak{z}_r = 2d_r$. Define ν_r by $a = \nu_r(z)$, i.e. by $\nu_r = \mu_r|_{\mathfrak{z}_r}$. Since $[x, y] = uz$ now $\mathrm{Pf}\,(a) = \mathrm{Pf}\,'(a, b)au = uc_r' a^{d_r}$, in particular $c_r = uc_r'$.

By induction on dimension, ξ_μ occurs in $L^2(Q_r/(\Gamma \cap Q_r))$ if and only if each $\mu_i \in \Lambda^* \cap \mathfrak{z}_i^*$ with $\mathrm{Pf}\,(b_{\mu_i}) \neq 0$ for $i < r$, both a and b are integers, and $a \neq 0$. To simplify the notation, fix the μ_i for $i < r$ and write $\xi(a, b)$ for the ξ_μ where μ_r has parameter (a, b). Then $\xi(a, b)$ has multiplicity $mult'(a, b) = |\mathrm{Pf}\,(b_{\mu_1}) \ldots \mathrm{Pf}\,(b_{\mu_{r-1}})\mathrm{Pf}\,'(b_{\mu_r})| = |\mathrm{Pf}\,(b_{\mu_1}) \ldots \mathrm{Pf}\,(b_{\mu_{r-1}})c_r' a^{d_r-1}|$. Thus c_r' is an integer, so $c_r = uc_r'$ is an integer as well.

Note that $\pi_\nu = \mathrm{Ind}_{Q_r}^{L_r}(\xi_\mu)$ whenever $\mu|_{\mathfrak{s}_r} = \nu$ and that $\pi_\nu|_{Q_r}$ is the direct integral of all such ξ_μ. Denote $A'(\nu) = \{\mu \mid \mu|_{\mathfrak{s}_r} = \nu$ and ξ_μ occurs in $L^2(Q_r/(\Gamma \cap Q_r))\}$. It consists of all $\xi(a, b)$ with fixed $a = \nu_r(z) \neq 0$ and integral b if a is an integer, the empty set if a is not integral. Fix a set $A(\nu)$ of representatives of the orbits of $\Gamma \cap N_r$ on $A'(\nu)$. As in the proof of [8, Theorem 7], the algorithm of [7, page 153] says that the multiplicity of π_ν in $L^2(L_r/(\Gamma \cap L_r))$ is $mult(\nu) = \sum_{\mu \in A(\nu)} mult'(\mu)$.

An immediate consequence: $mult(\nu) > 0$ if and only if each $\nu_i \in \Lambda^*$. That proves the first assertion of the Theorem.

We look at action of $\Gamma \cap N_r$ on $A'(\nu)$. First, $\Gamma \cap Q_r$ acts trivially, so the action is given by the cyclic group $(\Gamma \cap N_r)/(\Gamma \cap Q_r)$, which has generator $\overline{\gamma} = \exp(y)(\Gamma \cap Q_r)$. As $[x, y] = uz$ the action is $\overline{\gamma} : \xi(a, b) \mapsto \xi(a, b + au)$. So we can assume that $A(\nu)$ consists of the au elements $\xi(a, b+i)$ where i is integral with $0 \leqq i < au$. Each $mult'(a, b + i) = |\mathrm{Pf}\,(b_{\mu_1}) \ldots \mathrm{Pf}\,(b_{\mu_{r-1}})c_r' a^{d_r-1}|$, so now $mult(\nu) = |\mathrm{Pf}\,(b_{\nu_1}) \ldots \mathrm{Pf}\,(b_{\nu_{r-1}})| \cdot |auc_r' a^{d_r-1}| = |\mathrm{Pf}\,(b_{\nu_1}) \ldots \mathrm{Pf}\,(b_{\nu_{r-1}})| \cdot |\mathrm{Pf}\,(b_{\nu_r})|$. This completes the proof of the induction step, and thus of the Theorem.

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