A duality for the double fibration transform

Michael G. Eastwood and Joseph A. Wolf

ABSTRACT. We establish a duality within the spectral sequence that governs the holomorphic double fibration transform. It has immediate application to the questions of injectivity and range characterization for this transform. We discuss some key examples and an improved duality that holds in the Hermitian holomorphic case.

1. Double fibrations

In this article we shall always work in the holomorphic category. By a *double fibration* we shall mean a diagram of the form

(1.1)
$$\begin{array}{ccc} & \mathfrak{X}_D & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ D & & \mathcal{M}_D \end{array}$$

where

(1.2)
- D, $\mathfrak{X}_D$, and $\mathcal{M}_D$ are complex manifolds;
- μ is a holomorphic submersion with contractible fibers;
- ν is a holomorphic submersion with compact fibers;
- $\mathfrak{X}_D \xrightarrow{(\mu,\nu)} D \times \mathcal{M}_D$ is a holomorphic embedding;
- $\mathcal{M}$ is a contractible Stein manifold.

Examples of double fibrations arise naturally as follows. Let G be a complex semisimple (or even reductive) Lie group. There is a beautiful class of complex homogeneous spaces $Z = G/Q$ that can be characterized by any of the following equivalent conditions (see e.g. [**6**] for details).

- Z is a compact complex manifold;
- Z is a compact Kähler manifold;
- Z is a complex projective variety;
- Q is a parabolic subgroup of G.

We shall refer to such compact complex homogeneous spaces Z as *complex flag manifolds*. Now fix a complex flag manifold $Z = G/Q$ and consider a real form G_0 of G. Then it is known [**9**] that the natural action of G_0 on Z has only finitely many orbits and so there is at least one open orbit. If G_0 is compact, then it acts transitively on Z and there are few other exceptional cases when this happens.

Otherwise, an open G_0-orbit $D \subsetneq Z$ is known as a *flag domain*. As a simple example, let us take $G = \mathrm{SL}(4, \mathbb{C})$ acting on $Z = \mathbb{CP}_3$ in the usual fashion, namely

$$\mathrm{SL}(4, \mathbb{C}) \times \mathbb{CP}_3 \ni (A, [z]) \mapsto [Az] \in \mathbb{CP}_3,$$

where $z \in \mathbb{C}^4$ is regarded as a column vector. If we take

$$\begin{bmatrix} * \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{CP}_3 \text{ as basepoint, then } Q = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \in \mathrm{SL}(4, \mathbb{C}) \right\}.$$

If we take $G_0 = \mathrm{SU}(2,2)$, defined as preserving the Hermitian form

(1.3) $$\langle w, z \rangle \equiv w_1 \overline{z_1} + w_2 \overline{z_2} - w_3 \overline{z_3} - w_4 \overline{z_4}$$

on $\mathbb{C}^4$, then

$$D = \mathbb{CP}_3^+ \equiv \{ [z] \in \mathbb{CP}_3 \mid |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 > 0 \}$$

is a flag domain for the action of G_0 on Z.

In general, fixing $K_0 \subset G_0$ a maximal compact subgroup, it is known [9] that there is just one K_0-orbit C_0 in D that is a complex submanifold of Z. We regard C_0 as the basepoint of the *cycle space*

$$\mathcal{M}_D \equiv \text{connected component of } C_0 \text{ in } \{ gC_0 \mid g \in G \text{ and } gC_0 \subset D \}$$

of D. Evidently, $\mathcal{M}_D$ is an open subset of

$$\mathcal{M}_Z \equiv \{ gC_0 \mid g \in G \} = G/J, \quad \text{where } J \equiv \{ g \in G \mid gC_0 = C_0 \}$$

and hence is a complex manifold. Let us set

$$\mathfrak{X}_Z \equiv G/(Q \cap J) \quad \text{and} \quad \mathfrak{X}_D \equiv \{ (z, C) \in D \times \mathcal{M}_D \mid z \in C \}.$$

Then

(1.4)

$$\begin{array}{ccc}
\mathfrak{X}_D & & \mathfrak{X}_Z \\
{}^{\mu}\swarrow \quad \searrow^{\nu} & \text{open } \subset & \swarrow \quad \searrow \\
D \qquad \mathcal{M}_D & & Z \qquad \mathcal{M}_Z
\end{array}$$

and it is known for any flag domain (see e.g. [6] for details) that all the conditions (1.2) of a double fibration are satisfied.

In our example, we may take

(1.5) $$K_0 = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)) = \left\{ \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \in \mathrm{SU}(2,2) \right\}$$

whence

$$C_0 = \left\{ \begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix} \in \mathbb{CP}_3 \right\}, \quad J = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \in \mathrm{SL}(4, \mathbb{C}) \right\},$$

and $\mathcal{M}_Z = \mathrm{Gr}_2(\mathbb{C}^4)$, the Grassmannian of 2-planes in $\mathbb{C}^4$. The base cycle C_0 and, therefore, every other cycle is intrinsically a Riemann sphere $\mathbb{CP}_1$. Geometrically,

$$\mathcal{M}_D = \{ \Pi \in \mathrm{Gr}_2(\mathbb{C}^4) \mid \langle \cdot, \cdot \rangle |_\Pi \text{ is positive definite} \} \equiv \mathbb{M}^{++}$$

and analytically we may realize $\mathcal{M}_D$ as a convex tube domain in $\mathbb{C}^4$

$$\mathcal{M}_D \cong \{\zeta = x + iy \in \mathbb{C}^4 \mid x_1{}^2 > x_2{}^2 + x_3{}^2 + x_4{}^2 \text{ and } x_1 > 0\}$$

by means of

$$\mathbb{C}^4 \ni (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \longmapsto \Pi \equiv \operatorname{span} \left\{ \begin{bmatrix} 1 + \zeta_1 + \zeta_2 \\ \zeta_3 + i\zeta_4 \\ 1 - \zeta_1 - \zeta_2 \\ -\zeta_3 - i\zeta_4 \end{bmatrix}, \begin{bmatrix} \zeta_3 - i\zeta_4 \\ 1 + \zeta_1 - \zeta_2 \\ -\zeta_3 + i\zeta_4 \\ 1 - \zeta_1 + \zeta_2 \end{bmatrix} \right\}.$$

Notice that, in this particular case, the cycle space $\mathcal{M}_D$ is itself a flag domain (for the action of $\mathrm{SU}(2,2)$ on $\mathrm{Gr}_2(\mathbb{C}^4)$). This is unusual.

For our second example, let us start with another of the open orbits of $\mathrm{SU}(2,2)$ on $\mathrm{Gr}_2(\mathbb{C}^4)$, namely

$$D = \{\Pi \in \mathrm{Gr}_2(\mathbb{C}^4) \mid \langle \cdot\, ,\, \cdot \rangle|_\Pi \text{ is strictly indefinite}\} \equiv \mathbb{M}^{+-}.$$

With the same choice (1.5) of maximal compact subgroup K_0, the base cycle C_0 is

$$\left\{ \Pi \in \mathrm{Gr}_2(\mathbb{C}^4) \mid \Pi = \operatorname{span}\{\alpha, \beta\} \text{ for some } \left\{ \begin{array}{l} \alpha \text{ of the form } [*, *, 0, 0]^t \\ \beta \text{ of the form } [0, 0, *, *]^t \end{array} \right\} \right..$$

Hence the base cycle and, therefore, every other cycle is intrinsically $\mathbb{CP}_1 \times \mathbb{CP}_1$. By definition, we always have $J \supseteq K$, the complexification of K_0, but often they are equal, and this is the case here. The cycle space $\mathcal{M}_D$ is $\mathbb{M}^{++} \times \mathbb{M}^{--}$, where $\mathbb{M}^{--}$ denotes the set of planes in $\mathbb{C}^4$ on which $\langle \cdot\, ,\, \cdot \rangle$ is negative definite. As a product of two Stein manifolds it is Stein. For $(\Pi_1, \Pi_2) \in \mathcal{M}_D$, the corresponding cycle is

$$\left\{ \Pi \in \mathrm{Gr}_2(\mathbb{C}^4) \mid \Pi = \operatorname{span}\{\alpha, \beta\} \text{ for some } \alpha \in \Pi_1 \text{ and } \beta \in \Pi_2 \right\}.$$

2. The transform

Consider a general double fibration (1.1), satisfying the conditions (1.2), and suppose we are given a holomorphic vector bundle E on D and a cohomology class $\omega \in H^r(D; \mathcal{O}(E))$. We shall continue to refer to the compact complex submanifolds $\mu(\nu^{-1}(x))$ for $x \in \mathcal{M}_D$ as *cycles* in D and now consider the restriction of ω to these cycles:

$$\omega|_{\mu(\nu^{-1}(x))} \in H^r\big(\mu(\nu^{-1}(x)); \mathcal{O}(E|_{\mu(\nu^{-1}(x))})\big), \text{ as } x \in \mathcal{M}_D \text{ varies.}$$

As ν has compact fibers, these cohomology spaces are finite-dimensional and we shall suppose that their dimension is constant as $x \in \mathcal{M}_D$ varies (generically this is the case and in the homogeneous setting, as discussed above, this is manifest if one starts with E a G-homogeneous vector bundle). Then, as $x \in \mathcal{M}_D$ varies we obtain a vector bundle E' on $\mathcal{M}_D$ and a holomorphic section $\mathcal{P}\omega \in \Gamma(\mathcal{M}_D, \mathcal{O}(E'))$ thereof. This is the *double fibration transform* of ω. It is often most interesting starting with cohomology in the same degree as the dimension of the fibers of ν. Two natural questions associated with this transform are

- is it injective?
- what is its range?

There are clear parallels with the Radon transform and other transforms from real integral geometry, especially when integrating over compact cycles.

The complex version, however, benefits from the following general result.

THEOREM 2.1. *For any double fibration* (1.1), *and holomorphic vector bundle E on D, there is a spectral sequence*

$$(2.1) \qquad E_1^{p,q} = \Gamma(\mathcal{M}_D; \nu_*^q \Omega_\mu^p(E)) \Longrightarrow H^{p+q}(D; \mathcal{O}(E)),$$

where

- $\Omega_\mu^1 \equiv \Omega_{\mathfrak{X}_D}^1 / \mu^* \Omega_D^1$, *the holomorphic 1-forms along the fibers of μ;*
- $\Omega_\mu^p \equiv \Lambda^p \Omega_\mu^1$, *the holomorphic p-forms along the fibers of μ;*
- $\Omega_\mu^p(E) \equiv \Omega_\mu^p \otimes \mu^* E$.

PROOF. There are two stages to the proof, the details of which may be found in [**1**]. The first uses that the fibers of μ are contractible to conclude that

$$H^r(D; \mathcal{O}(E)) \cong H^r(\mathfrak{X}_D; \mu^{-1} \mathcal{O}(E))$$

where $\mu^{-1} \mathcal{O}(E)$ denotes the sheaf of germs of holomorphic sections of $\mu^* E$ on $\mathfrak{X}_D$ that are locally constant along the fibers of μ. The second stage uses the resolution $0 \to \mu^{-1} \mathcal{O}(E) \to \Omega_\mu^\bullet(E)$ to construct a spectral sequence

$$E_1^{p,q} = H^q(\mathfrak{X}_D; \Omega_\mu^p(E)) \Longrightarrow H^{p+q}(\mathfrak{X}_D; \mu^{-1} \mathcal{O}(E)),$$

which combines with the natural isomorphisms $H^q(\mathfrak{X}_D, \mathcal{O}(F)) \cong \Gamma(\mathcal{M}_D, \nu_*^q \mathcal{O}(F))$, valid for any holomorphic vector bundle F on $\mathfrak{X}_D$ because $\mathcal{M}_D$ is Stein. □

For the rest of this article we shall suppose that the direct images $\nu_*^q \Omega_\mu^p(E)$ are locally free and therefore may be regarded as holomorphic vector bundles on $\mathcal{M}_D$. From this viewpoint, the E_1-differentials become first order differential operators on $\mathcal{M}_D$ and, more generally, the spectral sequence ideally interprets the cohomology $H^r(D; \mathcal{O}(E))$ in terms of systems of holomorphic differential equations on $\mathcal{M}_D$. This is especially interesting when D is a flag domain, $\mathcal{M}_D$ is its cycle space, and E is G-homogeneous because then this double fibration transform can provide useful alternative realizations of the G_0-representations afforded by $H^r(D; \mathcal{O}(E))$.

3. Examples

Let us now return to the flag domains introduced in §1 and see how the spectral sequence (2.1) works out for the first of these domains, namely $D = \mathbb{CP}_3^+$. The main issue in executing (2.1) is in computing the direct images $\nu_*^q \Omega_\mu^p(E)$. We need a notation for the irreducible homogeneous vector bundles on the flag manifold Z. For this we shall follow [**1**], recording both the parabolic subgroup Q and the representation of Q by annotating the appropriate Dynkin diagram (it turns out to be most convenient to record the lowest weight of the representation). For our first domain, in which $Z = \mathbb{CP}_3$, the irreducible homogeneous vector bundles are

$$\overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\bullet} \qquad \text{for integers } a, b, c \text{ with } b, c \geq 0.$$

The details are in [**1**] but some particular cases are

$$\overset{0}{\times}\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!-\!\!\overset{0}{\bullet} = \text{the trivial bundle} \equiv \mathcal{O}$$

$$\overset{1}{\times}\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!-\!\!\overset{1}{\bullet} = \text{the holomorphic tangent bundle} \equiv \Theta$$

$$\overset{-2}{\times}\!\!-\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!-\!\!\overset{0}{\bullet} = \text{the holomorphic cotangent bundle} \equiv \Omega^1$$

$$\overset{-3}{\times}\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!-\!\!\overset{1}{\bullet} = \text{the bundle of holomorphic 2-forms} \equiv \Omega^2$$

$$\overset{1}{\times}\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!-\!\!\overset{0}{\bullet} = \text{the tautological bundle} \equiv \mathcal{O}(1)$$

$$\overset{k}{\times}\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!-\!\!\overset{0}{\bullet} = \text{the } k^{\text{th}} \text{ power of the tautological bundle} \equiv \mathcal{O}(k).$$

Similarly, the irreducible homogeneous vector bundles on $\mathfrak{X}_Z$, the flag manifold

$$\mathrm{F}_{1,2}(\mathbb{C}^4) = \{(L,\Pi) \in \mathbb{CP}_3 \times \mathrm{Gr}_2(\mathbb{C}^4) \mid L \subset \Pi\}$$

are given by

$$\overset{a}{\times}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{c}{\bullet} \qquad \text{for integers } a, b, c \text{ with } c \geq 0.$$

For computational purposes, it is always better to consider the diagram

$$(3.1) \qquad \begin{array}{ccc} & \mathfrak{X}_Z = G/(Q \cap J) & \\ {}^{\mu}\swarrow & & \searrow{}^{\nu} \\ G/Q = Z & & \mathcal{M}_Z = G/J \end{array}$$

from (1.4), where we have extended the definition of μ and ν to $\mathfrak{X}_Z$ as shown. The point is that, with this enhanced definition of ν, we have $\mathfrak{X}_D = \nu^{-1}(\mathcal{M}_D)$ and so the fibers over $\mathcal{M}_D$ are unchanged. In particular, the direct images $\nu_*^q \Omega_\mu^p(E)$, as required in the spectral sequence (2.1), can be computed from (3.1) and then simply restricted to the open Stein subset $\mathcal{M}_D \subset \mathcal{M}_Z$. The advantage of (3.1) is that all three spaces are G-homogeneous and the two mappings are G-equivariant. Hence, we may use representation theory to compute $\nu_*^q \Omega_\mu^p(E)$ etc.

With this enhanced viewpoint in place, the bundles of holomorphic forms along the fibers of μ are

$$(3.2) \qquad \Omega_\mu^0 = \overset{0}{\times}\!\!-\!\!\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet}, \qquad \Omega_\mu^1 = \overset{1}{\times}\!\!-\!\!\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}, \qquad \Omega_\mu^2 = \overset{2}{\times}\!\!-\!\!\overset{-3}{\times}\!\!-\!\!\overset{0}{\bullet}.$$

Now let us consider the double fibration transform for $H^r(D; \mathcal{O}(k))$. Line bundles are straightforward because

$$\mu^* \mathcal{O}(k) = \mu^* \overset{k}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet} = \overset{k}{\times}\!\!-\!\!\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet},$$

which is irreducible. Writing $\Omega_\mu^p(k)$ for $\Omega_\mu^p \otimes \mu^* \mathcal{O}(k)$, we have

$$(3.3) \qquad \Omega_\mu^0(k) = \overset{k}{\times}\!\!-\!\!\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet} \qquad \Omega_\mu^1(k) = \overset{k+1}{\times}\!\!-\!\!\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet} \qquad \Omega_\mu^2(k) = \overset{k+2}{\times}\!\!-\!\!\overset{-3}{\times}\!\!-\!\!\overset{0}{\bullet}.$$

The direct images are computed in accordance with the Bott-Borel-Weil Theorem along the fibers of ν, which reads

$$(3.4) \qquad \begin{aligned} \nu_* \overset{a}{\times}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{c}{\bullet} &= \overset{a}{\bullet}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{c}{\bullet} \quad \text{for } a \geq 0 \\ \nu_*^1 \overset{a}{\times}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{c}{\bullet} &= \overset{-a-2}{\bullet}\!\!-\!\!\overset{a+b+1}{\times}\!\!-\!\!\overset{c}{\bullet} \quad \text{for } a \leq -2, \end{aligned}$$

with all other direct images vanishing (see e.g. [1] for details). In particular, there are the spectral sequences (2.1) of the following form.

Let us say that the spectral sequence $E_1^{p,q}$ is *concentrated in degree zero* if and only if $E_1^{p,q} = 0$ for $q > 0$ and *strictly concentrated in degree zero* if and only if, in addition, $E_1^{p,0} \neq 0$, $\forall p$. Similarly, let us say we have *concentration in top degree* if and only if $E_1^{p,q} = 0$ for $q < s$, where $s = \dim_{\mathbb{C}}(\text{fibers of } \nu)$ and *strictly* so if and

only if $E_1^{p,s} \neq 0$, $\forall p$. Thus, strict concentration occurs in this example for $k \geq 0$ or $k \leq -4$. In fact, it is easily verified that

$$(3.5) \quad \begin{array}{rcl} k \geq 0 & \Longrightarrow & \text{strict concentration in degree zero,} \\ k = -1 & \Longrightarrow & \text{concentration in degree zero,} \\ k = -2 & \Longrightarrow & \text{no concentration,} \\ k = -3 & \Longrightarrow & \text{concentration in top degree } (s = 1), \\ k \leq -4 & \Longrightarrow & \text{strict concentration in top degree.} \end{array}$$

The double fibration transform in this case is known as the *Penrose transform* [**4**]. Always, the spectral sequence is most easily interpreted when it concentrates in top degree for then it collapses to yield, in particular, an isomorphism

$$\mathcal{P} : H^s(D; \mathcal{O}(E)) \xrightarrow{\simeq} \ker : \Gamma(\mathcal{M}_D; \nu_*^s \Omega_\mu^0(E)) \to \Gamma(\mathcal{M}_D; \nu_*^s \Omega_\mu^1(E)).$$

In our example

$$\mathcal{P} : H^s(D; \mathcal{O}(k)) \xrightarrow{\simeq} \ker : \Gamma(\mathcal{M}_D; \overset{-k-2}{\bullet}\!\!\overset{k+1}{\times}\!\!\overset{0}{\bullet}) \to \Gamma(\mathcal{M}_D; \overset{-k-3}{\bullet}\!\!\overset{k}{\times}\!\!\overset{1}{\bullet}),$$

for $k \leq -3$ and the right hand side has an interpretation in physics as so-called *massless fields of helicity* $-1 - k/2$ (see e.g. [**4**] for details).

The main aim of this article is to show that concentration in zero versus top degree are related by a duality. This will turn out to be useful because the spectral sequence has simple consequences when concentrated in top degree whereas criteria for concentration in degree zero are more easily determined.

4. The duality

THEOREM 4.1. *Let κ_D and $\kappa_{\mathcal{M}_D}$ denote the canonical bundles on D and $\mathcal{M}_D$, respectively. Let $d = \dim_\mathbb{C}(\text{fibers of } \mu)$ and recall that $s = \dim_\mathbb{C}(\text{fibers of } \nu)$. Then there are canonical isomorphisms*

$$(4.1) \qquad \nu_*^q \Omega_\mu^p(\kappa_D \otimes E^*) = \kappa_{\mathcal{M}_D} \otimes (\nu_*^{s-q} \Omega_\mu^{d-p}(E))^*, \quad \forall\, 0 \leq p \leq d,\, 0 \leq q \leq s.$$

The spectral sequence (2.1) for the vector bundle E is (strictly) concentrated in top degree if and only if the corresponding spectral sequence for $\kappa_D \otimes E^*$ is (strictly) concentrated in degree zero.

PROOF. Certainly, the last statement follows immediately from (4.1): as $\mathcal{M}_D$ is contractible and Stein, if $\nu_*^q \Omega_\mu^p(E)$ is non-zero then neither is $\Gamma(\mathcal{M}_D; \nu_*^q \Omega_\mu^p(E))$.

Notice that (4.1) generalizes Serre duality [**8**]. Specifically, if D is an arbitrary compact manifold, then we may take $\mathfrak{X}_D = D$ and $\mathcal{M}_D$ to be a point. Then $d = 0$, direct images revert to cohomology, and (4.1) becomes

$$H^q(D; \mathcal{O}(\kappa_D \otimes E^*)) = H^{s-q}(D; \mathcal{O}(E))^*.$$

Conversely, Serre duality along the fibers of ν is the essential ingredient in proving (4.1) as follows. Let $\kappa_{\mathfrak{X}_D}$ denote the canonical bundle on $\mathfrak{X}_D$. Since μ and ν are submersions, we can write $\kappa_{\mathfrak{X}_D}$ in two different ways:

$$(4.2) \qquad \kappa_{\mathfrak{X}_D} = \mu^*(\kappa_D) \otimes \kappa_\mu \quad \text{and} \quad \kappa_{\mathfrak{X}_D} = \nu^*(\kappa_{\mathcal{M}_D}) \otimes \kappa_\nu,$$

where κ_μ and κ_ν are the canonical bundles along the fibers of μ and ν, respectively. Thus, bearing in mind the Hodge isomorphism $\Omega_\mu^p = \kappa_\mu \otimes (\Omega_\mu^{d-p})^*$ along the fibers

of μ, we find that

$$
\begin{aligned}
\nu_*^q \Omega_\mu^p(\kappa_D \otimes E^*) &= \nu_*^q\big(\mu^*(\kappa_D) \otimes \Omega_\mu^p \otimes \mu^*(E^*)\big) \\
&= \nu_*^q\big(\kappa_{\mathfrak{X}_D} \otimes \kappa_\mu^* \otimes \Omega_\mu^p \otimes \mu^*(E^*)\big) \\
&= \nu_*^q\big(\nu^*(\kappa_{\mathcal{M}_D}) \otimes \kappa_\nu \otimes (\kappa_\mu^* \otimes \Omega_\mu^p) \otimes \mu^*(E^*)\big) \\
&= \kappa_{\mathcal{M}_D} \otimes \nu_*^q\big(\kappa_\nu \otimes (\Omega_\mu^{d-p})^* \otimes \mu^*(E^*)\big) \\
&= \kappa_{\mathcal{M}_D} \otimes \nu_*^q\big(\kappa_\nu \otimes (\Omega_\mu^{d-p} \otimes \mu^*(E))^*\big),
\end{aligned}
$$

which may be identified by Serre duality along the fibers of ν to give

$$
\nu_*^q \Omega_\mu^p(\kappa_D \otimes E^*) = \kappa_{\mathcal{M}_D} \otimes \big(\nu_*^{s-q}(\Omega_\mu^{d-p} \otimes \mu^*(E))\big)^* = \kappa_{\mathcal{M}_D} \otimes (\nu_*^{s-q}\Omega_\mu^{d-p}(E))^*,
$$

as required. $\qquad\square$

5. Applications

Let us firstly show how Theorem 4.1 yields (3.5) with minimal calculation. It is clear from (3.3) that strict concentration in degree zero occurs if $k \geq 0$. Indeed, since $\overset{-1\ \ 0\ \ \ 0}{\times\!\!-\!\!\times\!\!-\!\!\bullet}$ is singular along the fibers of ν it is also clear that concentration in degree zero also occurs when $k = -1$. But now

$$
\kappa_D \otimes \big(\overset{k\ \ 0\ \ 0}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}\big)^* = \overset{-4\ \ 0\ \ 0}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet} \otimes \overset{-k\ \ 0\ \ 0}{\times\!\!-\!\!\times\!\!-\!\!\bullet} = \overset{-k-4\ \ 0\ \ 0}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}
$$

and Theorem 4.1 tells us that we have strict concentration in top degree if and only if $-k - 4 \geq 0$, which gives $k \leq -4$ as expected. Similarly, $-k - 4 = -1$ if and only if $k = -3$.

To extend this analysis to vector bundles there are two issues to be overcome. The first is that the pullback $\mu^*(\overset{a\ \ \ b\ \ \ c}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet})$ is reducible in general. Specifically,

$$
\tag{5.1}
\mu^*(\overset{a\ \ \ b\ \ \ c}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}) = \overset{a\ \ \ b\ \ \ c}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet} + \begin{matrix} \overset{a+1\ \ b-2\ \ c+1}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \\ \oplus \\ \overset{a+1\ \ b-1\ \ c-1}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \end{matrix} + \begin{matrix} \overset{a+2\ \ b-4\ \ c+2}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \\ \oplus \\ \overset{a+2\ \ b-3\ \ c}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \\ \oplus \\ \overset{a+2\ \ b-2\ \ c-2}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \end{matrix} + \cdots + \overset{a+b+c\ \ -b-c\ \ b}{\times\!\!-\!\!\times\!\!-\!\!\bullet}.
$$

The second is that, even for an irreducible bundle $V = \overset{a\ \ \ b\ \ \ c}{\times\!\!-\!\!\times\!\!-\!\!\bullet}$ on $\mathrm{F}_{1,2}(\mathbb{C}^4)$, the bundle $\Omega_\mu^1 \otimes V$ may be reducible. For example, these two issues in combination imply that

$$
\Omega_\mu^1(\overset{1\ \ \ 0\ \ \ 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}) = \overset{1\ \ -2\ \ 1}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \otimes \big(\overset{1\ \ \ 0\ \ \ 1}{\times\!\!-\!\!\times\!\!-\!\!\bullet} + \overset{2\ \ -1\ \ 0}{\times\!\!-\!\!\times\!\!-\!\!\bullet}\big) = \begin{matrix} \overset{2\ \ -2\ \ 2}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \\ \oplus \\ \overset{2\ \ -1\ \ 0}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \end{matrix} + \overset{3\ \ -3\ \ 1}{\times\!\!-\!\!\times\!\!-\!\!\bullet}
$$

and the spectral sequence $\nu_*^q\Omega_\mu^p(\overset{1\ \ \ 0\ \ \ 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet})$ takes the form

$$
\tag{5.2}
$$

In particular, it is concentrated in degree zero. This is a general feature as follows.

THEOREM 5.1. *The spectral sequence for $H^r(D; \mathcal{O}(\overset{a\ \ b\ \ c}{\times\!-\!\bullet\!-\!\bullet}))$ associated to the double fibration*

(5.3)

$$
\begin{array}{ccccccc}
 & \mathfrak{X}_D & & & & \mathfrak{X}_Z = \mathrm{F}_{1,2}(\mathbb{C}^4) & \\
 & \mu\swarrow \quad \searrow\nu & & \text{open } \subset & & \mu\swarrow \quad \searrow\nu & \\
\mathbb{CP}_3^+ = D & & \mathcal{M}_D = \mathbb{M}^{++} & & \mathbb{CP}_3 = Z & & \mathcal{M}_Z = \mathrm{Gr}_2(\mathbb{C}^4)
\end{array}
$$

is strictly concentrated in degree zero if $a \geq 0$.

PROOF. Firstly, notice that all the composition factors occurring in (5.1) are dominant with respect to the first node if $a \geq 0$. Although clear by inspection, the underlying reason for this is that the composition factors are obtained from the leading term $\overset{a\ \ b\ \ c}{\times\!-\!\times\!-\!\bullet}$ by adding simple negative roots for $\times\!-\!\bullet\!-\!\bullet$, namely

(5.4)
$$
\overset{1\ \ -2\ \ 1}{\times\!-\!\bullet\!-\!\bullet} \quad \text{and} \quad \overset{0\ \ 1\ \ -2}{\times\!-\!\bullet\!-\!\bullet}
$$

(minus the second two rows of the Cartan matrix for $\mathfrak{sl}(4, \mathbb{C})$) both of which have a non-negative coefficient over the first node. Secondly, there is the question of tensoring these composition factors with Ω_μ^p from (3.2). Each of Ω_μ^p is dominant with respect to the first node and, more generally,

(5.5)
$$
\begin{aligned}
\Omega_\mu^0 \otimes \overset{a\ \ b\ \ c}{\times\!-\!\times\!-\!\bullet} &= \overset{a\ \ b\ \ c}{\times\!-\!\times\!-\!\bullet} \\
\Omega_\mu^1 \otimes \overset{a\ \ b\ \ c}{\times\!-\!\times\!-\!\bullet} &= \overset{a+1\ b-2\ c+1}{\times\!-\!\times\!-\!\bullet} \oplus \overset{a+1\ b-1\ c-1}{\times\!-\!\times\!-\!\bullet} \quad (\text{if } c \geq 1) \\
\Omega_\mu^2 \otimes \overset{a\ \ b\ \ c}{\times\!-\!\times\!-\!\bullet} &= \overset{a+2\ b-3\ c}{\times\!-\!\times\!-\!\bullet}
\end{aligned}
$$

(noting that it is $\overset{0\ \ 1\ \ -2}{\times\!-\!\times\!-\!\bullet}$ (cf. (5.4)) that is responsible for the second direct summand of $\Omega_\mu^1(\overset{a\ \ b\ \ c}{\times\!-\!\times\!-\!\bullet})$). Clearly, all the various composition factors occurring in $\Omega_\mu^p(\overset{a\ \ b\ \ c}{\times\!-\!\bullet\!-\!\bullet})$ are dominant with respect to the first node if $a \geq 0$ and, therefore, all direct images are concentrated in degree zero, as required. $\qquad\square$

COROLLARY 5.2. *The spectral sequence for $H^r(\mathbb{CP}_3^+; \mathcal{O}(\overset{a\ \ b\ \ c}{\times\!-\!\bullet\!-\!\bullet}))$ associated to the double fibration* (5.3) *is strictly concentrated in top degree (namely, first degree) if $a \leq -4 - b - c$.*

PROOF. By standard weight considerations

$$
(\overset{a\ \ b\ \ c}{\times\!-\!\bullet\!-\!\bullet})^* = \overset{-a-b-c\ \ c\ \ b}{\times\!-\!\bullet\!-\!\bullet}
$$

and so

$$
\kappa_{\mathbb{CP}_3} \otimes (\overset{a\ \ b\ \ c}{\times\!-\!\bullet\!-\!\bullet})^* = \overset{-4\ \ 0\ \ 0}{\times\!-\!\bullet\!-\!\bullet} \otimes \overset{-a-b-c\ \ c\ \ b}{\times\!-\!\bullet\!-\!\bullet} = \overset{-4-a-b-c\ \ c\ \ b}{\times\!-\!\bullet\!-\!\bullet},
$$

and we require $-4 - a - b - c \geq 0$ in accordance with Theorem 4.1. $\qquad\square$

Further discussion of this example is postponed until §6.

Now let us consider the other example from §1, namely the flag domain $\mathbb{M}^{+-}$. As usual, for computational purposes, we should extend the double fibration to the diagram (3.1). In this case we obtain

$$
\begin{array}{ccccccc}
 & \mathfrak{X}_D & & & & \mathfrak{X}_Z = G/(Q \cap K) & \\
 & \mu\swarrow \quad \searrow\nu & & \text{open } \subset & & \swarrow \quad \searrow & \\
\mathbb{M}^{+-} = D & & \mathcal{M}_D = \mathbb{M}^{++} \times \mathbb{M}^{--} & & \mathrm{Gr}_2(\mathbb{C}^4) = Z & & \mathcal{M}_Z = G/K
\end{array}
$$

where

$$G/K = \mathrm{SL}(4,\mathbb{C})/\left\{\begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}\right\} = \left\{(\Pi_1, \Pi_2) \in \mathrm{Gr}_2(\mathbb{C}^4) \times \mathrm{Gr}_2(\mathbb{C}^4) \mid \Pi_1 \pitchfork \Pi_2\right\}.$$

An additional difficulty in effecting the transform in this case is that, having taken $\mathrm{SU}(2,2)$ to preserve the standard Hermitian form (1.3), the usual basepoint for $\mathrm{Gr}_2(\mathbb{C}^4)$ is not in the domain $\mathbb{M}^{+-}$. Instead, as basepoints we may take

$$\left(\begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ * \\ * \end{bmatrix}\right) \in \mathcal{M}_Z \qquad \begin{bmatrix} * \\ 0 \\ * \\ 0 \end{bmatrix} \in Z \implies Q = \left\{\begin{bmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{bmatrix}\right\}.$$

On the other hand, we would like to denote the homogeneous vector bundles on $\mathrm{Gr}_2(\mathbb{C}^4)$ by $\overset{a}{\bullet}\!\!\overset{b}{-\!\!\times\!\!-}\!\!\overset{c}{\bullet}$ as usual. In order to reconcile these two viewpoints, notice that we may conjugate $Q \subset \mathrm{SL}(4,\mathbb{C})$ into standard form: explicitly,

$$(5.6) \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \equiv \widetilde{Q}.$$

We are therefore confronted with the diagram

$$(5.7)$$

$$\begin{array}{ccc} & & \mathfrak{X}_Z = G/(Q \cap K) \\ \overset{\widetilde{\mu}}{\diagup} & \overset{\mu}{\diagup} \quad \overset{\nu}{\diagdown} & \\ \mathrm{Gr}_2(\mathbb{C}^4) = G/\widetilde{Q} \cong G/Q = Z & & \mathcal{M}_Z = G/K \end{array}$$

as the computational key to the double fibration transform. The first consequence of this additional feature appears in pulling back an irreducible vector bundle from Z to $\mathfrak{X}_Z$. As already mentioned, we shall identify Z as $G/\widetilde{Q}$ and write the irreducible bundles thereon as $\overset{a}{\bullet}\!\!\overset{b}{-\!\!\times\!\!-}\!\!\overset{c}{\bullet}$. Irreducible bundles on

$$\mathfrak{X}_Z = G/(Q \cap K) = \mathrm{SL}(4,\mathbb{C})/\left\{\begin{bmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\},$$

however, are carried by representations of the diagonal subgroup of $Q \cap K$. Hence, pullback by $\widetilde{\mu}$ may be achieved by restriction to the subgroup

$$\widetilde{Q} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \supset \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \equiv B$$

followed by conjugation as in (5.6). Explicitly,

$$\widetilde{\mu}^*(\overset{a}{\bullet}\!\!\overset{b}{-\!\!\times\!\!-}\!\!\overset{0}{\bullet}) = \sigma_2\big(\overset{a}{\times}\!\!\overset{b}{-\!\!\times\!\!-}\!\!\overset{0}{\times} + \overset{a-2}{\times}\!\!\overset{b+1}{-\!\!\times\!\!-}\!\!\overset{0}{\times} + \overset{a-4}{\times}\!\!\overset{b+2}{-\!\!\times\!\!-}\!\!\overset{0}{\times} + \cdots + \overset{-a}{\times}\!\!\overset{a+b}{-\!\!\times\!\!-}\!\!\overset{0}{\times}\big)$$

where σ_2 denotes the effect of the conjugation (5.6) on weights. This is a simple Weyl group reflection, the effect of which is computed in [1], for example, to obtain

$$\widetilde{\mu}^*(\overset{a}{\bullet}\!\!\overset{b}{-\!\!\times\!\!-}\!\!\overset{0}{\bullet}) = \overset{a+b}{\times}\!\!\overset{-b}{-\!\!\times\!\!-}\!\!\overset{b}{\times} + \overset{a+b-1}{\times}\!\!\overset{-b-1}{-\!\!\times\!\!-}\!\!\overset{b+1}{\times} + \overset{a+b-2}{\times}\!\!\overset{-b-2}{-\!\!\times\!\!-}\!\!\overset{b+2}{\times} + \cdots + \overset{b}{\times}\!\!\overset{-a-b}{-\!\!\times\!\!-}\!\!\overset{a+b}{\times}.$$

More generally, we may write $\overset{a\ \ \ b\ \ \ c}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet} = \overset{a\ \ \ b\ \ \ 0}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet} \otimes \overset{0\ \ \ 0\ \ \ c}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet}$ to compute

$$(5.8) \quad \widetilde{\mu}^*(\overset{a\ \ \ b\ \ \ c}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet}) = \overset{a+b\ \ -b\ \ b+c}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times} + \begin{matrix} \overset{a+b-1\ \ -b-1\ \ b+c+1}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times} \\ \oplus \\ \overset{a+b+1\ \ -b-1\ \ b+c-1}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times} \end{matrix} + \cdots + \overset{b+c\ \ -a-b-c\ \ a+b}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times}.$$

The following proposition is almost immediate by inspection.

PROPOSITION 5.3. *The direct images* $\nu_*^q\widetilde{\mu}^*(\overset{a\ \ \ b\ \ \ c}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet})$ *are strictly concentrated in degree zero if* $b \geq 0$.

PROOF. It remains to observe that, recording the irreducible homogeneous vector bundles on $\mathcal{M}_Z$ by irreducible representations of K in the usual manner,

$$(5.9) \quad \begin{aligned} \nu_*(\overset{r\ \ \ s\ \ \ t}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times}) &= \overset{r\ \ \ s\ \ \ t}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet} && \text{if } r \geq 0 \quad \text{and } t \geq 0 \\ \nu_*^1(\overset{r\ \ \ s\ \ \ t}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times}) &= \overset{r\ \ s+t+1\ \ -t-2}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet} && \text{if } r \geq 0 \quad \text{and } t \leq -2 \\ \nu_*^1(\overset{r\ \ \ s\ \ \ t}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times}) &= \overset{-r-2\ \ r+s+1\ \ t}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet} && \text{if } r \leq -2 \quad \text{and } t \geq 0 \\ \nu_*^2(\overset{r\ \ \ s\ \ \ t}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times}) &= \overset{-r-2\ \ r+s+t+2\ \ -t-2}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet} && \text{if } r \leq -2 \quad \text{and } t \leq -2 \end{aligned}$$

and all other direct images vanish; i.e., the usual formulæ [1] pertain. □

LEMMA 5.4. *The holomorphic* 1-*forms along the fibers of* μ *from the diagram* (5.7) *are given by*

$$(5.10) \quad \Omega_\mu^1 = \left(\overset{1\ \ -2\ \ 1}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times} + \begin{matrix} \overset{-1\ \ -1\ \ 1}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times} \\ \oplus \\ \overset{1\ \ -1\ \ -1}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times} \end{matrix} \right) \oplus \left(\overset{1\ \ \ 0\ \ \ 1}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times} + \begin{matrix} \overset{-1\ \ \ 1\ \ \ 1}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times} \\ \oplus \\ \overset{1\ \ \ 1\ \ -1}{\times\!\!\!-\!\!\!\times\!\!\!-\!\!\!\times} \end{matrix} \right)$$

PROOF. This is simply a matter of identifying the weights of $\mathfrak{q}/(\mathfrak{q} \cap \mathfrak{k})$. □

THEOREM 5.5. *The spectral sequence for* $H^r(D; \mathcal{O}(\overset{a\ \ \ b\ \ \ c}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet}))$ *associated to the double fibration*

$$(5.11)$$

$$\begin{array}{ccc} & \mathfrak{X}_D & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ \mathrm{Gr}_2(\mathbb{C}^4) \supset^{\mathrm{open}} \mathbb{M}^{+-} = D & & \mathcal{M}_D = \mathbb{M}^{++} \times \mathbb{M}^{--} \end{array}$$

is strictly concentrated in degree zero if $b \geq 0$.

PROOF. As for Proposition 5.3, we should inspect the composition factors in $\Omega_\mu^p(\overset{a\ \ \ b\ \ \ c}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet}) = \Omega_\mu^p \otimes \widetilde{\mu}^*(\overset{a\ \ \ b\ \ \ c}{\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet})$ and determine, with respect to the first and last nodes, whether they are dominant or singular (i.e. whether the integer over that node is non-negative or -1, respectively) since, according to (5.9), such an inspection will determine whether we have (strict) concentration in degree zero. As the composition factors are all line bundles, this is straightforward arithmetic. If $b \geq 1$, then it is easy to check that the leading terms are dominant and the rest are mostly dominant but occasionally singular. In this case, the conclusion of the theorem is clear. When $b = 0$ there are just two exceptions, both of which occur

within $\Omega^2_\mu(\overset{a\quad b\quad c}{\bullet\!-\!\!\times\!-\!\bullet})$. More specifically, Lemma 5.4 yields

$$\Omega^2_\mu = \begin{pmatrix} \overset{0\ -3\ 2}{\times\!-\!\times\!-\!\times} \\ \oplus \\ \overset{2\ -3\ 0}{\times\!-\!\times\!-\!\times} \end{pmatrix} + \cdots \ \oplus\ \begin{pmatrix} \boxed{\overset{-2\ 0\ 2}{\times\!-\!\times\!-\!\times}} \\ \oplus \\ \cdots + 2\overset{0\ 0\ 0}{\times\!-\!\times\!-\!\times} \\ \oplus \\ \boxed{\overset{2\ 0\ -2}{\times\!-\!\times\!-\!\times}} \end{pmatrix} \ \oplus\ \begin{pmatrix} \overset{0\ 1\ 2}{\times\!-\!\times\!-\!\times} \\ \oplus \\ \overset{2\ 1\ 0}{\times\!-\!\times\!-\!\times} \end{pmatrix} + \cdots$$

and the two boxed bundles give rise to

(5.12) $\quad \overset{-2\ 0\ 2}{\times\!-\!\times\!-\!\times} \otimes \overset{0\ -a\ a+c}{\times\!-\!\times\!-\!\times} = \overset{-2\ -a\ a+c+2}{\times\!-\!\times\!-\!\times}$ and $\overset{a+c+2\ -c\ -2}{\times\!-\!\times\!-\!\times},$

respectively. These two cases require a more delicate analysis, as follows. Without loss of generality let us consider the first of them. As can be seen from (5.10), the line bundle in question arises as a subbundle of the rank two vector bundle

$$W \equiv \left(\overset{1\ -2\ 1}{\times\!-\!\times\!-\!\times} + \overset{-1\ -1\ 1}{\times\!-\!\times\!-\!\times}\right) \otimes \overset{-1\ 1\ 1}{\times\!-\!\times\!-\!\times} \otimes \overset{0\ -a\ a+c}{\times\!-\!\times\!-\!\times}.$$

The short exact sequence

$$0 \to \overset{-2\ -a\ a+c+2}{\times\!-\!\times\!-\!\times} \longrightarrow W \longrightarrow \overset{0\ -a-1\ a+c+2}{\times\!-\!\times\!-\!\times} \to 0$$

induces the exact sequence

$$\begin{array}{ccccccccc}
0 \to & \nu_* W & \to & \nu_*(\overset{0\ -a-1\ a+c+2}{\times\!-\!\times\!-\!\times}) & \overset{\simeq}{\longrightarrow} & \nu^1_*(\overset{-2\ -a\ a+c+2}{\times\!-\!\times\!-\!\times}) & \to & \nu^1_* W & \to 0 \\
& \| & & \| & & \| & & \| & \\
0 \to & \nu_* W & \to & \overset{0\ -a-1\ a+c+2}{\bullet\!-\!\times\!-\!\bullet} & \overset{\simeq}{\longrightarrow} & \overset{0\ -a-1\ a+c+2}{\bullet\!-\!\times\!-\!\bullet} & \to & \nu^1_* W & \to 0
\end{array}$$

on direct images and we claim that the middle arrows are, in fact, isomorphisms, as illustrated. To see this, we trace back the origin of $\overset{1\ -2\ 1}{\times\!-\!\times\!-\!\times} + \overset{-1\ -1\ 1}{\times\!-\!\times\!-\!\times}$ within W from (5.10) and discover that we can write

$$\overset{1\ -2\ 1}{\times\!-\!\times\!-\!\times} + \overset{-1\ -1\ 1}{\times\!-\!\times\!-\!\times} = \eta^*(\overset{1\ -2\ 1}{\bullet\!-\!\times\!-\!\times})$$

for the natural projection

$$\mathfrak{X}_Z = \mathrm{SL}(4,\mathbb{C})/\left\{\begin{bmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\} \overset{\eta}{\longrightarrow} \mathrm{SL}(4,\mathbb{C})/\left\{\begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\}.$$

Therefore,

$$\eta^q_* W = \eta^q_*\left(\eta^*(\overset{1\ -2\ 1}{\bullet\!-\!\times\!-\!\times}) \otimes \overset{-1\ -a+1\ a+c+1}{\times\!-\!\times\!-\!\times}\right) = \overset{1\ -2\ 1}{\bullet\!-\!\times\!-\!\times} \otimes \eta^q_*(\overset{-1\ -a+1\ a+c+1}{\times\!-\!\times\!-\!\times}),$$

all of which vanish owing to the -1 over the first node. Since ν factors through η it follows that $\nu^q_* W = 0$ for all q and the claimed isomorphisms follow. Exactly the same isomorphisms arise in computing $\nu^q_*\Omega^2_\mu(\overset{a\quad b\quad c}{\bullet\!-\!\times\!-\!\bullet})$ from the direct images of its composition factors and it follows that the only two non-zero first direct images arising from the two line bundles (5.12) are, in fact, absorbed into isomorphic zeroth direct images. All other contributions are certainly in degree zero. $\qquad\square$

COROLLARY 5.6. *The spectral sequence for* $H^r(\mathbb{M}^{+-}; \mathcal{O}(\overset{a\quad b\quad c}{\bullet\!-\!\times\!-\!\bullet}))$ *associated to the double fibration* (5.11) *is strictly concentrated in top degree (namely, second degree) if* $b \le -4 - a - c$.

PROOF. The longest element in the Weyl group of $\widetilde{Q}$ as a subgroup of the Weyl group of G is $\sigma_1\sigma_3$, the product of the two of simple reflections σ_1 and σ_3. As

$$\sigma_1\sigma_3(\overset{a\quad b\quad c}{\times\!\!-\!\!\times\!\!-\!\!\times}) = \sigma_1(\overset{a\quad b+c\quad -c}{\times\!\!-\!\!\times\!\!-\!\!\times}) = \overset{-a\quad a+b+c\quad -c}{\times\!\!-\!\!\times\!\!-\!\!\times}$$

it follows that

$$\kappa_{\mathrm{Gr}_2(\mathbb{C}^4)} \otimes (\overset{a\quad b\quad c}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet})^* = \overset{0\quad -4\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet} \otimes \overset{a\quad -a-b-c\quad c}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet} = \overset{a\quad -4-a-b-c\quad c}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}.$$

According to Theorems 4.1 and 5.5, we require $-4 - a - b - c \geq 0$. □

6. Improved duality in the Hermitian holomorphic setting

Now let us reconsider the double fibration transform of $H^r(\mathbb{CP}_3^+; \mathcal{O}(\overset{1\quad 0\quad 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}))$. According to the spectral sequence (5.2), this cohomology is transformed to the cohomology of the complex of differential operators

$$0 \to \begin{matrix}\overset{1\quad 0\quad 1}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}\\ +\\ \overset{2\quad -1\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}\end{matrix} \longrightarrow \begin{matrix}\overset{2\quad -1\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet} \oplus \overset{2\quad -2\quad 2}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}\\ +\\ \overset{3\quad -3\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}\end{matrix} \longrightarrow \begin{matrix}\overset{3\quad -3\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}\\ +\\ \overset{4\quad -4\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}\end{matrix} \to 0$$

on the bounded symmetric domain $\mathbb{M}^{++}$. But this complex may be replaced by an equivalent differential complex as follows. Arguing as in [3], the two operators

$$\begin{matrix}\overset{1\quad 0\quad 1}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}\\ +\\ \boxed{\overset{2\quad -1\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}}\end{matrix} \quad\nearrow\quad \begin{matrix}\boxed{\overset{2\quad -1\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}} \oplus \overset{2\quad -2\quad 2}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}\\ +\\ \boxed{\overset{3\quad -3\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}}\end{matrix} \quad\nearrow\quad \begin{matrix}\boxed{\overset{3\quad -3\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}}\\ +\\ \overset{4\quad -4\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}\end{matrix}$$

are not differential operators at all but, instead, linear isomorphisms of the vector bundles involved. A diagram chase, performed in [3], shows that we may cancel these particular bundles to obtain an alternative complex

$$(6.1) \qquad 0 \to \overset{1\quad 0\quad 1}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet} \longrightarrow \overset{2\quad -2\quad 2}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet} \longrightarrow \overset{4\quad -4\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet} \to 0$$

calculating the cohomology $H^r(\mathbb{CP}_3^+; \mathcal{O}(\overset{1\quad 0\quad 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}))$. In particular,

$$H^1(\mathbb{CP}_3^+; \mathcal{O}(\overset{1\quad 0\quad 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet})) \cong \frac{\ker : \Gamma(\mathbb{M}^{++}; \mathcal{O}(\overset{2\quad -2\quad 2}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet})) \to \Gamma(\mathbb{M}^{++}; \mathcal{O}(\overset{4\quad -4\quad 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}))}{\mathrm{im} : \Gamma(\mathbb{M}^{++}; \mathcal{O}(\overset{1\quad 0\quad 1}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet})) \to \Gamma(\mathbb{M}^{++}; \mathcal{O}(\overset{2\quad -2\quad 2}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}))}.$$

Explicit formulæ for the differential operators of (6.1) are given in [3]. In particular, they have orders 1 and 2, respectively.

An alternative route to the same conclusion is as follows. Let us recall that the spectral sequence in question arises from the double complex (5.3) and, in particular, from the coupled holomorphic de Rham sequence

$$(6.2) \qquad 0 \to \mu^*(\overset{1\quad 0\quad 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}) \to \Omega_\mu^1 \otimes \mu^*(\overset{1\quad 0\quad 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}) \to \Omega_\mu^2 \otimes \mu^*(\overset{1\quad 0\quad 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}) \to 0$$

along the fibers of $\mu : \mathrm{F}_{1,2}(\mathbb{C}^4) \to \mathbb{CP}_3$. Bearing in mind that

$$\mu^*(\overset{1\quad 0\quad 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}) = \overset{1\quad 0\quad 1}{\times\!\!-\!\!\times\!\!-\!\!\bullet} + \overset{2\quad -1\quad 0}{\times\!\!-\!\!\times\!\!-\!\!\bullet}$$

as a special case of (5.1) and writing out (6.2) more explicitly, gives

$$0 \;\rightarrow\; \Omega^0_\mu(\overset{1\quad 0\quad 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}) \;\rightarrow\; \Omega^1_\mu(\overset{1\quad 0\quad 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}) \qquad \rightarrow\; \Omega^2_\mu(\overset{1\quad 0\quad 1}{\times\!\!-\!\!\bullet\!\!-\!\!\bullet}) \;\rightarrow\; 0$$

$$\|\qquad\qquad\qquad\qquad\|\qquad\qquad\qquad\qquad\|$$

$$0 \;\rightarrow\; \begin{matrix}\overset{1\ \ 0\ \ 1}{\times\!-\!\times\!-\!\bullet}\\ +\\ \overset{2\ -1\ \ 0}{\times\!-\!\times\!-\!\bullet}\end{matrix} \;\rightarrow\; \begin{matrix}\overset{2\ -1\ \ 0}{\times\!-\!\times\!-\!\bullet}\oplus\overset{2\ -2\ \ 2}{\times\!-\!\times\!-\!\bullet}\\ +\\ \overset{3\ -3\ \ 0}{\times\!-\!\times\!-\!\bullet}\end{matrix} \;\rightarrow\; \begin{matrix}\overset{3\ -3\ \ 0}{\times\!-\!\times\!-\!\bullet}\\ +\\ \overset{4\ -4\ \ 0}{\times\!-\!\times\!-\!\bullet}\end{matrix} \;\rightarrow\; 0$$

and essentially the same cancellation as occurred down on $\mathrm{Gr}_2(\mathbb{C}^4)$ can be seen in this diagram on $\mathrm{F}_{1,2}(\mathbb{C}^4)$. It means that we may replace the coupled holomorphic de Rham sequence along the fibers of μ by the complex

$$(6.3)\quad 0\to\mu^{-1}\mathcal{O}(\overset{1\ \ 0\ \ 1}{\times\!-\!\bullet\!-\!\bullet})\to\mathcal{O}(\overset{1\ \ 0\ \ 1}{\times\!-\!\times\!-\!\bullet})\to\mathcal{O}(\overset{2\ -2\ \ 2}{\times\!-\!\times\!-\!\bullet})\to\mathcal{O}(\overset{4\ -4\ \ 0}{\times\!-\!\times\!-\!\bullet})\to 0$$

and obtain (6.1) directly by zeroth direct image. The complex (6.3) is an example of a Bernstein-Gelfand-Gelfand (BGG) complex. More specifically, the fibers of μ are intrinsically $\mathbb{CP}_2$ and along μ (6.3) is a BGG complex on projective space, a simple general construction for which is given in [2]. In summary, although the filtering (5.1) is quite complicated, there is a relatively simple alternative resolution of $\mu^{-1}\mathcal{O}(\overset{a\ \ b\ \ c}{\times\!-\!\bullet\!-\!\bullet})$, namely

$$(6.4)\quad \mathcal{O}(\overset{a\ \ b\ \ c}{\times\!-\!\times\!-\!\bullet})\to\mathcal{O}(\overset{a+b+1\ \ -b-2\ \ b+c+1}{\times\!-\!\times\!-\!\bullet})\to\mathcal{O}(\overset{a+b+c+2\ \ -b-c-3\ \ b}{\times\!-\!\times\!-\!\bullet})\to 0$$

obtained by cancelling all but three of the composition factors from the coupled de Rham complex $\Omega^\bullet_\mu(\overset{a\ \ b\ \ c}{\times\!-\!\bullet\!-\!\bullet})$. This is the general BGG complex along the fibers of μ. It is elementarily constructed in [2] and used in [1] to effect the double fibration transform in this case. There is a spectral sequence constructed from (6.4), which is clearly concentrated in degree zero if $a \geq 0$. This is the BGG counterpart to Theorem 5.1. Writing $\Delta^\bullet_\mu(E)$ for the BGG complex along the fibers of μ associated to the bundle $E = \overset{a\ \ b\ \ c}{\times\!-\!\bullet\!-\!\bullet}$ on $Z = \mathbb{CP}_3$, firstly we have the following BGG counterpart to Theorem 2.1.

THEOREM 6.1. *For the double fibration* (5.3)*, and any irreducible homogeneous vector bundle E on $\mathbb{CP}_3^+$, there is a spectral sequence*

$$(6.5)\qquad E_1^{p,q} = \Gamma(\mathbb{M}^{++};\nu^q_*\Delta^p_\mu(E)) \Longrightarrow H^{p+q}(\mathbb{CP}_3^+;\mathcal{O}(E)).$$

PROOF. A simple variation on the proof of Theorem 2.1: details are in [1]. $\qquad\square$

Secondly, we have the following BGG counterpart to Theorem 5.1.

THEOREM 6.2. *The spectral sequence* (6.5) *controlling $H^r(\mathbb{CP}_3^+;\mathcal{O}(\overset{a\ \ b\ \ c}{\times\!-\!\bullet\!-\!\bullet}))$ is strictly concentrated in degree zero if $a \geq 0$.*

PROOF. As already discussed, this is clear from (6.4). $\qquad\square$

Notice that the proof of Theorem 6.2 is considerably more straightforward than the proof of Theorem 5.1 because the BGG resolution (6.4) is considerably more straightforward than the coupled de Rham complex obtained by combining (5.1) with (5.5). Furthermore, the BGG complex on projective space may be constructed without too much difficulty [2].

The advantages of using the BGG complex in this setting continue with BGG counterparts to Theorem 4.1 and Corollary 5.2, combined as follows.

PROPOSITION 6.3. *Let* $\kappa_{\mathbb{CP}_3} = \overset{-4}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$ *and* $\kappa_{\mathrm{Gr}_2(\mathbb{C}^4)} = \overset{0}{\bullet}\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{0}{\bullet}$ *denote the canonical bundles on* $\mathbb{CP}_3$ *and* $\mathrm{Gr}_2(\mathbb{C}^4)$, *respectively. Then there are canonical isomorphisms*

$$(6.6) \quad \nu_*^q \Delta_\mu^p(\kappa_{\mathbb{CP}_3} \otimes E^*) = \kappa_{\mathrm{Gr}_2(\mathbb{C}^4)} \otimes (\nu_*^{1-q} \Delta_\mu^{2-p}(E))^*, \quad \forall 0 \le p \le 2,\, 0 \le q \le 1.$$

The spectral sequence (6.5) *for the vector bundle* E *is (strictly) concentrated in top degree if and only if the corresponding spectral sequence for* $\kappa_{\mathbb{CP}_3} \otimes E^*$ *is (strictly) concentrated in degree zero. The spectral sequence* (6.5) *is concentrated in top degree (namely, first degree) if* $a \le -4-b-c$ *and in this case yields an isomorphism between* $H^r(\mathbb{CP}_3^+; \mathcal{O}(\overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet}))$ *and*

$$(6.7) \qquad \ker : \Gamma(\mathbb{M}^{++}; \mathcal{O}(\overset{-a-2}{\bullet}\!\!-\!\!\overset{a+b+1}{\times}\!\!-\!\!\overset{c}{\bullet})) \to \Gamma(\mathbb{M}^{++}; \mathcal{O}(\overset{-a-b-3}{\bullet}\!\!-\!\!\overset{a}{\times}\!\!-\!\!\overset{b+c+1}{\bullet})).$$

PROOF. For a linear differential operator $\delta : V \to W$ between any two vector bundles, there is its formal adjoint $\delta^* : \kappa \otimes W^* \to \kappa \otimes V^*$ and the BGG resolution $\Delta_\mu^\bullet(E^*)$ is the formal adjoint of $\Delta_\mu^\bullet(E)$. For the bundles themselves, this is easily verified from (6.4). Specifically,

$$\Delta_\mu^p(E^*) = \kappa_\mu \otimes \Delta_\mu^{2-p}(E)^*, \quad \forall 0 \le p \le 2,$$

where recall that, as in the proof of Theorem 4.1, we denote by $\kappa_\mu (= \overset{2}{\times}\!\!-\!\!\overset{-3}{\times}\!\!-\!\!\overset{0}{\bullet})$ the canonical bundle along the fibers of μ. Also notice from (6.4) that for any line bundle $L = \overset{k}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$ on $\mathbb{CP}_3$, we have $\Delta_\mu^\bullet(L \otimes E) = \mu^* L \otimes \Delta_\mu^\bullet(E)$. As in the proof of Theorem 4.1 we now use (4.2) to write $\mu^*(\kappa_{\mathbb{CP}_3}) \otimes \kappa_\mu = \nu^*(\kappa_{\mathrm{Gr}_2(\mathbb{C}^4)}) \otimes \kappa_\nu$ and deduce that

$$\Delta_\mu^p(\kappa_{\mathbb{CP}_3} \otimes E^*) = \mu^*(\kappa_{\mathbb{CP}_3}) \otimes \kappa_\mu \otimes \Delta_\mu^{2-p}(E)^* = \nu^*(\kappa_{\mathrm{Gr}_2(\mathbb{C}^4)}) \otimes \kappa_\nu \otimes \Delta_\mu^{2-p}(E)^*$$

and, therefore, that

$$\nu_*^q \Delta_\mu^p(\kappa_{\mathbb{CP}_3} \otimes E^*) = \kappa_{\mathrm{Gr}_2(\mathbb{C}^4)} \otimes \nu_*^q(\kappa_\nu \otimes \Delta_\mu^{2-p}(E)^*).$$

Serre duality along the fibers of ν yields (6.6). The remaining statements in this proposition are straightforward save for the appearance of the specific differential (6.7) operator whose kernel identifies the range of the double fibration transform. The bundles may be obtained by direct computation from (3.4), (6.4), and (6.5). In fact, once the bundles are identified, the differential operator acting between them is characterized by its G-invariance [**3**], which it manifestly enjoys. $\square$

Continuing the theme of formal adjoints implicit in (6.6), it is interesting to note that the operator $\nu_*^1 \Delta_\mu^0(E) \to \nu_*^1 \Delta_\mu^1(E)$ of (6.7) may also be obtained as the formal adjoint of $\nu_* \Delta_\mu^1(\kappa_{\mathbb{CP}_3} \otimes E^*) \to \nu_* \Delta_\mu^2(\kappa_{\mathbb{CP}_3} \otimes E^*)$. This alternative derivation is available more generally [**5**].

We shall now discuss the generality in which the BGG resolution can be brought to bear, as above, in order to improve our understanding of the double fibration transform. In brief, our considerations of the de Rham-based spectral sequence (2.1) went as follows. Our viewpoint was to interpret $H^r(D; \mathcal{O}(E))$ as an analytic object on $\mathcal{M}_D$. Roughly speaking, the steps were as follows.

- Establish isomorphisms $H^r(D; \mathcal{O}(E)) \cong H^r(\mathfrak{X}_D; \mu^{-1}(\mathcal{O}(E)))$, $\forall r$.
- Let $\Omega_\mu^p(E) = \Omega_\mu^p \otimes_{\mathcal{O}} \mathcal{O}(\mu^* E)$, the holomorphic p-forms along the fibers of μ, coupled with the pullback $\mu^* E$ of E to $\mathfrak{X}_D$.

- Combine these steps to establish Theorem 2.1, namely that there is a spectral sequence $E_1^{p,q} = \Gamma(\mathcal{M}_D; \nu_*^q \Omega_\mu^p(E)) \implies H^{p+q}(D; \mathcal{O}(E))$.
- Interpret the direct images $\nu_*^q \Omega_\mu^p(E) = \mathcal{O}(V^{(q)})$ in this spectral sequence as certain $(\mathfrak{g}, G_0)$-homogeneous vector bundles $V^{(q)} \to \mathcal{M}_D$.
- If $E \to D$ is *sufficiently negative* then the $\nu_*^q \Omega_\mu^p(E)$ are concentrated in degree $s = \dim_\mathbb{C}$(fibers of ν), and the spectral sequence collapses to
- $H^s(D; \mathcal{O}(E)) \cong \ker : \Gamma(\mathcal{M}_D, \nu_*^s \Omega_\mu^0(E)) \xrightarrow{\nabla} \Gamma(\mathcal{M}_D, \nu_*^s \Omega_\mu^1(E))$.

The big problem here is to do the computations that result in the required sufficient negativity. In good cases, this is facilitated by use of a BGG-based spectral sequence (see Theorem 6.1 and [1] more generally). The idea is that the BGG resolution introduces considerable simplification when $\mathcal{M}_D$ is a *bounded symmetric domain*.

So suppose $\mathcal{M}_D$ is a bounded symmetric domain G_0/K_0 and P is the parabolic subgroup of G such that $\mathcal{M}_Z = G/P$ is the complex flag manifolds that is the dual of $\mathcal{M}_D$.

The BGG resolution is constructed from the holomorphic de Rham resolution essentially as follows.

- Observe that μ has fiber $Q/(P \cap Q) = Q^{ss}/(P \cap Q^{ss})$, where Q^{ss} indicates the semisimple part.
- Let $W_{(\mathfrak{p} \cap \mathfrak{q})}^\mathfrak{q}$ consist of the minimal length Weyl group elements $w_{(\mathfrak{p} \cap \mathfrak{q})}^\mathfrak{q}$ representing the right cosets $W_{(\mathfrak{p} \cap \mathfrak{q})} \backslash W_\mathfrak{q}$.
- Define $\Delta_\mu^r(\lambda) = \sum \{\mathcal{O}_{G/(P \cap Q)}(w_{(\mathfrak{p} \cap \mathfrak{q})}^\mathfrak{q}.\lambda) \mid \text{length } \ell(w_{(\mathfrak{p} \cap \mathfrak{q})}^\mathfrak{q}) = r\}$, where λ is the extremal weight corresponding to the homogeneous bundle E.

The BGG counterpart to the de Rham-based spectral sequence is the following result (generalizing Theorem 6.1): If $E \to D$ is an irreducible G_0-homogeneous holomorphic vector bundle then there is a spectral sequence

$$(6.8) \qquad E_1^{p,q} = \Gamma(\mathcal{M}_D; \nu_*^q \Delta_\mu^p(E)) \implies H^{p+q}(D; \mathcal{O}(E)).$$

See [1] for this theorem and for computation of the homogeneous bundles $\Delta_\mu^p(E)$. The point is that all flag domain computations can be carried out for the simpler (compactified) correspondence of flag manifolds.

The case of flag domains D for which $\mathcal{M}_D$ is a bounded symmetric domain G_0/K_0 is known as the *Hermitian holomorphic* case: see [6] for details and the rough division of cycle spaces into various other cases. The double fibration (5.3) is the prototype of this sort of cycle space and is the basic correspondence of *twistor theory* [7]. This is the case in which the BGG resolution can be used as indicated above. The details will appear elsewhere [5] but here is the final conclusion regarding sufficient negativity of the bundle $E \to D$ in this case.

THEOREM 6.4. *In the Hermitian holomorphic case the spectral sequences (2.1) and (6.8) are concentrated in degree zero if the highest weight for E is dominant for G. These spectral sequences are concentrated in top degree if the highest weight for $\kappa_D \otimes E^*$ is dominant for G.*

Of course, for this theorem to be useful we need an effective way to compute the highest weights of κ_D and E^* from that of E. An efficient algorithm for this purpose is also given in [5].

7. Outlook

As sketched in §6, we are able to say in the Hermitian holomorphic case, when the spectral sequence concentrates in top degree. An effective criterion is given in Theorem 6.4. Furthermore, by using the BGG resolution and the improved spectral sequence (6.8) we are able to identify the range of the transform quite explicitly: the details for the twistor correspondence are in §6 and in general will appear in [**5**].

Another important case is where $\mathcal{B} = G_0/K_0$ is a bounded symmetric domain but the cycle space $\mathcal{M}_D$ is not $\mathcal{B}$ itself but rather $\mathcal{B} \times \overline{\mathcal{B}}$. Our double fibration (5.11) falls into this category, generally known as the *Hermitian non-holomorphic case* [**6**]. In this case, the duality of Theorem 4.1 means that we need only determine when the spectral sequence is concentrated in degree zero in order to have an effective criterion for sufficient negativity. One is confronted by diagrams generalizing (5.7), which may be approached as in §5. The analogue of Proposition 5.3 follows without too much difficulty. The analogue of Theorem 5.5 is awkward but, nevertheless, we conjecture that Theorem 6.4 also holds in the Hermitian non-holomorphic case.

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Mathematical Sciences Institute, Australian National University, ACT 0200, Australia
    E-mail address: meastwoo@member.ams.org

Department of Mathematics, University of California, Berkeley, California 94720–3840
    E-mail address: jawolf@math.berkeley.edu