# Principal series representations of infinite-dimensional Lie groups, I: Minimal parabolic subgroups

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*To Nolan Wallach on the occasion of his seventieth birthday*

**Abstract** We study the structure of minimal parabolic subgroups of the classical infinite-dimensional real simple Lie groups, corresponding to the classical simple direct limit Lie algebras. This depends on the recently developed structure of parabolic subgroups and subalgebras that are not necessarily direct limits of finite-dimensional parabolics. We then discuss the use of that structure theory for the infinite-dimensional analog of the classical principal series representations. We look at the unitary representation theory of the classical lim-compact groups $U(\infty)$, $SO(\infty)$ and $Sp(\infty)$ in order to construct the inducing representations, and we indicate some of the analytic considerations in the actual construction of the induced representations.

**Key words:** Principle series representation • Infinite-dimensional Lie group • Minimal parabolic subgroup

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## 1 Introduction

This paper discusses some recent developments in a program of extending aspects of real semisimple group representation theory to infinite-dimensional real Lie groups. The finite-dimensional theory is entwined with the structure

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of parabolic subgroups, and that structure has recently been worked out for the classical direct limit groups such as $SL(\infty, \mathbb{R})$ and $Sp(\infty; \mathbb{R})$. Here we explore the consequences of that structure theory for the construction of the counterpart of various Harish-Chandra series of representations, specifically the principal series.

The representation theory of finite-dimensional real semisimple Lie groups is based on the now-classical constructions and Plancherel formula of Harish-Chandra. Let $G$ be a real semisimple Lie group, e.g., $SL(n; \mathbb{R})$, $SU(p, q)$, $SO(p, q)$, $\dots$. Then one associates a series of representations to each conjugacy class of Cartan subgroups. Roughly speaking, this goes as follows. Let $\mathrm{Car}(G)$ denote the set of conjugacy classes $[H]$ of Cartan subgroups $H$ of $G$. Choose $[H] \in \mathrm{Car}(G)$, $H \in [H]$, and an irreducible unitary representation $\chi$ of $H$. Then we have a "cuspidal" parabolic subgroup $P$ of $G$ constructed from $H$, and a unitary representation $\pi_\chi$ of $G$ constructed from $\chi$ and $P$. Let $\Theta_{\pi_\chi}$ denote the distribution character of $\pi_\chi$. The Plancherel Formula: if $f \in \mathcal{C}(G)$, the Harish-Chandra Schwartz space, then

$$f(x) = \sum_{[H] \in \mathrm{Car}(G)} \int_{\widehat{H}} \Theta_{\pi_\chi}(r_x f) d\mu_{[H]}(\chi) \tag{1.1}$$

where $r_x$ is right translation and $\mu_{[H]}$ is a Plancherel measure on the unitary dual $\widehat{H}$.

In order to consider any elements of this theory in the context of real semisimple direct limit groups, we have to look more closely at the construction of the Harish-Chandra series that enter into (1.1).

Let $H$ be a Cartan subgroup of $G$. It is stable under a Cartan involution $\theta$, an involutive automorphism of $G$ whose fixed point set $K = G^\theta$ is a maximal compactly embedded[1] subgroup. Then $H$ has a $\theta$-stable decomposition $T \times A$ where $T = H \cap K$ is the compactly embedded part and (using lower case gothic letters for Lie algebras) $\exp : \mathfrak{a} \to A$ is a bijection. Then $\mathfrak{a}$ is commutative and acts diagonalizably on $\mathfrak{g}$. Any choice of a positive $\mathfrak{a}$-root system defines a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ in $\mathfrak{g}$ and thus defines a parabolic subgroup $P = MAN$ in $G$. If $\tau$ is an irreducible unitary representation of $M$ and $\sigma \in \mathfrak{a}^*$, then $\eta_{\tau,\sigma} : man \mapsto e^{i\sigma(\log a)}\tau(m)$ is a well defined irreducible unitary representation of $P$. The equivalence class of the unitarily induced representation $\pi_{\tau,\sigma} = \mathrm{Ind}_P^G(\eta_{\tau,\sigma})$ is independent of the choice of a positive $\mathfrak{a}$-root system. The group $M$ has (relative) discrete series representations, and $\{\pi_{\tau,\sigma} \mid \tau \text{ is a discrete series rep of } M\}$ is the series of unitary representations associated to $\{\mathrm{Ad}(g)H \mid g \in G\}$.

One of the most difficult points here is dealing with the discrete series. In fact the possibilities of direct limit representations of direct limit groups are somewhat limited except in cases where one can pass cohomologies through

[1] A subgroup of $G$ is *compactly embedded* if it has compact image under the adjoint representation of $G$.

direct limits without change of cohomology degree. See [14] for limits of holomorphic discrete series, [15] for Bott–Borel–Weil theory in the direct limit context, [11] for some nonholomorphic discrete series cases, and [24] for principal series of classical type. The principal series representations in (1.1) are those for which $M$ is compactly embedded in $G$, equivalently the ones for which $P$ is a minimal parabolic subgroup of $G$.

Here we work out the structure of the minimal parabolic subgroups of the finitary simple real Lie groups and discuss construction of the associated principal series representations. As in the finite-dimensional case, a minimal parabolic has structure $P = MAN$. Here $M = P \cap K$ is a (possibly infinite) direct product of torus groups, compact classical groups such as $Spin(n)$, $SU(n)$, $U(n)$ and $Sp(n)$, and their classical direct limits $Spin(\infty)$, $SU(\infty)$, $U(\infty)$ and $Sp(\infty)$ (modulo intersections and discrete central subgroups).

Since this setting is not standard we start by sketching the background. In Section 2 we recall the classical simple real direct limit Lie algebras and Lie groups. There are no surprises. Section 3 sketches their relatively recent theory of complex parabolic subalgebras. It is a little bit complicated and there are some surprises. Section 4 carries those results over to real parabolic subalgebras. There are no new surprises. Then in Sections 5 and 6 we deal with Levi components and Chevalley decompositions. That completes the background.

In Section 7 we examine the real group structure of Levi components of real parabolics. Then we specialize this to minimal self-normalizing parabolics in Section 8. There the Levi components are locally isomorphic to direct sums in an explicit way of subgroups that are either the compact classical groups $SU(n)$, $SO(n)$ or $Sp(n)$, or their limits $SU(\infty)$, $SO(\infty)$ or $Sp(\infty)$. The Chevalley (maximal reductive part) components are slightly more complicated, for example involving extensions $1 \to SU(*) \to U(*) \to T^1 \to 1$ as well as direct products with tori and vector groups. The main result is Theorem 8.3, which gives the structure of the minimal self-normalizing parabolics in terms similar to those of the finite dimensional case. Proposition 8.12 then gives an explicit construction for minimal parabolics with a given Levi factor.

In Section 9 we discuss the various possibilities for the inducing representation. There are many good choices, for example tame representations or more generally representations that are factors of type $II$. The theory is at such an early stage that the best choice is not yet clear.

Finally, in Section 10 we indicate construction of the induced representations in our infinite-dimensional setting. Smoothness conditions do not introduce surprises, but unitarity is a problem, and we defer details of that construction to [26] and applications to [27].

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## 2 The classical simple real groups

In this section we recall the real simple countably infinite-dimensional locally finite ("finitary") Lie algebras and the corresponding Lie groups. This material follows from results in [1], [2] and [6].

We start with the three classical simple locally finite countable-dimensional Lie algebras $\mathfrak{g}_{\mathbb{C}} = \varinjlim \mathfrak{g}_{n,\mathbb{C}}$, and their real forms $\mathfrak{g}_{\mathbb{R}}$. The Lie algebras $\mathfrak{g}_{\mathbb{C}}$ are the classical direct limits,

$$\begin{aligned} &\mathfrak{sl}(\infty,\mathbb{C}) = \varinjlim \mathfrak{sl}(n;\mathbb{C}),\\ &\mathfrak{so}(\infty,\mathbb{C}) = \varinjlim \mathfrak{so}(2n;\mathbb{C}) = \varinjlim \mathfrak{so}(2n+1;\mathbb{C}),\\ &\mathfrak{sp}(\infty,\mathbb{C}) = \varinjlim \mathfrak{sp}(n;\mathbb{C}), \end{aligned} \tag{2.1}$$

where the direct systems are given by the inclusions of the form $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We will also consider the locally reductive algebra $\mathfrak{gl}(\infty;\mathbb{C}) = \varinjlim \mathfrak{gl}(n;\mathbb{C})$ along with $\mathfrak{sl}(\infty;\mathbb{C})$. The direct limit process of (2.1) defines the universal enveloping algebras

$$\begin{aligned} &\mathcal{U}(\mathfrak{sl}(\infty,\mathbb{C})) = \varinjlim \mathcal{U}(\mathfrak{sl}(n;\mathbb{C})) \text{ and } \mathcal{U}(\mathfrak{gl}(\infty,\mathbb{C})) = \varinjlim \mathcal{U}(\mathfrak{gl}(n;\mathbb{C})),\\ &\mathcal{U}(\mathfrak{so}(\infty,\mathbb{C})) = \varinjlim \mathcal{U}(\mathfrak{so}(2n;\mathbb{C})) = \varinjlim \mathcal{U}(\mathfrak{so}(2n+1;\mathbb{C})), \text{ and}\\ &\mathcal{U}(\mathfrak{sp}(\infty,\mathbb{C})) = \varinjlim \mathcal{U}(\mathfrak{sp}(n;\mathbb{C})). \end{aligned} \tag{2.2}$$

Of course each of these Lie algebras $\mathfrak{g}_{\mathbb{C}}$ has the underlying structure of a real Lie algebra. Besides that, their real forms are as follows ([1], [2], [6]).

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(\infty;\mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of $\mathfrak{sl}(\infty;\mathbb{R}) = \varinjlim \mathfrak{sl}(n;\mathbb{R})$, the real special linear Lie algebra; $\mathfrak{sl}(\infty;\mathbb{H}) = \varinjlim \mathfrak{sl}(n;\mathbb{H})$, the quaternionic special linear Lie algebra, given by $\mathfrak{sl}(n;\mathbb{H}) := \mathfrak{gl}(n;\mathbb{H}) \cap \mathfrak{sl}(2n;\mathbb{C})$; $\mathfrak{su}(p,\infty) = \varinjlim \mathfrak{su}(p,n)$, the complex special unitary Lie algebra of real rank $p$; or $\mathfrak{su}(\infty,\infty) = \varinjlim \mathfrak{su}(p,q)$, complex special unitary algebra of infinite real rank.

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(\infty;\mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of $\mathfrak{so}(p,\infty) = \varinjlim \mathfrak{so}(p,n)$, the real orthogonal Lie algebra of finite real rank $p$; $\mathfrak{so}(\infty,\infty) = \varinjlim \mathfrak{so}(p,q)$, the real orthogonal Lie algebra of infinite real rank; or $\mathfrak{so}^*(2\infty) = \varinjlim \mathfrak{so}^*(2n)$

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(\infty;\mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of $\mathfrak{sp}(\infty;\mathbb{R}) = \varinjlim \mathfrak{sp}(n;\mathbb{R})$, the real symplectic Lie algebra; $\mathfrak{sp}(p,\infty) = \varinjlim \mathfrak{sp}(p,n)$, the quaternionic unitary Lie algebra of real rank $p$; or $\mathfrak{sp}(\infty,\infty) = \varinjlim \mathfrak{sp}(p,q)$, quaternionic unitary Lie algebra of infinite real rank.

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(\infty;\mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one $\mathfrak{gl}(\infty;\mathbb{R}) = \varinjlim \mathfrak{gl}(n;\mathbb{R})$, the real general linear Lie algebra; $\mathfrak{gl}(\infty;\mathbb{H}) = \varinjlim \mathfrak{gl}(n;\mathbb{H})$, the quaternionic general linear Lie algebra; $\mathfrak{u}(p,\infty) = \varinjlim \mathfrak{u}(p,n)$, the complex unitary Lie algebra of finite real rank $p$; or $\mathfrak{u}(\infty,\infty) = \varinjlim \mathfrak{u}(p,q)$, the complex unitary Lie algebra of infinite real rank.

As in (2.2), given one of these Lie algebras $\mathfrak{g}_{\mathbb{R}} = \varinjlim \mathfrak{g}_{n,\mathbb{R}}$ we have the universal enveloping algebra. We will need it for the induced representation process. As in the finite-dimensional case, we use the universal enveloping algebra of the complexification. Thus when we write $\mathcal{U}(\mathfrak{g}_{\mathbb{R}})$ it is understood that we mean $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. The reason for this is that we will want our representations of real Lie groups to be representations on complex vector spaces.

The corresponding Lie groups are exactly what one expects. First, the complex groups, viewed either as complex groups or as real groups,

$$\begin{aligned} &SL(\infty;\mathbb{C}) = \varinjlim SL(n;\mathbb{C}) \text{ and } GL(\infty;\mathbb{C}) = \varinjlim GL(n;\mathbb{C}),\\ &SO(\infty;\mathbb{C}) = \varinjlim SO(n;\mathbb{C}) = \varinjlim SO(2n;\mathbb{C}) = \varinjlim SO(2n+1;\mathbb{C}),\\ &Sp(\infty;\mathbb{C}) = \varinjlim Sp(n;\mathbb{C}). \end{aligned} \tag{2.3}$$

The real forms of the complex special and general linear groups $SL(\infty;\mathbb{C})$ and $GL(\infty;\mathbb{C})$ are

$$\begin{aligned} &SL(\infty;\mathbb{R}) \text{ and } GL(\infty;\mathbb{R}): \text{ real special/general linear groups},\\ &SL(\infty;\mathbb{H}): \text{ quaternionic special linear group},\\ &(S)U(p,\infty): \text{ (special) unitary groups of real rank } p<\infty,\\ &(S)U(\infty,\infty): \text{ (special) unitary groups of infinite real rank.} \end{aligned} \tag{2.4}$$

The real forms of the complex orthogonal and spin groups $SO(\infty;\mathbb{C})$ and $Spin(\infty;\mathbb{C})$ are

$$\begin{aligned} &SO(p,\infty),\ Spin(p;\infty): \text{ real orth./spin groups of real rank } p<\infty,\\ &SO(\infty,\infty),\ Spin(\infty,\infty): \text{ real orthog./spin groups of real rank } \infty,\\ &SO^*(2\infty) = \varinjlim SO^*(2n), \text{ which doesn't have a standard name} \end{aligned} \tag{2.5}$$

Here

$$SO^*(2n) = \{g \in SL(n;\mathbb{H}) \mid g \text{ preserves the form } \kappa(x,y) := \sum x^\ell i \bar{y}^\ell = {}^t x i \bar{y}\}.$$

Alternatively, $SO^*(2n) = SO(2n;\mathbb{C}) \cap U(n,n)$ with

$$SO(2n;C) \text{ defined by } (u,v) = \sum (u_j v_{n+jr} + v_{n+j} w_j).$$

Finally, the real forms of the complex symplectic group $Sp(\infty;\mathbb{C})$ are

$$\begin{aligned} &Sp(\infty;\mathbb{R}): \text{ real symplectic group},\\ &Sp(p,\infty): \text{ quaternion unitary group of real rank } p<\infty, \text{ and}\\ &Sp(\infty,\infty): \text{ quaternion unitary group of infinite real rank.} \end{aligned} \tag{2.6}$$

## 3 Complex parabolic subalgebras

In this section we recall the structure of parabolic subalgebras of $\mathfrak{gl}(\infty;\mathbb{C})$, $\mathfrak{sl}(\infty);\mathbb{C})$, $\mathfrak{so}(\infty;\mathbb{C})$ and $\mathfrak{sp}(\infty;\mathbb{C})$. We follow Dan–Cohen and Penkov ([3], [4]).

We first describe $\mathfrak{g}_\mathbb{C}$ in terms of linear spaces. Let $V$ and $W$ be nondegenerately paired countably infinite-dimensional complex vector spaces. Then $\mathfrak{gl}(\infty,\mathbb{C}) = \mathfrak{gl}(V,W) := V \otimes W$ consists of all finite linear combinations of the rank 1 operators $v \otimes w : x \mapsto \langle w, x\rangle v$. In the usual ordered basis of $V = \mathbb{C}^\infty$, parameterized by the positive integers, and with the dual basis of $W = V^* = (\mathbb{C}^\infty)^*$, we can view $\mathfrak{gl}(\infty,\mathbb{C})$ as infinite matrices with only finitely many nonzero entries. However $V$ has more exotic ordered bases, for example parameterized by the rational numbers, where the matrix picture is not intuitive.

The rank 1 operator $v \otimes w$ has a well-defined trace, so trace is well defined on $\mathfrak{gl}(\infty,\mathbb{C})$. Then $\mathfrak{sl}(\infty,\mathbb{C})$ is the traceless part, $\{g \in \mathfrak{gl}(\infty;\mathbb{C}) \mid \operatorname{trace} g = 0\}$.

In the orthogonal case we can take $V = W$ using the symmetric bilinear form that defines $\mathfrak{so}(\infty;\mathbb{C})$. Then

$$\mathfrak{so}(\infty;\mathbb{C}) = \mathfrak{so}(V,V) = \Lambda\mathfrak{gl}(\infty;\mathbb{C}) \text{ where } \Lambda(v \otimes v') = v \otimes v' - v' \otimes v.$$

In other words, in an ordered orthonormal basis of $V = \mathbb{C}^\infty$ parameterized by the positive integers, $\mathfrak{so}(\infty;\mathbb{C})$ can be viewed as the infinite antisymmetric matrices with only finitely many nonzero entries.

Similarly, in the symplectic case we can take $V = W$ using the antisymmetric bilinear form that defines $\mathfrak{sp}(\infty;\mathbb{C})$, and then

$$\mathfrak{sp}(\infty;\mathbb{C}) = \mathfrak{sp}(V,V) = S\mathfrak{gl}(\infty;\mathbb{C}) \text{ where } S(v \otimes v') = v \otimes v' + v' \otimes v.$$

In an appropriate ordered basis of $V = \mathbb{C}^\infty$ parameterized by the positive integers, $\mathfrak{sp}(\infty;\mathbb{C})$ can be viewed as the infinite symmetric matrices with only finitely many nonzero entries.

In the finite-dimensional setting, Borel subalgebra means a maximal solvable subalgebra, and parabolic subalgebra means one that contains a Borel. It is the same here except that one must use *locally solvable* to avoid the prospect of an infinite derived series.

**Definition 3.1.** A *Borel subalgebra* of $\mathfrak{g}_\mathbb{C}$ is a maximal locally solvable subalgebra. A *parabolic subalgebra* of $\mathfrak{g}_\mathbb{C}$ is a subalgebra that contains a Borel subalgebra. $\diamondsuit$

In the finite-dimensional setting a parabolic subalgebra is the stabilizer of an appropriate nested sequence of subspaces (possibly with an orientation condition in the orthogonal group case). In the infinite-dimensional setting here, one must be very careful as to which nested sequences of subspaces are appropriate. If $F$ is a subspace of $V$, then $F^\perp$ denotes its annihilator in $W$.

Similarly if $'F$ is a subspace of $W$, then $'F^\perp$ denotes its annihilator in $V$. We say that $F$ (resp. $'F$) is *closed* if $F = F^{\perp\perp}$ (resp. $'F = {'F}^{\perp\perp}$). This is the closure relation in the Mackey topology [13], i.e., the weak topology for the functionals on $V$ from $W$ and on $W$ from $V$.

In order to avoid repeating the following definitions later on, we make them in somewhat greater generality than we need just now.

**Definition 3.2.** Let $V$ and $W$ be countable dimensional vector spaces over a real division ring $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, with a nondegenerate bilinear pairing $\langle \cdot, \cdot \rangle : V \times W \to \mathbb{D}$. A *chain* or $\mathbb{D}$-*chain* in $V$ (resp. $W$) is a set of $\mathbb{D}$-subspaces totally ordered by inclusion. An *generalized* $\mathbb{D}$-*flag* in $V$ (resp. $W$) is a $\mathbb{D}$-chain such that each subspace has an immediate predecessor or an immediate successor in the inclusion ordering, and every nonzero vector of $V$ (or $W$) is caught between an immediate predecessor successor (IPS) pair. An generalized $\mathbb{D}$-flag $\mathcal{F}$ in $V$ (resp. $'\mathcal{F}$ in $W$) is *semiclosed* if $F \in \mathcal{F}$ with $F \neq F^{\perp\perp}$ implies $\{F, F^{\perp\perp}\}$ is an IPS pair (resp. $'F \in {'\mathcal{F}}$ with $'F \neq' F^{\perp\perp}$ implies $\{'F,' F^{\perp\perp}\}$ is an IPS pair). $\diamondsuit$

**Definition 3.3.** Let $\mathbb{D}$, $V$ and $W$ be as above. Generalized $\mathbb{D}$-flags $\mathcal{F}$ in $V$ and $'\mathcal{F}$ in $W$ form a *taut couple* when (i) if $F \in \mathcal{F}$ then $F^\perp$ is invariant by the $\mathfrak{gl}$-stabilizer of $'\mathcal{F}$ and (ii) if $'F \in {'\mathcal{F}}$, then its annihilator $'F^\perp$ is invariant by the $\mathfrak{gl}$-stabilizer of $\mathcal{F}$. $\diamondsuit$

In the $\mathfrak{so}$ and $\mathfrak{sp}$ cases one can use the associated bilinear form to identify $V$ with $W$ and $\mathcal{F}$ with $'\mathcal{F}$. Then we speak of a generalized flag $\mathcal{F}$ in $V$ as *self-taut*. If $\mathcal{F}$ is a self-taut generalized flag in $V$, then [6] every $F \in \mathcal{F}$ is either isotropic or coisotropic.

**Example 3.4.** Here is a quick peek at an obvious phenomenon introduced by infinite dimensionality. Enumerate bases of $V = \mathbb{C}^\infty$ and $W = \mathbb{C}^\infty$ by $(\mathbb{Z}^+)^2$, say $\{v_i = v_{i_1,i_2}\}$ and $\{w_j = w_{j_1,j_2}\}$, with $\langle v_i, w_j \rangle = 1$ if both $i_1 = j_1$ and $i_2 = j_2$ and $\langle v_i, w_j \rangle = 0$ otherwise. Define $\mathcal{F} = \{F_i\}$ ordered by inclusion where one builds up bases of the $F_i$ first with the $v_{i_1,1},\ i_1 \geqq 1$ and then the $v_{i_1,2},\ i_1 \geqq 1$ and then the $v_{i_1,3},\ i_1 \geqq 1$, and so on. One does the same for $'\mathcal{F}$ using the $\{w_j\}$. Now these form a taut couple of semiclosed generalized flags whose ordering involves an infinite number of limit ordinals. That makes it hard to use matrix methods. $\diamondsuit$

**Theorem 3.5.** *The self-normalizing parabolic subalgebras of the Lie algebras $\mathfrak{sl}(V,W)$ and $\mathfrak{gl}(V,W)$ are the normalizers of taut couples of semiclosed generalized flags in $V$ and $W$, and this is a one-to-one correspondence. The self-normalizing parabolic subalgebras of $\mathfrak{sp}(V)$ are the normalizers of self-taut semiclosed generalized flags in $V$, and this too is a one-to-one correspondence.*

**Theorem 3.6.** *The self-normalizing parabolic subalgebras of $\mathfrak{so}(V)$ are the normalizers of self-taut semiclosed generalized flags $\mathcal{F}$ in $V$, and there are two possibilities:*

(1) *the flag $\mathcal{F}$ is uniquely determined by the parabolic, or*
(2) *there are exactly three self-taut generalized flags with the same stabilizer as $\mathcal{F}$.*

*The latter case occurs precisely when there exists an isotropic subspace $L \in \mathcal{F}$ with $\dim_{\mathbb{C}} L^{\perp}/L = 2$. The three flags with the same stabilizer are then*

$$\{F \in \mathcal{F} \mid F \subset L \text{ or } L^{\perp} \subset F\},$$
$$\{F \in \mathcal{F} \mid F \subset L \text{ or } L^{\perp} \subset F\} \cup M_1,$$
$$\{F \in \mathcal{F} \mid F \subset L \text{ or } L^{\perp} \subset F\} \cup M_2,$$

*where $M_1$ and $M_2$ are the two maximal isotropic subspaces containing $L$.*

**Example 3.7.** Before proceeding we indicate an example which shows that not all parabolics are equal to their normalizers. Enumerate bases of $V = \mathbb{C}^\infty$ and $W = \mathbb{C}^\infty$ by rational numbers with pairing

$$\langle v_q, w_r \rangle = 1 \text{ if } q > r, \quad = 0 \text{ if } q \leqq r.$$

Then $\mathrm{Span}\{v_q \otimes w_r \mid q \leqq r\}$ is a Borel subalgebra of $\mathfrak{gl}(\infty)$ contained in $\mathfrak{sl}(\infty)$. This shows that $\mathfrak{sl}(\infty)$ is parabolic in $\mathfrak{gl}(\infty)$. $\diamondsuit$

One pinpoints this situation as follows. If $\mathfrak{p}$ is a (real or complex) subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{q}$ is a quotient algebra isomorphic to $\mathfrak{gl}(\infty;\mathbb{C})$, say with quotient map $f : \mathfrak{p} \to \mathfrak{q}$, then we refer to the composition $\mathit{trace} \circ f : \mathfrak{p} \to \mathbb{C}$ as an *infinite trace* on $\mathfrak{g}_{\mathbb{C}}$. If $\{f_i\}$ is a finite set of infinite traces on $\mathfrak{g}_{\mathbb{C}}$ and $\{c_i\}$ are complex numbers, then we refer to the condition $\sum c_i f_i = 0$ as an *infinite trace condition* on $\mathfrak{p}$.

These quotients can exist. In Example 3.4 we can take $V_a$ to be the span of the $v_{i_1,a}$ and $W_a$ the span of the the dual $w_{i_1,a}$ for $a = 1, 2, ...$ and then the normalizer of the taut couple $(\mathcal{F}, {}'\mathcal{F})$ has infinitely many quotients $\mathfrak{gl}(V_a, W_a)$.

**Theorem 3.8.** *The parabolic subalgebras $\mathfrak{p}$ in $\mathfrak{g}_{\mathbb{C}}$ are the algebras obtained from self-normalizing parabolics $\widetilde{\mathfrak{p}}$ by imposing infinite trace conditions.*

As a general principle one tries to be explicit by constructing representations that are as close to irreducible as feasible. For this reason we will construct principal series representations by inducing from parabolic subgroups that are minimal among the self-normalizing parabolic subgroups. Still, one should be aware of the phenomenon of Example 3.7 and Theorem 3.8.

# 4 Real parabolic subalgebras and subgroups

In this section we discuss the structure of parabolic subalgebras of real forms of the classical $\mathfrak{sl}(\infty,\mathbb{C})$, $\mathfrak{so}(\infty,\mathbb{C})$, $\mathfrak{sp}(\infty,\mathbb{C})$ and $\mathfrak{gl}(\infty,\mathbb{C})$. In this section $\mathfrak{g}_{\mathbb{C}}$ will always be one of them and $G_{\mathbb{C}}$ will be the corresponding connected complex Lie group. Also, $\mathfrak{g}_{\mathbb{R}}$ will be a real form of $\mathfrak{g}_{\mathbb{C}}$, and $G_{\mathbb{R}}$ will be the corresponding connected real subgroup of $G_{\mathbb{C}}$.

**Definition 4.1.** Let $\mathfrak{g}_\mathbb{R}$ be a real form of $\mathfrak{g}_\mathbb{C}$. Then a subalgebra $\mathfrak{p}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ is a *parabolic subalgebra* if its complexification $\mathfrak{p}_\mathbb{C}$ is a parabolic subalgebra of $\mathfrak{g}_\mathbb{C}$. $\diamondsuit$

When $\mathfrak{g}_\mathbb{R}$ has two inequivalent defining representations, in other words when

$$\mathfrak{g}_\mathbb{R} = \mathfrak{sl}(\infty;\mathbb{R}),\ \mathfrak{gl}(\infty;\mathbb{R}),\ \mathfrak{su}(*,\infty),\ \mathfrak{u}(*,\infty),\ \text{ or }\ \mathfrak{sl}(\infty;\mathbb{H}),$$

we denote them by $V_\mathbb{R}$ and $W_\mathbb{R}$, and when $\mathfrak{g}_\mathbb{R}$ has only one defining representation, in other words when

$$\mathfrak{g}_\mathbb{R} = \mathfrak{so}(*,\infty),\ \mathfrak{sp}(*,\infty),\ \mathfrak{sp}(\infty;\mathbb{R}),\ \text{ or }\ \mathfrak{so}^*(2\infty) \text{ as quaternion matrices},$$

we denote it by $V_\mathbb{R}$. The commuting algebra of $\mathfrak{g}_\mathbb{R}$ on $V_\mathbb{R}$ is a real division algebra $\mathbb{D}$. The main result of [6] is

**Theorem 4.2.** *Suppose that $\mathfrak{g}_\mathbb{R}$ has two inequivalent defining representations. Then a subalgebra of $\mathfrak{g}_\mathbb{R}$ (resp. subgroup of $G_\mathbb{R}$) is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_\mathbb{R}$-stabilizer (resp. $G_\mathbb{R}$-stabilizer) of a taut couple of generalized $\mathbb{D}$-flags $\mathcal{F}$ in $V_\mathbb{R}$ and $'\mathcal{F}$ in $W_\mathbb{R}$.*

*Suppose that $\mathfrak{g}_\mathbb{R}$ has only one defining representation. A subalgebra of $\mathfrak{g}_\mathbb{R}$ (resp. subgroup) of $G_\mathbb{R}$ is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_\mathbb{R}$-stabilizer (resp. $G_\mathbb{R}$-stabilizer) of a self-taut generalized $\mathbb{D}$-flag $\mathcal{F}$ in $V_\mathbb{R}$.*

## 5 Levi components of complex parabolics

In this section we discuss Levi components of complex parabolic subalgebras, recalling results from [8], [9], [4], [10], [5] and [25]. We start with the definition.

**Definition 5.1.** *Let $\mathfrak{p}$ be a locally finite Lie algebra and $\mathfrak{r}$ its locally solvable radical. A subalgebra $\mathfrak{l} \subset \mathfrak{p}$ is a* Levi component *if $[\mathfrak{p},\mathfrak{p}]$ is the semidirect sum $(\mathfrak{r} \cap [\mathfrak{p},\mathfrak{p}]) \in\!\!\!\!+ \mathfrak{l}$.* $\diamondsuit$

Every finitary Lie algebra has a Levi component. Evidently, Levi components are maximal semisimple subalgebras, but the converse fails for finitary Lie algebras. In any case, parabolic subalgebras of our classical Lie algebras $\mathfrak{g}_\mathbb{C}$ have maximal semisimple subalgebras, and those are their Levi components.

**Definition 5.2.** *Let $X \subset V$ and $Y \subset W$ be paired subspaces, isotropic in the orthogonal and symplectic cases. The subalgebras*

$$\begin{aligned} &\mathfrak{gl}(X,Y) \subset \mathfrak{gl}(V,W) &&\textit{and } \mathfrak{sl}(X,Y) \subset \mathfrak{sl}(V,W),\\ &\Lambda\mathfrak{gl}(X,Y) \subset \Lambda\mathfrak{gl}(V,V) &&\textit{and } S\mathfrak{gl}(X,Y) \subset S\mathfrak{gl}(V,V) \end{aligned}$$

*are called* standard. ♢

**Proposition 5.3.** *A subalgebra $\mathfrak{l}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is the Levi component of a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ if and only if it is the direct sum of standard special linear subalgebras and at most one subalgebra $\Lambda\mathfrak{gl}(X,Y)$ in the orthogonal case, and at most one subalgebra $S\mathfrak{gl}(X,Y)$ in the symplectic case.*

The occurrence of "at most one subalgebra" in Proposition 5.3 is analogous to the finite-dimensional case, where it is seen by deleting some simple root nodes from a Dynkin diagram.

Let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{sl}(V,W)$ or $\mathfrak{gl}(V,W)$ defined by the taut couple $(\mathcal{F}, {}'\mathcal{F})$ of semiclosed generalized flags.

**Definition 5.4.** Define two sets $J$ and ${}'J$ by

$$\begin{aligned} J &= \{(F', F'') \text{ IPS pair in } \mathcal{F} \mid F' = (F')^{\perp\perp} \text{ and } \dim F''/F' > 1\},\\ {}'J &= \{({}'F', {}'F'') \text{ IPS pair in } {}'\mathcal{F} \mid {}'F' = ({}'F')^{\perp\perp}, \dim {}'F''/{}'F' > 1\}. \end{aligned}$$

Since $V \times W \to \mathbb{C}$ is nondegenerate, the sets $J$ and ${}'J$ are in one-to-one correspondence by $(F''/F') \times ({}'F''/{}'F') \to \mathbb{C}$ is nondegenerate. We use this to identify $J$ with $J'$, and we write $(F_j', F_j'')$ and $({}'F_j', {}'F_j'')$ treating $J$ as an index set.

**Theorem 5.5.** *Let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{sl}(V,W)$ or $\mathfrak{gl}(V,W)$ defined by the taut couple $\mathcal{F}$ and ${}'\mathcal{F}$ of semiclosed generalized flags. For each $j \in J$ choose a subspace $X_j \subset V$ and a subspace $Y_j \subset W$ such that $F_j'' = X_j + F_j'$ and ${}'F_j'' = Y_j + {}'F_j{}'$ Then $\bigoplus_{j\in J} \mathfrak{sl}(X_j, Y_j)$ is a Levi component of $\mathfrak{p}$. The inclusion relations of $\mathcal{F}$ and ${}'\mathcal{F}$ induce a total order on $J$.*

*Conversely, if $\mathfrak{l}$ is a Levi component of $\mathfrak{p}$ then there exist subspaces $X_j \subset V$ and $Y_j \subset W$ such that $\mathfrak{l} = \bigoplus_{j\in J} \mathfrak{sl}(X_j, Y_j)$.*

Now the idea of finite matrices with blocks down the diagonal suggests the construction of $\mathfrak{p}$ from the totally ordered set $J$ and the direct sum $\mathfrak{l} = \bigoplus_{j\in J} \mathfrak{sl}(X_j, Y_j)$ of standard special linear algebras. We outline the idea of the construction; see [5]. First, $\langle X_j, Y_{j'}\rangle = 0$ for $j \neq j'$ because the $\mathfrak{s}_j = \mathfrak{sl}(X_j, Y_j)$ commute with each other. Define $U_j := ((\bigoplus_{k\leqq j} X_k)^\perp \oplus Y_j)^\perp$. Then one proves $U_j = ((U_j \oplus X_j)^\perp \oplus Y_j)^\perp$. From that, one shows that there is a unique semiclosed generalized flag $\mathcal{F}_{\min}$ in $V$ with the same stabilizer as the set $\{U_j, U_j \oplus X_j \mid j \in J\}$. One constructs similar subspaces ${}'U_j \subset W$ and shows that there is a unique semiclosed generalized flag ${}'\mathcal{F}_{\min}$ in $W$ with the same stabilizer as the set $\{{}'U_j, {}'U_j \oplus Y_j \mid j \in J\}$. In fact $(\mathcal{F}_{\min}, {}'\mathcal{F}_{\min})$ is the minimal taut couple with IPS pairs $U_j \subset (U_j \oplus X_j)$ in $\mathcal{F}_0$ and $(U_j \oplus X_j)^\perp \subset ((U_j \oplus X_j)^\perp \oplus Y_j)$ in ${}'\mathcal{F}_0$ for $j \in J$. If $(\mathcal{F}_{\max}, {}'\mathcal{F}_{\max})$ is maximal among the taut couples of semiclosed generalized flags with IPS pairs $U_j \subset (U_j \oplus X_j)$ in $\mathcal{F}_{\max}$ and $(U_j \oplus X_j)^\perp \subset ((U_j \oplus X_j)^\perp \oplus Y_j)$ in ${}'\mathcal{F}_{\max}$, then the corresponding parabolic $\mathfrak{p}$ has Levi component $\mathfrak{l}$.

The situation is essentially the same for Levi components of parabolic subalgebras of $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(\infty;\mathbb{C})$ or $\mathfrak{sp}(\infty;\mathbb{C})$, except that we modify Definition 5.4 of $J$ to add the condition that $F''$ be isotropic, and we add the orientation aspect of the $\mathfrak{so}$ case.

**Theorem 5.6.** *Let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(V)$ or $\mathfrak{sp}(V)$, defined by the self-taut semiclosed generalized flag $\mathcal{F}$. Let $\widetilde{F}$ be the union of all subspaces $F''$ in IPS pairs $(F', F'')$ of $\mathcal{F}$ for which $F''$ is isotropic. Let $\widetilde{{}'F}$ be the intersection of all subspaces $F'$ in IPS pairs for which $F'$ is closed ($F' = (F')^{\perp\perp}$) and coisotropic. Then $\mathfrak{l}$ is a Levi component of $\mathfrak{p}$ if and only if there are isotropic subspaces $X_j, Y_j$ in $V$ such that*

$$F''_j = F'_j + X_j \text{ and } {}'F''_j = {}'F_j + Y_j \text{ for every } j \in J$$

*and a subspace $Z$ in $V$ such that $\widetilde{F} = Z + \widetilde{{}'F}$, where $Z = 0$ in case $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(V)$ and $\dim \widetilde{F}/\widetilde{{}'F} \leqq 2$, such that*

$$\mathfrak{l} = \mathfrak{sp}(Z) \oplus \bigoplus\nolimits_{j\in J} \mathfrak{sl}(X_j, Y_j) \text{ if } \mathfrak{g}_\mathbb{C} = \mathfrak{sp}(V),$$
$$\mathfrak{l} = \mathfrak{so}(Z) \oplus \bigoplus\nolimits_{j\in J} \mathfrak{sl}(X_j, Y_j) \text{ if } \mathfrak{g}_\mathbb{C} = \mathfrak{so}(V).$$

*Further, the inclusion relations of $\mathcal{F}$ induce a total order on $J$ which leads to a construction of $\mathfrak{p}$ from $\mathfrak{l}$.*

## 6 Chevalley decomposition

In this section we apply the extension [4] to our parabolic subalgebras, of the Chevalley decomposition for a (finite-dimensional) algebraic Lie algebra.

Let $\mathfrak{p}$ be a locally finite linear Lie algebra, in our case a subalgebra of $\mathfrak{gl}(\infty)$. Every element $\xi \in \mathfrak{p}$ has a Jordan canonical form, yielding a decomposition $\xi = \xi_{\mathrm{ss}} + \xi_{\mathrm{nil}}$ into semisimple and nilpotent parts. The algebra $\mathfrak{p}$ is *splittable* if it contains the semisimple and the nilpotent parts of each of its elements. Note that $\xi_{\mathrm{ss}}$ and $\xi_{\mathrm{nil}}$ are polynomials in $\xi$; this follows from the finite-dimensional fact. In particular, if $X$ is any $\xi$-invariant subspace of $V$, then it is invariant under both $\xi_{\mathrm{ss}}$ and $\xi_{\mathrm{nil}}$.

Conversely, parabolic subalgebras (and many others) of our classical Lie algebras $\mathfrak{g}$ are splittable.

The *linear nilradical* of a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is the set $\mathfrak{p}_{\mathrm{nil}}$ of all nilpotent elements of the locally solvable radical $\mathfrak{r}$ of $\mathfrak{p}$. It is a locally nilpotent ideal in $\mathfrak{p}$ and satisfies $\mathfrak{p}_{\mathrm{nil}} \cap [\mathfrak{p}, \mathfrak{p}] = \mathfrak{r} \cap [\mathfrak{p}, \mathfrak{p}]$.

If $\mathfrak{p}$ is splittable, then it has a well-defined maximal locally reductive subalgebra $\mathfrak{p}_{\rm red}$. This means that $\mathfrak{p}_{\rm red}$ is an increasing union of finite-dimensional reductive Lie algebras, each reductive in the next. In particular $\mathfrak{p}_{\rm red}$ maps isomorphically under the projection $\mathfrak{p} \to \mathfrak{p}/\mathfrak{p}_{\rm nil}$. That gives a semidirect sum decomposition $\mathfrak{p} = \mathfrak{p}_{\rm nil} \inplus \mathfrak{p}_{\rm red}$ analogous to the Chevalley decomposition mentioned above. Also, here,

$$\mathfrak{p}_{\rm red} = \mathfrak{l} \inplus \mathfrak{t} \quad \text{and} \quad [\mathfrak{p}_{\rm red}, \mathfrak{p}_{\rm red}] = \mathfrak{l}$$

where $\mathfrak{t}$ is a toral subalgebra and $\mathfrak{l}$ is the Levi component of $\mathfrak{p}$. A glance at $\mathfrak{u}(\infty)$ or $\mathfrak{gl}(\infty;\mathbb{C})$ shows that the semidirect sum decomposition of $\mathfrak{p}_{\rm red}$ need not be direct.

## 7 Levi and Chevalley components of real parabolics

Now we adapt the material of Sections 5 and 6 to study Levi and Chevalley components of real parabolic subalgebras in the real classical Lie algebras.

Let $\mathfrak{g}_\mathbb{R}$ be a real form of a classical locally finite complex simple Lie algebra $\mathfrak{g}_\mathbb{C}$. Consider a real parabolic subalgebra $\mathfrak{p}_\mathbb{R}$. It has form $\mathfrak{p}_\mathbb{R} = \mathfrak{p}_\mathbb{C} \cap \mathfrak{g}_\mathbb{R}$ where its complexification $\mathfrak{p}_\mathbb{C}$ is parabolic in $\mathfrak{g}_\mathbb{C}$. Let $\tau$ denote complex conjugation of $\mathfrak{g}_\mathbb{C}$ over $\mathfrak{g}_\mathbb{R}$. Then the locally solvable radical $\mathfrak{r}_\mathbb{C}$ of $\mathfrak{p}_\mathbb{C}$ is $\tau$-stable because $\mathfrak{r}_\mathbb{C} + \tau\mathfrak{r}_\mathbb{C}$ is a locally solvable ideal, so the locally solvable radical $\mathfrak{r}_\mathbb{R}$ of $\mathfrak{p}_\mathbb{R}$ is a real form of $\mathfrak{r}_\mathbb{C}$.

Let $\mathfrak{l}_\mathbb{R}$ be a maximal semisimple subalgebra of $\mathfrak{p}_\mathbb{R}$. Its complexification $\mathfrak{l}_\mathbb{C}$ is a maximal semisimple subalgebra, hence a Levi component, of $\mathfrak{p}_\mathbb{C}$. Thus $[\mathfrak{p}_\mathbb{C}, \mathfrak{p}_\mathbb{C}]$ is the semidirect sum $(\mathfrak{r}_\mathbb{C} \cap [\mathfrak{p}_\mathbb{C}, \mathfrak{p}_\mathbb{C}]) \inplus \mathfrak{l}_\mathbb{C}$. The elements of this formula all are $\tau$-stable, so we have proved

**Lemma 7.1.** *The Levi components of $\mathfrak{p}_\mathbb{R}$ are real forms of the Levi components of $\mathfrak{p}_\mathbb{C}$.*

If $\mathfrak{g}_\mathbb{C}$ is $\mathfrak{sl}(V,W)$ or $\mathfrak{gl}(V,W)$ as in Theorem 5.5, then $\mathfrak{l}_\mathbb{C} = \bigoplus_{j \in J} \mathfrak{sl}(X_j, Y_j)$ as indicated there. Initially the possibilities for the action of $\tau$ are

- $\tau$ preserves $\mathfrak{sl}(X_j, Y_j)$ with fixed point set $\mathfrak{sl}(X_{j,\mathbb{R}}, Y_{j,\mathbb{R}}) \cong \mathfrak{sl}(*;\mathbb{R})$,
- $\tau$ preserves $\mathfrak{sl}(X_j, Y_j)$ with fixed point set $\mathfrak{sl}(X_{j,\mathbb{H}}, Y_{j,\mathbb{H}}) \cong \mathfrak{sl}(*;\mathbb{H})$,
- $\tau$ preserves $\mathfrak{sl}(X_j, Y_j)$ with f.p. set $\mathfrak{su}(X_j', X_j'') \cong \mathfrak{su}(*,*)$, $X_j = X_j' + X_j''$, and
- $\tau$ interchanges two summands $\mathfrak{sl}(X_j, Y_j)$ and $\mathfrak{sl}(X_{j'}, Y_{j'})$ of $\mathfrak{l}_\mathbb{C}$, with fixed point set the diagonal ($\cong \mathfrak{sl}(X_j, Y_j)$) of their direct sum.

If $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(V)$ as in Theorem 5.6, $\mathfrak{l}_\mathbb{C}$ can also have a summand $\mathfrak{so}(Z)$, or if $\mathfrak{g}_\mathbb{C} = \mathfrak{sp}(V)$ it can also have a summand $\mathfrak{sp}(V)$. Except when $A_4 = D_3$ occurs, these additional summands must be $\tau$-stable, resulting in fixed point sets

- when $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(V)$: $\mathfrak{so}(Z)^\tau$ is $\mathfrak{so}(*,*)$ or $\mathfrak{so}^*(2\infty)$,

- when $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(V)$: $\mathfrak{sp}(Z)^\tau$ is $\mathfrak{sp}(*,*)$ or $\mathfrak{sp}(*;\mathbb{R})$.

## 8 Minimal parabolic subgroups

We describe the structure of minimal parabolic subgroups of the classical real simple Lie groups $G_{\mathbb{R}}$.

**Proposition 8.1.** *Let $\mathfrak{p}_{\mathbb{R}}$ be a parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{l}_{\mathbb{R}}$ a Levi component of $\mathfrak{p}_{\mathbb{R}}$. If $\mathfrak{p}_{\mathbb{R}}$ is a minimal parabolic subalgebra, then $\mathfrak{l}_{\mathbb{R}}$ is a direct sum of finite-dimensional compact algebras $\mathfrak{su}(p)$, $\mathfrak{so}(p)$ and $\mathfrak{sp}(p)$, and their infinite-dimensional limits $\mathfrak{su}(\infty)$, $\mathfrak{so}(\infty)$ and $\mathfrak{sp}(\infty)$. If $\mathfrak{l}_{\mathbb{R}}$ is a direct sum of finite-dimensional compact algebras $\mathfrak{su}(p)$, $\mathfrak{so}(p)$ and $\mathfrak{sp}(p)$ and their limits $\mathfrak{su}(\infty)$, $\mathfrak{so}(\infty)$ and $\mathfrak{sp}(\infty)$, then $\mathfrak{p}_{\mathbb{R}}$ contains a minimal parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ with the same Levi component $\mathfrak{l}_{\mathbb{R}}$.*

*Proof.* Suppose that $\mathfrak{p}_{\mathbb{R}}$ is a minimal parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$. If a direct summand $\mathfrak{l}'_{\mathbb{R}}$ of $\mathfrak{l}_{\mathbb{R}}$ has a proper parabolic subalgebra $\mathfrak{q}_{\mathbb{R}}$, we replace $\mathfrak{l}'_{\mathbb{R}}$ by $\mathfrak{q}_{\mathbb{R}}$ in $\mathfrak{l}_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}$. In other words we refine the flag(s) that define $\mathfrak{p}_{\mathbb{R}}$. The refined flag defines a parabolic $\mathfrak{q}_{\mathbb{R}} \subsetneqq \mathfrak{p}_{\mathbb{R}}$. This contradicts minimality. Thus no summand of $\mathfrak{l}_{\mathbb{R}}$ has a proper parabolic subalgebra. Theorems 5.5 and 5.6 show that $\mathfrak{su}(p)$, $\mathfrak{so}(p)$ and $\mathfrak{sp}(p)$, and their limits $\mathfrak{su}(\infty)$, $\mathfrak{so}(\infty)$ and $\mathfrak{sp}(\infty)$, are the only possibilities for the simple summands of $\mathfrak{l}_{\mathbb{R}}$.

Conversely suppose that the summands of $\mathfrak{l}_{\mathbb{R}}$ are $\mathfrak{su}(p)$, $\mathfrak{so}(p)$ and $\mathfrak{sp}(p)$ or their limits $\mathfrak{su}(\infty)$, $\mathfrak{so}(\infty)$ and $\mathfrak{sp}(\infty)$. Let $(\mathcal{F}, {}'\mathcal{F})$ or $\mathcal{F}$ be the flag(s) that define $\mathfrak{p}_{\mathbb{R}}$. In the discussion between Theorems 5.5 and 5.6 we described a minimal taut couple $(\mathcal{F}_{\min}, {}'\mathcal{F}_{\min})$ and a maximal taut couple $(\mathcal{F}_{\max}, {}'\mathcal{F}_{\max})$ (in the $\mathfrak{sl}$ and $\mathfrak{gl}$ cases) of semiclosed generalized flags which define parabolics that have the same Levi component $\mathfrak{l}_{\mathbb{C}}$ as $\mathfrak{p}_{\mathbb{C}}$. By construction $(\mathcal{F}, {}'\mathcal{F})$ refines $(\mathcal{F}_{\min}, {}'\mathcal{F}_{\min})$ and $(\mathcal{F}_{\max}, {}'\mathcal{F}_{\max})$ refines $(\mathcal{F}, {}'\mathcal{F})$. As $(\mathcal{F}_{\min}, {}'\mathcal{F}_{\min})$ is uniquely defined from $(\mathcal{F}, {}'\mathcal{F})$ it is $\tau$-stable. Now the maximal $\tau$-stable taut couple $(\mathcal{F}^*_{\max}, {}'\mathcal{F}^*_{\max})$ of semiclosed generalized flags defines a $\tau$-stable parabolic $\mathfrak{q}_{\mathbb{C}}$ with the same Levi component $\mathfrak{l}_{\mathbb{C}}$ as $\mathfrak{p}_{\mathbb{C}}$, and $\mathfrak{q}_{\mathbb{R}} := \mathfrak{q}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{R}}$ is a minimal parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ with Levi component $\mathfrak{l}_{\mathbb{R}}$.

The argument is the same when $\mathfrak{g}_{\mathbb{C}}$ is $\mathfrak{so}$ or $\mathfrak{sp}$. □

Proposition 8.1 says that the Levi components of the minimal parabolics are the compact real forms, in the sense of [21], of the complex $\mathfrak{sl}$, $\mathfrak{so}$ and $\mathfrak{sp}$. We extend this notion.

The group $G_{\mathbb{R}}$ has the natural *Cartan involution* $\theta$ such that $d\theta((\mathfrak{p}_{\mathbb{R}})_{\mathrm{red}}) = (\mathfrak{p}_{\mathbb{R}})_{\mathrm{red}}$, defined as follows. Every element of $\mathfrak{l}_{\mathbb{R}}$ is elliptic, and $(\mathfrak{p}_{\mathbb{R}})_{\mathrm{red}} = \mathfrak{l}_{\mathbb{R}} \subsetplus \mathfrak{t}_{\mathbb{R}}$ where $\mathfrak{t}_{\mathbb{R}}$ is toral, so every element of $(\mathfrak{p}_{\mathbb{R}})_{\mathrm{red}}$ is semisimple. (This is where we use minimality of the parabolic $\mathfrak{p}_{\mathbb{R}}$.) Thus $(\mathfrak{p}_{\mathbb{R}})_{\mathrm{red}} \cap \mathfrak{g}_{n,\mathbb{R}}$ is reductive in $\mathfrak{g}_{m,\mathbb{R}}$ for every $m \geqq n$. Consequently we have Cartan involutions $\theta_n$ of the groups $G_{n,\mathbb{R}}$ such that $\theta_{n+1}|_{G_{n,\mathbb{R}}} = \theta_n$ and $d\theta_n((\mathfrak{p}_{\mathbb{R}})_{\mathrm{red}} \cap \mathfrak{g}_{n,\mathbb{R}}) = (\mathfrak{p}_{\mathbb{R}})_{\mathrm{red}} \cap \mathfrak{g}_{n,\mathbb{R}}$.

Now $\theta = \varinjlim \theta_n$ (in other words $\theta|_{G_{n,\mathbb{R}}} = \theta_n$) is the desired Cartan involution of $\mathfrak{g}_\mathbb{R}$. Note that $\mathfrak{l}_\mathbb{R}$ is contained in the fixed point set of $d\theta$.

The Lie algebra $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} + \mathfrak{s}_\mathbb{R}$ where $\mathfrak{k}_\mathbb{R}$ is the $(+1)$-eigenspace of $d\theta$ and $\mathfrak{s}_\mathbb{R}$ is the $(-1)$-eigenspace. The fixed point set $K_\mathbb{R} = G_\mathbb{R}^\theta$ is the direct limit of the maximal compact subgroups $K_{n,\mathbb{R}} = G_{n,\mathbb{R}}^{\theta_n}$. We will refer to $K_\mathbb{R}$ as a *maximal lim-compact subgroup* of $G_\mathbb{R}$ and to $\mathfrak{k}_\mathbb{R}$ as a maximal *lim-compact subalgebra* of $\mathfrak{g}_\mathbb{R}$. By construction $\mathfrak{l}_\mathbb{R} \subset \mathfrak{k}_\mathbb{R}$, as in the case of finite-dimensional minimal parabolics. Also as in the finite-dimensional case (and using the same proof), $[\mathfrak{k}_\mathbb{R}, \mathfrak{k}_\mathbb{R}] \subset \mathfrak{k}_\mathbb{R}$, $[\mathfrak{k}_\mathbb{R}, \mathfrak{s}_\mathbb{R}] \subset \mathfrak{s}_\mathbb{R}$ and $[\mathfrak{s}_\mathbb{R}, \mathfrak{s}_\mathbb{R}] \subset \mathfrak{k}_\mathbb{R}$.

**Lemma 8.2.** *Decompose* $(\mathfrak{p}_\mathbb{R})_{\mathrm{red}} = \mathfrak{m}_\mathbb{R} + \mathfrak{a}_\mathbb{R}$ *where* $\mathfrak{m}_\mathbb{R} = (\mathfrak{p}_\mathbb{R})_{\mathrm{red}} \cap \mathfrak{k}_\mathbb{R}$ *and* $\mathfrak{a}_\mathbb{R} = (\mathfrak{p}_\mathbb{R})_{\mathrm{red}} \cap \mathfrak{s}_\mathbb{R}$*. Then* $\mathfrak{m}_\mathbb{R}$ *and* $\mathfrak{a}_\mathbb{R}$ *are ideals in* $(\mathfrak{p}_\mathbb{R})_{\mathrm{red}}$ *with* $\mathfrak{a}_\mathbb{R}$ *commutative. In particular* $(\mathfrak{p}_\mathbb{R})_{\mathrm{red}} = \mathfrak{m}_\mathbb{R} \oplus \mathfrak{a}_\mathbb{R}$*, direct sum of ideals.*

*Proof.* Since $\mathfrak{l}_\mathbb{R} = [(\mathfrak{p}_\mathbb{R})_{\mathrm{red}}, (\mathfrak{p}_\mathbb{R})_{\mathrm{red}}]$ we compute $[\mathfrak{m}_\mathbb{R}, \mathfrak{a}_\mathbb{R}] \subset \mathfrak{l}_\mathbb{R} \cap \mathfrak{a}_\mathbb{R} = 0$. In particular $[[\mathfrak{a}_\mathbb{R}, \mathfrak{a}_\mathbb{R}], \mathfrak{a}_\mathbb{R}] = 0$. So $[\mathfrak{a}_\mathbb{R}, \mathfrak{a}_\mathbb{R}]$ is a commutative ideal in the semisimple algebra $\mathfrak{l}_\mathbb{R}$, in other words $\mathfrak{a}_\mathbb{R}$ is commutative. □

The main result of this section is the following generalization of the standard decomposition of a finite-dimensional real parabolic. We have formulated it to emphasize the parallel with the finite-dimensional case. However some details of the construction are rather different; see Proposition 8.12 and the discussion leading up to it.

**Theorem 8.3.** *The minimal parabolic subalgebra* $\mathfrak{p}_\mathbb{R}$ *of* $\mathfrak{g}_\mathbb{R}$ *decomposes as* $\mathfrak{p}_\mathbb{R} = \mathfrak{m}_\mathbb{R} + \mathfrak{a}_\mathbb{R} + \mathfrak{n}_\mathbb{R} = \mathfrak{n}_\mathbb{R} \Subset (\mathfrak{m}_\mathbb{R} \oplus \mathfrak{a}_\mathbb{R})$*, where* $\mathfrak{a}_\mathbb{R}$ *is commutative, the Levi component* $\mathfrak{l}_\mathbb{R}$ *is an ideal in* $\mathfrak{m}_\mathbb{R}$ *, and* $\mathfrak{n}_\mathbb{R}$ *is the linear nilradical* $(\mathfrak{p}_\mathbb{R})_{\mathrm{nil}}$*. On the group level,* $P_\mathbb{R} = M_\mathbb{R} A_\mathbb{R} N_\mathbb{R} = N_\mathbb{R} \ltimes (M_\mathbb{R} \times A_\mathbb{R})$ *where* $N_\mathbb{R} = \exp(\mathfrak{n}_\mathbb{R})$ *is the linear unipotent radical of* $P_\mathbb{R}$*,* $A_\mathbb{R} = \exp(\mathfrak{a}_\mathbb{R})$ *is isomorphic to a vector group, and* $M_\mathbb{R} = P_\mathbb{R} \cap K_\mathbb{R}$ *is limit–compact with Lie algebra* $\mathfrak{m}_\mathbb{R}$*.*

*Proof.* The algebra level statements come out of Lemma 8.2 and the semidirect sum decomposition $\mathfrak{p}_\mathbb{R} = (\mathfrak{p}_\mathbb{R})_{\mathrm{nil}} \Subset (\mathfrak{p}_\mathbb{R})_{\mathrm{red}}$.

For the group level statements, we need only check that $K_\mathbb{R}$ meets every topological component of $P_\mathbb{R}$. Even though $P_\mathbb{R} \cap G_{n,\mathbb{R}}$ need not be parabolic in $G_{n,\mathbb{R}}$, the group $P_\mathbb{R} \cap \theta P_\mathbb{R} \cap G_{n,\mathbb{R}}$ is reductive in $G_{n,\mathbb{R}}$ and $\theta_n$-stable, so $K_{n,\mathbb{R}}$ meets each of its components. Now $K_\mathbb{R}$ meets every component of $P_\mathbb{R} \cap \theta P_\mathbb{R}$. The linear unipotent radical of $P_\mathbb{R}$ has Lie algebra $\mathfrak{n}_\mathbb{R}$ and thus must be equal to $\exp(\mathfrak{n}_\mathbb{R})$, so it does not effect components. Thus every component of $P_{\mathrm{red}}$ is represented by an element of $K_\mathbb{R} \cap P_\mathbb{R} \cap \theta P_\mathbb{R} = K_\mathbb{R} \cap P_\mathbb{R} = M_\mathbb{R}$. That derives $P_\mathbb{R} = M_\mathbb{R} A_\mathbb{R} N_\mathbb{R} = N_\mathbb{R} \ltimes (M_\mathbb{R} \times A_\mathbb{R})$ from $\mathfrak{p}_\mathbb{R} = \mathfrak{m}_\mathbb{R} + \mathfrak{a}_\mathbb{R} + \mathfrak{n}_\mathbb{R} = \mathfrak{n}_\mathbb{R} \Subset (\mathfrak{m}_\mathbb{R} \oplus \mathfrak{a}_\mathbb{R})$. □

The reductive part of the group $\mathfrak{p}_\mathbb{R}$ can be constructed explicitly. We do this for the cases where $\mathfrak{g}_\mathbb{R}$ is defined by a hermitian form $f : V_\mathbb{F} \times V_\mathbb{F} \to \mathbb{F}$ where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The idea is the same for the other cases. See Proposition 8.12 below.

Write $V_{\mathbb{F}}$ for $V_{\mathbb{R}}$, $V_{\mathbb{C}}$ or $V_{\mathbb{H}}$, as appropriate, and similarly for $W_{\mathbb{F}}$. We use $f$ for an $\mathbb{F}$-conjugate-linear identification of $V_{\mathbb{F}}$ and $W_{\mathbb{F}}$. We are dealing with a minimal Levi component $\mathfrak{l}_{\mathbb{R}} = \bigoplus_{j\in J} \mathfrak{l}_{j,\mathbb{R}}$ where the $\mathfrak{l}_{j,\mathbb{R}}$ are simple. Let $X_{\mathbb{F}}$ denote the sum of the corresponding subspaces $(X_j)_{\mathbb{F}} \subset V_{\mathbb{F}}$ and $Y_{\mathbb{F}}$ the analogous sum of the $(Y_j)_{\mathbb{F}} \subset W_{\mathbb{F}}$. Then $X_{\mathbb{F}}$ and $Y_{\mathbb{F}}$ are nondegenerately paired. Of course they may be small, even zero. In any case,

$$\begin{aligned} &V_{\mathbb{F}} = X_{\mathbb{F}} \oplus Y_{\mathbb{F}}^{\perp}\,, W_{\mathbb{F}} = Y_{\mathbb{F}} \oplus X_{\mathbb{F}}^{\perp},\ \text{and}\\ &X_{\mathbb{F}}^{\perp} \text{ and } Y_{\mathbb{F}}^{\perp} \text{ are nondegenerately paired.}\end{aligned} \tag{8.4}$$

These direct sum decompositions (8.4) now become

$$V_{\mathbb{F}} = X_{\mathbb{F}} \oplus X_{\mathbb{F}}^{\perp} \quad \text{and} \quad f \text{ is nondegenerate on each summand.} \tag{8.5}$$

Let $X'$ and $X''$ be paired maximal isotropic subspaces of $X_{\mathbb{F}}^{\perp}$. Then

$$V_{\mathbb{F}} = X_{\mathbb{F}} \oplus (X'_{\mathbb{F}} \oplus X''_{\mathbb{F}}) \oplus Q_{\mathbb{F}} \text{ where } Q_{\mathbb{F}} := (X_{\mathbb{F}} \oplus (X'_{\mathbb{F}} \oplus X''_{\mathbb{F}}))^{\perp}. \tag{8.6}$$

The subalgebra $\{\xi \in \mathfrak{g}_{\mathbb{R}} \mid \xi(X_{\mathbb{F}} \oplus Q_{\mathbb{F}}) = 0\}$ of $\mathfrak{g}_{\mathbb{R}}$ has a maximal toral subalgebra $\mathfrak{a}\dagger_{\mathbb{R}}$, contained in $\mathfrak{s}_{\mathbb{R}}$, in which every element has all eigenvalues real. One example, which is diagonalizable (in fact diagonal) over $\mathbb{R}$, is

$$\begin{aligned} \mathfrak{a}_{\mathbb{R}}^{\dagger} = \bigoplus\nolimits_{\ell\in C} \mathfrak{gl}(x'_\ell\mathbb{R}, x''_\ell\mathbb{R}) \text{ where}&\\ \{x'_\ell \mid \ell \in C\} \text{ is a basis of } X'_{\mathbb{F}} \text{ and}&\\ \{x''_\ell \mid \ell \in C\} \text{ is the dual basis of } X''_{\mathbb{F}}.&\end{aligned} \tag{8.7}$$

We interpolate the self-taut semiclosed generalized flag $\mathcal{F}$ defining $\mathfrak{p}$ with the subspaces $x'_\ell\mathbb{R} \oplus x''_\ell\mathbb{R}$. Any such interpolation (and usually there will be infinitely many) gives a self-taut semiclosed generalized flag $\mathcal{F}^{\dagger}$ and defines a minimal self-normalizing parabolic subalgebra $\mathfrak{p}_{\mathbb{R}}^{\dagger}$ of $\mathfrak{g}_{\mathbb{R}}$ with the same Levi component as $\mathfrak{p}_{\mathbb{R}}$. The decompositions corresponding to (8.4), (8.5) and (8.6) are given by $X_{\mathbb{F}}^{\dagger} = X_{\mathbb{F}} \oplus (X'_{\mathbb{F}} \oplus X''_{\mathbb{F}})$ and $Q_{\mathbb{F}}^{\dagger} = Q_{\mathbb{F}}$.

In addition, the subalgebra $\{\xi \in \mathfrak{p}_{\mathbb{R}} \mid \xi(X_{\mathbb{F}} \oplus (X'_{\mathbb{F}} \oplus X''_{\mathbb{F}})) = 0\}$ has a maximal toral subalgebra $\mathfrak{t}'_{\mathbb{R}}$ in which every eigenvalue is pure imaginary, because $f$ is definite on $Q_{\mathbb{F}}$. It is unique because it has derived algebra zero and is given by the action of the $\mathfrak{p}_{\mathbb{R}}$-stabilizer of $Q_{\mathbb{F}}$ on the definite subspace $Q_{\mathbb{F}}$. This uniqueness tell us that $\mathfrak{t}'_{\mathbb{R}}$ is the same for $\mathfrak{p}_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}^{\dagger}$.

Let $\mathfrak{t}''_{\mathbb{R}}$ denote the maximal toral subalgebra in $\{\xi \in \mathfrak{p}_{\mathbb{R}} \mid \xi(X_{\mathbb{F}} \oplus Q_{\mathbb{F}})) = 0\}$. It stabilizes each $\mathrm{Span}(x'_\ell, x''_\ell)$ in (8.7) and centralizes $\mathfrak{a}_{\mathbb{R}}^{\dagger}$, so it vanishes if $\mathbb{F} \neq \mathbb{C}$. The $\mathfrak{p}_{\mathbb{R}}^{\dagger}$ analog of $\mathfrak{t}''_{\mathbb{R}}$ is 0 because $X_{\mathbb{F}}^{\dagger} \oplus Q_{\mathbb{F}} = 0$. In any case we have

$$\mathfrak{t}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}}^{\dagger} := \mathfrak{t}'_{\mathbb{R}} \oplus \mathfrak{t}''_{\mathbb{R}}\,. \tag{8.8}$$

For each $j \in J$ we define an algebra that contains $\mathfrak{l}_{j,\mathbb{R}}$ and acts on $(X_j)_{\mathbb{F}}$ by: if $\mathfrak{l}_{j,\mathbb{R}} = \mathfrak{su}(*)$, then $\widetilde{\mathfrak{l}_{j,\mathbb{R}}} = \mathfrak{u}(*)$ (acting on $(X_j)_{\mathbb{C}}$); otherwise $\widetilde{\mathfrak{l}_{j,\mathbb{R}}} = \mathfrak{l}_{j,\mathbb{R}}$.

Define

$$\widetilde{\mathfrak{l}_{\mathbb{R}}} = \bigoplus_{j \in J} \widetilde{\mathfrak{l}_{j,\mathbb{R}}} \quad \text{and} \quad \mathfrak{m}^{\dagger}_{\mathbb{R}} = \widetilde{\mathfrak{l}_{\mathbb{R}}} + \mathfrak{t}_{\mathbb{R}}. \tag{8.9}$$

Then, by construction, $\mathfrak{m}^{\dagger}_{\mathbb{R}} = \mathfrak{m}_{\mathbb{R}}$. Thus $\mathfrak{p}^{\dagger}_{\mathbb{R}}$ satisfies

$$\mathfrak{p}^{\dagger}_{\mathbb{R}} := \mathfrak{m}_{\mathbb{R}} + \mathfrak{a}^{\dagger}_{\mathbb{R}} + \mathfrak{n}^{\dagger}_{\mathbb{R}} = \mathfrak{n}^{\dagger}_{\mathbb{R}} \subsetplus (\mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}^{\dagger}_{\mathbb{R}}). \tag{8.10}$$

Let $\mathfrak{z}_{\mathbb{R}}$ denote the centralizer of $\mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ and let $\mathfrak{z}^{\dagger}_{\mathbb{R}}$ denote the centralizer of $\mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}^{\dagger}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$. We claim

$$\mathfrak{m}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}} = \widetilde{\mathfrak{l}_{\mathbb{R}}} + \mathfrak{z}_{\mathbb{R}} \text{ and } \mathfrak{m}_{\mathbb{R}} + \mathfrak{a}^{\dagger}_{\mathbb{R}} = \widetilde{\mathfrak{l}_{\mathbb{R}}} + \mathfrak{z}^{\dagger}_{\mathbb{R}} \tag{8.11}$$

For by construction $\mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}} = \widetilde{\mathfrak{l}_{\mathbb{R}}} + \mathfrak{t}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}} \subset \widetilde{\mathfrak{l}_{\mathbb{R}}} + \mathfrak{z}_{\mathbb{R}}$. Conversely, if $\xi \in \mathfrak{z}_{\mathbb{R}}$ it preserves each $X_{j,\mathbb{F}}$, each joint eigenspace of $\mathfrak{a}_{\mathbb{R}}$ on $X'_{\mathbb{F}} \oplus X''_{\mathbb{F}}$, and each joint eigenspace of $\mathfrak{t}_{\mathbb{R}}$, so $\xi \subset \widetilde{\mathfrak{l}_{\mathbb{R}}} + \mathfrak{t}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}}$. Thus $\mathfrak{m}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}} = \widetilde{\mathfrak{l}_{\mathbb{R}}} + \mathfrak{z}_{\mathbb{R}}$. The same argument shows that $\mathfrak{m}_{\mathbb{R}} + \mathfrak{a}^{\dagger}_{\mathbb{R}} = \widetilde{\mathfrak{l}_{\mathbb{R}}} + \mathfrak{z}^{\dagger}_{\mathbb{R}}$.

If $\mathfrak{a}_{\mathbb{R}}$ is diagonalizable as in the definition (8.7) of $\mathfrak{a}^{\dagger}_{\mathbb{R}}$, in other words if it is a sum of standard $\mathfrak{gl}(1;\mathbb{R})$'s, then we could choose $\mathfrak{a}^{\dagger}_{\mathbb{R}} = \mathfrak{a}_{\mathbb{R}}$, hence we could construct $\mathcal{F}^{\dagger}$ equal to $\mathcal{F}$, resulting in $\mathfrak{p}_{\mathbb{R}} = \mathfrak{p}^{\dagger}_{\mathbb{R}}$. In summary:

**Proposition 8.12.** *Let $\mathfrak{g}_{\mathbb{R}}$ be defined by a hermitian form and let $\mathfrak{p}_{\mathbb{R}}$ be a minimal self-normalizing parabolic subalgebra. In the notation above, $\mathfrak{p}^{\dagger}_{\mathbb{R}}$ is a minimal self-normalizing parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ with $\mathfrak{m}^{\dagger}_{\mathbb{R}} = \mathfrak{m}_{\mathbb{R}}$. In particular $\mathfrak{p}^{\dagger}_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}$ have the same Levi component. Further we can take $\mathfrak{p}_{\mathbb{R}} = \mathfrak{p}^{\dagger}_{\mathbb{R}}$ if and only if $\mathfrak{a}_{\mathbb{R}}$ is the sum of commuting standard $\mathfrak{gl}(1;\mathbb{R})$'s.*

Similar arguments give the construction behind Proposition 8.12 for the other real simple direct limit Lie algebras.

## 9 The inducing representation

In this section $P_{\mathbb{R}}$ is a self-normalizing minimal parabolic subgroup of $G_{\mathbb{R}}$. We discuss representations of $P_{\mathbb{R}}$ and the induced representations of $G_{\mathbb{R}}$. The latter are the *principal series* representations of $G_{\mathbb{R}}$ associated to $\mathfrak{p}_{\mathbb{R}}$, or more precisely to the pair $(\mathfrak{l}_{\mathbb{R}}, J)$ where $\mathfrak{l}_{\mathbb{R}}$ is the Levi component and $J$ is the ordering on the simple summands of $\mathfrak{l}_{\mathbb{R}}$.

We must first choose a class $\mathcal{C}_{M_{\mathbb{R}}}$ of representations of $M_{\mathbb{R}}$. Reasonable choices include various classes of unitary representations (we will discuss this in a moment) and continuous representations on nuclear Fréchet spaces, but "tame" (essentially the same as $II_1$) may be the best with which to start. In any case, given a representation $\kappa$ in our chosen class and a linear functional $\sigma : \mathfrak{a}_{\mathbb{R}} \to \mathbb{R}$ we have the representation $\kappa \otimes e^{i\sigma}$ of $M_{\mathbb{R}} \times A_{\mathbb{R}}$. Here $e^{i\sigma}(a)$ means

$e^{i\sigma(\log a)}$ where $\log : A_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}$ inverts $\exp : \mathfrak{a}_{\mathbb{R}} \to A_{\mathbb{R}}$. We write $E_\kappa$ for the representation space of $\kappa$.

We discuss some possibilities for $\mathcal{C}_{M_{\mathbb{R}}}$. Note that $\mathfrak{l}_{\mathbb{R}} = [(\mathfrak{p}_{\mathbb{R}})_{\mathrm{red}}, (\mathfrak{p}_{\mathbb{R}})_{\mathrm{red}}] = [\mathfrak{m}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}}, \mathfrak{m}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}}] = [\mathfrak{m}_{\mathbb{R}}, \mathfrak{m}_{\mathbb{R}}]$. Define

$$L_{\mathbb{R}} = [M_{\mathbb{R}}, M_{\mathbb{R}}] \text{ and } T_{\mathbb{R}} = M_{\mathbb{R}}/L_{\mathbb{R}}\,.$$

Then $T_{\mathbb{R}}$ is a real toral group with all eigenvalues pure imaginary, and $M_{\mathbb{R}}$ is an extension $1 \to L_{\mathbb{R}} \to M_{\mathbb{R}} \to T_{\mathbb{R}} \to 1$. Examples indicate that $M_{\mathbb{R}}$ is the product of a closed subgroup $T'_{\mathbb{R}}$ of $T_{\mathbb{R}}$ with factors of the group $L'_{\mathbb{R}}$ indicated in the previous section. That was where we replaced summands $\mathfrak{su}(*)$ of $\mathfrak{l}_{\mathbb{R}}$ by slightly larger algebras $\mathfrak{u}(*)$, hence subgroups $SU(*)$ of $L_{\mathbb{R}}$ by slightly larger groups $U(*)$. There is no need to discuss the representations of the classical finite-dimensional $U(n)$, $SO(n)$ or $Sp(n)$, where we have the Cartan highest weight theory and other classical combinatorial methods. So we look at $U(\infty)$.

**Tensor Representations of** $U(\infty)$. In the classical setting, one can use the action of the symmetric group $\mathfrak{S}_n$, permuting factors of $\otimes^n(\mathbb{C}^p)$. This gives a representation of $U(p) \times \mathfrak{S}_n$. Then we have the action of $U(p)$ on tensors picked out by an irreducible summand of that action of $\mathfrak{S}_n$. These summands occur with multiplicity 1. See Weyl's book [23]. Segal [17], Kirillov [12], and Strătilă & Voiculescu [18] developed and proved an analog of this for $U(\infty)$. However those "tensor representations" form a small class of the continuous unitary representations of $U(\infty)$. They are factor representations of type $II_\infty$, but they are somewhat restricted in that they do not even extend to the class of unitary operators of the form $1 +$ (compact). See [19, Section 2] for a summary of this topic. Because of this limitation one may also wish to consider other classes of factor representations of $U(\infty)$.

**Type** $II_1$ **Representations of** $U(\infty)$. Let $\pi$ be a continuous unitary finite-factor representation of $U(\infty)$. It has a character $\chi_\pi(x) = \mathrm{trace}\ \pi(x)$ (normalized trace). Voiculescu [22] worked out the parameter space for these finite-factor representations. It consists of all bilateral sequences $\{c_n\}_{-\infty<n<\infty}$ such that (i) $\det((c_{m_i+j-i})_{1\leqq i,j\leqq N} \geqq 0$ for $m_i \in \mathbb{Z}$ and $N \geqq 0$ and (ii) $\sum c_n = 1$. The character corresponding to $\{c_n\}$ and $\pi$ is $\chi_\pi(x) = \prod_i p(z_i)$ where $\{z_i\}$ is the multiset of eigenvalues of $x$ and $p(z) = \sum c_n z^n$. Here $\pi$ extends to the group of all unitary operators $X$ on the Hilbert space completion of $\mathbb{C}^\infty$ such that $X - 1$ is of trace class. See [19, Section 3] for a more detailed summary. This may be the best choice of class $\mathcal{C}_{M_{\mathbb{R}}}$. It is closely tied to the Olshanskii–Vershik notion (see [16]) of tame representation.

**Other Factor Representations of** $U(\infty)$. Let $\mathcal{H}$ be the Hilbert space completion of $\varinjlim \mathcal{H}_n$ where $\mathcal{H}_n$ is the natural representation space of $U(n)$. Fix a bounded hermitian operator $B$ on $\mathcal{H}$ with $0 \leqq B \leqq I$. Then

$$\psi_B : U(\infty) \to \mathbb{C}\,,\ \text{defined by } \psi_B(x) = \det((1 - B) + Bx)$$

is a continuous function of positive type on $U(\infty)$. Let $\pi_B$ denote the associated cyclic representation of $U(\infty)$. Then ([20, Theorem 3.1], or see [19, Theorem 7.2]),

(1) $\psi_B$ is of type $I$ if and only if $B(I-B)$ is of trace class. In that case $\pi_B$ is a direct sum of irreducible representations.
(2) $\psi_B$ is factorial and type $I$ if and only if $B$ is a projection. In that case $\pi_B$ is irreducible.
(3) $\psi_B$ is factorial but not of type $I$ if and only if $B(I-B)$ is not of trace class. In that case
   (i) $\psi_B$ is of type $II_1$ if and only if $B-tI$ is Hilbert–Schmidt where $0<t<1$; then $\pi_B$ is a factor representation of type $II_1$.
   (ii) $\psi_B$ is of type $II_\infty$ if and only if (a) $B(I-B)(B-pI)^2$ is trace class where $0<t<1$ and (b) the essential spectrum of $B$ contains 0 or 1; then $\pi_B$ is a factor representation of type $II_\infty$.
   (iii) $\psi_B$ is of type $III$ if and only if $B(I-B)(B-pI)^2$ is not of trace class whenever $0<t<1$; then $\pi_B$ is a factor representation of type $III$.

Similar considerations hold for $SU(\infty)$, $SO(\infty)$ and $Sp(\infty)$. This gives an indication of the delicacy in choice of type of representations of $M_{\mathbb{R}}$. Clearly factor representations of type $I$ and $II_1$ will be the easiest to deal with.

It is worthwhile to consider the case where the inducing representation $\kappa\otimes e^{i\sigma}$ is trivial on $M_{\mathbb{R}}$, in other words it is a unitary character on $P_{\mathbb{R}}$. In the finite-dimensional case this leads to a $K_{\mathbb{R}}$-fixed vector, spherical functions on $G_{\mathbb{R}}$ and functions on the symmetric space $G_{\mathbb{R}}/K_{\mathbb{R}}$. In the infinite dimensional case it leads to open problems, but there are a few examples ([7], [24]) that may give accurate indications.

## 10 Parabolic induction

We view $\kappa\otimes e^{i\sigma}$ as a representation $man \mapsto e^{i\sigma}(a)\kappa(m)$ of $P_{\mathbb{R}} = M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ on $E_\kappa$. It is well defined because $N_{\mathbb{R}}$ is a closed normal subgroup of $P_{\mathbb{R}}$. Let $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. The *algebraically induced representation* is given on the Lie algebra level as the left multiplication action of $\mathfrak{g}_{\mathbb{C}}$ on $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\otimes_{\mathfrak{p}_{\mathbb{R}}} E_\kappa$,

$$d\pi_{\kappa,\sigma,\mathrm{alg}}(\xi) : \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\otimes_{\mathfrak{p}_{\mathbb{R}}} E_\kappa \to \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\otimes_{\mathfrak{p}_{\mathbb{R}}} E_\kappa \text{ by } \eta\otimes e \mapsto (\xi\eta)\otimes e.$$

If $\xi\in\mathfrak{p}_{\mathbb{R}}$, then $d\pi_{\kappa,\sigma,\mathrm{alg}}(\xi)(\eta\otimes e) = \mathrm{Ad}(\xi)\eta\otimes e + \eta\otimes d(\kappa\otimes e^{i\sigma})(\xi)e$. To obtain the associated representation $\pi_{\kappa,\sigma}$ of $G_{\mathbb{R}}$ we need a $G_{\mathbb{R}}$-invariant completion of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\otimes_{\mathfrak{p}_{\mathbb{R}}} E_\kappa$ so that the $\pi_{\kappa,\sigma,alg}(\exp(\xi)) := \exp(d\pi_{\kappa,\sigma,\mathrm{alg}}(\xi))$ are well defined. For example we could use a $C^k$ completion, $k\in\{0,1,2,\dots,\infty,\omega\}$, representation of $G_{\mathbb{R}}$ on $C^k$ sections of the vector bundle $\mathbb{E}_{\kappa\otimes e^{i\sigma}} \to G_{\mathbb{R}}/P_{\mathbb{R}}$

associated to the action $\kappa \otimes e^{i\sigma}$ of $P_{\mathbb{R}}$ on $E_\kappa$. The representation space is

$$\{\varphi : G_{\mathbb{R}} \to E_\kappa \mid \varphi \text{ is } C^k \text{ and } \varphi(xman) = e^{i\sigma}(a)^{-1}\kappa(m)^{-1}f(x)\}$$

where $m \in M_{\mathbb{R}}$ , $a \in A_{\mathbb{R}}$ and $n \in N_{\mathbb{R}}$, and the action of $G_{\mathbb{R}}$ is

$$[\pi_{\kappa,\sigma,C^k}(x)(\varphi)](z) = \varphi(x^{-1}z).$$

In some cases one can unitarize $d\pi_{\kappa,\sigma,alg}$ by constructing a Hilbert space of sections of $\mathbb{E}_{\kappa\otimes e^{i\sigma}} \to G_{\mathbb{R}}/P_{\mathbb{R}}$. This has been worked out explicitly when $P_{\mathbb{R}}$ is a direct limit of minimal parabolic subgroups of the $G_{n,\mathbb{R}}$ [24], and more generally it comes down to transitivity of $K_{\mathbb{R}}$ on $G_{\mathbb{R}}/P_{\mathbb{R}}$ [26]. In any case the resulting representations of $G_{\mathbb{R}}$ depend on the choice of class $\mathcal{C}_{M_{\mathbb{R}}}$ of representations of $M_{\mathbb{R}}$.

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