http://dx.doi.org/10.1090/conm/544/10749

Contemporary Mathematics
Volume **544**, 2011

Extension of Symmetric Spaces and Restriction of Weyl Groups and Invariant Polynomials

Gestur Ólafsson and Joseph A. Wolf

Abstract. Polynomial invariants are fundamental objects in analysis on Lie groups and symmetric spaces. Invariant differential operators on symmetric spaces are described by Weyl group invariant polynomial. In this article we give a simple criterion that ensure that the restriction of invariant polynomials to subspaces is surjective. In another paper we will apply our criterion to problems in Fourier analysis on projective/injective limits, specifically to theorems of Paley–Wiener type.

Introduction

Invariant polynomials play a fundamental role in several branches of mathematics. In this paper we set up the invariant theory needed for our paper [**13**] on Paley–Wiener theory for injective limits of Riemannian symmetric spaces. We also describe that theory, leaving the proofs of our Paley–Wiener theorems to [**13**].

Let G be a connected semisimple real Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Then the algebra of G–invariant polynomials on $\mathfrak{g}$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}$, and restriction of invariant polynomials to $\mathfrak{h}$ is an isomorphism onto the algebra of Weyl group invariant polynomials on $\mathfrak{h}$. Replace G by a Riemannian symmetric space $M = G/K$ corresponding to a Cartan involution θ and replace $\mathfrak{h}$ by a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{s} := \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Then the Weyl group invariant polynomials correspond to the invariant differential operators on M. They are therefore closely related to harmonic analysis on M, in particular to the determination of the spherical functions on M.

In general we need $\mathfrak{a} \subset \mathfrak{h}$ and $\theta\mathfrak{h} = \mathfrak{h}$. For this, of course, we need only choose $\mathfrak{h}$ to be a Cartan subalgebra of the centralizer of $\mathfrak{a}$.

2010 *Mathematics Subject Classification.* 43A85, 53C35, 22E46.

Key words and phrases. Invariant polynomials; Extension of symmetric spaces; Spherical Fourier transform; Paley-Wiener theorem; Invariant differential operators.

The research of G. Ólafsson was supported by NSF grants DMS-0402068 and DMS-0801010.

The research of J. A. Wolf was partially supported by NSF grant DMS-0652840.

Denote by $W(\mathfrak{g}, \mathfrak{h})$ the Weyl group of $\mathfrak{g}$ relative to $\mathfrak{h}$, $W(\mathfrak{g}, \mathfrak{a})$ the "baby" Weyl group of $\mathfrak{g}$ relative to $\mathfrak{a}$, $W_{\mathfrak{a}}(\mathfrak{g}, \mathfrak{h}) = \{w \in W(\mathfrak{g}, \mathfrak{h}) \mid w(\mathfrak{a}) = \mathfrak{a}\}$, $\mathrm{I}(\mathfrak{g})$ the algebra of $W(\mathfrak{g}, \mathfrak{h})$–invariant polynomials on $\mathfrak{h}$ and finally $\mathrm{I}(\mathfrak{a})$ the algebra of $W(\mathfrak{g}, \mathfrak{a})$–invariant polynomials on $\mathfrak{a}$. It is well known for all semisimple Lie algebras that $W_{\mathfrak{a}}(\mathfrak{g}, \mathfrak{h})|_{\mathfrak{a}} = W(\mathfrak{g}, \mathfrak{a})$. In [**6**] Helgason showed for all classical semisimple Lie algebras that $\mathrm{I}(\mathfrak{h})|_{\mathfrak{a}} = \mathrm{I}(\mathfrak{a})$. As an application, this shows that in most cases the invariant differential operators on M come from elements in the center of the universal enveloping algebra of $\mathfrak{g}$.

In this article we discuss similar restriction problems for the case of pairs of Lie groups $G_n \subset G_k$ and symmetric spaces $M_n \subset M_k$. We use the above notation with indices $_n$ respectively $_k$. The first question is about restriction from $\mathfrak{h}_k$ to $\mathfrak{h}_n$. It is clear that neither does the group $W_{\mathfrak{h}_n}(\mathfrak{g}_k, \mathfrak{h}_k)$ restrict to $W(\mathfrak{g}_n, \mathfrak{h}_n)$ in general, nor is $\mathrm{I}(\mathfrak{h}_k)|_{\mathfrak{h}_n} = \mathrm{I}(\mathfrak{h}_n)$. To make this work, we introduce the notion that $\mathfrak{g}_k$ is a propagation of $\mathfrak{g}_n$ using the Dynkin diagram of simple Lie classical Lie algebras.

In terms of restricted roots, propagation means that either the rank and restricted root system of the large and the small symmetric spaces are the same, or roots are added to the left end of the Dynkin diagram. The result is that both symmetric spaces have the same type of root system but the larger one can have higher rank. In that case the restriction result above holds for all cases except when the restricted root systems are of type D. This includes all the cases of classical Lie groups of the same type. If G_k is a propagation of G_n, then $W_{\mathfrak{h}_n}(\mathfrak{g}_k, \mathfrak{h}_k)|_{\mathfrak{h}_n} = W(\mathfrak{g}_n, \mathfrak{h}_n)$ and $\mathrm{I}(\mathfrak{h}_n)|_{\mathfrak{h}_n} = \mathrm{I}(\mathfrak{h}_n)$, except in the case of simple algebras of type D, where a parity condition is needed, i.e., we have to extend the Weyl group by incorporating odd sign changes for simple factors of type D. The resulting finite group is denoted by $\widetilde{W}(\mathfrak{g}, \mathfrak{h})$. Then, in all classical cases, the $\widetilde{W}(\mathfrak{g}_k, \mathfrak{h}_k)$-invariant polynomials restrict to $\widetilde{W}(\mathfrak{g}_n, \mathfrak{h}_n)$-invariant polynomials. We also show that $\widetilde{W}_{\mathfrak{a}}(\mathfrak{g}, \mathfrak{h})|_{\mathfrak{a}} = \widetilde{W}(\mathfrak{g}, \mathfrak{a})$.

In Section 1 we introduce the notion of propagation and examine the corresponding invariants explicitly for each type of root system. The main result, Theorem 1.7, summarizes the facts on restriction of Weyl groups for propagation of symmetric spaces. The proof is by case by case consideration of each simple root system.

In Section 2 we prove surjectivity of Weyl group invariant polynomials for propagation of symmetric spaces. As mentioned above, this is analogous to Helgason's result on restriction of invariants from the full Cartan $\mathfrak{h}$ of $\mathfrak{g}$ to the Cartan $\mathfrak{a}$ of $(\mathfrak{g}, \mathfrak{k})$.

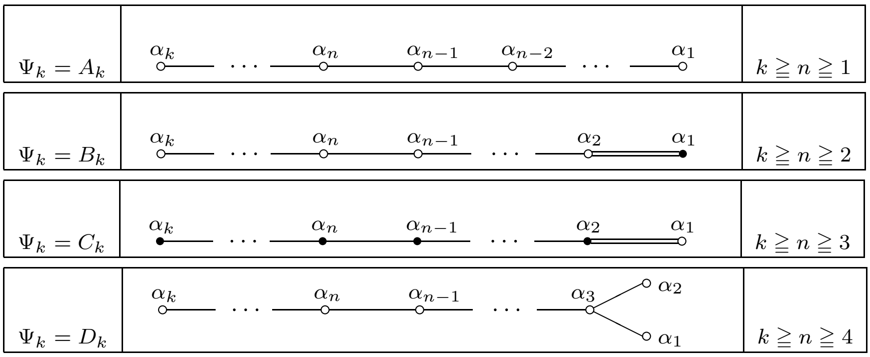
In Section 3 we indicate some applications of our results on Weyl group invariants to Fourier analysis on Riemannian symmetric spaces of noncompact type. This includes applications to the Fourier transform of compactly supported functions and the Paley-Wiener theorem as well as applications to invariant differential operators and related differential equations on symmetric spaces and their inductive limits.

1. Restriction of Invariants for Classical Simple Lie Algebras

In this section we discuss restriction of polynomial functions invariant under a Weyl group of classical type, i.e., a finite reflection group associated to a classical root system. Those can be concretely realized as permutation groups extended by a group of sign changes.

Let $\mathfrak{g}_n$ be a simple Lie algebra of classical type and let $\mathfrak{h}_n \subset \mathfrak{g}_n$ be a Cartan subalgebra. Let $\Delta_n = \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$ be the set of roots of $\mathfrak{h}_{n,\mathbb{C}}$ in $\mathfrak{g}_{n,\mathbb{C}}$ and $\Psi_n = \Psi(\mathfrak{g}_n, \mathfrak{h}_n)$ a set of simple roots. We label the corresponding Dynkin diagram so that α_1 is the *right* endpoint. If $\mathfrak{g}_n \subseteqq \mathfrak{g}_k$ then we chose $\mathfrak{h}_n$ and $\mathfrak{h}_k$ so that $\mathfrak{h}_n = \mathfrak{g}_n \cap \mathfrak{h}_k$. We say that $\mathfrak{g}_k$ *propagates* $\mathfrak{g}_n$, if Ψ_k is constructed from Ψ_n by adding simple roots to the *left* end of the Dynkin diagrams:

(1.1)



Let $\mathfrak{g}$ and $'\mathfrak{g} \subset \mathfrak{g}$ be semisimple Lie algebras. Then $\mathfrak{g}$ *propagates* $'\mathfrak{g}$ if we can number the simple ideals $\mathfrak{g}_j$, $j = 1, 2, \ldots, r$, in $\mathfrak{g}$ and the simple ideals $'\mathfrak{g}_i$, $i = 1, 2, \ldots, s$, in $'\mathfrak{g}$, so that $\mathfrak{g}_j$ propagates $'\mathfrak{g}_j$ for $j = 1, \ldots, s$.

When $\mathfrak{g}_k$ propagates $\mathfrak{g}_n$ as above, they have Cartan subalgebras $\mathfrak{h}_k$ and $\mathfrak{h}_n$ such that $\mathfrak{h}_n \subseteqq \mathfrak{h}_k$, and we have choices of root order such that

$$\text{if } \alpha \in \Psi_n \text{ then there is a unique } \alpha' \in \Psi_k \text{ such that } \alpha'|_{\mathfrak{h}_n} = \alpha.$$

It follows that

$$\Delta_n \subseteqq \{\alpha|_{\mathfrak{h}_n} \mid \alpha \in \Delta_k \text{ and } \alpha|_{\mathfrak{h}_n} \neq 0\}\,.$$

For a Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ in a semisimple complex Lie algebra $\mathfrak{g}_\mathbb{C}$ denote by $\mathfrak{h}_\mathbb{R}$ the Euclidean vector space

$$\mathfrak{h}_\mathbb{R} = \{X \in \mathfrak{h}_\mathbb{C} \mid \alpha(X) \in \mathbb{R} \text{ for all } \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})\}\,.$$

We now discuss case by case the classical simple Lie algebras and how the Weyl group and the invariants behave under propagation. The result will be collected in Theorem 1.7 below. The corresponding result for Riemannian symmetric spaces is Theorem 2.13.

For $s \in \mathbb{N}$ identify $\mathbb{R}^s$ with its dual. Let $f_1 = (0, 0, \ldots, 0, 1)$, $\ldots$, $f_s = (1, 0, 0, \ldots, 0)$ be the standard basis for $\mathbb{R}^s$. This enumeration is opposite to the usual one. We write

$$x = x_1 f_1 + \ldots + x_s f_s = (x_s, \ldots, x_1)$$

to indicate that in the following we will be adding zeros to the left to adjust for our numbering in the Dynkin diagrams. We use the discussion in [**16**, p. 293] as a reference for the realization of the classical Lie algebras.

When $\mathfrak{g}$ is a classical simple Lie algebra of rank n we write π_n for the defining representation and

$$F_n(t, X) := \det(t + \pi_n(X))\,.$$

We denote by the same letter the restriction of $F_n(t, \cdot)$ to $\mathfrak{h}_n$. In this section only we use the following simplified notation: $W_k = W(\mathfrak{g}_k, \mathfrak{h}_k)$ denotes the usual Weyl group of the pair $(\mathfrak{g}_k, \mathfrak{h}_k)$ and $W_{k,n} = W_{\mathfrak{h}_{n,\mathbb{R}}}(\mathfrak{g}_k, \mathfrak{h}_k) = \{w \in W_k \mid w(\mathfrak{h}_{n,\mathbb{R}}) = \mathfrak{h}_{n,\mathbb{R}}\}$ is the subgroup with well defined restriction to $\mathfrak{h}_n$.

The case $\mathbf{A_k}$, where $\mathfrak{g}_\mathbf{k} = \mathfrak{sl}(\mathbf{k+1}, \mathbb{C})$. In this case

$$\mathfrak{h}_{k,\mathbb{R}} = \{(x_{k+1}, \dots, x_1) \in \mathbb{R}^{k+1} \mid x_1 + \dots + x_{k+1} = 0\}\,, \tag{1.2}$$

where $x \in \mathbb{R}^{k+1}$ corresponds to the diagonal matrix

$$x \leftrightarrow \operatorname{diag}(x) := \begin{pmatrix} x_{k+1} & 0 & \dots & 0 \\ 0 & x_k & & \\ & & \ddots & \\ & & & x_1 \end{pmatrix}$$

Then $\Delta = \{f_i - f_j \mid 1 \leqq i \neq j \leqq k+1\}$ where f_ℓ maps a diagonal matrix to its ℓ^{th} diagonal element. Here $W(\mathfrak{g}_k, \mathfrak{h}_k)$ is the symmetric group $\mathfrak{S}_{k+1}$, all permutations of $\{1, \dots, k+1\}$, acting on the $\mathfrak{h}_k$ by

$$\sigma \cdot (x_{k+1}, \dots, x_1) = (x_{\sigma^{-1}(k+1)}, \dots, x_{\sigma^{-1}(1)})\,.$$

We will use the simple root system

$$\Psi(\mathfrak{g}_k, \mathfrak{h}_k) = \{f_j - f_{j-1} \mid j = 2, \dots, k+1\}\,.$$

The analogous notation will be used for A_n. In particular, denoting the zero vector of length j by 0_j, we have

$$\mathfrak{h}_{n,\mathbb{R}} = \left\{ (0_{k-n}, x_{n+1}, \dots, x_1) \mid x_j \in \mathbb{R} \quad \text{and} \quad \sum_{j=1}^{n+1} x_j = 0 \right\} \subset \mathfrak{h}_{k,\mathbb{R}}\,. \tag{1.3}$$

This corresponds to the embedding

$$\mathfrak{sl}(n, \mathbb{C}) \hookrightarrow \mathfrak{sl}(k, \mathbb{C})\,, \quad X \mapsto \left(\begin{smallmatrix} 0_{k-n,k-n} & 0 \\ 0 & X \end{smallmatrix}\right)\,.$$

It follows that

$$W_{k,n} = \mathfrak{S}_{k-n} \times \mathfrak{S}_{n+1}\,.$$

Hence $W_{k,n}|_{\mathfrak{h}_{n,\mathbb{R}}} = W(\mathfrak{g}_n, \mathfrak{h}_n)$ and the kernel of the restriction map is the first factor $\mathfrak{S}_{k-n}$.

According to [**16**, Exercise 58, p. 410] we have

$$F_k(t, X) = \prod_{j=1}^{k+1} (t + x_j) = t^{k+1} + \sum_{\nu=1}^{k+1} p_{k,\nu}(X) t^{\nu-1}\,.$$

The polynomials $p_{k,\nu}$ generate $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$. By (1.3), if $X = (0_{k-n}, x) \in \mathfrak{h}_{n,\mathbb{R}}$, then

$$\begin{aligned} F_k(t, (0_{k-n}, x)) &= t^{k+1} + \sum_{\nu=1}^{k+1} p_{k,\nu}(X) t^{\nu-1} = t^{k-n} \det(t + \pi_n(x)) \\ &= t^{k-n}(t^{n+1} + \sum_{\nu=1}^{n+1} p_{n,\nu}(x) t^{\nu-1}) = t^{k+1} + \sum_{\nu=k-n+1}^{k+1} p_{n,\nu+n-k}(x) t^{\nu-1}\,. \end{aligned}$$

Hence

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = p_{n,\nu+n-k} \text{ for } k-n+1 \leqq \nu \leqq k$$

and

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = 0 \text{ for } 1 \leqq \nu \leqq k-n\,.$$

In particular the restriction map $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})$ is surjective.

The case $\mathbf{B_k}$, where $\mathfrak{g}_\mathbf{k} = \mathfrak{so}(\mathbf{2k+1}, \mathbb{C})$. In this case $\mathfrak{h}_{k,\mathbb{R}} = \mathbb{R}^k$ where $\mathbb{R}^k$ is embedded into $\mathfrak{so}(2k+1,\mathbb{C})$ by

$$x \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{diag}(x) & 0 \\ 0 & 0 & -\mathrm{diag}(x) \end{pmatrix}. \tag{1.4}$$

Here $\Delta_k = \{\pm(f_i \pm f_j) \mid 1 \leqq j < i \leqq k\} \bigcup \{\pm f_1, \dots, \pm f_k\}$ and we have the positive system $\Delta_k^+ = \{f_i \pm f_j \mid 1 \leqq j < i \leqq k\} \bigcup \{f_1, \dots, f_k\}$. The simple root system is $\Psi = \Psi(\mathfrak{g}_k, \mathfrak{h}_k) = \{\alpha_1, \dots, \alpha_k\}$ where

$$\text{the simple root } \alpha_1 = f_1, \text{ and } \alpha_j = f_j - f_{j-1} \text{ for } 2 \leqq j \leqq k.$$

In this case the Weyl group $W(\mathfrak{g}_k, \mathfrak{h}_k)$ is the semidirect product $\mathfrak{S}_k \rtimes \{1,-1\}^k$, where $\mathfrak{S}_k$ acts as before and

$$\{1,-1\}^k \cong (\mathbb{Z}/2\mathbb{Z})^k = \{\epsilon = (\epsilon_k, \dots, \epsilon_1) \mid \epsilon_j = \pm 1\}$$

acts by sign changes, $\epsilon \cdot x = (\epsilon_k x_k, \dots, \epsilon_1 x_1)$. Similar notation holds for $\mathfrak{h}_{n,\mathbb{R}}$. Our embedding of $\mathfrak{h}_{n,\mathbb{R}} \hookrightarrow \mathfrak{h}_{k,\mathbb{R}}$ corresponds to the (non-standard) embedding of $\mathfrak{so}(2n+1,\mathbb{C})$ into $\mathfrak{so}(2k+1,\mathbb{C})$ given by

$$\begin{pmatrix} 0 & a & b \\ -b^t & A & B \\ -a^t & C & -A^t \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0_{k-n} & a & 0_{k-n} & b \\ 0_{k-n}^t & 0 & 0 & 0 & 0 \\ -b^t & 0 & A & 0 & B \\ 0_{k-n}^t & 0 & 0 & 0 & 0 \\ -a^t & 0 & C & 0 & -A^t \end{pmatrix}$$

where the zeros stands for the zero matrix of the obvious size and we use the realization from [**16**, p. 303]. Here we see that

$$W_{k,n} = (\mathfrak{S}_{k-n} \rtimes \{1,-1\}^{k-n}) \times (\mathfrak{S}_n \rtimes \{1,-1\}^n).$$

Thus $W_{k,n}|_{\mathfrak{h}_{n,\mathbb{R}}} = W(\mathfrak{g}_n, \mathfrak{h}_n)$ and the kernel of the restriction map is $\mathfrak{S}_{k-n} \rtimes \{1,-1\}^{k-n}$.

For the invariant polynomials we have, again using [**16**, Exercise 58, p. 410], that

$$F_k(t,X) = \det(t + \pi_k(X)) = t^{2k+1} + \sum_{\nu=1}^{k} p_{k,\nu}(X) t^{2\nu-1}$$

and the polynomials $p_{k,\nu}$ freely generate $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$. For $X \in \mathfrak{h}_k$, $F_k(t,X)$ is given by $t\prod_{j=1}^n (t+x_j)(t-x_j) = t\prod_{j=1}^n (t^2 - x_j^2)$. Arguing as above we have for $X = (0_{k-n}, x) \in \mathfrak{h}_{n,\mathbb{R}} \subseteqq \mathfrak{h}_{k,\mathbb{R}}$:

$$\begin{aligned} F_k(t,(0_{k-n},x)) &= t^{2k+1} + \sum_{\nu=1}^{k} p_{k,\nu}(X) t^{2\nu-1} = t^{2(k-n)} \det(t + \pi_n(x)) \\ &= t^{2(k-n)}(t^{2n+1} + \sum_{\nu=1}^{n} p_{n,\nu}(x) t^{2\nu-1}) = t^{2k+1} + \sum_{\nu=k-n+1}^{k} p_{n,\nu+n-k}(x) t^{2\nu-1}. \end{aligned}$$

Hence

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = p_{n,\nu+n-k} \text{ for } k-n+1 \leqq \nu \leqq k$$

and

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = 0 \text{ for } 1 \leqq \nu \leqq k-n.$$

In particular, the restriction map $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})$ is surjective.

The case $\mathbf{C_k}$, where $\mathfrak{g}_\mathbf{k} = \mathfrak{sp}(\mathbf{k}, \mathbb{C})$. Again $\mathfrak{h}_{k,\mathbb{R}} = \mathbb{R}^k$ embedded in $\mathfrak{sp}(k, \mathbb{C})$ by

$$x \mapsto \begin{pmatrix} \mathrm{diag}(x) & 0 \\ 0 & -\mathrm{diag}(x) \end{pmatrix} . \tag{1.5}$$

In this case $\Delta_k = \{\pm(f_i \pm f_j) \mid 1 \leqq j < i \leqq k\} \bigcup \{\pm 2f_1, \ldots, \pm 2f_k\}$. Take $\Delta_k^+ = \{f_i - f_j \mid 1 \leqq j < i \leqq k\} \cup \{2f_1, \ldots, 2f_k\}$ as a positive system. Then the simple root system $\Psi = \Psi(\mathfrak{g}_k, \mathfrak{h}_k) = \{\alpha_1, \ldots, \alpha_k\}$ is given by

$$\text{the simple root } \alpha_1 = 2f_1, \text{ and } \alpha_j = f_j - f_{j-1} \text{ for } 2 \leqq j \leqq k.$$

The Weyl group $W(\mathfrak{g}_k, \mathfrak{h}_k)$ is again $\mathfrak{S}_k \rtimes \{1, -1\}^k$ and

$$W_{k,n} = (\mathfrak{S}_{k-n} \rtimes \{1, -1\}^{k-n}) \times (\mathfrak{S}_n \rtimes \{1, -1\}^n) .$$

Thus, $W_{k,n}|_{\mathfrak{h}_{n,\mathbb{R}}} = W(\mathfrak{g}_n, \mathfrak{h}_n)$ and the restriction map has kernel $\mathfrak{S}_{k-n} \rtimes \{1, -1\}^{k-n}$.

For the invariant polynomials we have, again using [**16**, Exercise 58, p. 410], that

$$F_k(t, X) = t^{2k} + \sum_{\nu=1}^{k} p_{k,\nu}(X) t^{2(\nu-1)} = \prod_{j=1}^{n} (t^2 - x_j^2)$$

and the $p_{k,\nu}$ freely generate $\mathrm{I}_{W(\mathfrak{g}_k, \mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$. We embed $\mathfrak{sp}(n, \mathbb{C})$ into $\mathfrak{sp}(k, \mathbb{C})$ by

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mapsto \begin{pmatrix} 0_{k-n,k-n} & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 0_{k-n,k-n} & 0 \\ 0 & C & 0 & -A^t \end{pmatrix}$$

where as usual 0 stands for a zero matrix of the correct size. Then

$$\begin{aligned} F_k(t, (0_{k-n}, x)) &= t^{2k} + \sum_{\nu=1}^{k} p_{k,\nu}(X) t^{2(\nu-1)} = t^{2(k-n)} \det(t + \pi_n(x)) \\ &= t^{2(k-n)} (t^{2n} + \sum_{\nu=1}^{n} p_{n,\nu}(x) t^{2(\nu-1)}) = t^{2k} + \sum_{\nu=k-n+1}^{k} p_{n,\nu+n-k}(x) t^{2(\nu-1)} . \end{aligned}$$

Hence

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = p_{n,\nu+n-k} \text{ for } k - n + 1 \leqq \nu \leqq k$$

and

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = 0 \text{ for } 1 \leqq \nu \leqq k - n .$$

In particular, the restriction map $\mathrm{I}_{W(\mathfrak{g}_k, \mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}_{W(\mathfrak{g}_n, \mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})$ is surjective.

The case $\mathbf{D_k}$, where $\mathfrak{g}_\mathbf{k} = \mathfrak{so}(\mathbf{2k}, \mathbb{C})$. We embed $\mathfrak{h}_{k,\mathbb{R}} = \mathbb{R}^k$ in $\mathfrak{so}(2k, \mathbb{C})$ by

$$x \mapsto \begin{pmatrix} \mathrm{diag}(x) & 0 \\ 0 & -\mathrm{diag}(x) \end{pmatrix} . \tag{1.6}$$

Then $\Delta_k = \{\pm(f_i \pm f_j) \mid 1 \leqq j < i \leqq k\}$ and we use the simple root system $\Psi(\mathfrak{g}_k, \mathfrak{h}_k) = \{\alpha_1, \ldots, \alpha_k\}$ given by

$$\alpha_1 = f_1 + f_2, \text{ and } \alpha_i = f_i - f_{i-1} \text{ for } 2 \leqq i \leqq k$$

The Weyl group is $W(\mathfrak{g}_k, \mathfrak{h}_k) = \mathfrak{S}_k \rtimes \{\epsilon \in \{1, -1\}^k \mid \epsilon_1 \cdots \epsilon_= 1\}$. In other words the elements of $W(\mathfrak{g}_k, \mathfrak{h}_k)$ contain only an *even* number of sign-changes. The invariants are given by

$$F_k(t, X) = t^{2k} + \sum_{\nu=2}^{k} p_{k,\nu}(X) t^{2(\nu-1)} + p_{k,1}(X)^2 = \prod_{\nu=1}^{n} (t^2 - x_j^2)$$

where p_1 is the Pfaffian, $p_1(X) = (-1)^{k/2}x_1 \dots x_k$, so $p_1(X)^2 = \det(X)$. The polynomials $p_{k,1}, \dots, p_{k,k}$ freely generate $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$.

We embed $\mathfrak{h}_{n,\mathbb{R}}$ in $\mathfrak{h}_{k,\mathbb{R}}$ in the same manner as before. This corresponds to

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mapsto \begin{pmatrix} 0_{k-n,k-n} & & 0_{k-n,k-n} & \\ 0 & A & 0 & B \\ 0_{k-n,k-n} & 0 & 0 & \\ 0 & C & 0 & -A^t \end{pmatrix} .$$

It is then clear that

$$W_{k,n} = (\mathfrak{S}_{k-n} \rtimes \{1,-1\}^{k-n}) \times_* (\mathfrak{S}_n \rtimes \{1,-1\}^n)$$

where the $_*$ indicates that $\epsilon_1 \cdots \epsilon_n = 1$. Therefore, the restrictions of elements of $W_{k,n}$, $k > n$, contain all sign changes, and

$$\mathfrak{S}_n \rtimes \{\epsilon \in \{1,-1\}^{n-1} \mid \epsilon_1 \dots \epsilon_n = 1\} = W(\mathfrak{g}_n, \mathfrak{h}_n) \subsetneqq W_{k,n}|_{\mathfrak{h}_{n,\mathbb{R}}} = \mathfrak{S}_n \rtimes \{1,-1\}^n .$$

The Pfaffian $p_{k,1}(0, X) = 0$ and

$$\begin{aligned} F_k(t,(0,x)) &= t^{2k} + \sum_{\nu=2}^{k} p_{k,\nu}(0,x) t^{2(\nu-1)} \\ &= t^{2(k-n)} F_n(t,x) = t^{2(k-n)}(t^{2n} + \sum_{\nu=2}^{n} p_{n,\nu}(x) t^{2(\nu-1)} + p_{n,1}(x)^2) \\ &= t^{2k} + \sum_{\nu=k-n+2}^{k} p_{n,\nu+n-k}(x) t^{2(\nu-1)} + p_{n,1}(x)^2 t^{2(k-n)} . \end{aligned}$$

Hence

$$\begin{aligned} &p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = p_{n,\nu+n-k} \text{ for } k-n+2 \leqq \nu \leqq k , \\ &p_{k,k-n+1}|_{\mathfrak{h}_{n,\mathbb{R}}} = p_{n,1}(x)^2 , \text{ and} \\ &p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = 0 , \quad \nu = 1, \dots, k-n . \end{aligned}$$

In particular the elements in $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})|_{\mathfrak{h}_{n,\mathbb{R}}}$ are polynomials in even powers of x_j and $p_{n,1}$ is not in the image of the restriction map. Thus

$$\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})|_{\mathfrak{h}_{n,\mathbb{R}}} \subsetneqq \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}}) .$$

Let σ_k be the involution of the Dynkin diagram for D_k given by $\sigma(\alpha_1) = \alpha_2$, $\sigma(\alpha_2) = \alpha_1$ and $\sigma_k(\alpha_j) = \alpha_j$ for $3 \leq j \leq k$. Then $\sigma_k|_{\mathfrak{h}_n} = \sigma_n$, $\sigma_k(\mathfrak{h}_{n,\mathbb{R}})$ and σ_k normalizes $W(\mathfrak{g}_k, \mathfrak{h}_k)$. The group $\widetilde{W}_k = \widetilde{W}(\mathfrak{g}_k, \mathfrak{h}_k) := W(\mathfrak{g}_k, \mathfrak{h}_k) \rtimes \{1, \sigma_k\}$ is the group $\mathfrak{S}_k \rtimes \{1,-1\}^k$. Hence

$$\widetilde{W}(\mathfrak{g}_n, \mathfrak{h}_n) = W_{\mathfrak{h}_n}(\mathfrak{g}_k, \mathfrak{h}_k)|_{\mathfrak{h}_{n,\mathbb{R}}} = \widetilde{W}_{\mathfrak{h}_n}(\mathfrak{g}_k, \mathfrak{h}_k)|_{\mathfrak{h}_n} .$$

We also note that $\widetilde{W}(\mathfrak{g}_k, \mathfrak{h}_k)$ is isomorphic to the Weyl group of the root system B_k and hence is a finite reflection group.

The algebra $\mathrm{I}_{\widetilde{W}_k}(\mathfrak{g}_k, \mathfrak{h}_k)$ is the algebra of all *even* elements in $\mathrm{I}_{W_k}(\mathfrak{g}_k, \mathfrak{h}_k)$. Denote it by $\mathrm{I}^{\text{even}}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$. The above calculations shows that

$$\mathrm{I}^{\text{even}}_{W_n}(\mathfrak{g}_n, \mathfrak{h}_n) = \mathrm{I}_{W_k}(\mathfrak{g}_k, \mathfrak{h}_k)|_{\mathfrak{h}_n} = \mathrm{I}^{\text{even}}_{W_k}(\mathfrak{g}_k, \mathfrak{h}_k)|_{\mathfrak{h}_n} .$$

We put these results together in the following theorem.

THEOREM 1.7. *Assume $\mathfrak{g}_n$ and $\mathfrak{g}_k$ are simple complex Lie algebras of ranks n and k, respectively, and that $\mathfrak{g}_k$ propagates $\mathfrak{g}_n$.*

(1) *If* $\mathfrak{g}_n \neq \mathfrak{so}(2n,\mathbb{C})$ *and* $\mathfrak{g}_k \neq \mathfrak{so}(2k,\mathbb{C})$ *then*

$$W(\mathfrak{g}_n,\mathfrak{h}_n) = W_{\mathfrak{h}_n}(\mathfrak{g}_k,\mathfrak{h}_k)|_{\mathfrak{h}_n} = \{w|_{\mathfrak{h}_n} \mid w \in W(\mathfrak{g}_k,\mathfrak{h}_k) \text{ with } w(\mathfrak{h}_n) = \mathfrak{h}_n\}$$

and the restriction map

$$\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})$$

is surjective.

(2) *If* $\mathfrak{g}_n = \mathfrak{so}(2n,\mathbb{C}) \subset \mathfrak{g}_k = \mathfrak{so}(2k,\mathbb{C})$*, then*

$$W_{\mathfrak{h}_n}(\mathfrak{g}_k,\mathfrak{h}_k)|_{\mathfrak{h}_n} = \{w|_{\mathfrak{h}_n} \mid w \in W(\mathfrak{g}_k,\mathfrak{h}_k) \text{ with } w(\mathfrak{h}_n) = \mathfrak{h}_n\} = \mathfrak{S}_n \rtimes \{1,-1\}^n$$

contains all sign changes, but the elements of $W(\mathfrak{g}_n,\mathfrak{h}_n)$ *contain only even numbers of sign changes, so* $W(\mathfrak{g}_n,\mathfrak{h}_n) \subsetneqq W_{\mathfrak{h}_n}(\mathfrak{g}_k,\mathfrak{h}_k)|_{\mathfrak{h}_n}$*. The elements of* $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})|_{\mathfrak{h}_{n,\mathbb{R}}}$ *are polynomials in the* x_j^2*, and the Pfaffian* (*square root of the determinant*) *is not in the image of the restriction map* $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})$.

(3) *With the assumptions from* (2) *let* σ_k *be as above and let* $\widetilde{W}_k(\mathfrak{g}_k,\mathfrak{h}_k) = W(\mathfrak{g}_k,\mathfrak{h}_k) \ltimes \{1,\sigma_k\}$*. Then* $\widetilde{W}(\mathfrak{g}_k,\mathfrak{h}_k)$ *is a finite reflection group, and*

$$\widetilde{W}(\mathfrak{g}_n,\mathfrak{h}_n) = W_{\mathfrak{h}_n}(\mathfrak{g}_k,\mathfrak{h}_k)|_{\mathfrak{h}_{n,\mathbb{R}}} = \widetilde{W}_{\mathfrak{h}_n}(\mathfrak{g}_k,\mathfrak{h}_k)|_{\mathfrak{h}_n}\,.$$

(4) *With the assumptions from* (2) *we have*

$$\mathrm{I}^{even}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) = \mathrm{I}_{\widetilde{W}(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$$

and

$$\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})|_{\mathfrak{h}_{n,\mathbb{R}}} = \mathrm{I}_{\widetilde{W}(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})|_{\mathfrak{h}_{n,\mathbb{R}}} = \mathrm{I}_{\widetilde{W}(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})\,.$$

REMARK 1.8. If $\mathfrak{g}_k = \mathfrak{sl}(k+1,\mathbb{C})$ and $\mathfrak{g}_n$ is constructed from $\mathfrak{g}_k$ by removing any $n-k$ simple roots from the Dynkin diagram of $\mathfrak{g}_k$, then Theorem 1.7(1) remains valid because all the Weyl groups are permutation groups. On the other hand, if $\mathfrak{g}_k$ is of type B_k, C_k, or D_k ($k \geqq 3$) and if $\mathfrak{g}_n$ is constructed from $\mathfrak{g}_k$ by removing at least one simple root α_i with $k-i \geqq 2$, then $\mathfrak{g}_n$ contains at least one simple factor $\mathfrak{l}$ of type A_ℓ, $\ell \geqq 2$. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{l}$. Then the restriction of the Weyl group of $\mathfrak{g}_k$ to $\mathfrak{a}_\mathbb{R}$ will contain $-\mathrm{id}$. But $-\mathrm{id}$ is not in the Weyl group $W(\mathfrak{sl}(\ell+1,\mathbb{C}))$, and the restriction of the invariants will only contain even polynomials. Hence the conclusion Theorem 1.7(1) fails in this case. $\diamondsuit$

We also note the following consequence of the definition of propagation. It is implicit in the diagrams following that definition.

LEMMA 1.9. *Assume that* $\mathfrak{g}_k$ *propagates* $\mathfrak{g}_n$*. Let* $\mathfrak{h}_k$ *be a Cartan subalgebra of* $\mathfrak{g}_k$ *such that* $\mathfrak{h}_n = \mathfrak{h}_k \cap \mathfrak{g}_n$ *is a Cartan subalgebra of* $\mathfrak{g}_n$*. Recursively choose positive systems* $\Delta^+(\mathfrak{g}_k,\mathfrak{h}_k) \subset \Delta(\mathfrak{g}_k,\mathfrak{h}_k)$ *and* $\Delta^+(\mathfrak{g}_n,\mathfrak{h}_n) \subset \Delta(\mathfrak{g}_n,\mathfrak{h}_n)$ *aligned so that* $\Delta^+(\mathfrak{g}_n,\mathfrak{h}_n) \subseteqq \Delta^+(\mathfrak{g}_k,\mathfrak{h}_k)|_{\mathfrak{h}_n}$*. Then we can number the simple roots such that* $\alpha_{n,j} = \alpha_{k,j}|_{\mathfrak{h}_n}$ *for* $j = 1,\dots,\dim\mathfrak{h}_n$.

2. Symmetric Spaces

Now we discuss restriction of invariant polynomials related to Riemannian symmetric spaces. Let $M = G/K$ be a Riemannian symmetric space of compact or noncompact type. Thus G is a connected semisimple Lie group with an involution θ such that

$$(G^\theta)_o \subseteqq K \subseteqq G^\theta$$

where $G^\theta = \{x \in G \mid \theta(x) = x\}$ and the subscript $_o$ denotes the connected component containing the identity element. If G is simply connected then G^θ is connected and $K = G^\theta$. If G is noncompact and with finite center, then $K \subset G$ is a *maximal compact* subgroup of G, K is connected, and G/K is simply connected.

Denote the Lie algebra of G by $\mathfrak{g}$. Then θ defines an involution $\theta : \mathfrak{g} \to \mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ is the Lie algebra of K and $\mathfrak{s} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$.

Cartan Duality is the bijection between simply connected symmetric spaces of noncompact type and those of compact type defined by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \leftrightarrow \mathfrak{k} \oplus i\mathfrak{s} = \mathfrak{g}^d$. We denote it by $M \leftrightarrow M^d$.

Fix a maximal abelian subset $\mathfrak{a} \subset \mathfrak{s}$. If $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$ we write $\mathfrak{g}_{\mathbb{C},\alpha} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}_{\mathbb{C}}\}$, and if $\mathfrak{g}_{\mathbb{C},\alpha} \neq \{0\}$ then α is a (restricted) root. Denote by $\Sigma(\mathfrak{g}, \mathfrak{a})$ the set of roots. If M is of noncompact type, then all the roots are in the real dual space $\mathfrak{a}^*$ and $\mathfrak{g}_{\mathbb{C},\alpha} = \mathfrak{g}_\alpha + i\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \mathfrak{g}_{\mathbb{C},\alpha} \cap \mathfrak{g}$. If M is of compact type, then the roots take pure imaginary values on $\mathfrak{a}$, $\Sigma(\mathfrak{g}, \mathfrak{a}) \subset i\mathfrak{a}^*$, and $\mathfrak{g}_{\mathbb{C},\alpha} \cap \mathfrak{g} = \{0\}$. The set of roots is preserved under duality where we view those roots as $\mathbb{C}$–linear functionals on $\mathfrak{a}_{\mathbb{C}}$.

Let $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \frac{1}{2}\alpha \notin \Sigma(\mathfrak{g}, \mathfrak{a})\}$. Then $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$ is a root system in the usual sense and the Weyl group corresponding to $\Sigma(\mathfrak{g}, \mathfrak{a})$ is the same as the Weyl group generated by the reflections s_α, $\alpha \in \Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. Furthermore, M is irreducible if and only if $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$ is irreducible, i.e., can not be decomposed into two mutually orthogonal root systems.

Let $\Sigma^+(\mathfrak{g}, \mathfrak{a}) \subset \Sigma(\mathfrak{g}, \mathfrak{a})$ be a positive system and denote $\Sigma^+_{1/2}(\mathfrak{g}, \mathfrak{a}) = \Sigma^+(\mathfrak{g}, \mathfrak{a}) \cap \Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. Then $\Sigma^+_{1/2}(\mathfrak{g}, \mathfrak{a})$ is a positive root system in $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. Denote by $\Psi_{1/2}(\mathfrak{g}, \mathfrak{a})$ the set of simple roots in $\Sigma^+_{1/2}(\mathfrak{g}, \mathfrak{a})$. Then $\Psi_{1/2}(\mathfrak{g}, \mathfrak{a})$ is a basis for $\Sigma(\mathfrak{g}, \mathfrak{a})$.

The list of irreducible symmetric spaces is given by the following table. The indices j and k are related by $k = 2j + 1$. In the fifth column we list the realization of K as a subgroup of the compact real form. The second column indicates the type of the root system $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. (More detailed information is given by the Satake–Tits diagram for M; see [**1**] or [**9**, pp. 530–534]. In that classification the case $\mathrm{SU}(p, 1)$, $p \geqq 1$, is denoted by AIV, but here it appears in $AIII$. The case $\mathrm{SO}(p, q)$, $p + q$ odd, $p \geq q > 1$, is denoted by BI as in this case the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(p + q, \mathbb{C})$ is of type B. The case $\mathrm{SO}(p, q)$, with $p + q$ even, $p \geq q > 1$ is denoted by DI as in this case $\mathfrak{g}_{\mathbb{C}}$ is of type D. Finally, the case $\mathrm{SO}(p, 1)$, p even, is denoted by BII and $\mathrm{SO}(p, 1)$, p odd, is denoted by DII.)

(2.10)

Irreducible Riemannian Symmetric $M = G/K$, G classical, K connected

		G noncompact	G compact	K	Rank$M$	Dim$M$
1	A_j	$\mathrm{SL}(j, \mathbb{C})$	$\mathrm{SU}(j) \times \mathrm{SU}(j)$	$\mathrm{diag}\,\mathrm{SU}(j)$	$j - 1$	$j^2 - 1$
2	B_j	$\mathrm{SO}(2j + 1, \mathbb{C})$	$\mathrm{SO}(2j + 1) \times \mathrm{SO}(2j + 1)$	$\mathrm{diag}\,\mathrm{SO}(2j + 1)$	j	$2j^2 + j$
3	D_j	$\mathrm{SO}(2j, \mathbb{C})$	$\mathrm{SO}(2j) \times \mathrm{SO}(2j)$	$\mathrm{diag}\,\mathrm{SO}(2j)$	j	$2j^2 - j$
4	C_j	$\mathrm{Sp}(j, \mathbb{C})$	$\mathrm{Sp}(j) \times \mathrm{Sp}(j)$	$\mathrm{diag}\,\mathrm{Sp}(j)$	j	$2j^2 + j$
5	$AIII$	$\mathrm{SU}(p, q)$	$\mathrm{SU}(p + q)$	$\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$	$\min(p, q)$	$2pq$
6	AI	$\mathrm{SL}(j, \mathbb{R})$	$\mathrm{SU}(j)$	$\mathrm{SO}(j)$	$j - 1$	$\frac{(j-1)(j+2)}{2}$
7	AII	$\mathrm{SU}^*(2j) = SL(j, \mathbb{H})$	$\mathrm{SU}(2j)$	$\mathrm{Sp}(j)$	$j - 1$	$2j^2 - j - 1$
8	BDI	$\mathrm{SO}_o(p, q)$	$\mathrm{SO}(p + q)$	$\mathrm{SO}(p) \times \mathrm{SO}(q)$	$\min(p, q)$	pq
9	$DIII$	$\mathrm{SO}^*(2j)$	$\mathrm{SO}(2j)$	$\mathrm{U}(j)$	$[\frac{j}{2}]$	$j(j - 1)$
10	CII	$\mathrm{Sp}(p, q)$	$\mathrm{Sp}(p + q)$	$\mathrm{Sp}(p) \times \mathrm{Sp}(q)$	$\min(p, q)$	$4pq$
11	CI	$\mathrm{Sp}(j, \mathbb{R})$	$\mathrm{Sp}(j)$	$\mathrm{U}(j)$	j	$j(j + 1)$

Only in the following cases do we have $\Sigma_{1/2}(\mathfrak{g},\mathfrak{a}) \neq \Sigma(\mathfrak{g},\mathfrak{a})$:

- $AIII$ for $1 \leqq p < q$,
- CII for $1 \leqq p < q$, and
- $DIII$ for j odd.

In those three cases there is exactly one simple root with $2\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$ and this simple root is at the right end of the Dynkin diagram for $\Psi_{1/2}(\mathfrak{g},\mathfrak{a})$. Also, either $\Psi_{1/2}(\mathfrak{g},\mathfrak{a}) = \{\alpha\}$ contains one simple root or $\Psi_{1/2}(\mathfrak{g},\mathfrak{a})$ is of type B_r where $r = \dim \mathfrak{a}$ is the rank of M.

Finally, the only two cases where $\Psi_{1/2}(\mathfrak{g},\mathfrak{a})$ is of type D are

$$\mathrm{SO}(2j,\mathbb{C})/\mathrm{SO}(2j) \text{ and the split case } \mathrm{SO}_o(p,p)/\mathrm{SO}(p)\times\mathrm{SO}(p).$$

In particular, if $\Psi_{1/2}(\mathfrak{g},\mathfrak{a})$ is of type D then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$.

Let G/K be an irreducible symmetric space of compact or non-compact type. As before let $\mathfrak{a} \subset \mathfrak{s}$ be maximal abelian. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{h} = (\mathfrak{h}\cap\mathfrak{k})\oplus\mathfrak{a}$. Let $\Delta(\mathfrak{g},\mathfrak{h})$, $\Sigma(\mathfrak{g},\mathfrak{a})$, and $\Sigma_{1/2}(\mathfrak{g},\mathfrak{a})$ denote the corresponding root systems and $W(\mathfrak{g},\mathfrak{h})$ respectively $W(\mathfrak{g},\mathfrak{a})$ the Weyl group corresponding to $\Delta(\mathfrak{g},\mathfrak{h})$ respectively $\Sigma(\mathfrak{g},\mathfrak{a})$. We define an extension of those Weyl groups $\widetilde{W}(\mathfrak{g},\mathfrak{h})$ and $\widetilde{W}(\mathfrak{g},\mathfrak{a})$ as before.

Note that $\widetilde{W}(\mathfrak{g},\mathfrak{a}) = W(\mathfrak{g},\mathfrak{a})$ with only two exceptions: (i) the cases where M locally isomorphic to $\mathrm{SO}(2j,\mathbb{C})/\mathrm{SO}(2j)$ (with $\mathfrak{h} = \mathfrak{a}_{\mathbb{C}}$) or to its compact dual $(SO(2j)\times SO(2j))/\mathrm{diag}\, SO(2j)$ (with $\mathfrak{h} \cong \mathfrak{a}\oplus\mathfrak{a}$), and (ii) the cases where M locally isomorphic to $\mathrm{SO}_o(j,j)/\mathrm{SO}(j)\times\mathrm{SO}(j)$ or to its compact dual $\mathrm{SO}(2j)/\mathrm{SO}(j)\times\mathrm{SO}(j)$ with $\mathfrak{h} = \mathfrak{a}$.

Theorem 2.11. *Let G/K be a symmetric space of compact or non-compact type (no Euclidean factors). In the above notation, $\widetilde{W}(\mathfrak{g},\mathfrak{a}) = \widetilde{W}_{\mathfrak{a}}(\mathfrak{g},\mathfrak{h})|_{\mathfrak{a}}$ and the restriction map $I_{\widetilde{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{h}_{\mathbb{R}}) \to I_{\widetilde{W}(\mathfrak{g},\mathfrak{a})}(\mathfrak{a})$ is surjective.*

Proof. We can assume that G/K is irreducible. If neither $\Delta(\mathfrak{g},\mathfrak{h})$ nor $\Sigma(\mathfrak{g},\mathfrak{a})$ is of type D this is Theorem 5 from [**6**]. According to the above discussion, the only cases where $\Sigma(\mathfrak{g},\mathfrak{a})$ is of type D are where $\Delta(\mathfrak{g},\mathfrak{h})$ is also of type D and $\mathfrak{a} = \mathfrak{h}_{\mathbb{R}}$, or $\mathfrak{a}$ is the diagonal in $\mathfrak{h} \cong \mathfrak{a}\oplus\mathfrak{a}$, or $\mathfrak{a} = \mathfrak{h}$. The statement is clear when $\mathfrak{a}$ is $\mathfrak{h}$ or $\mathfrak{h}_{\mathbb{R}}$. If $\mathfrak{a}$ is the diagonal in $\mathfrak{h} \cong \mathfrak{a}\oplus\mathfrak{a}$ then $\widetilde{W}_{\mathfrak{a}}(\mathfrak{g},\mathfrak{h})$ is the diagonal in $\widetilde{W}(\mathfrak{g},\mathfrak{h}) \cong \widetilde{W}(\mathfrak{g},\mathfrak{a}) \times \widetilde{W}(\mathfrak{g},\mathfrak{a})$, hence again is $\widetilde{W}(\mathfrak{g},\mathfrak{a})$.

Now suppose that neither $\Delta(\mathfrak{g},\mathfrak{h})$ nor $\Sigma_{1/2}(\mathfrak{g},\mathfrak{a})$ is of type D. Then $\widetilde{W}(\mathfrak{g},\mathfrak{a}) = W(\mathfrak{g},\mathfrak{a})$ consists of all permutations with sign changes (with respect to the correct basis). The claim now follows from the explicit calculations in [**6**, pp. 594, 596]. □

Let $M_k = G_k/K_k$ and $M_n = G_n/K_n$ be irreducible symmetric spaces of compact or noncompact type. We say that M_k *propagates* M_n, if $G_n \subseteqq G_k$, $K_n = K_k \cap G_n$, and either $\mathfrak{a}_k = \mathfrak{a}_n$ or choosing $\mathfrak{a}_n \subseteqq \mathfrak{a}_k$ we only add simple roots to the left end of the Dynkin diagram for $\Psi_{1/2}(\mathfrak{g}_n,\mathfrak{a}_n)$ to obtain the Dynkin diagram for $\Psi_{1/2}(\mathfrak{g}_k,\mathfrak{a}_k)$. So, in particular $\Psi_{1/2}(\mathfrak{g}_n,\mathfrak{a}_n)$ and $\Psi_{1/2}(\mathfrak{g}_k,\mathfrak{a}_k)$ are of the same type. In general, if M_k and M_n are Riemannian symmetric spaces of compact or noncompact type, with universal covering $\widetilde{M_k}$ respectively $\widetilde{M_n}$, then M_k *propagates* M_n if we can enumerate the irreducible factors of $\widetilde{M_k} = M_k^1 \times \ldots \times M_k^j$ and $\widetilde{M_n} = M_n^1 \times \ldots \times M_n^i$, $i \leqq j$ so that M_k^s propagates M_n^s for $s = 1, \ldots, i$. Thus, each M_n is, up to covering, a product of irreducible factors listed in Table 2.10.

In general we can construct infinite sequences of propagations by moving along each row in Table 2.10. But there are also some propagations that do not fit easily into sequences, such as $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n) \subset \mathrm{SL}(k,\mathbb{C})/\mathrm{SU}(k)$ which satisfy the definition of propagation.

When $\mathfrak{g}_k$ propagates $\mathfrak{g}_n$, and θ_k and θ_n are the corresponding involutions with $\theta_k|_{\mathfrak{g}_n} = \theta_n$, the corresponding eigenspace decompositions $\mathfrak{g}_k = \mathfrak{k}_k \oplus \mathfrak{s}_k$ and $\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{s}_n$ give us

$$\mathfrak{k}_n = \mathfrak{k}_k \cap \mathfrak{g}_n\,, \quad \text{and} \quad \mathfrak{s}_n = \mathfrak{g}_n \cap \mathfrak{s}_k\,.$$

We recursively choose maximal commutative subspaces $\mathfrak{a}_k \subset \mathfrak{s}_k$ such that $\mathfrak{a}_n \subseteqq \mathfrak{a}_k$ for $k \geqq n$. Denote by $W(\mathfrak{g}_n, \mathfrak{a}_n)$ and $W(\mathfrak{g}_k, \mathfrak{a}_k)$ the corresponding Weyl groups. The extensions $\widetilde{W}(\mathfrak{g}_k, \mathfrak{a}_k)$ and $\widetilde{W}(\mathfrak{g}_n, \mathfrak{a}_n)$ are defined as just before Theorem 2.11. Let $\mathrm{I}(\mathfrak{a}_n) = \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n)$, $\mathrm{I}_{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n)$, and $\mathrm{I}_{\widetilde{W}(\mathfrak{g}_k,\mathfrak{a}_k)}(\mathfrak{a}_k)$ denote the respective sets of Weyl group invariant or $\widetilde{W}$–invariant polynomials on $\mathfrak{a}_n$ and $\mathfrak{a}_k$. As before we let

$$W_{\mathfrak{a}_n}(\mathfrak{g}_k, \mathfrak{a}_k) := \{w \in W(\mathfrak{g}_k, \mathfrak{a}_k) \mid w(\mathfrak{a}_n) = \mathfrak{a}_n\} \tag{2.12}$$

and define $\widetilde{W}_{\mathfrak{a}_n}(\mathfrak{g}_k, \mathfrak{a}_k)$ in the same way.

Theorem 2.13. *Assume that M_k and M_n are symmetric spaces of compact or noncompact type and that M_k propagates M_n.*
(1) *If M_n does not contain any irreducible factor with $\Psi_{1/2}(\mathfrak{g}_n, \mathfrak{a}_n)$ of type D, then*

$$W_{\mathfrak{a}_n}(\mathfrak{g}_k, \mathfrak{a}_k)|_{\mathfrak{a}_n} = W(\mathfrak{g}_n, \mathfrak{a}_n) \tag{2.14}$$

and the restriction map $\mathrm{I}(\mathfrak{a}_k) \to \mathrm{I}(\mathfrak{a}_n)$ is surjective.
(2) *If $\Psi_{1/2}(\mathfrak{g}_n, \mathfrak{a}_n)$ is of type D then*

$$W(\mathfrak{g}_n, \mathfrak{a}_n) \subsetneqq W_{\mathfrak{a}_n}(\mathfrak{g}_k, \mathfrak{a}_k)|_{\mathfrak{a}_n} \ \textit{and}\ \mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{a}_k)}(\mathfrak{a}_k)|_{\mathfrak{a}_n} \subsetneqq \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n).$$

On the other hand

$$\widetilde{W}(\mathfrak{g}_n, \mathfrak{a}_n) = \widetilde{W}_{\mathfrak{a}_n}(\mathfrak{g}_k, \mathfrak{a}_k)|_{\mathfrak{a}_n} \ \textit{and}\ \mathrm{I}_{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n) = \mathrm{I}_{\widetilde{W}(\mathfrak{g}_k,\mathfrak{a}_k)}(\mathfrak{a}_k)|_{\mathfrak{a}_n}.$$

(3) *In all cases $\widetilde{W}(\mathfrak{g}_n, \mathfrak{a}_n) = \widetilde{W}_{\mathfrak{a}_n}(\mathfrak{g}_k, \mathfrak{a}_k)|_{\mathfrak{a}_n}$ and $\mathrm{I}_{\widetilde{W}(\mathfrak{g}_k,\mathfrak{a}_k)}(\mathfrak{a}_k)|_{\mathfrak{a}_n} = \mathrm{I}_{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n)$.*

Proof. It suffices to prove this for each irreducible component of M_n. The argument of Theorem 1.7 is valid here as well, and our assertion follows. □

3. Applications

Our interest in restriction of Weyl groups and polynomial invariants came from the study of projective limits of function of exponential growth. It turned out that the main step in showing that that the projective limit is non zero one needed to understand the restriction of invariant polynomials and Weyl groups. We refer to [**13**] for those applications. Some of those results are also mentioned in [**4**] in this volume and will use the notation from that article. We assume that $M = G/K$ is a symmetric space of the noncompact type. We keep the notation from the previous sections. In particular, $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{a})$ is a positive system of restricted roots. Let

$$\mathfrak{n} := \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha \text{ and } \mathfrak{p} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}\,.$$

Then $\mathfrak{n}$ is a nilpotent Lie algebra and $\mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{n})$ is a minimal parabolic subalgebra. The corresponding minimal parabolic subgroup is $P = MAN$ with $M = Z_K(\mathfrak{a})$,

$A = \exp \mathfrak{a}$, and $N = \exp(\mathfrak{n})$. We have the Iwasawa decomposition $G = KAN \simeq K \times A \times N$. Write $x = k(x)a(x)n(x)$ for the unique decomposition of x. This implies that $B := G/P = K/M$ and G acts on B by $x \cdot kM = k(xk)M$.

If $a = \exp(H) \in A$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ then $(man)^\lambda = a^\lambda := e^{\lambda(H)}$. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Sigma^+} \dim \mathfrak{g}_\alpha \alpha \in \mathfrak{a}^*$. We normalize the invariant measures so that K has measure one, $\int_N a(\theta(n))^{-2\rho}\, dn = 1$, and the measure on A and $\mathfrak{a}^*$ are normalized so that the Fourier inversion holds without constant. Finally $\int_G f(g)\, dg = \int_K \int_A \int_N f(kan)a^{-2\rho}\, dndadk$, $f \in C_c(G)$. The spherical function with spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is defined by

$$\varphi_\lambda(x) = \int_G a(x^{-1}k)^{-\lambda-\rho}\, dk\,. \tag{3.1}$$

We have $\varphi_\lambda = \varphi_\mu$ if and only if there exists $w \in W$ such that $w\lambda = \mu$.

The spherical Fourier transform is defined by

$$\widehat{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)\, dx\,, \quad f \in C_c^\infty(G/K)^K\,.$$

Then $\widehat{f}$ is a holomorphic Weyl group invariant function on $\mathfrak{a}_{\mathbb{C}}^*$. Furthermore

$$f(x) = \int_{i\mathfrak{a}^*} \widehat{f}(\lambda)\varphi_\lambda(x)\, \frac{d\lambda}{\#W|c(\lambda)|^2}$$

and the Fourier transform extends to an unitary isomorphism

$$L^2(M)^K \simeq L^2\left(i\mathfrak{a}^*, \frac{d\lambda}{\#W|c(\lambda)|^2}\right)\,.$$

Here $c(\lambda)$ denotes the Harish-Chandra c-function. We will also write $\mathcal{F}(f)$ for $\widehat{f}$.

We start with the following lemma.

A connected semisimple Lie group G is *algebraically simply connected* if it is an analytic subgroup of the connected simply connected group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Then the analytic subgroup K of G for $\mathfrak{k}$ is compact, and every automorphism of $\mathfrak{g}$ integrates to an automorphism of G.

LEMMA 3.2. *Let G/K be a Riemannian symmetric space of noncompact type with G simple and algebraically simply connected. Suppose that $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, i.e., that $\mathfrak{g}$ is a split real form of $\mathfrak{g}_{\mathbb{C}}$. If $\sigma : \mathfrak{a} \to \mathfrak{a}$ is a linear isomorphism such that σ' defines an automorphism of the Dynkin diagram of $\Psi(\mathfrak{g}, \mathfrak{a})$, then there exists a automorphism $\widetilde{\sigma} : G \to G$ such that*

(1) *$\widetilde{\sigma}|_{\mathfrak{a}} = \sigma$ where by abuse of notation we write $\widetilde{\sigma}$ for $d\widetilde{\sigma}$,*
(2) *$\widetilde{\sigma}$ commutes with the the Cartan involution θ, and in particular $\widetilde{\sigma}(K) = K$,*
(3) *$\widetilde{\sigma}(N) = N$.*

PROOF. The complexification of $\mathfrak{a}$ is a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a}$. Let $\{Z_\alpha\}_{\alpha\in\Sigma(\mathfrak{g},\mathfrak{a})}$ be a Weyl basis for $\mathfrak{g}_{\mathbb{C}}$ (see, for example, [**16**, page 285]). Then (see, for example, [**16**, Theorem 4.3.26]),

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{h})} \mathbb{R}Z_\alpha$$

is a real form of $\mathfrak{g}_{\mathbb{C}}$. Denote by B the Killing form of $\mathfrak{g}_{\mathbb{C}}$. Then $B(Z_\alpha, Z_{-\alpha}) = -1$ and it follows that B is positive definite on $\mathfrak{a}$ and on $\bigoplus_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})} \mathbb{R}(Z_\alpha - Z_{-\alpha})$, and negative definite on $\bigoplus_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})}\mathbb{R}(Z_\alpha+Z_{-\alpha})$. Hence, the map

$$\theta|_{\mathfrak{a}} = -\mathrm{id} \text{ and } \theta(Z_\alpha) = Z_{-\alpha}$$

defines a Cartan involution on $\mathfrak{g}_0$ such that the Cartan subalgebra $\mathfrak{a}$ is contained in the corresponding -1 eigenspace $\mathfrak{s}$. As there is (up to isomorphism) only one real form of $\mathfrak{g}_{\mathbb{C}}$ with Cartan involution such that $\mathfrak{a}\subset\mathfrak{s}$ we can assume that $\mathfrak{g}=\mathfrak{g}_0$ and that the above Cartan involution θ is the the one we started with.

Going back to the proof of [**16**, Lemma 4.3.24] the map defined by

$$\widetilde{\sigma}|_{\mathfrak{a}} = \sigma \quad \text{and} \quad \widetilde{\sigma}(Z_\alpha) = Z_{\sigma\alpha}$$

is a Lie algebra isomorphism $\widetilde{\sigma}:\mathfrak{g}\to\mathfrak{g}$. But then

$$\widetilde{\sigma}(\theta(Z_\alpha)) = \widetilde{\sigma}(Z_{-\alpha}) = Z_{\sigma(-\alpha)} = Z_{-\sigma(\alpha)} = \theta(\widetilde{\sigma}(Z_\alpha)).$$

Finally, $\theta|_{\mathfrak{a}} = -\mathrm{id}$ and it follows that $\widetilde{\sigma}$ and θ commute. As

$$\mathfrak{k} = \bigoplus_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})} \mathbb{R}(Z_\alpha+\theta(Z_\alpha))$$

and $\sigma(\Sigma^+(\mathfrak{g},\mathfrak{a})) = \Sigma^+(\mathfrak{g},\mathfrak{a})$ it follows that $\widetilde{\sigma}(\mathfrak{k}) = \mathfrak{k}$.

As $\sigma(\Sigma^+(\mathfrak{g},\mathfrak{a})) = \Sigma^+(\mathfrak{g},\mathfrak{a})$ it follows that $\widetilde{\sigma}(\mathfrak{n}) = \mathfrak{n}$.

As G is assumed to be algebraically simply connected, there is an automorphism of G with differential $\widetilde{\sigma}$. Denote this automorphism also by $\widetilde{\sigma}$. It is clear that $\widetilde{\sigma}$ satisfies the assertions of the lemma. ☐

Define an involution $\widetilde{\sigma}$ on G in the following way: If G_j/K_j is an irreducible factor of $M = G/K$ then $\widetilde{\sigma}|_{G_j}$ is the identity if G_j/K_j is not of type D, otherwise it is the involution from Lemma 3.2. Then we define $\widetilde{G} = G\rtimes\{1,\widetilde{\sigma}\}$ and $\widetilde{K} = K\rtimes\{1,\widetilde{\sigma}\}$. Note that $M = G/K = \widetilde{G}/\widetilde{K}$.

THEOREM 3.3. *Let $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ and $x\in M$. Then*

$$\varphi_\lambda(\widetilde{\sigma}(x)) = \varphi_{\sigma(\lambda)}(x)\,.$$

If $f\in L^2(M)^{\widetilde{K}}$ then $\widehat{f}$ is σ-invariant.

PROOF. Write $x = kan$, then $\widetilde{\sigma}(x) = \widetilde{\sigma}(k)\widetilde{\sigma}(a)\widetilde{\sigma}(n)$. Thus $a(\widetilde{\sigma}(x)) = \widetilde{\sigma}(a(x))$. By (3.1) and the fact that $\sigma(\rho)=\rho$ and that the invariant measure on K is $\widetilde{\sigma}$-invariant we get

$$\begin{aligned}\varphi_\lambda(\widetilde{\sigma}(x)) &= \int_K a(\widetilde{\sigma}(x^{-1})k)^{-\lambda-\rho}\,dk\\ &= \int_K (\widetilde{\sigma}(a(x^{-1}k)))^{-\lambda-\rho}\,dk\\ &= \int_K a(x^{-1}k)^{-\sigma\lambda-\rho}\,dk\\ &= \varphi_{\sigma(\lambda)}(x)\,.\end{aligned}$$

The remaining statements are now clear. ☐

Fix a positive definite K–invariant bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{s}$. It defines an invariant Riemannian structure on M and hence also an invariant metric $d(x,y)$. Let $x_o = eK\in M$ and for $r>0$ denote by $B_r = B_r(x_o)$ the closed ball

$$B_r = \{x\in M \mid d(x,x_o)\leqq r\}\,.$$

Note that B_r is $\widetilde{K}$–invariant. Denote by $C_r^\infty(M)^{\widetilde{K}}$ the space of smooth $\widetilde{K}$–invariant functions on M with support in B_r. The restriction map $f \mapsto f|_A$ is a bijection from $C_r^\infty(M)^{\widetilde{K}}$ onto $C_r^\infty(A)^{\widetilde{W}}$ (using the obvious notation).

For a finite dimensional Euclidean vector space E and a closed subgroup W of $\mathrm{O}(E)$ let $\mathrm{PW}_r(E_\mathbb{C})^W$ be the space of holomorphic functions $F : E_\mathbb{C} \to \mathbb{C}$ such that for all $k \in \mathbb{N}$

$$\sup_{z \in E_\mathbb{C}} (1 + |z|)^k e^{-r|\operatorname{Im} z|} |F(z)| < \infty$$

and $F(w \cdot z) = F(z)$ for all $z \in E_\mathbb{C}$ and $w \in W$. In particular $\mathrm{PW}_r(\mathfrak{a}_\mathbb{C}^*)^{\widetilde{W}}$ is well defined. The following is a simple modification of the Paley-Wiener theorem of Helgason [**7, 10**] and Gangolli [**5**]; see [**11**] for a short overview.

THEOREM 3.4 (The Paley-Wiener Theorem). *The Fourier transform defines bijections*

$$C_r^\infty(M)^{\widetilde{K}} \cong \mathrm{PW}_r(\mathfrak{a}_\mathbb{C}^*)^{\widetilde{W}} .$$

We assume now that M_k propagates M_n, $k \geqq n$. The index j refers to the symmetric space M_j, for a function F on $\mathfrak{a}_{k,\mathbb{C}}^*$ let $P_{k,n}(F) := F|_{\mathfrak{a}_{n,\mathbb{C}}}$. We fix a compatible K–invariant inner products on $\mathfrak{s}_n$ and $\mathfrak{s}_k$, i.e., for all $X, Y \in \mathfrak{s}_n \subsetneqq \mathfrak{s}_k$ we have

$$\langle X, Y\rangle_k = \langle X, Y\rangle_n .$$

We refer to [**13**] for the application to injective sequences of symmetric spaces, for the injective limit of symmetric spaces of the noncompact type, see also the overview [**4**] in this volume.

THEOREM 3.5 ([**13**]). *Assume that M_k propagates M_n. Let $r > 0$. Then the following holds:*

1. *The map $P_{k,n} : \mathrm{PW}_r(\mathfrak{a}_{k,\mathbb{C}}^*)^{\widetilde{W}(\mathfrak{g}_k,\mathfrak{a}_k)} \to \mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}}^*)^{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}$ is surjective.*
2. *The map $C_{k,n} = \mathcal{F}_n^{-1} \circ P_{k,n} \circ \mathcal{F}_k : C_r^\infty(M_k)^{\widetilde{K}_k} \to C_r^\infty(M_n)^{\widetilde{K}_n}$ is surjective.*

Let us explain the connection with Theorem 2.13. For that let $F \in \mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}}^*)^{\widetilde{W}_n}$, where $\widetilde{W}_n = \widetilde{W}(\mathfrak{g}_n, \mathfrak{a}_n)$. Then, according to a result of Cowling [**3**] there exists a $G \in \mathrm{PW}_r(\mathfrak{a}_{k,\mathbb{C}}^*)^{\widetilde{W}_k}$ such that $G|_{\mathfrak{a}_{n,\mathbb{C}}^*} = F$. We can assume that G is invariant under $\widetilde{W}_{k,n} = \{w \in \widetilde{W}_k \mid w(\mathfrak{a}_n) = \mathfrak{a}_n\}$. As $\widetilde{W}_k$ is a finite reflection group it follows by [**14**] that there exists $G_1, \ldots, G_{r_k} \in \mathrm{PW}_r(\mathfrak{a}_{k,\mathbb{C}}^*)^{\widetilde{W}_k}$ and $p_1, \ldots, p_{r_k} \in \mathrm{I}_{\widetilde{W}_{k,n}}(\mathfrak{a}_k)$ such that

$$G = p_1 G_1 + \ldots + p_{r_n} G_{r_n} .$$

As $p_j|_{\mathfrak{a}_n} \in \mathrm{I}_{\widetilde{W}_n}(\mathfrak{a}_n)$ Theorem 2.13 there exists $q_j \in \mathrm{I}_{\widetilde{W}_k}(\mathfrak{a}_k)$ such that $q_j|_{\mathfrak{a}_n} = p_j|_{\mathfrak{a}_n}$. But then $H := q_1 G_1 + \ldots + q_{r_k} G_{r_k} \in \mathrm{PW}_r(\mathfrak{a}_{k,\mathbb{C}}^*)^{\widetilde{W}_k}$ and $H|_{\mathfrak{a}_{n,\mathbb{C}}^*} = F$ showing that the restriction map is surjective.

It is well known, [**10**, Thm 5.13,p.300], that if $M = G/K$ is a Riemannian symmetric space of the noncompact type then there exists an algebra isomorphism $\Gamma : \mathbb{D}(M) \to \mathrm{I}_W(\mathfrak{a})$, where $\mathbb{D}(M)$ is the algebra of invariant differential operators, such that

$$D\varphi_\lambda = \Gamma(\lambda)\varphi_\lambda \quad \text{for all } \lambda \in \mathfrak{a}_{n,\mathbb{C}}^* .$$

Restricting Γ to $\widetilde{\mathbb{D}}(M)$, the algebra of $\widetilde{G}$-invariant differential operators on M then gives:

LEMMA 3.6. *There exists an algebra isomorphism* $\widetilde{\Gamma} : \widetilde{\mathbb{D}}(M) \to \mathrm{I}_{\widetilde{W}}(\mathfrak{a}_n)$ *such that for all* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *and* $D \in \widetilde{\mathbb{D}}(M)$ *we have*

$$D\varphi_\lambda = \widetilde{\Gamma}(D)\varphi_\lambda\,.$$

THEOREM 3.7. *Assume that* M_k *propagates* M_n. *There exists a surjective algebra homomorphism* $\Gamma_{k,n} : \widetilde{\mathbb{D}}(M_k) \to \widetilde{\mathbb{D}}(M_n)$ *such that for all* $f \in C_c^\infty(M_k)$ *we have*

$$C_{k,n}(Df) = \Gamma_{k,n}(D)C_{k,n}(f)\,.$$

PROOF. For $D \in \widetilde{\mathbb{D}}(M_k)$ define $\Gamma_{k,n}(D) := \Gamma_n^{-1}(\Gamma_k(D)|_{\mathfrak{a}_n})$. Then $\Gamma_{k,n}(D) \in \widetilde{\mathbb{D}}(M_n)$ and by Theorem 2.13 $\Gamma_{k,n} : \widetilde{\mathbb{D}}(M_k) \to \widetilde{\mathbb{D}}(M_n)$ is a surjective homomorphism.

Let $f \in C_c^\infty(M_k)$. Then

$$\begin{aligned} C_{k,n}(Df) &= \mathcal{F}_n^{-1}(P_{k,n}(\mathcal{F}_k(Df))) \\ &= \mathcal{F}_n^{-1}(\Gamma_k(D)|_{\mathfrak{a}_n} P_{k,n}(f) \\ &= \Gamma_n^{-1}(\Gamma_k(D)|_{\mathfrak{a}_n})C_{k,n}(f) \\ &= \Gamma_{k,n}(D)C_{k,n}(f) \end{aligned}$$

proving the theorem. □

Note, if we take D to be the Laplacian Δ_k on M_k then $\Gamma_k(D) = \lambda^2 - |\rho_k|^2$ where $\lambda^2 = \lambda_1^2 + \ldots + \lambda_{r_k}^2$ where we write $\lambda = \lambda_1 e_1^* + \ldots + \lambda_{r_k} e_{r_k}^*$ with respect to an orthonormal basis of $\mathfrak{a}_k^*$. Thus

$$\Gamma_{k,n}(\Delta_k) = \Delta_n - (|\rho_k|^2 - |\rho_n|^2)\,.$$

Let $M_k = \mathrm{SO}_o(1,k)/\mathrm{SO}(k)$ and $M_n = \mathrm{SO}(1,n)/\mathrm{SO}(n)$ then $\mathfrak{a}_k = \mathfrak{a}_n$, $\Sigma = \{\alpha, -\alpha\}$, $\rho_k = \frac{k}{2}\alpha$, and $\rho_n = \frac{n}{2}\alpha$. Normalizing the inner product so that $|\alpha| = 1$ we get

$$|\rho_k|^2 - |\rho_n|^2 = \frac{1}{4}(k^2 - n^2) \to \infty \text{ as } n, k \to \infty\,.$$

Hence in the limit Δ_∞ does not exists. However, the shifted Laplacian $\Delta_k - |\rho_k|^2$ has a limit as $k \to \infty$. It should be noted, that it is exactly this shifted Laplacian that plays a role in the wave equation on symmetric spaces of the noncompact type, see [**2, 12**] and the reference therein. It is also interesting to note that in [**15**] the same ρ-shift was used in the spherical functions to study the heat equation on inductive limits of a class of symmetric spaces of the noncompact type.

References

[1] S. Araki, On root systems and an infinitesimal classification of irreducible symmetric spaces, J. Math. Osaka City Univ. **13** (1962), 1–34.

[2] T. Branson, G. Ólafsson, and H. Schlichtkrull, Huygens' principle in Riemannian symmetric spaces. *Math. Ann.* **301** (1995), 445–462

[3] M. Cowling, On the Paley–Wiener theorem, *Invent. Math.* **83** (1986), 403–404.

[4] S. Dann and G. Ólafsson, Paley-Wiener Theorems with respect to spectral parameters. This volume.

[5] R. Gangolli, On the Plancherel formula and the Paley–Wiener theorem for spherical functions on semisimple Lie groups. *Ann. of Math.* (2) **93** (1971), 150–165.

[6] S. Helgason, Fundamental solutions of invariant differential operators on symmetric spaces. *Amer. J. Math.* **86** 1964 565–601.

[7] ———, An analog of the Paley–Wiener theorem for the Fourier transform on certain symmetric spaces, *Math. Ann.* **165** (1966), 297–308.

[8] ______ Some results on invariant differential operators on symmetric spaces. Amer. J. Math. **114** (1992), no. 4, 789–811.

[9] ______, *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press, 1978.

[10] ______, *Groups and Geometric Analysis*, Academic Press, 1984.

[11] G. Ólafsson, and A. Pasquale, Paley–Wiener theorems for the Θ–spherical transform: an overview. *Acta Appl. Math.* **81** (2004), no. 1–3, 275–309.

[12] ______, Wave propagation on Riemannian symmetric spaces. *J. Funct. Anal.* **107** (1992), 270–278.

[13] G. Ólafsson and J. A. Wolf, The Paley-Wiener Theorem and Limits of Symmetric Spaces, to appear.

[14] M. Rais, Groupes linéaires compacts et fonctions C^∞ covariantes, *Bull. Sc. Math.* **107** (1983), 93–111.

[15] A. R. Sinton, The spherical transform on projective limits of symmetric spaces. *J. Lie Theory* **17** (2007), 869–898.

[16] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Prentice–Hall, 1974.

[17] J. A. Wolf, Infinite dimensional multiplicity free spaces III: Matrix coefficients and regular function, *Math. Anallen* **349** (2011), 263-299. {arXiv:0909.1735 (math.RT, math.DG).}

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.

E-mail address: olafsson@math.lsu.edu

Department of Mathematics, University of California, Berkeley, CA 94720–3840, U.S.A.

E-mail address: jawolf@math.berkeley.edu