THE PALEY-WIENER THEOREM AND LIMITS OF SYMMETRIC SPACES

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Abstract. We extend the Paley–Wiener theorem for riemannian symmetric spaces to an important class of infinite dimensional symmetric spaces. For this we define a notion of propagation of symmetric spaces and examine the direct (injective) limit symmetric spaces defined by propagation. This relies on some of our earlier work on invariant differential operators and the action of Weyl group invariant polynomials under restriction.

Introduction

We start with the notion of prolongation for symmetric spaces. In essence, a symmetric space M_k is a prolongation of another, say M_n, when M_n sits in M_k in the simplest possible way. For example, if $M_\ell = SU(\ell+1)$, compact group manifold, then M_n sits in M_k as an upper left hand corner.

Suppose that M_k is a prolongation M_n where both are of compact type or both of noncompact type. We prove surjectivity for restriction of Weyl group invariant holomorphic functions of exponential growth r. We discuss the conditions on r in a moment. This gives a corresponding restriction result on the Fourier transform spaces and then a sujective map $C^\infty_r(M_k) \to C^\infty_r(M_n)$. Using results on conjugate and cut locus of compact symmetric spaces we show that the radius of injectivity for compact symmetric spaces forming a direct system, related by prolongation, is constant. If R is that radius then the condition on the exponential growth size r is a function of R, thus constant for the direct system. This, together with the results of [17], allows us to carry the finite dimensional Paley–Wiener theorem to the limit. See Theorems 3.5, 4.6 and 7.12 below.

The classical Paley–Wiener Theorem describes the growth of the Fourier transform of a function $f \in C^\infty_c(\mathbb{R}^n)$ in terms of the size of its support. Helgason and Gangolli generalized it to riemannian symmetric spaces of noncompact type, Arthur extended it to semisimple Lie groups, van den Ban and Schlichtkrull made the extension to pseudo-riemannian reductive symmetric spaces, and finally Ólafsson and Schlichtkrull worked out the corresponding result for compact riemannian symmetric spaces. Here we extend these results to a class of infinite dimensional riemannian symmetric spaces, the classical direct limits compact symmetric spaces. The main idea is to combine the results of Ólafsson and Schlichtkrull with Wolf's results on direct limits $\varinjlim M_n$ of riemannian symmetric spaces and limits of the corresponding function spaces on the M_n.

Of course compact support in the Paley–Wiener Theorem is irrelevant for functions on a compact symmetric space. There one concentrates on the radius of the support. The Fourier transform space is interpreted as the parameter space for spherical functions. It is linear dual space of the complex span of the restricted roots. When we pass to direct limits it is crucial that these ingredients be properly

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normalized. In order to do this we introduce the notion of propagation for pairs of root systems, pairs of groups, and pairs of symmetric spaces.

In Section 1 we recall some basic facts concerning Paley–Wiener theorems on Euclidean spaces and their behavior under the action of finite symmetry groups. In this setting we give surjectivity criteria for restriction of Paley–Wiener spaces.

In Section 2 we discuss the structural results, both for symmetric spaces of compact type and of noncompact type, that we will need later. In order to do this we recall our notion of propagation from [17] and examine the corresponding Weyl group invariants explicitly for each type of root system. The key there is the main result of [17], which summarizes the facts on restriction of Weyl groups for propagation of symmetric spaces.

In Section 3 we apply our results on Weyl group invariants to Fourier analysis on riemannian symmetric spaces of noncompact type. The main result is Theorem 3.7, the Paley–Wiener Theorem for classical direct limits of those spaces. As indicated earlier, a $\mathbb{Z}_2$ extension of the Weyl group is needed in case of root systems of type D. The extension can be realized by an automorphism σ of the of the Dynkin diagram. We show that there exists an automorphism $\widetilde{\sigma}$ of G or a double cover such that $d\widetilde{\sigma}|_{\mathfrak{a}} = \sigma$ and the spherical function with spectral parameter λ satisfies $\varphi_\lambda(\widetilde{\sigma}(x)) = \varphi_{\sigma'(\lambda)}(x)$.

In Section 4 we set up the basic surjectivity of the direct limit Paley–Wiener Theorem for the classical sequences $\{SU(n)\}$, $\{SO(2n)\}$, $\{SO(2n+1)\}$ and $\{Sp(2n)\}$. The key tool is Theorem 4.1, the calculation of the injectivity radius. That radius turns out to be a simple constant ($\sqrt{2}\,\pi$ or 2π) for each of the series. The main result is Theorem 4.7, which sets up the projective systems of functions used in the Paley–Wiener Theorem for $SU(\infty)$, $SO(\infty)$ and $Sp(\infty)$. All this is needed when we go to limits of symmetric spaces.

In Section 5 we examine limits of spherical representations of compact symmetric spaces. Theorem 5.10 is the main result. It sets up the sequence of function spaces corresponding to a direct system $\{M_n\}$ of compact riemannian symmetric spaces in which M_k propagates M_n for $k \geqq n$. We use this in Section 6 to show that a certain surjective map $Q : C^\infty(G)^G \to C^\infty(G/K)^K$ is in fact surjective as a map $C^\infty_r(G)^G \to C^\infty_r(G/K)^K$. Here $Q(f)(xK) := \int_K f(xk)\,dk$ and the subscript $_r$ denotes the size of the support.

Then in Section 6, we relate the spherical Fourier transforms for the sequence $\{M_n\}$, show how the injectivity radii remain constant on the sequence. We then prove the Paley–Wiener Theorem 6.7 for compact symmetric spaces in a form that is applicable to direct limits $M_\infty = \varinjlim M_n$ of compact riemannian symmetric spaces in which M_k propagates M_n for $k \geqq n$. Along the way we obtain a stronger form, Theorem 6.9, of one of the key ingredients in the proof of the surjectivity.

Finally in Section 7 we introduce and discuss a K–invariant domain in M that behaves well under propagation. This leads to a corresponding restriction theorem, Theorem 7.12, and another result of Paley–Wiener type, Theorem 7.15.

Our discussion of direct limit Paley–Wiener Theorems involves function space maps that have a somewhat indirect relation [23] to the L^2 theory of [22]. This is discussed in Section 8, where we compare our maps with the partial isometries of [22].

1. Polynomial Invariants and Restriction of Paley-Wiener spaces

In this section we recall and refine some results of Cowling and Rais that will be used later in this article.

Let $E \cong \mathbb{R}^n$ be a finite dimensional Euclidean space. Let $\langle x, y\rangle_E = \langle x, y\rangle = x \cdot y$ denote the inner product on E and its $\mathbb{C}$–bilinear extension to the complexification $E_{\mathbb{C}} \cong \mathbb{C}^n$. Let $|\cdot|$ denote the corresponding norm on E and $E_{\mathbb{C}}$. Note that $\langle\,\cdot\,,\,\cdot\,\rangle$ defines an bilinear form and a norm on E^* and $E_{\mathbb{C}}^*$.

Denote by $C_r^\infty(E)$ the space of smooth functions on E with support in a closed ball $\overline{B_r(0)}$ of radius $r > 0$. Write $\mathrm{PW}_r(E_{\mathbb{C}}^*)$ for the space of holomorphic function on $E_{\mathbb{C}}^*$ with the property that for each $n \in \mathbb{Z}^+$ there exists a constant $C_n > 0$ such that

$$\nu_n(F) := \sup_{\lambda \in E_{\mathbb{C}}} (1 + |\lambda|^2)^n e^{-r|\operatorname{Im}\lambda|} |F(\lambda)| < \infty\,. \tag{1.1}$$

Consider a G-module V. The action on functions is given as usual by $L_w f(v) := f(w^{-1}v)$ and we denote the fixed point set by

$$V^G = \{v \in V \mid g \cdot v = v \text{ for all } g \in G\}\,. \tag{1.2}$$

In particular, given a closed subgroup $G \subset O(E)$, the spaces $\mathrm{PW}_r(E_{\mathbb{C}}^*)^G$ and $C_r^\infty(E)^G$ are well defined. We normalize the Fourier transform on E as

$$\mathcal{F}_E(f)(\lambda) = \widehat{f}(\lambda) = (2\pi)^{-n/2} \int_E f(x) e^{-i\lambda(x)}\, dx\,, \quad \lambda \in E_{\mathbb{C}}^* \text{ and } n = \dim E. \tag{1.3}$$

The Paley–Wiener Theorem says that $\mathcal{F}_E : C_r^\infty(E)^G \to \mathrm{PW}_r(E_{\mathbb{C}}^*)^G$ is an isomorphism.

From now on we assume that F is another Euclidean space and that $E \subseteqq F$. We always assume that the inner products on E and F are chosen so that $\langle x, y\rangle_E = \langle x, y\rangle_F$ for all $x, y \in E$. Furthermore, if $W(E)$ and $W(F)$ are closed subgroups of the respective orthogonal groups acting on E and F, then set

$$W_E(F) = \{w \in W(F) \mid w(E) = E\}\,.$$

We always assume that $W(E)$ and $W(F)$ are generated by reflections $s_\alpha : v \mapsto v - \frac{2\alpha(v)}{\langle\alpha,\alpha\rangle} h_\alpha$, for α in a root system in E^* (respectively F^*). However the Cowling result below holds for arbitrary closed subgroup of $\mathrm{O}(E)$ (respectively $\mathrm{O}(F)$).

Theorem 1.4 (Cowling)**.** *The restriction map* $\mathrm{PW}_r(F_{\mathbb{C}}^*)^{W_E(F)} \to \mathrm{PW}_r(E_{\mathbb{C}}^*)^{W_E(F)|_{E_{\mathbb{C}}}}$*, given by* $F \mapsto F|_{E_{\mathbb{C}}^*}$*, is surjective.*

Denote by $\mathrm{S}(E)$ the symmetric algebra of E. It can be identified with the algebra of polynomial functions on E^*. We use similar notation for F^*.

Theorem 1.5 (Rais)**.** *Let* $P_1, \ldots, P_n$ *be a basis for* $\mathrm{S}(F)$ *over* $\mathrm{S}(F)^{W(F)}$*. If* $F \in \mathrm{PW}_r(F_{\mathbb{C}}^*)$ *there exist* $\Phi_1, \ldots, \Phi_n \in \mathrm{PW}_r(F_{\mathbb{C}}^*)^{W(F)}$ *such that*

$$F = P_1\Phi_1 + \ldots + P_n\Phi_n\,.$$

If $W_E(F)|_E = W(E)$ then Cowling's Theorem implies that the restriction map

$$\mathrm{PW}_r(F_{\mathbb{C}}^*)^{W_E(F)} \to \mathrm{PW}_r(E_{\mathbb{C}}^*)^{W(E)}\,, \quad F \mapsto F|_{E_{\mathbb{C}}^*}\,,$$

is surjective, but in general $\mathrm{PW}_r(F_{\mathbb{C}}^*)^{W(F)}$ is smaller than $\mathrm{PW}_r(F_{\mathbb{C}}^*)^{W_E(F)}$, so one would in general not expect the restriction map to remain surjective. The following theorem gives a sufficient condition for that to happen.

Theorem 1.6. *Let the notation be as above. Assume that* $W_E(F)|_E = W(E)$ *and that the restriction map* $\mathrm{S}(F)^{W(F)} \to \mathrm{S}(E)^{W(E)}$ *is surjective. Then the restriction map*

$$\mathrm{PW}_r(F_{\mathbb{C}}^*)^{W(F)} \to \mathrm{PW}_r(E_{\mathbb{C}}^*)^{W(E)}\,, \text{ given by } F \mapsto F|_{E_{\mathbb{C}}^*}\,,$$

is surjective.

Proof. It is clear that if $F \in \mathrm{PW}_r(F^*_{\mathbb{C}})^{W(F)}$ then $F|_{E^*_{\mathbb{C}}} \in \mathrm{PW}_r(E^*_{\mathbb{C}})^{W(E)}$. For the surjectivity let $G \in \mathrm{PW}_r(E^*_{\mathbb{C}})^{W(E)}$. By Theorem 1.4 and our assumption on the reflection groups there exists a function $\widetilde{G} \in \mathrm{PW}_r(F^*_{\mathbb{C}})^{W_E(F)}$ such that $\widetilde{G}|_{E^*_{\mathbb{C}}} = G$. By Theorem 1.5, there exist $\Phi_1, \ldots, \Phi_n \in \mathrm{PW}_r(F^*_{\mathbb{C}})^{W(F)}$ and polynomials $P_1, \ldots, P_n \in \mathrm{S}(F)$ such that $\widetilde{G} = P_1\Phi_1 + \ldots + P_n\Phi_n$ and $G = \widetilde{G}|_{E^*_{\mathbb{C}}} = (P_1|_{E^*_{\mathbb{C}}})(\Phi_1|_{E^*_{\mathbb{C}}}) + \ldots + (P_n|_{E^*_{\mathbb{C}}})(\Phi_n|_{E^*_{\mathbb{C}}})$. As $W(E) = W_E(F)|_E$, G is $W(E)$–invariant and the functions Φ_j are $W(F)$–invariant, we can average the polynomials P_j over $W_E(F)$ and thus assume that $P_j|_{E^*_{\mathbb{C}}} \in \mathrm{S}(E)^{W(E)}$. But then there exists $Q_j \in \mathrm{S}(F)^{W(F)}$ such that $Q_j|_{E^*_{\mathbb{C}}} = P_j|_{E^*_{\mathbb{C}}}$. Let $\Phi := Q_1\Phi_1 + \ldots + Q_r\Phi_r$. Then $\Phi \in \mathrm{PW}_r(F^*_{\mathbb{C}})^{W(F)}$ and $\Phi|_{E^*_{\mathbb{C}}} = G$. Hence the restriction map is surjective. □

Let $n = \dim E$ and $m = \dim F$. Denote by $\mathcal{F}_E$ respectively $\mathcal{F}_F$ the Euclidean Fourier transforms on E and F. The following map C was denoted by P in [3].

Corollary 1.7 (Cowling)**.** *Let the assumptions be as above. Then the map*

$$C : C^\infty_r(F)^{W(F)} \to C^\infty_r(E)^{W(E)}\,, \text{ given by } Cf(x) = \int_{E^\perp} f(x, y)\, dy,$$

is surjective.

Proof. Let $c = (2\pi)^{(n-m)/2}$. For $g \in C^\infty_r(E)^{W(E)}$ let $G = \mathcal{F}_E(g) \in \mathrm{PW}_r(E^*_{\mathbb{C}})^{W(E)}$. Choice $F \in \mathrm{PW}_r(F^*_{\mathbb{C}})^{W(F)}$ such that $F|_{E^*_{\mathbb{C}}} = c^{-1}G$. With $f := \mathcal{F}_F^{-1}(G|_F) \in C^\infty_r(F)^{W(E)}$ a simple calculation shows that $C(f) = g$. □

Theorem 1.8. *Let $\{E_j\}$ be a sequence of Euclidean spaces, $E_j \subseteqq E_{j+1}$, that satisfies the hypotheses of* Theorem 1.6 *for each pair (E_j, E_k), $k \geqq j$. Denote the restriction maps by $P^k_j : \mathrm{PW}_r(E^*_{k,\mathbb{C}})^{W(E_k)} \to \mathrm{PW}_r(E^*_{j,\mathbb{C}})^{W(E_j)}$. Then $\{\mathrm{PW}_r(E^*_{j,\mathbb{C}})^{W(E_j)}, P^k_j\}$ is a projective system whose limit $P^\infty_n : \varprojlim \mathrm{PW}_r(E^*_{j,\mathbb{C}})^{W(E_j)} \to \mathrm{PW}_r(E^*_{n,\mathbb{C}})^{W(E_n)}$ is surjective for all n. In particular, $\varprojlim \mathrm{PW}_r(E^*_{j,\mathbb{C}})^{W(E_j)} \neq \{0\}$.*

Proof. It is clear that $\{\mathrm{PW}_r(E^*_{j,\mathbb{C}})^{W(E_j)}, P^k_j\}$ is a projective system. Given n and a nonzero $F \in \mathrm{PW}_r(E^*_{n,\mathbb{C}})^{W(E_n)}$, recursively choose $F_k \in \mathrm{PW}_r(E^*_{k,\mathbb{C}})^{W(E_k)}$ for $k \geqq n$ such that $F_{k+1}|_{E^*_{k,\mathbb{C}}} = F_k$. Then the sequence $\{F_k\}$ is a non-zero element of $\varprojlim \mathrm{PW}_r(E^*_{j,\mathbb{C}})^{W(E_j)}$ and $P^\infty_n(\{F_k\}) = F$. □

Theorem 1.9. *Given the conditions of* Theorem 1.8 *define $C^k_j : C^\infty_r(E_k)^{W(E_k)} \to C^\infty_r(E_j)^{W(E_j)}$ by*

$$[C^k_j(f)](x) = \int_{E_j^\perp} f(x, y)\, dy\,.$$

Then the maps C^k_j are surjective, $\{C^\infty_r(E_j)^{W(E_j)}, C^k_j\}$ is a projective system, and its limit $C^\infty_n : \varprojlim C^\infty_r(E_j)^{W(E_j)} \to C^\infty_r(E_n)^{W(E_n)}$ is surjective for all n. In particular, $\varprojlim C^\infty_r(E_j)^{W(E_j)} \neq \{0\}$.

Proof. The proof is the same as that of Theorem 1.8, making use of Corollary 1.7. □

Remark 1.10. *The last two theorems remain valid if the assumptions holds for a cofinite subsequence of $\{E_j\}_{j\in J}$.* ♢

The elements in $\varprojlim \mathrm{PW}_r(E^*_{j,\mathbb{C}})^{W(E_j)}$ can be viewed as functions on the injective limit $E^*_\infty = \varinjlim E^*_j = \bigcup E^*_j$. To see that let $F \in \varprojlim \mathrm{PW}_r(E^*_{j,\mathbb{C}})^{W(E_j)}$ and $v \in E_\infty$. Let n be such that $v \in E_n$ and define $F(v) := P^\infty_n(F)(v)$. The definition is clearly independent of n. Finally, as the Fourier transform $\mathcal{F}_j : C^\infty_r(E_j)^{W(E_j)} \to \mathrm{PW}_r(E^*_{j,\mathbb{C}})^{W(E_j)}$ is an isomorphism on each level and for $k \geqq n$ $\mathcal{F}_n \circ C^k_n = P^k_n \circ \mathcal{F}_k$, we get the following theorem:

Theorem 1.11. *There exists an unique isomorphism*

$$\mathcal{F}_\infty : \varprojlim C_r^\infty(E_j)^{W(E_j)} \to \varprojlim \mathrm{PW}_r(E_{j,\mathbb{C}}^*)^{W(E_j)}$$

such that for all n *we have* $\mathcal{F}_n \circ C_n^\infty = P_n^\infty \circ \mathcal{F}_\infty$.

2. Symmetric Spaces

In this section we apply the results of Section 1 to harmonic analysis on symmetric spaces of noncompact type. We start with some general considerations that are valid for symmetric spaces both of compact and noncompact type.

Let $M = G/K$ be a riemannian symmetric space of compact or noncompact type. Thus G is a connected semisimple Lie group with an involution θ such that

$$(G^\theta)_o \subseteqq K \subseteqq G^\theta$$

where $G^\theta = \{x \in G \mid \theta(x) = x\}$ and the subscript $_o$ denotes the connected component containing the identity element. If G is simply connected then G^θ is connected and $K = G^\theta$. If G is without compact factors and with finite center, then $K \subset G$ is a *maximal compact* subgroup of G, K is connected, and G/K is simply connected.

Denote the Lie algebra of G by $\mathfrak{g}$. Then θ defines an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ is the Lie algebra of K and $\mathfrak{s} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$.

Cartan Duality is a bijection between the classes of simply connected symmetric spaces of noncompact type and of compact type. On the Lie algebra level this isomorphism is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \leftrightarrow \mathfrak{k} \oplus i\mathfrak{s} = \mathfrak{g}^d$. We denote this bijection by $M \leftrightarrow M^d$.

Fix a maximal abelian subset $\mathfrak{a} \subset \mathfrak{s}$. For $\alpha \in \mathfrak{a}_\mathbb{C}^*$ let

$$\mathfrak{g}_{\mathbb{C},\alpha} = \{X \in \mathfrak{g}_\mathbb{C} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}_\mathbb{C}\}\,.$$

If $\mathfrak{g}_{\mathbb{C},\alpha} \neq \{0\}$ then α is called a (restricted) root. Denote by $\Sigma(\mathfrak{g}, \mathfrak{a})$ the set of roots. If M is of noncompact type, then $\Sigma(\mathfrak{g}, \mathfrak{a}) \subset \mathfrak{a}^*$ and $\mathfrak{g}_{\mathbb{C},\alpha} = \mathfrak{g}_\alpha + i\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \mathfrak{g}_{\mathbb{C},\alpha} \cap \mathfrak{g}$. If M is of compact type, then the roots are purely imaginary on $\mathfrak{a}$, $\Sigma(\mathfrak{g}, \mathfrak{a}) \subset i\mathfrak{a}^*$, and $\mathfrak{g}_{\mathbb{C},\alpha} \cap \mathfrak{g} = \{0\}$. The set of roots is preserved under duality, $\Sigma(\mathfrak{g}, \mathfrak{a}) = \Sigma(\mathfrak{g}^d, i\mathfrak{a})$, where we view those roots as $\mathbb{C}$–linear functionals on $\mathfrak{a}_\mathbb{C}$.

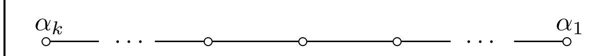
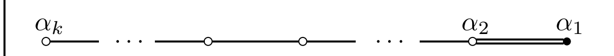
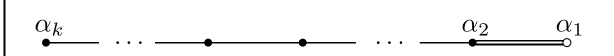
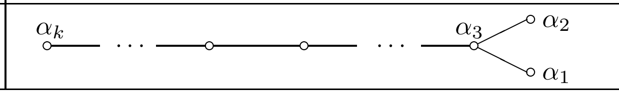
If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ it can happen that $\frac{1}{2}\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ or $2\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ (but not both). Define

$$\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \tfrac{1}{2}\alpha \notin \Sigma(\mathfrak{g}, \mathfrak{a})\}\,.$$

Then $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$ is a root system in the usual sense and the Weyl group corresponding to $\Sigma(\mathfrak{g}, \mathfrak{a})$ is the same as the Weyl group generated by the reflections s_α, $\alpha \in \Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. Furthermore, M is irreducible if and only if $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$ is irreducible, i.e., can not be decomposed into two mutually orthogonal root systems.

Let $\Sigma^+(\mathfrak{g}, \mathfrak{a}) \subset \Sigma(\mathfrak{g}, \mathfrak{a})$ be a positive system and $\Sigma^+_{1/2}(\mathfrak{g}, \mathfrak{a}) = \Sigma^+(\mathfrak{g}, \mathfrak{a}) \cap \Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. Then $\Sigma^+_{1/2}(\mathfrak{g}, \mathfrak{a})$ is a positive system in $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. Denote by $\Psi_{1/2}(\mathfrak{g}, \mathfrak{a}) = \{\alpha_1, \ldots, \alpha_r\}$, $r = \dim \mathfrak{a}$, the set of simple roots in $\Sigma^+_{1/2}(\mathfrak{g}, \mathfrak{a})$. Then $\Psi_{1/2}(\mathfrak{g}, \mathfrak{a})$ is a basis for $\Sigma(\mathfrak{g}, \mathfrak{a})$. We will always assume that $\Psi_{1/2}$ is not one of the exceptional root system and we number the simple roots in the following way:

(2.1)

$\Psi_{1/2} = A_k$		$k \geqq 1$
$\Psi_{1/2} = B_k$		$k \geqq 2$
$\Psi_{1/2} = C_k$		$k \geqq 3$
$\Psi_{1/2} = D_k$		$k \geqq 4$

Later on we will also need the root system $\Sigma_2(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid 2\alpha \notin \Sigma(\mathfrak{g}, \mathfrak{a})\}$. Following the above discussion, this will only change the simple root at the right end of the Dynkin diagram. If $\Psi_2(\mathfrak{g}, \mathfrak{a})$ is of type B the root system $\Sigma_2(\mathfrak{g}, \mathfrak{a})$ will be of type C.

The classical irreducible symmetric spaces are given by the following table.[1] The fifth column lists K as a subgroup of the compact real form. The second column indicates the type of the root system $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$.

(2.2)

Irreducible Riemannian Symmetric $M = G/K$, G classical, K connected						
		G noncompact	G compact	K	Rank$M$	Dim$M$
1	A_j	$\mathrm{SL}(j, \mathbb{C})$	$\mathrm{SU}(j) \times \mathrm{SU}(j)$	$\mathrm{diag}\,\mathrm{SU}(j)$	$j-1$	j^2-1
2	B_j	$\mathrm{SO}(2j+1, \mathbb{C})$	$\mathrm{SO}(2j+1) \times \mathrm{SO}(2j+1)$	$\mathrm{diag}\,\mathrm{SO}(2j+1)$	j	$2j^2+j$
3	D_j	$\mathrm{SO}(2j, \mathbb{C})$	$\mathrm{SO}(2j) \times \mathrm{SO}(2j)$	$\mathrm{diag}\,\mathrm{SO}(2j)$	j	$2j^2-j$
4	C_j	$\mathrm{Sp}(j, \mathbb{C})$	$\mathrm{Sp}(j) \times \mathrm{Sp}(j)$	$\mathrm{diag}\,\mathrm{Sp}(j)$	j	$2j^2+j$
5	$AIII$	$\mathrm{SU}(p, q)$	$\mathrm{SU}(p+q)$	$\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$	$\min(p, q)$	$2pq$
6	AI	$\mathrm{SL}(j, \mathbb{R})$	$\mathrm{SU}(j)$	$\mathrm{SO}(j)$	$j-1$	$\frac{(j-1)(j+2)}{2}$
7	AII	$\mathrm{SU}^*(2j)$	$\mathrm{SU}(2j)$	$\mathrm{Sp}(j)$	$j-1$	$2j^2-j-1$
8	BDI	$\mathrm{SO}_o(p, q)$	$\mathrm{SO}(p+q)$	$\mathrm{SO}(p) \times \mathrm{SO}(q)$	$\min(p, q)$	pq
9	$DIII$	$\mathrm{SO}^*(2j)$	$\mathrm{SO}(2j)$	$\mathrm{U}(j)$	$[\frac{j}{2}]$	$j(j-1)$
10	CII	$\mathrm{Sp}(p, q)$	$\mathrm{Sp}(p+q)$	$\mathrm{Sp}(p) \times \mathrm{Sp}(q)$	$\min(p, q)$	$4pq$
11	CI	$\mathrm{Sp}(j, \mathbb{R})$	$\mathrm{Sp}(j)$	$\mathrm{U}(j)$	j	$j(j+1)$

Only in the following cases do we have $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a}) \neq \Sigma(\mathfrak{g}, \mathfrak{a})$:

- $AIII$ for $1 \leqq p < q$,
- CII for $1 \leqq p < q$, and
- $DIII$ for j odd.

In those three cases there is exactly one simple root with $2\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ and this simple root is at the right end of the Dynkin diagram for $\Psi_{1/2}(\mathfrak{g}, \mathfrak{a})$. Also, either $\Psi_{1/2}(\mathfrak{g}, \mathfrak{a}) = \{\alpha\}$ contains one simple root or $\Psi_{1/2}(\mathfrak{g}, \mathfrak{a})$ is of type B_r where $r = \dim \mathfrak{a}$ is the rank of M.

Finally, the only two cases where $\Psi_{1/2}(\mathfrak{g}, \mathfrak{a})$ is of type D are the case $\mathrm{SO}(2j, \mathbb{C})/\mathrm{SO}(2j)$ or the split case $\mathrm{SO}_o(p, p)/\mathrm{SO}(p) \times \mathrm{SO}(p)$.

[1] More detailed information is given by the Satake–Tits diagram for M; see [1] or [9, pp. 530–534]. In that classification the case $\mathrm{SU}(p, 1)$, $p \geqq 1$, is denoted by AIV, but here it appears in $AIII$. The case $\mathrm{SO}(p, q)$, $p+q$ odd, $p \geqq q > 1$, is denoted by BI as in this case the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(p+q, \mathbb{C})$ is of type B. The case $\mathrm{SO}(p, q)$, with $p+q$ even, $p \geqq q > 1$ is denoted by DI as in this case $\mathfrak{g}_{\mathbb{C}}$ is of type D. Finally, the case $\mathrm{SO}(p, 1)$, p even, is denoted by BII and $\mathrm{SO}(p, 1)$, p odd, is denoted by DII.

Let $M_k = G_k/K_k$ and $M_n = G_n/K_n$ be irreducible symmetric spaces, both of compact type or both of noncompact type. We write Σ_n, Σ_n^+ and W_n for $\Sigma(\mathfrak{g}_n, \mathfrak{a}_n)$, $\Sigma^+(\mathfrak{g}_n, \mathfrak{a}_n)$ and $W(\mathfrak{g}_n, \mathfrak{a}_n)$. We say that M_k *propagates* M_n, if $G_n \subsetneqq G_k$, $K_n = K_k \cap G_n$, and either $\mathfrak{a}_k = \mathfrak{a}_n$ or choosing $\mathfrak{a}_n \subsetneqq \mathfrak{a}_k$ we only add simple roots to the left end of the Dynkin diagram for $\Psi_{n,1/2}$ to obtain the Dynkin diagram for $\Psi_{k,1/2}$. So, in particular $\Psi_{n,1/2}$ and $\Psi_{k,1/2}$ are of the same type. In general, if M_k and M_n are riemannian symmetric spaces of compact or noncompact type, with universal covering $\widetilde{M_k}$ respectively $\widetilde{M_n}$, then M_k *propagates* M_n if we can enumerate the irreducible factors of $\widetilde{M}_k = M_k^1 \times \ldots \times M_k^j$ and $\widetilde{M}_n = M_n^1 \times \ldots \times M_n^i$, $i \leqq j$ so that M_k^s propagates M_n^s for $s = 1, \ldots, i$. Thus, each M_n is, up to covering, a product of irreducible factors listed in Table 2.2.

In general we can construct infinite sequences of propagations by moving along each row in Table 2.2. But there are also inclusions like $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n) \subset \mathrm{SL}(k, \mathbb{C})/\mathrm{SU}(k)$ which satisfy the definition of propagation.

When $\mathfrak{g}_k$ propagates $\mathfrak{g}_n$, and θ_k and θ_n are the corresponding involutions with $\theta_k|_{\mathfrak{g}_n} = \theta_n$, the corresponding eigenspace decompositions $\mathfrak{g}_k = \mathfrak{k}_k \oplus \mathfrak{s}_k$ and $\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{s}_n$ give us

$$\mathfrak{k}_n = \mathfrak{k}_k \cap \mathfrak{g}_n\,, \quad \text{and} \quad \mathfrak{s}_n = \mathfrak{g}_n \cap \mathfrak{s}_k\,.$$

We recursively choose maximal commutative subspaces $\mathfrak{a}_k \subset \mathfrak{s}_k$ such that $\mathfrak{a}_n \subsetneqq \mathfrak{a}_k$ for $k \geqq n$. Assume for the moment that M_j is irreducible. Define an extended Weyl group $\widetilde{W}_n = \widetilde{W}(\mathfrak{g}_n, \mathfrak{a}_n)$ in the following way. If $\Psi_{n,1/2}$ is not of type D then $\widetilde{W}_n = W_n$. If $\Psi_{n,1/2}$ is of type D, then W_n is the group of permutations of $\{1, \ldots, r_n\}$, $r_n = \dim \mathfrak{a}_n$, and *even* number of sign changes. Let $\widetilde{W}_n$ be the extension of W_n by allowing all sign changes. $\widetilde{W}_n$ can be written as $W_n \rtimes \{1, \sigma\}$ where σ corresponds to the involution on the Dynkin diagram given by $\sigma(\alpha_1) = \alpha_2$, $\sigma(\alpha_2) = \alpha_1$ and $\sigma(\alpha_i) = \alpha_i$ for $i \geqq 3$. We note that $\widetilde{W}_n$ is isomorphic to the Weyl group generated by a root system of type B and hence a finite reflection group. For general symmetric spaces we define $\widetilde{W}_n$ as the product of the $\widetilde{W}$s for each irreducible factor. Let $k \geqq n$. As before we let

$$\widetilde{W}_{k,\mathfrak{a}_n} = \widetilde{W}_{\mathfrak{a}_n}(\mathfrak{g}_k, \mathfrak{a}_k) := \{w \in \widetilde{W}_k \mid w(\mathfrak{a}_n) = \mathfrak{a}_n\}\,. \tag{2.3}$$

Without loss of generality, if $\Psi_{n,1/2}$ is of type D we only consider propagation for $r_k \geqq r_n \geqq 4$. As we only add simple roots at the left end and those roots are orthogonal to α_1 and α_2 and fixed by σ_k it follows that $\sigma_k|_{\mathfrak{a}_n} = \sigma_n$.

Theorem 2.4. *Assume that M_k and M_n are symmetric spaces of compact or noncompact type and that M_k propagates M_n. Then*

$$W_{\mathfrak{a}_n}(\mathfrak{g}_k, \mathfrak{a}_k)|_{\mathfrak{a}_n} = \widetilde{W}_{\mathfrak{a}_n}(\mathfrak{g}_k, \mathfrak{a}_k)|_{\mathfrak{a}_n} = \widetilde{W}(\mathfrak{g}_n, \mathfrak{a}_n)$$

and the restriction maps are surjective:

$$\mathrm{S}(\mathfrak{a}_k)^{W_k}|_{\mathfrak{a}_n} = \mathrm{S}(\mathfrak{a}_k)^{\widetilde{W}_k}|_{\mathfrak{a}_n} = \mathrm{S}(\mathfrak{a}_n)^{\widetilde{W}_n}\,.$$

Proof. The proof is a case by case inspection of the classical root systems, see [17]. □

3. Application to Fourier Analysis on Symmetric Spaces of the noncompact Type

In this section we apply the above results to harmonic analysis. We first recall the main ingredients for the Helgason Fourier transform on a riemannian symmetric space $M = G/K$ of the noncompact type.

The material is standard and we refer to [10] for details. Retain the notation of the previous section: $\Sigma(\mathfrak{g},\mathfrak{a})$ is the set of (restricted) roots of $\mathfrak{a}$ in $\mathfrak{g}$ and $\Sigma^+(\mathfrak{g},\mathfrak{a}) \subset \Sigma(\mathfrak{g},\mathfrak{a})$ is a positive system. Let

$$\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha, \quad \mathfrak{m} = \mathfrak{z}_\mathfrak{k}(\mathfrak{a}), \quad \text{and} \quad \mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}.$$

Denote by N (respectively A) the analytic subgroup of G with Lie algebra $\mathfrak{n}$ (respectively $\mathfrak{a}$). Let $M = Z_K(\mathfrak{a})$ and $P = MAN$. Then M and P are closed subgroup of G and P is a *minimal parabolic subgroup*. Note, that we are using M in two different ways, once as the symmetric space M and also as a subgroup of G. The meaning will always be clear from the context.

We have the Iwasawa decomposition

$$G = KAN: \ C^\omega\text{–diffeomorphic to } K\times A\times N \text{ under } (k,a,n)\mapsto kan\,.$$

For $x\in G$ define $k(x)\in K$ and $a(x)\in A$ by $x\in k(x)a(x)N$. For $a\in A$ define $\log(a)\in\mathfrak{a}$ by $a=\exp(\log(a))$. Then $x\mapsto k(x)$ and $x\mapsto a(x)$ are analytic. For $\lambda\in\mathfrak{a}_\mathbb{C}^*$ let $a^\lambda := e^{\lambda(\log(a))}$. Then

$$man \mapsto \chi_\lambda(man) := a^\lambda$$

defines a character χ_λ of the group P, and χ_λ is unitary if and only if $\lambda\in i\mathfrak{a}^*$. Let $m_\alpha = \dim\mathfrak{g}_\alpha$ and

$$\rho = \tfrac{1}{2}\sum_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})} m_\alpha\,\alpha\,.$$

Denote by π_λ the representation of G induced from χ_λ. It can be realized as acting on $L^2(K/M)$ by

$$\pi_\lambda(x)f(kM) = a(x^{-1}k)^{-\lambda-\rho}f(k(x^{-1}k)M)\,.$$

The constant function $\mathbf{1}(kM)=1$ is a K-fixed vector and the corresponding spherical function is

$$\varphi_\lambda(x) = (\pi_\lambda(x)\mathbf{1},\mathbf{1}) = \int_K a(x^{-1}k)^{-\lambda-\rho}\,dk = \int_K a(xk)^{\lambda-\rho}\,dk \tag{3.1}$$

where the Haar measure dk on K is normalized by $\int_K dk = 1$. We have $\varphi_\lambda=\varphi_\mu$ if and only if $\mu\in W(\mathfrak{g},\mathfrak{a})\cdot\lambda$, and every spherical function on G is equal to some φ_λ.

The *spherical Fourier transform* on M is given by

$$\mathcal{F}(f)(\lambda) = \widehat{f}(\lambda) := \int_M f(x)\varphi_{-\lambda}(x)\,dx \qquad f\in C_c^\infty(M)^K\,.$$

The invariant measure dx on M can be normalized so that the spherical Fourier transform extends to an unitary isomorphism

$$f\mapsto\widehat{f}\,,\quad L^2(M)^K \cong L^2\left(i\mathfrak{a}^*, \tfrac{d\lambda}{\#W|c(\lambda)|^2}\right)^W$$

where $c(\lambda)$ denotes the Harish-Chandra c–function. For $f\in C_c^\infty(M)^K$ the inversion is given by

$$f(x) = \frac{1}{\#W}\int_{i\mathfrak{a}^*}\widehat{f}(\lambda)\varphi_\lambda(x)\frac{d\lambda}{|c(\lambda)|^2}\,.$$

Recall the involution σ on $\mathfrak{a}$ (and $\mathfrak{a}^*$) that corresponds to the non-trivial involution of the Dynkin diagram defined above in case $\Psi_{1/2}$ is of type D.

Lemma 3.2. *Let M be one of the irreducible symmetric spaces of type D. Then there exists an involution $\widetilde{\sigma}: G\to G$ such that*

1. *$\widetilde{\sigma}|_\mathfrak{a} = \sigma$ where by abuse of notation we write $\widetilde{\sigma}$ for $d\widetilde{\sigma}$,*
2. *$\widetilde{\sigma}$ commutes with the the Cartan involution θ, and in particular $\widetilde{\sigma}(K)=K$,*
3. *$\widetilde{\sigma}(N)=N$.*

Proof. One can prove this using a Weyl basis for $\mathfrak{g}_{\mathbb{C}}$ (see, for example, [20, page 285]). But the simplest proof is to note that we can replace $\mathrm{SO}(2j,\mathbb{C})/SO(2j)$ by $\mathrm{O}(2j,\mathbb{C})/\mathrm{O}(2j)$. Take

$$\mathfrak{a} = \left\{ \begin{pmatrix} t_1 X & & \\ & \ddots & \\ & & t_n X \end{pmatrix} \middle|\, t_1,\ldots,t_n \in \mathbb{R} \right\} \text{ where } X = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and then then $\widetilde{\sigma}$ is conjugation by $\mathrm{diag}(1,\ldots,1,-1)$. Similar construction can also be done for the other case $\mathrm{SO}_o(p,p)/\mathrm{SO}(p)\times\mathrm{SO}(p)$ by replacing $\mathrm{SO}_o(p,p)$ by $\mathrm{O}(p,p)$. □

In the general case we let $\widetilde{\sigma}$ be the identity on factors not of type D and the above constructed involution $\widetilde{\sigma}$ on factors of type D. Similar for the involution σ on $\mathfrak{a}$ and $\mathfrak{a}^*$. We need to extend K to a group $\widetilde{K}$ acting on M. In case the irreducible factor is not of type D then the corresponding $\widetilde{K}$-factor is just K and otherwise $K \rtimes \{1,\widetilde{\sigma}\}$. Note that $\widetilde{W}(\mathfrak{g},\mathfrak{a}) = N_{\widetilde{K}}(A)/Z_{\widetilde{K}}(A)$.

Theorem 3.3. *We have $\varphi_\lambda(\widetilde{\sigma}(x)) = \varphi_{\sigma(\lambda)}(x)$ and $\mathcal{F}(f\circ\widetilde{\sigma})(\lambda) = \mathcal{F}(f)(\sigma(\lambda))$ whenever $f \in C_c(M)^K$. In particular, $f \in C_c(M)^{\widetilde{K}}$ if and only if $\mathcal{F}(f)$ is σ-invariant.*

Proof. This follows from

$$\begin{aligned} k(\widetilde{\sigma}(x))a(\widetilde{\sigma}(x))n(\widetilde{\sigma}(x)) &= \sigma(x) = \widetilde{\sigma}(k(x)a(x)n(x)) \\ &= \widetilde{\sigma}(k(x))\widetilde{\sigma}(a(x))\widetilde{\sigma}(n(x)) \end{aligned}$$

and hence $a(\widetilde{\sigma}(x)) = \widetilde{\sigma}(a(x))$. The claim for the spherical function φ_λ follows now from the integral formula (3.1). That $\mathcal{F}(f\circ\widetilde{\sigma})(\lambda) = \mathcal{F}(f)(\sigma(\lambda))$ follows from the invariance of the invariant measure on M under $\widetilde{\sigma}$. The last statements follows then from the fact that the Fourier transform is injective on $C_c^\infty(M)^K$. □

Fix a positive definite K–invariant bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{s}$. It defines an invariant riemannian structure on M and hence an invariant metric $d(x,y)$. Let $x_o = eK \in M$ and for $r > 0$ denote by $B_r = B_r(x_o)$ the closed ball

$$B_r = \{x \in M \mid d(x,x_o) \leqq r\}\,.$$

Note that B_r is $\widetilde{K}$–invariant. Denote by $C_r^\infty(M)^{\widetilde{K}}$ the space of smooth $\widetilde{K}$–invariant functions on M with support in B_r. The restriction map $f \mapsto f|_A$ is a bijection from $C_r^\infty(M)^{\widetilde{K}}$ onto $C_r^\infty(A)^{\widetilde{W}}$ (using the obvious notation).

The following is a simple modification of the Paley-Wiener theorem of Helgason [8, 10] and Gangolli [5]; see [13] for a short overview.

Theorem 3.4 (The Paley-Wiener Theorem)**.** *The Fourier transform defines bijections*

$$C_r^\infty(M)^K \cong \mathrm{PW}_r(\mathfrak{a}_{\mathbb{C}}^*)^W \text{ and } C_r^\infty(M)^{\widetilde{K}} \cong \mathrm{PW}_r(\mathfrak{a}_{\mathbb{C}}^*)^{\widetilde{W}}\,.$$

Proof. This follows from the Helgason-Gangolli Paley-Wiener theorem and Theorem 3.3. □

We assume now that M_k propagates M_n, $k \geqq n$. The index j refers to the symmetric space M_j, for a function F on $\mathfrak{a}_{k,\mathbb{C}}^*$ let $P_n^k(F) := F|_{\mathfrak{a}_{n,\mathbb{C}}^*}$. We fix a compatible K–invariant inner products on $\mathfrak{s}_n$ and $\mathfrak{s}_k$, i.e., $\langle X,Y\rangle_k = \langle X,Y\rangle_n$ for all $X,Y \in \mathfrak{s}_n \subseteqq \mathfrak{s}_k$.

Theorem 3.5 (Paley-Wiener Isomorphisms)**.** *Assume that M_k propagates M_n. Let $r > 0$. Then the following hold:*

(1) *The map* $P_n^k : \mathrm{PW}_r(\mathfrak{a}_{k,\mathbb{C}}^*)^{\widetilde{W}_k} \to \mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}}^*)^{\widetilde{W}_n}$ *is surjective.*

(2) *The map* $C_n^k = \mathcal{F}_n^{-1} \circ P_n^k \circ \mathcal{F}_k : C_r^\infty(M_k)^{\widetilde{K}_k} \to C_r^\infty(M_n)^{\widetilde{K}_n}$ *is surjective.*

Proof. This follows from Theorem 1.6, Theorem 2.4 and Theorem 3.4 as $\widetilde{W}$ is a finite reflection group. □

We assume now that $\{M_n, \iota_{k,n}\}$ is a injective system of symmetric spaces such that M_k propagates M_n. Here $\iota_{k,n} : M_n \to M_k$ is the injection. Let

$$M_\infty = \varinjlim M_n \,.$$

We have also, in a natural way, injective systems $\mathfrak{g}_n \hookrightarrow \mathfrak{g}_k$, $\mathfrak{k}_n \hookrightarrow \mathfrak{k}_k$, $\mathfrak{s}_n \hookrightarrow \mathfrak{s}_k$, and $\mathfrak{a}_n \hookrightarrow \mathfrak{a}_k$ giving rise to corresponding injective systems. Let

$$\mathfrak{g}_\infty := \varinjlim \mathfrak{g}_n \,, \quad \mathfrak{k}_\infty := \varinjlim \mathfrak{k}_n \,, \quad \mathfrak{s}_\infty := \varinjlim \mathfrak{s}_n \,, \quad \text{and} \qquad \mathfrak{a}_\infty := \varinjlim \mathfrak{a}_n .$$

Then $\mathfrak{g}_\infty = \mathfrak{k}_\infty \oplus \mathfrak{s}_\infty$ is the eigenspace decomposition of $\mathfrak{g}_\infty$ with respect to the involution $\theta_\infty := \varinjlim \theta_n$, $\mathfrak{a}_\infty$ is a maximal abelian subspace of $\mathfrak{s}_\infty$.

The restriction maps $\mathrm{res}_n^k : \mathrm{S}(\mathfrak{a}_k)^{\widetilde{W}_k} \to \mathrm{S}(\mathfrak{a}_n)^{\widetilde{W}_n}$ and the maps from Theorem 3.5 define projective systems $\{\mathrm{S}(\mathfrak{a}_n)^{\widetilde{W}_n}\}_n$, $\{\mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}}^*)^{\widetilde{W}_n}\}_n$, and $\{C_r(M_n)^{\widetilde{K}_n}\}_n$.

Write $\Psi_{n,1/2} = \{\alpha_{n,1}, \ldots, \alpha_{n,r_n}\}$. There is a canonical inclusion $\widetilde{W}_n \overset{\iota_{k,n}}{\hookrightarrow} \widetilde{W}_{k,\mathfrak{a}_n}$ given by $s_{\alpha_{n,j}} \mapsto s_{\alpha_{k,j}}$, $1 \leqq j \leqq r_n$ and $\sigma_n \mapsto \sigma_k$. This map can also be constructed by realizing the extended Weyl groups as permutation group extended by sign changes. We have $\iota_{k,n}(s)|_{\mathfrak{a}_n} = s$. In this way, we get an injective system $\{\widetilde{W}(\mathfrak{g}_n, \mathfrak{a}_n)\}_n$. We also have a natural injective system $\{\widetilde{K}_n\}$. The restriction maps $\mathfrak{a}_{k,\mathbb{C}}^* \to \mathfrak{a}_{n,\mathbb{C}}^*$ lead to a projective system. Let $\mathfrak{a}_{\infty,\mathbb{C}}^* := \varprojlim \mathfrak{a}_{n,\mathbb{C}}^*$ and set

$$\begin{aligned}
\widetilde{W}_\infty &:= \varinjlim \widetilde{W}_n \\
\widetilde{K}_\infty &:= \varinjlim \widetilde{K}_n \\
\mathrm{S}_\infty(\mathfrak{a}_\infty)^{\widetilde{W}_\infty} &:= \varprojlim \mathrm{S}(\mathfrak{a}_n)^{\widetilde{W}_n} \\
\mathrm{PW}_r(\mathfrak{a}_{\infty,\mathbb{C}}^*)^{\widetilde{W}_\infty} &:= \varprojlim \mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}}^*)^{\widetilde{W}_n} \\
C_r^\infty(M_\infty)^{\widetilde{K}_\infty} &:= \varprojlim C_r^\infty(M_n)^{\widetilde{K}_n} \,.
\end{aligned}$$

We can view $\mathrm{S}_\infty(\mathfrak{a}_\infty)^{\widetilde{W}_\infty}$ as $\widetilde{W}_\infty$–invariant polynomials on $\mathfrak{a}_{\infty,\mathbb{C}}^*$ and $\mathrm{PW}_r(\mathfrak{a}_{\infty,\mathbb{C}}^*)^{\widetilde{W}_\infty}$ as $\widetilde{W}_\infty$–invariant functions on $\mathfrak{a}_{\infty,\mathbb{C}}^*$. The projective limit $C_{r,\infty}^\infty(M_\infty)^{K_\infty}$ consists of functions on on $A_\infty = \varinjlim A_n$, where $A_n = \exp \mathfrak{a}_n$. In Section 8 we discuss a direct limit function space on M_∞ that is more closely related to the representation theory of G_∞.

For $\mathbf{f} = (f_n)_n \in C_{r,\infty}^\infty(M_\infty)^{K_\infty}$ define $\mathcal{F}_\infty(\mathbf{f}) \in \mathrm{PW}_r(\mathfrak{a}_{\infty,\mathbb{C}}^*)^{\widetilde{W}_\infty}$ by

$$\mathcal{F}_\infty(\mathbf{f}) := \{\mathcal{F}_n(f_n)\} \,. \tag{3.6}$$

Then $\mathcal{F}_\infty(\mathbf{f})$ is well defined by Theorem 3.5 and we have a commutative diagram

$$\begin{tikzcd}
\cdots & C_r^\infty(M_n)^{\widetilde{K}_n} \arrow[d, "\mathcal{F}_n"'] & C_r^\infty(M_{n+1})^{\widetilde{K}_{n+1}} \arrow[l, "C_n^{n+1}"'] \arrow[d, "\mathcal{F}_{n+1}"'] & {} \arrow[l, "C_{n+1}^{n+2}"'] & \cdots & C_r^\infty(M_\infty)^{\widetilde{K}_\infty} \arrow[d, "\mathcal{F}_\infty"'] \\
\cdots & \mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}}^*)^{\widetilde{W}_n} & \mathrm{PW}_r(\mathfrak{a}_{n+1,\mathbb{C}}^*)^{\widetilde{W}_{n+1}} \arrow[l, "P_n^{n+1}"] & {} \arrow[l, "P_{n+1}^{n+2}"] & \cdots & \mathrm{PW}_r(\mathfrak{a}_{\infty,\mathbb{C}}^*)^{\widetilde{W}_\infty}
\end{tikzcd}$$

Then the maps

$$C_n^\infty : C_r^\infty(M_\infty)^{\widetilde{K}_\infty} \to C_r^\infty(M_n)^{\widetilde{K}_n} \text{ and } P_n^\infty : \mathrm{PW}_r(\mathfrak{a}_{\infty,\mathbb{C}}^*)^{\widetilde{W}_\infty} \to \mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}}^*)^{\widetilde{W}_n}$$

are well defined.

Theorem 3.7 (Infinite dimensional Paley-Wiener Theorem)**.** *Let the notation be as above. Then the projection maps C_n^∞ and P_n^∞ are surjective. In particular, $C_r^\infty(M_\infty)^{\widetilde{K}_\infty} \neq \{0\}$ and $\mathrm{PW}_r(\mathfrak{a}_{\infty,\mathbb{C}}^*)^{\widetilde{W}_\infty} \neq \{0\}$. Furthermore,*

$$\mathcal{F}_\infty : C_r^\infty(M_\infty)^{\widetilde{K}_\infty} \to \mathrm{PW}_r(\mathfrak{a}_{\infty,\mathbb{C}}^*)^{\widetilde{W}_\infty}$$

is a linear isomorphism.

4. Central Functions on Compact Lie Groups

The following results on compact Lie groups are a special case of the more general statements on compact symmetric spaces discussed in the next section, as every group can be viewed as a symmetric space $G \times G/\mathrm{diag}(G)$ via the map

$$(g,1)\mathrm{diag}(G) \mapsto g, \text{ in other words } (a,b)\mathrm{diag}(G) \mapsto ab^{-1}$$

corresponding to the involution $\tau(a,b) = (b,a)$. The action of $G\times G$ is the left-right action $(L\times R)(a,b)\cdot x = axb^{-1}$ and the $\mathrm{diag}(G)$–invariant functions are the *central* functions $f(axa^{-1}) = f(x)$ for all $a, x \in G$. Thus f is central if and only if $f \circ \mathrm{Ad}(a) = f$ for all $a \in G$, where as usual $\mathrm{Ad}(a)(x) = axa^{-1}$. But it is still worth treating this case separately, first because the normalization of the Fourier transform on G viewed as a group is different from the normalization as a symmetric space, and second because the proof of the Paley-Wiener Theorem for compact symmetric spaces in [14] was by reduction to this case, as was originally done in [6].

In this section G, G_n and G_k will denote compact connected semisimple Lie groups. For simplicity, we will assume that those groups are simply connected. For the general case one needs to change the semi-lattice of highest weights of irreducible representations and the injectivity radius, whose numerical value does not play an important rule in the following. The invariant measures on compact groups and homogeneous spaces are normalized to total mass one.

We say that G_k propagates G_n if $\mathfrak{g}_k$ propagates $\mathfrak{g}_n$. This is the same as saying that G_k propagates G_n as a symmetric space. We fix a Cartan subalgebra $\mathfrak{h}_k$ of $\mathfrak{g}_k$ such that $\mathfrak{h}_n := \mathfrak{h}_k \cap \mathfrak{g}_n$ is a Cartan subalgebra of $\mathfrak{g}_n$. We use the notation from the previous section. The index n respectively k will then denote the corresponding object for G_n respectively G_k. We fix inner products $\langle\cdot,\cdot\rangle_n$ on $\mathfrak{g}_n$ and $\langle\cdot,\cdot\rangle_k$ on $\mathfrak{g}_k$ such that $\langle X, Y\rangle_n = \langle X, Y\rangle_k$ for $X, Y \in \mathfrak{g}_n \subseteqq \mathfrak{g}_k$. This can be done by viewing $G_n \subset G_k$ as locally isomorphic to linear groups and use the trace form $X, Y \mapsto -\mathrm{Tr}\,(XY)$. We denote by R the injectivity radius. Theorem 4.1 below shows that the injectivity radius is the same for G_n and G_k.

The following is a reformulation of results of Crittenden [4]. A case by case inspection of each of the root systems gives us

Theorem 4.1. *The injectivity radius of the classical compact simply connected Lie groups G, in the riemannian metric given by the inner product $\langle X, Y\rangle = -\mathrm{Tr}\,(XY)$ on $\mathfrak{g}$, is $\sqrt{2}\,\pi$ for $SU(m+1)$ and $Sp(m)$, 2π for $SO(2m)$ and $SO(2m+1)$. In particular for each of the four classical series the injectivity radius R is independent of m.*

Denote by $\Lambda^+(G) \subset i\mathfrak{h}^*$ the set of dominant integral weights,

$$\Lambda^+(G) = \left\{ \mu \in i\mathfrak{h}^*) \;\middle|\; \tfrac{2\langle \mu,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \right\} .$$

For $\mu \in \Lambda^+(G)$ denote by π_μ the corresponding representation with highest weight μ. As G is assumed simply connected $\mu \mapsto \pi_\mu$, is a bijection from $\Lambda^+(G)$ onto $\widehat{G}$. The representation space for π_μ is denoted by V_μ. Let $\chi_\mu = \mathrm{Tr} \circ \pi_\mu$ be the character of π_μ and $\deg(\mu) = \dim V_\mu$ its dimension. Then $\deg(\mu)$ is a polynomial function on $\mathfrak{h}_{\mathbb{C}}^*$. The space $L^2(G)^G := \{f \in L^2(G) \mid f \circ \mathrm{Ad}(g) = f \text{ for all } g \in G\}$ contains the set $\{\chi_\mu\}_{\mu \in \Lambda^+(G)}$ of characters as a complete orthonormal set.

For $f \in C(G)^G$ define the Fourier transform $\mathcal{F}(f) = \widehat{f} : \Lambda^+(G) \to \mathbb{C}$ by

$$\widehat{f}(\mu) = (f, \chi_\mu) = \int_G f(x)\overline{\chi_\mu(x)}\, dx, \quad \mu \in \Lambda^+(G)\,,$$

where (f, χ_μ) is the inner product in $L^2(G)$. The Fourier transform extends to an unitary isomorphism $\mathcal{F} : L^2(G)^G \to \ell^2(\Lambda^+(G))$ and

$$f = \sum_{\mu \in \Lambda^+(G)} \widehat{f}(\mu)\chi_\mu$$

in $L^2(G)^G$. If f is smooth the Fourier series converges in the topology of $C^\infty(G)^G$.

If not otherwise stated we will assume that G does not contain any simple factor of exceptional type. As before $W(\mathfrak{g}, \mathfrak{h})$ denotes the Weyl group of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, and $\widetilde{W} = \widetilde{W}(\mathfrak{g}, \mathfrak{h})$ denotes the extension of $W(\mathfrak{g}, \mathfrak{h})$ by σ. Similarly, $\widetilde{K}$ and $\widetilde{G}$ denote the extensions of K and G, respectively, by $\widetilde{\sigma}$. For $r > 0$ let $\mathrm{PW}_r^\rho(\mathfrak{h}_{\mathbb{C}}^*)^{\widetilde{W}}$ denote the space of holomorphic functions Φ on $\mathfrak{h}_{\mathbb{C}}^*$ such that

(1) For each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that
$$|\Phi(\lambda)| \leqq C_k(1 + |\lambda|)^{-k} e^{r|\mathrm{Re}\lambda|} \text{ for all } \lambda \in \mathfrak{h}_{\mathbb{C}}^*,$$
(2) $\Phi(w(\lambda + \rho) - \rho) = \det(w)\Phi(\lambda)$ for all $w \in \widetilde{W}$, $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$.

Let $H = \exp(\mathfrak{h})$. For $0 < r < R$ denote by $C_r^\infty(G)^{\widetilde{G}}$ the space of smooth function on G that are invariant under conjugation by $\widetilde{G}$ and are supported in the closed geodesic ball $B_r(e)$ of radius r. We have that $f \in C_r^\infty(G)^{\widetilde{G}}$ if and only if $f|_H \in C_r^\infty(H)^{\widetilde{W}}$. In this terminology the theorem of Gonzalez [6] reads as follows.

Theorem 4.2. *Let G be an arbitrary connected simply connected compact Lie group. Let $0 < r < R$ and let $f \in C^\infty(G)^G$ be given. Then f belongs to $C_r^\infty(G)^{\widetilde{G}}$ if and only if the Fourier transform $\mu \mapsto \widehat{f}(\mu)$ extends to a holomorphic function Φ_f on $\mathfrak{h}_{\mathbb{C}}^*$ such that $\Phi_f \in \mathrm{PW}_r^\rho(\mathfrak{h}_{\mathbb{C}}^*)^{\widetilde{W}}$.*

Proof. We only have to check that $f \in C_{r,\widetilde{W}}^\infty(G)^G$ if and only if $\widehat{f}(w(\mu + \rho) - \rho) = \widehat{f}(\mu)$. For factors not of type D_n that follows from Gonzalez's theorem. For factors of type D_n it follows Weyl's character formula. □

In [14] it is shown that the extension Φ_f is unique whenever r is sufficiently small. In that case Fourier transform, followed by holomorphic extension, is a bijection $C_r^\infty(G)^{\widetilde{G}} \cong \mathrm{PW}_r^\rho(\mathfrak{h}_{\mathbb{C}}^*)^{\widetilde{W}}$.

We will now extend these results to projective limits. We start with two simple lemmas.

Lemma 4.3. *Let $\Phi \in \mathrm{PW}_r^\rho(\mathfrak{h}_{\mathbb{C}}^*)^{\widetilde{W}}$. Assume that $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ is such that $\langle \lambda, \alpha \rangle = 0$ for some $\alpha \in \Delta$. Then $\Phi(\lambda - \rho) = 0$.*

Proof. Let s_α be the reflection in the hyper plane perpendicular to α. Then

$$\begin{aligned}\Phi(\lambda-\rho) &= \Phi(s_\alpha(\lambda)-\rho)\\ &= \Phi(s_\alpha(\lambda-\rho+\rho)-\rho) = \det(s_\alpha)\Phi(\lambda-\rho)\,.\end{aligned}$$

The claim now follows as $\det(s_\alpha)=-1$. □

Lemma 4.4. *Let $r>0$ and let $\widetilde{W}$ be as before. For $\Phi\in \mathrm{PW}_r^\rho(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}}$ define*

$$T(\Phi)(\lambda)=F_\Phi(\lambda):=\tfrac{\varpi(\rho)}{\varpi(\lambda)}\Phi(\lambda-\rho)\ \textit{where}\ \varpi(\lambda)=\prod_{\alpha\in\Delta^+}\langle\lambda,\alpha\rangle\,.$$

Then $T(\Phi)\in\mathrm{PW}_r(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}}$ and $T:\mathrm{PW}_r^\rho(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}}\to\mathrm{PW}_r(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}}$ is a linear isomorphism.

Proof. Let $\alpha\in\Delta^+$. Then $\lambda\mapsto\frac{1}{(\lambda,\alpha)}\Phi(\lambda)$ is holomorphic by Lemma 4.3. According to [11], Lemma 5.13 on page 288, it follows that this function is also of exponential type r. Iterating this for each root it follows that F_Φ is holomorphic of exponential type r. As $\varpi(w(\lambda))=\det(w)\varpi(\lambda)$ it follows using the same arguments as in the proof of Lemma 4.3 that F_Φ is $\widetilde{W}$–invariant. The surjectivity follow as $F\mapsto\varpi(\lambda)F(\cdot+\rho)$ maps $\mathrm{PW}_r(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}}$ into $\mathrm{PW}_r^\rho(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}}$. □

Theorem 4.5. *Let $r>0$ and assume that G_k propagates G_n. Then the map*

$$\Phi\mapsto P_n^k(\Phi):=T_n^{-1}(T_k(\Phi)|_{\mathfrak{h}_{n,\mathbb{C}}^*})=\frac{\varpi_n(\bullet)}{\varpi_n(\rho_n)}\left(\frac{\varpi_k(\rho_k)}{\varpi_k(\bullet)}\Phi(\bullet-\rho_k)|_{\mathfrak{h}_{n,\mathbb{C}}^*}\right)(\bullet+\rho_n)$$

from $\mathrm{PW}_r^{\rho_k}(\mathfrak{h}_{k,\mathbb{C}}^*)^{\widetilde{W}_k}\to\mathrm{PW}_r^{\rho_n}(\mathfrak{h}_{n,\mathbb{C}}^*)^{\widetilde{W}_n}$ is surjective.

Proof. This follows from Lemma 4.4 and Theorem 1.6. □

Recall from Theorem 4.1 that the injectivity radii R are the same for G_k and G_n. For $0<r<R$ we now define a map $C_n^k: C_r^\infty(G_k)^{\widetilde{G}_k}\to C_r^\infty(G_n)^{\widetilde{G}_n}$ by the commutative diagram using Gonzalez' theorem:

$$\begin{tikzcd} C_r^\infty(G_k)^{\widetilde{G}_k} \arrow[r, "C_n^k"] \arrow[d, "\mathcal{F}_k"'] & C_r^\infty(G_n)^{\widetilde{G}_n} \arrow[d, "\mathcal{F}_n"] \\ \mathrm{PW}_r^{\rho_k}(\mathfrak{h}_{k,\mathbb{C}})^{\widetilde{W}_k} \arrow[r, "P_n^k"'] & \mathrm{PW}_r^{\rho_n}(\mathfrak{h}_{n,\mathbb{C}})^{\widetilde{W}_n} \end{tikzcd}\ .$$

Theorem 4.6. *If G_k propagates G_n and $0<r<R$ then*

$$C_n^k: C_r^\infty(G_k)^{\widetilde{G}_k}\to C_r^\infty(G_n)^{\widetilde{G}_n}$$

is surjective.

Proof. This follows from Theorem 4.2 and Theorem 4.5. □

Theorem 4.7. *Let $r>0$ and assume that G_k propagates G_n. Then the sequences $(\mathrm{PW}_r^{\rho_n}(\mathfrak{h}_{n,\mathbb{C}}^*)^{\widetilde{W}_n},P_n^k)$ and $(C_r^\infty(G_n)^{\widetilde{G}_n},C_n^k)$ form projective systems and*

$$\mathrm{PW}_r^{\rho_\infty}(\mathfrak{h}_{\infty,\mathbb{C}})^{\widetilde{W}_\infty}:=\varprojlim \mathrm{PW}_r^{\rho_n}(\mathfrak{h}_{n,\mathbb{C}}^*)^{\widetilde{W}_n}\ \textit{and}\ C_r^\infty(G_\infty)^{\widetilde{G}_\infty}:=\varprojlim C_r^\infty(G_n)^{\widetilde{G}_n}$$

are nonzero.

Proof. This follows from Theorem 4.5 and Theorem 4.6. □

Remark 4.8. We can view elements $\Phi \in \mathrm{PW}_r^{\rho_\infty}(\mathfrak{h}_{\infty,\mathbb{C}})^{\widetilde{W}_\infty}$ as holomorphic functions on $\mathfrak{h}_{\infty,\mathbb{C}}^*$ when we view $\mathfrak{h}_{\infty,\mathbb{C}}^*$ as the spectrum of $\varprojlim \mathrm{PW}_r^{\rho_n}(\mathfrak{h}_{n,\mathbb{C}}^*)$. Furthermore, we have a commutative diagram where all maps are surjective

$$\begin{tikzcd}
\cdots & C_r^\infty(G_n)^{\widetilde{G}_n} \arrow[d, "\mathcal{F}_n"'] & C_r^\infty(G_{n+1})^{\widetilde{G}_{n+1}} \arrow[l, "C_n^{n+1}"'] \arrow[d, "\mathcal{F}_{n+1}"'] & \arrow[l, "C_{n+1}^{n+2}"'] \cdots & C_r^\infty(G_\infty)^{\widetilde{G}_\infty} \arrow[d, "\mathcal{F}_\infty"'] \\
\cdots & \mathrm{PW}_r^{\rho_n}(\mathfrak{h}_{n,\mathbb{C}}^*)^{\widetilde{W}_n} & \mathrm{PW}_r^{\rho_{n+1}}(\mathfrak{h}_{n+1,\mathbb{C}}^*)^{\widetilde{W}_{n+1}} \arrow[l, "P_n^{n+1}"] & \arrow[l, "P_{n+1}^{n+2}"] \cdots & \mathrm{PW}_r^{\rho_\infty}(\mathfrak{h}_{\mathbb{C}}^{\infty *})^{\widetilde{W}_\infty}
\end{tikzcd} .$$

$\diamondsuit$

5. Spherical Representations of Compact Groups

In the next sections we discuss theorems of Paley-Wiener type for compact symmetric spaces. We start by an overview over spherical representations, spherical functions and the spherical Fourier transform. Most of the material can be found in [22] and [23] but in part with different proofs. The notation will be as in Section 2, and G or G_n will always stand for a compact group. In particular, $M_n = G_n/K_n$ where G_n is a connected compact semisimple Lie group with Lie algebra $\mathfrak{g}_n$, which for simplicity we assume is simply connected. The result can easily be formulated for arbitrary compact symmetric spaces by following the arguments in [14]. We will assume that M_k propagates M_n. We denote by r_k and r_n the respective real ranks of M_k and M_n. As always we fix compatible K_k– and K_n–invariant inner products on $\mathfrak{s}_k$ respectively $\mathfrak{s}_n$.

As in Section 2 let $\Sigma_n = \Sigma_n(\mathfrak{g}_n, \mathfrak{a}_n)$ denote the system of restricted roots of $\mathfrak{a}_{n,\mathbb{C}}$ in $\mathfrak{g}_{n,\mathbb{C}}$. Let $\mathfrak{h}_n$ be a θ_n-stable Cartan subalgebra such that $\mathfrak{h}_n \cap \mathfrak{s}_n = \mathfrak{a}_n$. Let $\Delta_n = \Delta(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})$. Recall that $\Sigma_n \subset i\mathfrak{a}_n^*$. We choose positive subsystems Δ_n^+ and Σ_n^+ so that $\Sigma_n^+ \subseteqq \Delta_n^+|_{\mathfrak{a}_n}$, $\Delta_n^+ \subseteqq \Delta_k^+|_{\mathfrak{h}_{n,\mathbb{C}}}$, and $\Sigma_n^+ \subset \Sigma_k^+|_{\mathfrak{a}_n}$. Consider the reduced root system

$$\Sigma_{n,2} = \{\alpha \in \Sigma_n \mid 2\alpha \notin \Sigma_n\}$$

and its positive subsystem $\Sigma_{n,2}^+ := \Sigma_{n,2} \cap \Sigma_n^+$. Let

$$\Psi_{n,2} = \Psi_2(\mathfrak{g}_n, \mathfrak{a}_n) = \{\alpha_{n,1}, \ldots, \alpha_{n,r_n}\}$$

denote the set of simple roots for $\Sigma_{n,2}^+$. We note the following simple facts; they follow from the explicit realization (2.1) of the root systems discussed in [17, Lemma 1.9].

Lemma 5.1. *Suppose that the M_n are irreducible. Let $r_n = \dim \mathfrak{a}_n$, the rank of M_n. Number the simple root systems $\Psi_{n,2}$ as in (2.1). Suppose that M_k propagates M_n. If $j \leqq r_n$ then $\alpha_{k,j}$ is the unique element of $\Psi_{k,2}$ whose restriction to $\mathfrak{a}_n$ is $\alpha_{n,j}$.*

Since M_k propagates M_n each irreducible factor of M_k contains at most one simple factor of M_n. In particular if M_n is not irreducible then M_k is not irreducible, but we still can number the simple roots so that Lemma 5.1 applies.

We denote the positive Weyl chamber in $\mathfrak{a}_n$ by $\mathfrak{a}_n^+$ and similarly for $\mathfrak{a}_k$. For $\mu \in \Lambda^+(G_n)$ let

$$V_\mu^{K_n} = \{v \in V_\mu \mid \pi_\mu(k)v = v \text{ for all } k \in K_n\}.$$

We identify $i\mathfrak{a}_n^*$ with $\{\mu \in i\mathfrak{h}_n^* \mid \mu|_{\mathfrak{h}_n \cap \mathfrak{k}_n} = 0\}$ and similar for $\mathfrak{a}_n^*$ and $\mathfrak{a}_{n,\mathbb{C}}^*$. With this identification in mind set

$$\Lambda^+(G_n, K_n) = \left\{\mu \in i\mathfrak{a}_n^* \,\middle|\, \tfrac{(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+ \right\}.$$

Most of the time we will simply write Λ_n^+ instead of $\Lambda^+(G_n, K_n)$.

Since G_n is connected and M_n is simply connected it follows that K_n is connected. As K_n is compact there exists a unique G_n–invariant measure μ_{M_n} on M_n with $\mu_{M_n}(M_n) = 1$. For brevity we sometimes write dx instead of $d\mu_{M_n}$.

Theorem 5.2 (Cartan-Helgason)**.** *Assume that G_n is compact and simply connected. Then the following are equivalent.*

1. $\mu \in \Lambda_n^+$,
2. $V_\mu^{K_n} \neq 0$,
3. *π_μ is a subrepresentation of the representation of G_n on $L^2(M_n)$.*

When those conditions hold, $\dim V_\mu^{K_n} = 1$ *and π_μ occurs with multiplicity 1 in the representation of G_n on $L^2(M_n)$.*

Proof. See [10, Theorem 4.1, p. 535]. □

Remark 5.3. If G_n is compact but not simply connected one has to replace Λ_n^+ by sub semi–lattices of weights μ such that the group homomorphism $\exp(X) \mapsto e^{\mu(X)}$ is well defined on the maximal torus H_n, and then the proof of Theorem 5.2 remains valid. ♢

Define linear functionals $\xi_{n,j} \in i\mathfrak{a}_n^*$ by

$$\frac{\langle \xi_{n,i}, \alpha_{n,j}\rangle}{\langle \alpha_{n,j}, \alpha_{n,j}\rangle} = \delta_{i,j} \text{ for } 1 \leqq j \leqq r_n \quad . \tag{5.4}$$

Then for $\alpha \in \Sigma_{n,2}^+$

$$\frac{\langle \xi_{n,i}, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}^+ .$$

If $\alpha \in \Sigma^+ \setminus \Sigma_{n,2}^+$, then $2\alpha \in \Sigma_{n,2}^+$ and

$$\tfrac{\langle \xi_{n,i}, \alpha\rangle}{\langle \alpha, \alpha\rangle} = 2\tfrac{\langle \xi_{n,i}, 2\alpha\rangle}{\langle 2\alpha, 2\alpha\rangle} \in \mathbb{Z}^+ .$$

Hence $\xi_{n,i} \in \Lambda_n^+$. The weights $\xi_{n,j}$ are the *class 1 fundamental weights for* $(\mathfrak{g}_n, \mathfrak{k}_n)$. We set

$$\Xi_n = \{\xi_{n,1}, \ldots, \xi_{n,r_n}\} .$$

For $I = (k_1, \ldots, k_{r_n}) \in (\mathbb{Z}^+)^{r_n}$ define $\mu_I := \mu(I) = k_1\xi_{n,1} + \ldots + k_{r_n}\xi_{n,r_n}$.

Lemma 5.5. *If $\mu \in i\mathfrak{a}_n^*$ then $\mu \in \Lambda_n^+$ if and only if $\mu = \mu_I$ for some $I \in (\mathbb{Z}^+)^{r_n}$.*

Proof. This follows directly from the definition of $\xi_{n,j}$. □

Lemma 5.6. *Suppose that M_k is a propagation of M_n. Let $I_k = (m_1, \ldots, m_k) \in (\mathbb{Z}^+)^{r_k}$ and $\mu = \mu_{I_k}$. Then $\mu|_{\mathfrak{a}_n} \in \Lambda_n^+$. In particular $\xi_{k,j}|_{\mathfrak{a}_n} \in \Lambda_n^+$ for $1 \leqq j \leqq r_k$.*

Proof. Let $v_\mu \in V_\mu$ be a nonzero highest weight vector and $e_\mu \in V_\mu$ a K_k–fixed unit vector. Denote by $W = \langle \pi_\mu(G_n)v_\mu\rangle$ the cyclic G_n-module generated by v_μ and let $\mu_n = \mu|_{\mathfrak{a}_n}$.

Write $W = \bigoplus_{j=1}^s W_j$ with W_j irreducible. If W_j has highest weight $\nu_j \neq \mu$ then $v_\mu \perp W_j$ so $\langle \pi_\mu(G_n)v_\mu\rangle \perp W_j$, contradicting $W_j \subset W = \bigoplus W_i$. Now each W_j has highest weight μ. Write $v_\mu = v_1 + \ldots + v_s$ with $0 \neq v_j \in W_j$. As $(v_\mu, e_\mu) \neq 0$ it follows that $(v_j, e_\mu) \neq 0$ for some j. But then the projection of e_μ onto W_j is a non-zero K_n fixed vector in $W_j^{K_n} \neq 0$ and hence $\mu|_{\mathfrak{a}_n} \in \Lambda_n^+$. □

Lemma 5.7 ([22], Lemma 6)**.** *Assume that M_k is a propagation of M_n. If $1 \leqq j \leqq r_n$ then $\xi_{k,j}$ is the unique element of Ξ_k whose restriction of $\mathfrak{a}_n$ is $\xi_{n,j}$.*

Proof. This is clear when $\mathfrak{a}_k = \mathfrak{a}_n$. If $r_n < r_k$ it follows from the explicit construction of the fundamental weights for classical root system; see [7, p. 102]. $\square$

Lemma 5.8. *Assume that $\mu_k \in \Lambda_k^+$ is a combination of the first r_n fundamental weights, $\mu = \sum_{j=1}^{r_n} k_j \xi_{k,j}$. Let $\mu_n := \mu|_{\mathfrak{a}_n} = \sum_{j=1}^{r_n} k_j \xi_{n,j}$. If v is a nonzero highest weight vector in V_{μ_k} then $\langle \pi_{\mu_k}(G_n)v\rangle$ is irreducible and isomorphic to V_{μ_n}. Furthermore, π_{μ_n} occurs with multiplicity one in $\pi_{\mu_k}|_{G_n}$.*

Proof. Each G_n–irreducible summand W in $\langle \pi_{\mu_k}(G_n)v\rangle$ has highest weight μ_n. Fix one such G_n–submodule W and let $w \in W$ be a nonzero highest weight vector. Write $w = w_1 + \ldots + w_s$ where each w_j is of some $\mathfrak{h}_k$–weight $\mu_k - \sum_i k_{j,i}\beta_i$ and where each β_i is a simple root in $\Sigma^+(\mathfrak{g}_k, \mathfrak{h}_k)$ and each $k_{j,i} \in \mathbb{Z}^+$. As $\mu_k|_{\mathfrak{h}_n} = \mu_n$ it follows that $\langle \sum_i k_{j,i}\beta_i|_{\mathfrak{h}_n}, \alpha\rangle = 0$ for all $\alpha \in \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$. Thus $\sum_i k_{j,i}\beta_i|_{\mathfrak{h}_n} = 0$. In view of (2.1) each $\langle \beta_i, \alpha_j\rangle \leqq 0$ for $\alpha_j \in \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$ simple (specifically $\langle \beta_i, \alpha_j\rangle = 0$ unless $\beta_i = f_{c+1} - f_c$ and $\alpha_j = f_c - f_{c-1}$, for some c, in which case $\langle \beta_i, \alpha_j\rangle = -1$). Since every $k_{j,i} \in \mathbb{Z}^+$ now $\langle \beta_i, \alpha_j\rangle = 0$ for each $\alpha_j \in \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$ simple. Thus $\beta_i|_{\mathfrak{h}_n} = 0$.

Because of the compatibility of the positive systems $\Delta^+(\mathfrak{g}_{k,\mathbb{C}}, \mathfrak{h}_{k,\mathbb{C}})$ and $\Delta^+(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})$ there exists a $\beta \in \Delta^+(\mathfrak{g}_{k,\mathbb{C}}, \mathfrak{h}_{k,\mathbb{C}})$, $\beta|_{\mathfrak{h}_n} = 0$, such that $\mu_k - \beta$ is a weight in V_{μ_n}. Writing β as a sum of simple roots, we see that each of the simple roots has to vanish on $\mathfrak{a}_n$ and hence the restriction to $\mathfrak{a}_k$ can not contain any of the simple roots $\alpha_{k,j}$, $j = 1, \ldots, r_n$. But then β is perpendicular to the fundamental weights $\xi_{k,j}$, $j = 1, \ldots, r_n$. Hence $s_\beta(\mu_n - \beta) = \mu_n + \beta$ is also a weight, contradicting the fact that μ_n is the highest weight. (Here s_β is the reflection in the hyperplane $\beta = 0$.) This shows that π_{μ_n} can only occur once in $\langle \pi_{\mu_k}(G_n)v\rangle$. In particular, $\langle \pi_{\mu_k}(G_n)v\rangle$ is irreducible. $\square$

Lemma 5.8 allows us to form direct system of representations, as follows. For $\ell \in \mathbb{N}$ denote by $0_\ell = (0, \ldots, 0)$ the zero vector in $\mathbb{R}^\ell$. For $I_n = (k_1, \ldots, k_{r_n}) \in (\mathbb{Z}^+)^{r_n}$ let

(5.9)
- $\mu_{I,n} = \sum_{j=1}^{r_n} k_j \xi_{n,j} \in \Lambda_n^+$;
- $\pi_{I,n} = \pi_{\mu_{I,n}}$ the corresponding spherical representation;
- $V_{I,n} = V_{\mu_{I,n}}$ a fixed Hilbert space for the representation $\pi_{I,n}$;
- $v_{I,n} = v_{\mu_{I,n}}$ a highest weight unit vector in $V_{I,n}$;
- $e_{I,n} = e_{\mu_{I,n}}$ a K_n–fixed unit vector in $V_{I,n}$.

We collect our results in the following Theorem. Compare [22, Section 3].

Theorem 5.10. *Let M_k propagate M_n and let $\pi_{I,n}$ be an irreducible representation of G_n with highest weight $\mu_{I,n} \in \Lambda_n^+$. Let $I_k = (I_n, 0_{r_k - r_n})$. Then the following hold.*

(1) *$\mu_{I,k} \in \Lambda_k^+$ and $\mu_{I,k}|_{\mathfrak{a}_n} = \mu_{I,n}$.*
(2) *The G_n-submodule of $V_{I,k}$ generated by $v_{I,k}$ is irreducible.*
(3) *The multiplicity of $\pi_{I,n}$ in $\pi_{I,k}|_{G_n}$ is 1, in other words there is an unique G_n–intertwining operator $T_k^n : V_{I,n} \to V_{I,k}$ such that $T_k^n(\pi_{I,n}(g)v_{I,n}) = \pi_{I,k}(g)v_{I,k}$.*

Remark 5.11. From this point on, when $m \leqq q$ we will always assume that the Hilbert space $V_{I,m}$ is realized inside $V_{I,q}$ as $\langle \pi_{I,q}(G_m)v_{I,q}\rangle$. $\diamondsuit$

6. Spherical Fourier Analysis and the Paley-Wiener Theorem

In this section we give a short description of the spherical functions and Fourier analysis on compact symmetric spaces. Then we state and prove results for limits of compact symmetric spaces analogous to those of Section 3.

For the moment let $M = G/K$ be a compact symmetric space. We use the same notation as in the last section but without the index n. As usual we view functions on M as right K–invariant functions on G via $f(g) = f(g \cdot x_o)$, $x_o = eK$. For $\mu \in \Lambda^+$ denote by $\deg(\mu)$ the dimension of the irreducible representation π_μ. We note that $\mu \mapsto \deg(\mu)$ extends to a polynomial function on $\mathfrak{a}_{\mathbb{C}}^*$. Fix a unit K-fixed vector e_μ and define

$$\psi_\mu(g) = (e_\mu, \pi_\mu(g)e_\mu)\,.$$

Then ψ_μ is positive definite spherical function on G, and every positive definite spherical function is obtained in this way for a suitable representation π. Define

$$\ell^2_d(\Lambda^+) = \left\{ \{a_\mu\}_{\mu\in\Lambda^+} \;\middle|\; a_\mu \in \mathbb{C} \text{ and } \sum\nolimits_{\mu\in\Lambda^+} \deg(\mu)|a_\mu|^2 < \infty \right\}\,. \tag{6.1}$$

Then $\ell^2_d(\Lambda^+)$ is a Hilbert space with inner product

$$((a(\mu))_\mu, (b(\mu))_\mu) = \sum\nolimits_{\mu\in\Lambda^+} \deg(\mu)a(\mu)\overline{b(\mu)}\,.$$

For $f \in C^\infty(M)$ define the spherical Fourier transform of f, $\mathcal{S}(f) = \widehat{f} : \Lambda^+ \to \mathbb{C}$ by

$$\widehat{f}(\mu) = (f, \psi_\mu) = \int_M f(g)(\pi_\mu(g)e_\mu, e_\mu)\,dg = (\pi_\mu(f)e_\mu, e_\mu)$$

where $\pi_\mu(f)$ denotes the operator valued Fourier transform of f, $\pi_\mu(f) = \int_G f(g)\pi_\mu(g)\,dg$. Then the sequence $\mathcal{S}(f) = (\mathcal{S}(f)(\mu))_\mu$ is in $\ell^2_d(\Lambda^+(G,K))$ and $\|f\|^2 = \|\mathcal{S}(f)\|^2$. Finally, $\mathcal{S}$ extends by continuity to an unitary isomorphism

$$\mathcal{S} : L^2(M)^K \to \ell^2_d(\Lambda^+)\,.$$

We denote by $\mathcal{S}_\rho$ the map

$$\mathcal{S}_\rho(f)(\mu) = \mathcal{S}(f)(\mu - \rho)\,, \quad \mu \in \Lambda^+ + \rho\,. \tag{6.2}$$

If f is smooth, then f is given by

$$f(x) = \sum\nolimits_{\mu\in\Lambda^+} \deg(\mu)\mathcal{S}(f)(\mu)\psi_\mu(x) = \sum\nolimits_{\mu\in\Lambda^+} \deg(\mu)\mathcal{S}_\rho(f)(\mu+\rho)\psi_\mu(x)\,.$$

and the series converges in the usual Fréchet topology on $C^\infty(M)^K$. In general, the sum has to be interpreted as an L^2 limit.

Let

$$\Omega := \{X \in \mathfrak{a} \mid |\alpha(X)| < \pi/2 \text{ for all } \alpha \in \Sigma\}\,.$$

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let φ_λ denote the spherical function on the dual symmetric space of noncompact type G^d/K, where the Lie algebra of G^d is given by $\mathfrak{g}^d := \mathfrak{k} + i\mathfrak{s}$. Then φ_λ has a holomorphic extension as $K_{\mathbb{C}}$–invariant function to $K_{\mathbb{C}} \exp(2\Omega) \cdot x_o \subset G_{\mathbb{C}}/K_{\mathbb{C}}$, cf. [18, Theorem 3.15], see also [2] and [12]. Furthermore

$$\overline{\psi_\mu(x)} = \varphi_{\mu+\rho}(x^{-1}) = \varphi_{-\mu-\rho}(x)$$

for $x \in K_{\mathbb{C}} \exp(2\Omega) \cdot x_o$. We can therefore define a holomorphic function $\lambda \mapsto \mathcal{S}_\rho(f)(\lambda)$ by

$$\mathcal{S}_\rho(f)(\lambda) = \int_M f(x)\varphi_\lambda(x^{-1})\,dx \tag{6.3}$$

as long as f has support in $K_{\mathbb{C}} \exp(2\Omega) \cdot x_o$. $\mathcal{S}_\rho(f)$ is $W(\mathfrak{g}, \mathfrak{a})$ invariant and $\mathcal{S}_\rho(f)(\mu) = \mathcal{S}(f)(\mu - \rho)$ for all $\mu \in \Lambda^+(G,K) + \rho$.

Denote by R the injectivity radius of the riemannian exponential map $\mathrm{Exp} : \mathfrak{s} \to M$. Following the arguments in [4] we get:

Theorem 6.4. *The injectivity radius R of the classical compact simply connected riemannian symmetric spaces $M = G/K$, in the riemannian metric given by the inner product $\langle X, Y\rangle = -\mathrm{Tr}\,(XY)$ on $\mathfrak{s}$, depends only on the type of the restricted reduced root system $\Sigma_2(\mathfrak{g}_\mathbb{C}, \mathfrak{a}_\mathbb{C})$. It is $\sqrt{2}\,\pi$ for $\Sigma_2(\mathfrak{g}_\mathbb{C}, \mathfrak{a}_\mathbb{C})$ of type A or C and is 2π for $\Sigma_2(\mathfrak{g}_\mathbb{C}, \mathfrak{a}_\mathbb{C})$ of type B or D.*

Remark 6.5. Since Ω is given by $|\alpha(X)| < \pi/2$ and the interior of the injectivity radius disk is given by $|\alpha(X)| < 2\pi$ the set Ω is contained in the open disk in $\mathfrak{s}$ of center 0 and radius $R/4$. $\diamondsuit$

Essentially as before, B_r denotes the closed metric ball in M with center x_o and radius r, and $C_r^\infty(M)^{\widetilde{K}}$ denotes the space of $\widetilde{K}$-invariant smooth functions on M supported in B_r.

Remark 6.6. Theorem 6.7 below is, modulo a ρ-shift and $\widetilde{W}$-invariance, Theorem 4.2 and Remark 4.3 of [14]. As pointed out in [14, Remark 4.3], the known value for the constant S can be different in each part of the theorem. In Theorem6.7(1) we need that $S < R$ and the closed ball in $\mathfrak{s}$ with center zero and radius S has to be contained in $K_\mathbb{C} \exp(i\Omega) \cdot x_o$ to be able to use the estimates from [18] for the spherical functions to show that we actually end up in the Paley-Wiener space.

In Theorem 6.7(2) we need only that $S < R$. Thus the constant in (1) is smaller than the one in (2). That is used in part (3). For Theorem 6.7(4) we also need $\|X\| \leqq \pi/\|\xi_j\|$ for $j = 1, \ldots, r$. $\diamondsuit$

Theorem 6.7 (Paley-Wiener Theorem for Compact Symmetric Spaces)**.** *Let the notation be as above. Then the following hold.*

1. *There exists a constant $S > 0$ such that, for each $0 < r < S$ and $f \in C_r^\infty(M)^{\widetilde{K}}$, the ρ-shifted spherical Fourier transform $\mathcal{S}_\rho(f) : \Lambda_n^+ + \rho \to \mathbb{C}$ extends to a function in $\mathrm{PW}_r(\mathfrak{a}_\mathbb{C}^*)^{\widetilde{W}}$.*
2. *There exists a constant $S > 0$ such that if $F \in \mathrm{PW}_r(\mathfrak{a}_\mathbb{C}^*)^{\widetilde{W}}$, $0 < r < S$, the function*

$$f(x) := \sum\nolimits_{\mu\in\Lambda^+} \deg(\mu) F(\mu + \rho)\psi_\mu(x) \tag{6.8}$$

   is in $C_r^\infty(M)^{\widetilde{K}}$ and $\mathcal{S}_\rho f(\mu) = F(\mu)$.
3. *For S as in (1.) define $\mathcal{I}_\rho : \mathrm{PW}_r(\mathfrak{a}_\mathbb{C})^{\widetilde{W}} \to C_r^\infty(M)^{\widetilde{K}}$ by (6.8). Then $\mathcal{I}_\rho$ is surjective for all $0 < r < S$.*
4. *There exists a constant $S > 0$ such that for all $0 < r < S$ the map $\mathcal{S}_\rho$ followed by holomorphic extension defines a bijection $C_r(M)^{\widetilde{K}} \cong \mathrm{PW}_r(\mathfrak{a}_\mathbb{C})^{\widetilde{W}}$.*

Proof. This follows from [14], (6.3) and Theorem 3.3. $\square$

A weaker version of the following theorem was used in [14, Section 11]. It used an operator Q which we will define shortly, and some differentiation, to prove the surjectivity part of local Paley–Wiener Theorem. Denote the Fourier transform of $f \in C(G)^G$ by $\mathcal{F}(f)$. Recall the operator $T : \mathrm{PW}_r^\rho(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}(\mathfrak{g},\mathfrak{h})} \to \mathrm{PW}_r(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}(\mathfrak{g},\mathfrak{h})}$ from Theorem 4.4. Finally, for $f \in C(G)$ let $f^\vee(x) = f(x^{-1})$. Then ${}^\vee : C_r^\infty(G)^{\widetilde{G}} \to C_r^\infty(G)^{\widetilde{G}}$ is a bijection. We will identify $\mathfrak{a}_\mathbb{C}^*$ with the subspace $\{\lambda \in \mathfrak{h}_\mathbb{C}^* \mid \lambda|_{\mathfrak{h}_\mathbb{C}\cap\mathfrak{k}_\mathbb{C}} = 0\}$ without comment in the following.

Theorem 6.9. *Let $S > 0$ be as in* Theorem 6.7(1) *and let $0 < r < S$. Then the the restriction map $\mathrm{PW}_r(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}(\mathfrak{g},\mathfrak{h})} \to \mathrm{PW}_r(\mathfrak{a}_\mathbb{C}^*)^{\widetilde{W}(\mathfrak{g},\mathfrak{a})}$ is surjective. Furthermore, the map $C_r^\infty(G)^{\widetilde{G}} \to C_r^\infty(M)^{\widetilde{K}}$, given by*

$$Q(\varphi)(g \cdot x_o) = \int_K \varphi(gk)\, dk,$$

is surjective, and $\mathcal{S}_\rho \circ Q(f^\vee) = T \circ \mathcal{F}(f)$ *on* $\Lambda^+(G,K) + \rho$.

Proof. Surjectivity of the restriction map follows from Theorem 1.6 and Theorem 2.2 in [17] stating that $\widetilde{W}(\mathfrak{g},\mathfrak{h})|_{\mathfrak{a}} = \widetilde{W}(\mathfrak{g},\mathfrak{a})$ and $\mathrm{S}(\mathfrak{h})^{\widetilde{W}(\mathfrak{g},\mathfrak{h})}|_{\mathfrak{a}} = \mathrm{S}(\mathfrak{a})^{\widetilde{W}(\mathfrak{g},\mathfrak{a})}$.

Next, we have $Q(\chi_\mu^\vee)(x) = \int_K \chi_\mu(x^{-1}k)\,dk$. As $\int_K \pi_\mu(k)\,dk$ is the orthogonal projection onto V_μ^K it follows that $Q(\chi_\mu^\vee) = 0$ if $\mu \notin \Lambda^+(G,K)$ and

$$Q(\chi_\mu^\vee)(x) = (\pi_\mu(x^{-1})e_\mu, e_\mu) = (e_\mu, \pi_\mu(x)e_\mu) = \psi_\mu(x)$$

for $\mu \in \Lambda^+(G,K)$. Thus, if $f = \sum_\mu \mathcal{F}(f)(\mu)\chi_\mu$ we have

$$Q(f^\vee)(x) = \sum\nolimits_{\mu\in\Lambda^+(G,K)} \mathcal{F}(f)(\mu)\psi_\mu(x) = \sum\nolimits_{\mu\in\Lambda^+(G,K)} \deg(\mu)\tfrac{\mathcal{F}(f)(\mu)}{\deg(\mu)}\psi_\mu(x).$$

Using the Weyl dimension formula for finite dimensional representations, $\deg(\mu) = \frac{\varpi(\mu+\rho)}{\varpi(\rho)}$, we get

$$\mathcal{S}_\rho(Q(f^\vee))(\mu+\rho) = \tfrac{\varpi(\mu+\rho)}{\varpi(\rho)}\,\mathcal{F}(f)(\mu) = T(\mathcal{F}(f))|_{\mathfrak{a}}(\mu+\rho)$$

for $\mu \in \Lambda^+(G,K)$. Hence $\mathcal{S}_\rho \circ Q(f^\vee)|_{\Lambda^+(G,K)} = (T\circ\mathcal{F}(f)|_{\mathfrak{a}_\mathbb{C}})|_{\Lambda^+(G,K)}$.

Assume that $f \in C_r^\infty(G/K)^{\widetilde{K}}$. Then, by the Paley-Wiener Theorem, Theorem 6.7, there exists a $\Phi \in \mathrm{PW}_r(\mathfrak{a}_\mathbb{C}^*)^{\widetilde{W}(\mathfrak{g},\mathfrak{a})}$ such that $\Phi = \mathcal{S}_\rho(f)$ on $\Lambda^+(G,K)$. Then, by what we just proved, there exists $\Psi \in \mathrm{PW}_r(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}(\mathfrak{g},\mathfrak{h})}$ such that $\Psi|_{\mathfrak{a}_\mathbb{C}} = \Phi$. By Theorem 4.2 there exists $F \in C_r(G)^{\widetilde{G}}$ such that $T\circ\mathcal{F}(F) = \Psi$. By the above calculation we have

$$\mathcal{S}(f)(\mu) = \mathcal{S}(Q(F^\vee))(\mu) \quad \text{for all} \quad \mu \in \Lambda^+(G,K)\,.$$

As clearly $Q(F^\vee)$ is smooth, it follows that $Q(F^\vee) = f$ and hence Q is surjective. □

7. A K-invariant Domain in M and the Projective Limit

In this section we introduce an $\widetilde{K}$-invariant domain in $\mathfrak{s}$ that behaves well under propagation of symmetric spaces. We use the notation from [17] for the simple roots.

Let $\sigma = 2(\alpha_1 + \ldots + \alpha_\ell)$ where the $\alpha_j \in \Sigma_2^+$ are the simple roots. For M irreducible let

$$\begin{aligned} &\Omega^* := \Omega \text{ if } \Sigma_2 \text{ is of type } A_\ell \text{ or } C_\ell, \\ &\Omega^* := \bigcap_{w\in W} \{X \in \mathfrak{a} \mid |\sigma(w(X))| < \pi/2\} \text{ if } \Sigma_2 \text{ is of type } B_\ell \text{ or } D_\ell. \end{aligned} \tag{7.1}$$

In general, we define Ω^* to be the product of the Ω^*'s for all the irreducible factors. Then Ω^* is a convex Weyl group invariant polygon in $\mathfrak{a}$. We also have $\Omega^* = -\Omega^*$. This is easy to check and in any case will follow from our explicit description of Ω^*.

$\mathbf{A_n}$**:** We have $\mathfrak{a} = \{x \in \mathbb{R}^{n+1} \mid \sum x_j = 0\}$, $n \geqq 1$, and the roots are the $f_i - f_j : x \mapsto x_i - x_j$ for $i \neq j$. Hence

$$\Omega^* = \Omega = \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, \sum x_j = 0 \quad \text{and } |x_i - x_j| < \tfrac{\pi}{2} \text{ for } 1 \leqq i \neq j \leqq n+1 \right\}. \tag{7.2}$$

$\mathbf{B_n}$**:** We have $\mathfrak{a} = \mathbb{R}^n$, $n \geqq 2$ and $\sigma = 2(f_1 + (f_2 - f_1) + \ldots + (f_n - f_{n-1})) = 2f_n$. The Weyl group consists of all permutations and sign changes on the f_i. Hence

$$\Omega^* = \{x \in \mathbb{R}^n \mid |x_j| < \tfrac{\pi}{4} \text{ for } j = 1, \ldots, n\}\,. \tag{7.3}$$

$\mathbf{C_n}$: Again $\mathfrak{a} = \mathbb{R}^n$, $n \geqq 3$, and the roots are the $\pm(f_i \pm f_j)$ and $\pm 2f_j$. If $|x_i|, |x_j| < \pi/4$ then $|x_i \pm x_j| < \pi/2$. Hence

$$\Omega^* = \Omega = \{x \in \mathbb{R}^n \mid |x_j| < \tfrac{\pi}{4} \text{ for } j = 1, \dots, n\}\,. \tag{7.4}$$

$\mathbf{D_n}$: Also in this case $\mathfrak{a} = \mathbb{R}^n$ with $n \geqq 4$. We have $\sigma = 2(f_1+f_2+(f_2-f_1)+\ldots+(f_n-f_{n-1})) = 2(f_2+f_n)$. As the Weyl group is given by all permutations and even sign changes on the f_i, we get

$$\Omega^* = \{x \in \mathbb{R}^n \mid |x_i \pm x_j| < \tfrac{\pi}{4} \text{ for } i, j = 1, \dots, n\,,\ i \neq j\}. \tag{7.5}$$

Lemma 7.6. *We have $\Omega^* \subseteqq \Omega$.*

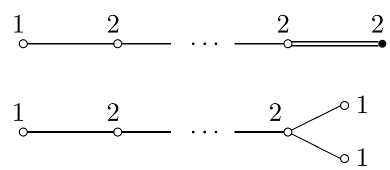
Proof. Let δ be the highest root in Σ^+. Then

$$\Omega = W\{X \in \overline{\mathfrak{a}^+} \mid \delta(X) < \pi/2\}\,.$$

For the classical Lie algebras, the coefficients of the simple roots in the highest root are all 1 or 2. Hence $\Omega^* \subseteqq \Omega$ and the claim follows. □

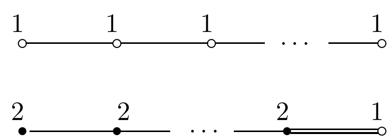
Remark 7.7. The distinction between Ω and Ω^* is caused by change in the coefficient in the highest root of the simple root on the left. Thus in cases B_n and D_n it goes from 1 to 2 as we move up in the rank of M:

B_ℓ : D_ℓ :



while in cases A_n and C_n it doesn't change:

A_ℓ : C_ℓ :



♢

Lemma 7.8. *If $S > 0$ such that $\{X \in \mathfrak{s} \mid \|X\| \leqq S\} \subset \mathrm{Ad}(K)\Omega^*\}$, then we can use S as the constant in* Theorem 6.7(1).

Proof. Recall from [14, Remark 4.3] that Theorem 6.7(1) holds when $0 < S < R$ and

$$\{X \in \mathfrak{s} \mid \|X\| \leqq S\} \subseteqq \mathrm{Ad}(K)\Omega\,. \tag{7.9}$$

But $\mathrm{Ad}(K)\Omega$ is open in $\mathfrak{s}$, and $\mathrm{Exp} : \mathrm{Ad}(K)\Omega \to M$ is injective by Theorem 6.4. Hence, if (7.9) holds then $S < R$, and the claim follows from the first part of Remark 6.6. □

We will now apply this to sequences $\{M_n\}$ where M_k is a propagation of M_n for $k \geqq n$. We use the same notation as before and add the index n (or k) to indicate the dependence of the space M_n (or M_k). We start with the following lemma.

Lemma 7.10. *If $k \geqq n$ then $\Omega_n^* = \Omega_k^* \cap \mathfrak{a}_n$.*

Proof. We can assume that M is irreducible. As M_k propagates M_n it follows that we are only adding simple roots to the left on the Dynkin diagram for Σ_2. Let r_n denote the rank of M_n and r_k the rank of M_k. We can assume that $r_n < r_k$, as the claim is obvious for $r_n = r_k$. We use the above explicit description Ω^* given above and case by case inspection:

Assume that $\Sigma_{n,2}$ is of type A_{r_n} and $\Sigma_{k,2}$ is of type A_{r_k} with $r_n < r_k$. It follows from (7.2) that $\Omega_n^* \subseteqq \Omega_k^* \cap \mathfrak{a}_n$. Let $(0,x) \in \Omega_n^*$. For $j > i$ we have

$$\pm (f_j - f_i)((0,x)) = \begin{cases} \pm(x_j - x_i) & \text{for} \quad j \leqq r_n + 1 \\ \mp(-x_i) & \text{for} \quad j > r_n + 1 \geqq i \\ 0 & \text{for} \quad j, i > r_n + 1 \end{cases} \tag{7.11}$$

Let $i \leqq r_n + 1$. Then, using that $x_i = -\sum_{j\neq i} x_j$ and $|x_i - x_j| < \pi/2$, we get

$$-r_k \tfrac{\pi}{2} < \sum_{i\neq j}(x_i - x_j) = r_k x_i - \sum_{j\neq i} x_j = (r_k + 1)x_i < r_k \tfrac{\pi}{2}\,.$$

Hence

$$-\tfrac{\pi}{2} < -\tfrac{r_k}{r_k+1}\,\tfrac{\pi}{2} < x_i < \tfrac{r_k}{r_k+1}\,\tfrac{\pi}{2} < \tfrac{\pi}{2}\,.$$

It follows now from (7.11) that $(0,x) \in \Omega_k^* \cap \mathfrak{a}_n$.

The cases of types B and C are obvious from (7.3) and (7.4). For the case of type D we note that $|x_i \pm x_j| < \frac{\pi}{4}$ implies both -$\frac{\pi}{4} < x_i - x_j < \frac{\pi}{4}$ and -$\frac{\pi}{4} < x_i + x_j < \frac{\pi}{4}$. Adding, -$\frac{\pi}{2} < 2x_i < \frac{\pi}{2}$, so $|x_i| < \frac{\pi}{4}$. Hence $(0,x) \in \Omega_k^* \cap \mathfrak{a}_n$ if and only if $x \in \Omega_n^*$ by (7.5). ☐

We can now proceed as in Section 3. We will always assume that $S > 0$ is small enough that Ω^* contains the closed ball in $\mathfrak{s}$ of radius S. Define $C_n^k : C_r^\infty(M_k)^{\widetilde{K}_k} \to C_r^\infty(M_n)^{\widetilde{K}_n}$ by $C_n^k := \mathcal{I}_{n,\rho_n} \circ P_n^k \circ \mathcal{S}_{k,\rho_k}$, in other words

$$C_n^k(f)(x) = \sum_{I\in(\mathbb{Z}^+)^{r_n}} \deg(\mu_{I,n})\widehat{f}(\mu_{I,k} - \rho_k + \rho_n)\psi_{\mu_{I,n}}(x)\,.$$

Theorem 7.12 (Paley-Wiener Isomorphism-II)**.** *If M_k propagates M_n and $0 < r < S$ then*

1. *the map $P_n^k : \mathrm{PW}_r(\mathfrak{a}_{k,\mathbb{C}}^*)^{\widetilde{W}_k} \to \mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}}^*)^{\widetilde{W}_n}$ is surjective, and*
2. *the map $C_n^k : C_r^\infty(M_k)^{\widetilde{K}_k} \to C_r^\infty(M_n)^{\widetilde{K}_n}$ is surjective.*

Proof. This follows from Theorem 1.6, Lemma 7.8, and Lemma 7.10. ☐

We now assume that $\{M_n, \iota_{k,n}\}$ is a injective system of riemannian symmetric spaces of compact type such that the direct system maps $\iota_{k,n} : M_n \to M_k$ are injections and M_k is a propagation of M_n along a cofinite subsequence. Passing to that cofinite subsequence we may assume that M_k is a propagation of M_n whenever $k \geqq n$. Denote $M_\infty = \varinjlim M_n$.

The compact symmetric spaces of Table 2.2 give rise to the following injective limits of symmetric spaces.

(7.13)

1. $\big(\mathrm{SU}(\infty)\times\mathrm{SU}(\infty)\big)/\mathrm{diag}\,\mathrm{SU}(\infty)$, group manifold $\mathrm{SU}(\infty)$,
2. $\big(\mathrm{Spin}(\infty)\times\mathrm{Spin}(\infty)\big)/\mathrm{diag}\,\mathrm{Spin}(\infty)$, group manifold $\mathrm{Spin}(\infty)$,
3. $\big(\mathrm{Sp}(\infty)\times\mathrm{Sp}(\infty)\big)/\mathrm{diag}\,\mathrm{Sp}(\infty)$, group manifold $\mathrm{Sp}(\infty)$,
4. $\mathrm{SU}(p+\infty)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(\infty))$, $\mathbb{C}^p$ subspaces of $\mathbb{C}^\infty$,
5. $\mathrm{SU}(2\infty)/[\mathrm{S}(\mathrm{U}(\infty)\times\mathrm{U}(\infty))]$, $\mathbb{C}^\infty$ subspaces of infinite codim in $\mathbb{C}^\infty$,
6. $\mathrm{SU}(\infty)/\mathrm{SO}(\infty)$, real forms of $\mathbb{C}^\infty$
7. $\mathrm{SU}(2\infty)/\mathrm{Sp}(\infty)$, quaternion vector space structures on $\mathbb{C}^\infty$,
8. $\mathrm{SO}(p+\infty)/[\mathrm{SO}(p)\times\mathrm{SO}(\infty)]$, oriented $\mathbb{R}^p$ subspaces of $\mathbb{R}^\infty$,
9. $\mathrm{SO}(2\infty)/[\mathrm{SO}(\infty)\times\mathrm{SO}(\infty)]$, $\mathbb{R}^\infty$ subspaces of infinite codim in $\mathbb{R}^\infty$,
10. $\mathrm{SO}(2\infty)/\mathrm{U}(\infty)$, complex vector space structures on $\mathbb{R}^\infty$,
11. $\mathrm{Sp}(p+\infty)/[\mathrm{Sp}(p)\times\mathrm{Sp}(\infty)]$, $\mathbb{H}^p$ subspaces of $\mathbb{H}^\infty$,
12. $\mathrm{Sp}(2\infty)/[\mathrm{Sp}(\infty)\times\mathrm{Sp}(\infty)]$, $\mathbb{H}^\infty$ subspaces of infinite codim in $\mathbb{H}^\infty$,
13. $\mathrm{Sp}(\infty)/\mathrm{U}(\infty)$, complex forms of $\mathbb{H}^\infty$.

We also have as before injective systems $\mathfrak{g}_n \hookrightarrow \mathfrak{g}_k$, $\mathfrak{k}_n \hookrightarrow \mathfrak{k}_k$, $\mathfrak{s}_n \hookrightarrow \mathfrak{s}_k$, and $\mathfrak{a}_n \hookrightarrow \mathfrak{a}_k$ giving rise to corresponding injective systems. Let

$$\mathfrak{g}_\infty := \varinjlim \mathfrak{g}_n\,,\quad \mathfrak{k}_\infty := \varinjlim \mathfrak{k}_n\,,\quad \mathfrak{s}_\infty := \varinjlim \mathfrak{s}_n\,,\quad \mathfrak{a}_\infty := \varinjlim \mathfrak{a}_n\,,\quad \text{and,}\quad \mathfrak{h}_\infty := \varinjlim \mathfrak{h}_n\,.$$

Then $\mathfrak{g}_\infty = \mathfrak{k}_\infty \oplus \mathfrak{s}_\infty$ is the eigenspace decomposition of $\mathfrak{g}_\infty$ with respect to the involution $\theta_\infty := \varinjlim \theta_n$, $\mathfrak{a}_\infty$ is a maximal abelian subspace of $\mathfrak{s}_\infty$.

Further, we have also projective systems $\{\mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}})^{\widetilde{W}_n}\}$ and $\{C_r(M_n)^{\widetilde{K}_n}\}$ with surjective projections, and their limits.

$$\mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})^{\widetilde{W}_\infty} := \varprojlim \mathrm{PW}_r(\mathfrak{a}^*_{n,\mathbb{C}})^{\widetilde{W}_n} \text{ and } C_r(M_\infty)^{\widetilde{K}_\infty} := \varprojlim C_r(M_n)^{\widetilde{K}_n}\,.$$

As before we view the elements of $\mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})^{\widetilde{W}_\infty}$ as $\widetilde{W}_\infty$–invariant functions on $\mathfrak{a}^*_{\infty,\mathbb{C}}$. For $\mathbf{f} = (f_n)_n \in C_r(M_\infty)^{\widetilde{K}_\infty}$ define $\mathcal{S}_{\rho,\infty}(\mathbf{f}) \in \mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})^{\widetilde{W}_\infty}$ by

$$\mathcal{S}_{\rho,\infty}(\mathbf{f}) := \{\mathcal{S}_{\rho,n}(f_n)\}\,. \tag{7.14}$$

Then $\mathcal{S}_{\rho,\infty}(\mathbf{f})$ is well defined by Theorem 7.12 and we have a commutative diagram

$$\begin{tikzcd}
\cdots & C_r^\infty(M_n)^{\widetilde{K}_n} \arrow[d, "\mathcal{S}_{\rho,n}"'] & C_r^\infty(M_{n+1})^{\widetilde{K}_{n+1}} \arrow[l, "C_n^{n+1}"'] \arrow[d, "\mathcal{S}_{\rho,n+1}"'] & \arrow[l, "C_{n+1}^{n+2}"'] & \cdots & C_r(M_\infty)^{\widetilde{K}_\infty} \arrow[d, "\mathcal{S}_{\rho,\infty}"'] \\
\cdots & \mathrm{PW}_r(\mathfrak{a}^*_{n,\mathbb{C}})^{\widetilde{W}_n} & \mathrm{PW}_r(\mathfrak{a}^*_{n+1,\mathbb{C}})^{\widetilde{W}_{n+1}} \arrow[l, "P_n^{n+1}"] & \arrow[l, "P_{n+1}^{n+2}"] & \cdots & \mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})^{\widetilde{W}_\infty}
\end{tikzcd}$$

Also see [15, 21] for the spherical Fourier transform and direct limits.

Theorem 7.15 (Infinite dimensional Paley-Wiener Theorem-II)**.** *In the above notation,* $\mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})^{\widetilde{W}_\infty} \neq \{0\}$, $C_r(M_\infty)^{\widetilde{K}_\infty} \neq \{0\}$, *and the spherical Fourier transform*

$$\mathcal{F}_\infty : C_r(M_\infty)^{\widetilde{K}_\infty} \to \mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})^{\widetilde{W}_\infty}$$

is injective.

8. Comparison with the L^2 Theory

Theorem 7.15 is based on limits of C^∞ and C_c^∞ spaces, rather than isometric immersions, L^2 spaces, and unitary representation theory. Just as the L^2 space of a compact symmetric space is the Hilbert space completion of the corresponding C^∞ space, it is now known [24, Proposition 3.27] that the same is true for inductive limits of compact symmetric spaces. Here we discuss those inductive limit L^2 spaces, clarifying the connection between Paley–Wiener theory and L^2 Fourier transform theory.

Any consideration of the projective limit of L^2 spaces follows similar lines by replacing the the maps of the inductive limit by the corresponding orthogonal projections, because inductive and projective limits are the same in the Hilbert space category.

The material of this section is taken from [22, Section 3] and [24, Section 3] and adapted to our setting. We assume without further comments that all extensions are propagations.

There are three steps to the comparison. First, we describe the construction of a direct limit Hilbert space $L^2(M_\infty) := \varinjlim\{L^2(M_n), L_{m,n}\}$ that carries a natural multiplicity–free unitary action of G_∞. Then we describe the ring $\mathcal{A}(M_\infty) := \varinjlim\{\mathcal{A}(M_n), \nu_{m,n}\}$ of regular functions on M_∞ where $\mathcal{A}(M_n)$ consists of the finite linear combinations of the matrix coefficients of the π_μ with $\mu \in \Lambda_n^+(G_n, K_n)$ and such that $\nu_{m,n}(f)|_{M_n} = f$. Thus $\mathcal{A}(M_\infty)$ is a (rather small) G_∞–submodule of the projective limit $\varprojlim\{\mathcal{A}(M_n),\ \text{restriction}\}$. Third, we describe a $\{G_n\}$–equivariant morphism $\{\mathcal{A}(M_n), \nu_{m,n}\} \rightsquigarrow \{L^2(M_n), L_{m,n}\}$ of direct systems that embeds $\mathcal{A}(M_\infty)$ as a dense G–submodule of $L^2(M_\infty)$, so that $L^2(M_\infty)$ is G_∞–isomorphic to a Hilbert space completion of the function space $\mathcal{A}(M_\infty)$.

We recall first some basic facts about the vector valued Fourier transform on M_n as well as the decomposition of $L^2(M_n)$ into irreducible summands. To simplify notation write Λ_n^+ for $\Lambda^+(G_n, K_n)$. Let $\mu \in \Lambda_n^+$ and let $V_{n,\mu}$ denote the irreducible G_n–module of highest weight μ. Recursively in n, we choose a highest weight vector $v_{n,\mu} \in V_{n,\mu}$ and and a K_n–invariant unit vector $e_{n,\mu} \in V_\mu^{K_n}$ such that (i) $V_{n-1,\mu} \hookrightarrow V_{n,\mu}$ is isometric and G_{n-1}–equivariant and sends $v_{n-1,\mu}$ to a multiple of $v_{n,\mu}$, (ii) orthogonal projection $V_{n,\mu} \to V_{n-1,\mu}$ sends $e_{n,\mu}$ to a non–negative real multiple $c_{n,n-1,\mu}e_{n-1,\mu}$ of $e_{n-1,\mu}$, and (iii) $\langle v_{n,\mu}, e_{n,\mu}\rangle = 1$. (Then $0 < c_{n,n-1,\mu} \leqq 1$.) Note that orthogonal projection $V_{m,\mu} \to V_{n,\mu}, m \geqq n$, sends $e_{m,\mu}$ to $c_{m,n,\mu}e_{n,\mu}$ where $c_{m,n,\mu} = c_{m,m-1,\mu} \cdots c_{n+1,n,\mu}$.

The Hermann Weyl degree formula provides polynomial functions on $\mathfrak{a}_\mathbb{C}^*$ that map μ to $\deg(\pi_{n,\mu}) = \dim V_{n,\mu}$. Earlier in this paper we had written $\deg(\mu)$ for that degree when n was fixed, but here it is crucial to track the variation of $\deg(\pi_{n,\mu})$ as n increases. Define a map $v \mapsto f_{n,\mu,v}$ from $V_{n,\mu}$ into $L^2(M_n)$ by

$$f_{n,\mu,v}(x) = \langle v, \pi_{n,\mu}(x)e_\mu\rangle\,. \tag{8.1}$$

It follows by the Frobenius–Schur orthogonality relations that $v \mapsto \deg(\pi_{n,\mu})^{1/2} f_{\mu,v}$ is a unitary G_n map from V_μ onto its image in $L^2(M_n)$.

The operator valued Fourier transform

$$L^2(G_n) \to \bigoplus_{\mu\in\Lambda_n^+} \mathrm{Hom}(V_{n,\mu}, V_{n,\mu}) \cong \bigoplus_{\mu\in\Lambda_n^+} V_{n,\mu} \otimes V_{n,\mu}^*$$

is defined by $f \mapsto \bigoplus_{\mu\in\Lambda_n^+} \pi_{n,\mu}(f)$ where $\pi_{n,\mu}(f) \in \mathrm{Hom}(V_{n,\mu}, V_{n,\mu})$ is given by

$$\pi_{n,\mu}(f)v := \int_{G_n} f(x)\pi_{n,\mu}(x)v \ \text{ for } f \in L^2(G_n)\,. \tag{8.2}$$

Denote by $P_\mu^{K_n}$ the orthogonal projection $V_{n,\mu} \to V_{n,\mu}^{K_n}$. Then $P_\mu^{K_n}(v) = \int_{K_n} \pi_{n,\mu}(k)v\,dk$, and if f is right K_n–invariant, then

$$\pi_{n,\mu}(f) = \pi_{n,\mu}(f)P_\mu^{K_n}\,.$$

That gives us the vector valued Fourier transform $f \mapsto \widehat{f} : \Lambda_n^+ \to \bigoplus_{\mu\in\Lambda_n^+} V_{n,\mu}$,

$$L^2(M_n) \to \bigoplus_{\mu\in\Lambda_n^+} V_{n,\mu} \text{ defined by } f \mapsto \widehat{f}(\mu) := \pi_{n,\mu}(f)e_{n,\mu}\,. \tag{8.3}$$

Then the Plancherel formula for $L^2(M_n)$ states that

$$f = \sum_{\mu\in\Lambda_n^+} \deg(\pi_{n,\mu}) f_{\mu,\widehat{f}(\mu)} = \sum_{\mu\in\Lambda_n^+} \deg(\pi_{n,\mu}) \langle \widehat{f}(\mu), \pi_{n,\mu}(\,\cdot\,)e_{n,\mu}\rangle \tag{8.4}$$

in $L^2(M_n)$ and

$$\|f\|_{L^2}^2 = \sum_{\mu\in\Lambda_n^+} \deg(\pi_{n,\mu}) \|\widehat{f}(\mu)\|_{HS}^2\,. \tag{8.5}$$

If f is smooth, then the series in (8.4) converges in the C^∞ topology of $C^\infty(M_n)$.

For $n \leqq m$ and $\mu = \mu_{I,n} \in \Lambda_n^+$ consider the following diagram of unitary G_n-maps, adapted from [24, Equation 3.21]:

$$\begin{array}{ccc} V_{\mu_{I,n}} & \xrightarrow{v\mapsto v} & V_{\mu_{I,m}} \\ {\scriptstyle v\mapsto \deg(\pi_{n,\mu})^{1/2} f_{\mu_{I,n},v}} \downarrow & & \downarrow {\scriptstyle v\mapsto \deg(\pi_{m,\mu})^{1/2} f_{\mu_{I,m},v}} \\ L^2(M_n) & \xrightarrow[L_{m,n}]{} & L^2(M_m) \end{array}$$

where $L_{m,n} : L^2(M_n) \to L^2(M_k)$ is the G_n–equivariant partial isometry defined by

$$L_{k,n} : \sum_{I_n} f_{\mu_{I,n},w_I} \mapsto \sum_{I_m} c_{m,n,\mu} \sqrt{\frac{\deg(\pi_{m,\mu})}{\deg(\pi_{n,\mu})}}\, f_{\mu_{I,m},w_I}\,, \quad w_I \in V_{n,\mu}\,. \tag{8.6}$$

As in [24, Section 4] we have

Theorem 8.7. *The map $L_{k,n}$ of* (8.6) *is a G_n–equivariant partial isometry with image*

$$\mathrm{Im}(L_{m,n}) \cong \bigoplus\nolimits_{I\in(\mathbb{Z}^+)^{r_k},\ k_{r_n+1}=\ldots=k_{r_k}=0} V_{\mu_I}\,.$$

If $n \leqq m \leqq k$ then

$$L_{k,n} = L_{m,n} \circ L_{k,m}$$

making $\{L^2(M_n), L_{k,n}\}$ into a direct system of Hilbert spaces.

Define

$$L^2(M_\infty) := \varinjlim L^2(M_n), \tag{8.8}$$

direct limit in the category of Hilbert spaces and unitary injections.

From construction of the $L_{m,n}$ we now have

Theorem 8.9 ([22], Theorem 13)**.** *The left regular representation of G_∞ on $L^2(M_\infty)$ is a multiplicity free discrete direct sum of irreducible representations. Specifically, that left regular representation is $\sum_{I\in\mathcal{I}} \pi_I$ where $\pi_I = \varinjlim \pi_{I,n}$ is the irreducible representation of G_∞ with highest weight $\xi_I := \sum k_r\xi_r$. This applies to all the direct systems of* (7.13).

The problem with the partial isometries $L_{m,n}$ is that they do not work well with restriction of functions, because of rescaling and because $L_{m,n}(L^2(M_n)^{K_n}) \not\subset L^2(M_m)^{K_m}$ for $n < m$. In particular the spherical functions $\psi_{I,n}(g) := \langle e_{I,n}, \pi_{I,n}(g)e_{I,n}) \rangle$ do not map forward, in other words $L_{m,n}(\psi_{I,n}) \neq \psi_{I,m}$.

We deal with this by viewing $L^2(M_\infty)$ as a Hilbert space completion of the ring $\mathcal{A}(M_\infty) := \varinjlim \mathcal{A}(M_n)$ of regular functions on M_∞. Adapting [24, Section 3] to our notation, we define

$$\begin{aligned} &\mathcal{A}(\pi_{n,\mu})^{K_n} = \{\text{finite linear combinations of the } f_{\mu,I_n,w_I} \text{ where } w_I \in V_{n,\mu}\}, \\ &\nu_{m,n,\mu} : \mathcal{A}(\pi_{n,\mu})^{K_n} \hookrightarrow \mathcal{A}(\pi_{m,\mu})^{K_m} \text{ by } f_{\mu,I_n,w_I} \mapsto f_{\mu,I_m,w_I}\ . \end{aligned} \tag{8.10}$$

Thus [24, Lemma 2.30] says that if $f \in \mathcal{A}(\pi_{n,\mu})^{K_n}$ then $\nu_{m,n,\mu}(f)|_{M_n} = f$.

The ring of regular functions on M_n is $\mathcal{A}(M_n) := \mathcal{A}(G_n)^{K_n} = \sum_\mu \mathcal{A}(\pi_{n,\mu})$, and the $\nu_{m,n,\mu}$ sum to define a direct system $\{\mathcal{A}(M_n), \nu_{m,n}\}$. Its limit is

$$\mathcal{A}(M_\infty) := \mathcal{A}(G_\infty)^{K_\infty} = \varinjlim \{\mathcal{A}(M_n), \nu_{m,n}\}. \tag{8.11}$$

As just noted, the maps of the direct system $\{\mathcal{A}(M_n), \nu_{m,n}\}$ are inverse to restriction of functions, so $\mathcal{A}(M_\infty)$ is a G_∞–submodule of the inverse limit $\varprojlim \{\mathcal{A}(M_n),\ \text{restriction}\}$.

For each n, $\mathcal{A}(M_n)$ is a dense subspace of $L^2(M_n)$ but, because the $\nu_{m,n}$ distort the Hilbert space structure, $\mathcal{A}(M_\infty)$ does not sit naturally as a subspace of $L^2(M_\infty)$. Thus we use the G_n–equivariant maps

$$\eta_{n,\mu} : \mathcal{A}(\pi_{n,\mu})^{K_n} \to \mathcal{H}_{\pi_n} \widehat{\otimes} (w_{n,\mu^*}\mathbb{C}) \text{ by } f_{\mu,I_n,w_I} \mapsto c_{n,1,\mu} \sqrt{\deg \pi_{n,\mu}}\, f_{\mu,I_n,w_I}. \tag{8.12}$$

where $c_{m,n,\mu}$ is the length of the projection of $e_{m,\mu}$ to $V_{n,\mu}$. Now [24, Proposition 3.27] says

Proposition 8.13. *The maps $L_{m,n,\mu}$ of* (8.6)*, $\nu_{m,n,\mu}$ of* (8.10) *and $\eta_{n,\mu}$ of* (8.12) *satisfy*

$$(\eta_{m,\mu} \circ \nu_{m,n,\mu})(f_{\mu,I_n,w_I}) = (L_{m,n,\mu} \circ \eta_{n,\mu})(f_{\mu,I_n,w_I})$$

for $f_{u,v,n} \in \mathcal{A}(\pi_{n,\mu})^{K_n}$. Thus they inject the direct system $\{\mathcal{A}(M_n), \nu_{m,n}\}$ into the direct system $\{L^2(M_n), L_{m,n}\}$. That map of direct systems defines a G_∞–equivariant injection

$$\widetilde{\eta} : \mathcal{A}(M_\infty) \to L^2(M_\infty)$$

with dense image. In particular η defines a pre Hilbert space structure on $\mathcal{A}(M_\infty)$ with completion isometric to $L^2(M_\infty)$.

This describes $L^2(M_\infty)$ as an ordinary Hilbert space completion of a natural function space on M_∞.

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